# MATH 325 - Lecture 3

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## 1 Logical Operators, Cont.

Implication: $\implies$, "IF ... THEN ..."
Biconditional: $\iff$, "...IF AND ONLY IF...", "...IFF..."
Conjunction: $\wedge$, "...AND..."
Disjunction: $\vee$, "OR"
Universal Quanitifer: $\forall$, "FOR ALL..."
Existential Quantifier: $\exists$, "THERE EXISTS..."
Unique Quantifier: $\exists!$, "THERE EXISTS ONLY ONE..."

$\forall x, \exists y \ni P(x,y) \not\Longleftrightarrow \exists y \ni \forall x \ni P(x,y)$
In $\forall x, \exists y \ni P(x,y)$ each $y$ can depend on $x$
In $\exists y \ni \forall x \ni P(x,y)$ all $x$ must be true for a single $y$

## 2 Negation With Quantifiers

Suppose that

$$\begin{aligned} &x: && \text{A person in class} \\ &P(x): && x \text{ is awake} \end{aligned}$$

$\forall x, P(x) \iff$ Everyone in class is awake.
What is the negation? $\exists x \ni \neg P(x) \iff$ Atleast one person in class is asleep.

### 2.1 Truth Table

| $\forall x, P(x)$ | $\exists x \ni \neg P(x)$ |
|---|---|
| T | F |
| F | T |

## 2.2 Negation w/ Quantifier Rules

$\neg[\forall x, P(x)] \iff \exists x \ni \neg P(x)$
$\neg[\exists x, P(x)] \iff \forall x, \neg P(x)$

## 2.3 Example 9 (Prob. 3,5)

| | |
|---|---|
| $A, B$ : | Sets of Real Numbers ($\mathbb{R}$) |
| $h$ : | Natural Number($\mathbb{N}$) |
| $f_n, f, g$ : | $\mathbb{R}$-valued functions of real numbers |

## 2.4 Notation:

Membership: $\in$, "IN"
Indicates that the set on the right contains the element on the left.

- For every $x$ in $A$, $f(x) > 8$. Let's negate that.
  $\neg[\forall x \in A, f(x) > 8] \iff \exists x \in A \ni f(x) <= 8$
  In words, There exists an $x$ in $A$ such that $f(x) <= 8$.
  Basically, you change the quantifier and negate the statement

- There is a positive real number $y$ s.t. $0 <= g(y) < 1$

| | |
|---|---|
| Statement: | $\exists y \in \mathbb{R} \ni y > 0 \wedge 0 <= g(y) < 1$ |
| Negation: | $\forall y \in \mathbb{R}, y <= 0 \vee g(y) < 0 \vee g(y) >= 1$ |
| In words: | For all real numbers $y$, either $y <= 0$ or $g(y) < 0$ or $g(y) >= 1$ |
| Alternate Statement: | $\exists y \in \mathbb{R}, y > 0, \ni 0 <= g(y) < 1$ |
| Negation: | $\forall y \ni \mathbb{R}, y > 0, g(y) < 0 \vee g(y) >= 1$ |
| In words: | For all positive real numbers $y$, either $g(y) < 0$ or $g(y) >= 1$ |

The previous example demonstrates that readers can interpret statements differently. Some readers regard the $y > 0$ as describing the universe of discourse, i.e. $\exists y \in (0, \infty)$. Another valid interpretation is that $y \in \mathbb{R}$ and $y > 0$ is part of the hypothesis. In the later case, the negation affects the $y > 0$.

- Given $\epsilon > 0$, There is an $N$ s.t. for any $x$, we find $|f_n(x) - f(x)| < \epsilon$, whenever $n > N$.

| | |
|---|---|
| Statement: | $\forall \epsilon \in \mathbb{R}, \epsilon > 0, \exists N \in \mathbb{R} \ni \forall x \in \mathbb{R}, n > N \implies \|f_n(x) - f(x)\| < \epsilon$ |
| $P(n, N):$ | $n > N$ |
| $Q(n, x, \epsilon):$ | $\|f_n(x) - f(x)\| < \epsilon$ |
| Negation: | $\exists \epsilon \in \mathbb{R}, \epsilon > 0, \ni \forall N \in \mathbb{R}, \exists x \in \mathbb{R} \ni \neg[P(n, N) \implies Q(n, x, \epsilon)]$ |
| | $\exists \epsilon \in \mathbb{R}, \epsilon > 0, \ni \forall N \in \mathbb{R}, \exists x \in \mathbb{R} \ni P(n, N) \wedge \neg Q(n, x, \epsilon)$ |
| | $\exists \epsilon \in \mathbb{R}, \epsilon > 0, \ni \forall N \in \mathbb{R}, \exists x \in \mathbb{R} \ni n > N \wedge \|f_n(x) - f(x)\| >= \epsilon$ |
| In words: | There is an $\epsilon > 0$ s.t. no matter how $N \in \mathbb{R}$ is picked, |
| | there is some $x \in \mathbb{R}$ and some $n$ s.t. $n > N$ and $\|f_n(x) - f(x)\| >= \epsilon$ |

Recall that if a variable appears in the "IF" portion of an implication and has no quantifier, use the universal quantifier.

$$P(x) \implies Q(x) \iff \forall x, P(x) \implies Q(x)$$

# 3 Proof Construction

## 3.1 Basic Rules

<u>Definition:</u> Make sure everything in the proof has a precise definition. If a new term or notation is used, define it. Ensure the 'universe' for a variable is clear.
<u>Standard Notation:</u> Try to use standard notation, ensure that expressions have an unambiguous interpretation.
<u>Provide Details:</u> The reader needs to be able to follow your reasoning. Obvious facts and well-known results can be assumed. This skill requires practice & feedback. Use proper English including complete setences and correct grammar.