MATH 325 - Lecture 4

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1 Proofs Continued

1.1 Basic Rules (p.28-31)

- Use precise definitions
- Define new symbols & notation
- Include details
- Use proper English
  - Use complete sentences
  - Use correct grammar & punctuation
- Check for correctness

1.2 Grading

- Mathematical correctness & logical consistency
- Sufficient details
- Correctness of language

1.3 Main Types

- Deductive
  - Applying a general principle to a particular case
  - All men are mortal. Socrates is a man. Socrates is mortal.
- Inductive
  - Establish a general principle from individual cases
  - Usually not good for a proof [1]

[1] This refers more to inductive reasoning. Inductive proofs are fine, but inductive reasoning just generates hypothesis

Conjectures are unestablished statements. Must get proved with formal logical reasoning.

2 Example 10:

The function

$$f(n) = n^2 + n + 17$$

seems to produce primes when $n = 0, 1, 2, ...$

$$\begin{aligned}
&f(0) = 17 \\
&f(1) = 19 \\
&f(2) = 23 \\
&. \\
&. \\
&. \\
&f(10) = 127 \\
&. \\
&. \\
&. \\
&f(15) = 257
\end{aligned}$$

Induction leads to the <u>conjecture</u>

$$P : \forall n \in \mathbb{N}, f(n) \text{ is prime}$$

However, the cases observed do not prove our statement. Instead, we have shown

$$Q : \exists n = 0, 1, 2, 3... \ni f(n) \text{ is a prime}$$

It turns out that our conjecture (P) is False. To demonstrate this, we will prove its negation ($\neg P$) is True

$$\neg P : \exists n \in \mathbb{N} \ni f(n) \text{ is not prime}$$

This technique is known as a <u>counter example</u>. One can either produce one or show that it exists. Here's a counter example for P

$$f(17) = 17^2 + 17 + 17 = 17 * 19$$

3 Example 11:

Consider

$$f(m,n) = m^2 + m + n$$

It seems that $f(m, m+1)$ produces perfect squares for $1, 2, 3, ...$

$$\begin{aligned}
&f(1,2) = 4 = 2^2 \\
&f(2,3) = 9 = 3^2 \\
&. \\
&. \\
&. \\
&f(12,13) = 169 = 13^2
\end{aligned}$$

Inductive reasoning leads to the conjecture

$$P : \forall m = 1, 2, 3, ..., f(m, m+1) = (m+1)^2$$

To prove, we use general principles (deductive reasoning).

Let $m = 1, 2, 3, ...$	Given
$f(m, m+1) = m^2 + m + (m+1)$	Definition of f
$= m^2 + (2m) + 1$	Addition is associative
$= (m+1) * (m+1)$	Factoring
$= (m+1)^2$	Definition of a Square

Since m was never fixed to a particular value, the statement P is true.

4 Structure of Theorems

Most math theorems have the structure form

$$P \implies Q$$

This means that whenever P is true, Q is also True.

$$\begin{aligned}
&P \text{ is the hypothesis} \\
&Q \text{ is the conclusion}
\end{aligned}$$

What is included in P is not always uniquely identified.

5 Example 12 (Problem 1, Assignment 2):

5.1 Theorem:

Suppose that I is a closed, bounded interval in $\mathbb{R}$. Let $f : I \to \mathbb{R}$ [2] be given. If f is continous on I, then there is a number $x_0 \in I$ s.t. the minimum of f over I is $f(x_0)$.

5.2 Hypothesis:

(Notation, definitions and context of theorem)

- I is closed
- I is bounded
- I is an interval in $\mathbb{R}$
- f is an $\mathbb{R}$-valued funciton on I

(Main hypothesis)

- f is continous

5.3 Conclusion:

$$\exists x_0 \in I \ni f(x_0) \text{ is a minimum}$$

Depending on what is included in the P and Q of $P \implies Q$ affects different strategies for proof.

5.3.1 Proving the Contrapositive

$$(\neg Q \implies \neg P)$$

We want to show:
Suppose I is a closed and bounded, then
$f : I \in \mathbb{R}$
If $\forall x_0 \in I, f(x_0)$ is <u>not</u> the minimum of f, then f is <u>not</u> continous on I

[2] This means that f is an $\mathbb{R}$-valued function defined on I

5.3.2 Proof by <u>Contradiction</u>

Assume hypothesis, assume negation of conclusion. Show $((P \wedge \neg Q) \implies \text{False})$
If negation is false, then the original statement is true.
I is a closed, bounded interval and $f : I \to \mathbb{R}$ is continous on I and that for every $x_0 \in I, f(x_0)$ is not the min of f over I
Show this leads to something that is always false. (e.g. $1 > 1$, $\mathbb{R}$ is finite, I is open)