MATH 325 - Lecture 16

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September 23, 2017

1 Administrative Concerns

Assignment 4 due next Wednesday, October 4.

1.1 Exam Monday, October 9:

- Problem 1 (10pts)
  - State a definition or result from class.
- Problem 2 (20pts)
  - Negate a statement
  - Write it logically
  - Explain its meaning
- Problems 3 and 4 (35pts each) (Pick 2 of 3 possible)
  - Logic, sets, functions
  - Real number system, upper/lower bounds, supremums/infinimums, completeness axiom, closure/limit and isolated pts
  - Bolzane-Weierstrass Sequences
  - Subsequences
  - Cauchy Sequences

2 Theorem (Limits Preserve Order)

Suppose $\{a_n\}\{b_n\} \subset \mathbb{R}$ are given with

$$\lim_{n\to\infty} a_n = L \wedge \lim_{n\to\infty} b_n = M$$

and

$$\forall n \in \mathbb{N}, a_n > b_n$$

then

$$L \leq M$$

2.1 Scratch Work:

Problem 3, assignment 3:

$$\forall \epsilon > 0, x < y + \epsilon \implies x \leq y$$

It is enough to show

$$\begin{aligned}
&\forall \epsilon > 0, L \leq M + \epsilon \\
&\iff L - a_n + a_n < M - b_n + b_n + \epsilon \\
&\iff a_n < b_n + (M - b_n) - (L - a_n) + \epsilon
\end{aligned}$$

We want to show that $(M - b_n) - (L - a_n) + \epsilon$ is bigger than or equal to zero.

$$\begin{aligned}
&(M - b_n) - (L - a_n) + \epsilon \geq 0 \\
&\iff (L - a_n) - (M - b_n) \leq \epsilon \\
&\impliedby |L - a_n| + |M - b_n| \leq \epsilon
\end{aligned}$$

We want to make $|L - a_n| \leq \frac{\epsilon}{2}$ and $|M - b_n| \leq \frac{\epsilon}{2}$. Choose $N \in \mathbb{N}$ such that

$$\begin{aligned}
|a_n - L| < \frac{\epsilon}{2} \\
|b_n - M| < \frac{\epsilon}{2}
\end{aligned}$$

2.2 Proof:

We will show that $\forall \epsilon > 0, L < M + \epsilon$. Then, Problem 3(a) from homework assignment 3 proves the result $L \leq M$.
Let $\epsilon > 0$ be given. Since $a_n \to L$ and $b_n \to M$ as $n \to \infty$, there is

$$\begin{aligned}
(1)\exists N_1 \in \mathbb{N} \ni n > N_1 \implies |a_n - L| < \frac{\epsilon}{2} \\
(2)\exists N_2 \in \mathbb{N} \ni n > N_2 \implies |b_n - M| < \frac{\epsilon}{2}
\end{aligned}$$

Select $N = \max(N_1, N_2)$

$\forall n > N, (1) \wedge (2)$

$$
\begin{aligned}
&|a_n - L| < \frac{\epsilon}{2} \wedge |b_n - M| < \frac{\epsilon}{2} \\
&\implies (L - a_n) + (b_n - M) < \epsilon \\
&\implies (M - b_n) - (L - a_n) + \epsilon > 0 \\
&\implies (M - b_n) - (L - a_n) + \epsilon + b_n > a_n && \text{Since } a_n \leq b_n \\
&\implies L - a_n + a_n < b_n + M - b_n + \epsilon \\
&\implies L < M + \epsilon \\
&\implies L \leq M && \text{Since } \epsilon \text{ was arbitrary, the result is proved}
\end{aligned}
$$

3 Theorem 54: (Squeeze Theorem/Sandwich Theorem)

(Theorem 2.6 in text) Suppose we have

$$\{a_n\}, \{b_n\}, \{c_n\} \subset \mathbb{R} \ni \forall n \in \mathbb{N}, a_n \leq b_n \leq c_n$$

If a_n converges to L and c_n converges to L, then b_n is convergent and $b_n \to \infty$ as $n \to \infty$.

3.1 Proof: (Sketch)

Select $N \in \mathbb{N}$ such that $\forall n > N, |a_n - L| < \epsilon$ and $|c_n - L| < \epsilon$. Then

$$
\begin{aligned}
&-\epsilon < a_n - L \leq b_n - L \leq c_n - L < \epsilon \\
&\implies |b_n - L| < \epsilon
\end{aligned}
$$

Not full proof, but underlying mechanics. Details on pg. 87 (Keep these in mind for problems 2 and 4 from the assignment)

4 Infinite Limits:

4.1 Defintion 55: (p.88)

A sequence $\{a_n\} \subset \mathbb{R}$

- Diverges to $+\infty$, denoted by $\lim_{n\to\infty} a_n = +\infty$ or $a_n \to \infty$ as $n \to \infty$ if

$$\forall M \in \mathbb{R}, \exists N \ni \forall n \in \mathbb{N}, n > N \implies a_n > M$$

- Diverges to $-\infty$, denoted by $\lim_{n\to\infty} a_n = -\infty$ or $a_n \to -\infty$ as $n \to \infty$ if

$$\forall m \in \mathbb{R}, \exists N \ni \forall n \in \mathbb{N}, n > N \implies a_n < m$$

4.2 Remark 56

- $\lim_{n\to\infty} a_n = +\infty$ and $\lim_{n\to\infty} a_n = -\infty$ indicates that the sequcence is not convergent. The sequence diverges by either growing without bound or become more negative without bounds.
- $\lim_{n\to\infty} a_n = +\infty$, for each M, there is a tail of a_n that stays above M. Similar remark for $\lim_{n\to\infty} a_n = -\infty$

5 Example 57: (Problem 2)

Find

$$\lim_{n\to\infty} \frac{n^2 - 4n}{n+3}$$

If it exists.

5.1 Initial Thoughts

As $n \to \infty, n^2 - 4n$ is dominated by n^2 and $n+3$ is dominated by n, so $\frac{n^2-4n}{n+3}$ looks like $\frac{n^2}{n} = n$ as n gets large.

5.2 Conjecture

$$\frac{n^2 - 4n}{n+3} = +\infty$$

5.3 Scratch

$$
\begin{aligned}
&\frac{n^2-4n}{n+3}\\
&=\frac{n^2-4n}{n+3}-n+n\\
&=\frac{n^2-4n-n^2-3n}{n+3}+n\\
&=\frac{-7n}{n+3}+n\\
&=-7(\frac{n}{n+3})+n\\
&\frac{n}{n+3}<1
\end{aligned}
$$

Pick $N = M + 7$, so $n > N \implies n - 7 > M$.

5.4 Proof

Let $M \in \mathbb{R}$ be given. Select $N \in mathbbN \ni N \geq M + 7$. Then

$$
\begin{aligned}
&\forall n > N, \frac{n^2-4n}{n+3}\\
&=\frac{n^2+4n}{n+3}-n+n\\
&=(-7)(\frac{n}{n+e})+n > n-7 > N-7 \geq M
\end{aligned}
$$

Since $M \in \mathbb{R}$ was arbitrary, $\lim_{n\to\infty} \frac{n^2-4n}{n+3} = +\infty$.

5.5 Useful Fact:

$$
\lim_{n\to\infty} n^\alpha = \begin{cases} +\infty, \alpha > 0\\ 1, \alpha = 0\\ 0, \alpha < 0 \end{cases}
$$

6 Theorem 58: (Monotone Convergence Theorem)

(Theorem 2.3 in the book)
Let $\{a_n\}_{n=1}^\infty \subset \mathbb{R}$ be given.
If $\{a_n\}_{n=1}^\infty$ is bounded and eventually monotone, then the sequence converges.