GNSS Introduction

Fabio Dovis

Politecnico di Torino - Electronics Department

Part II

Basic principles

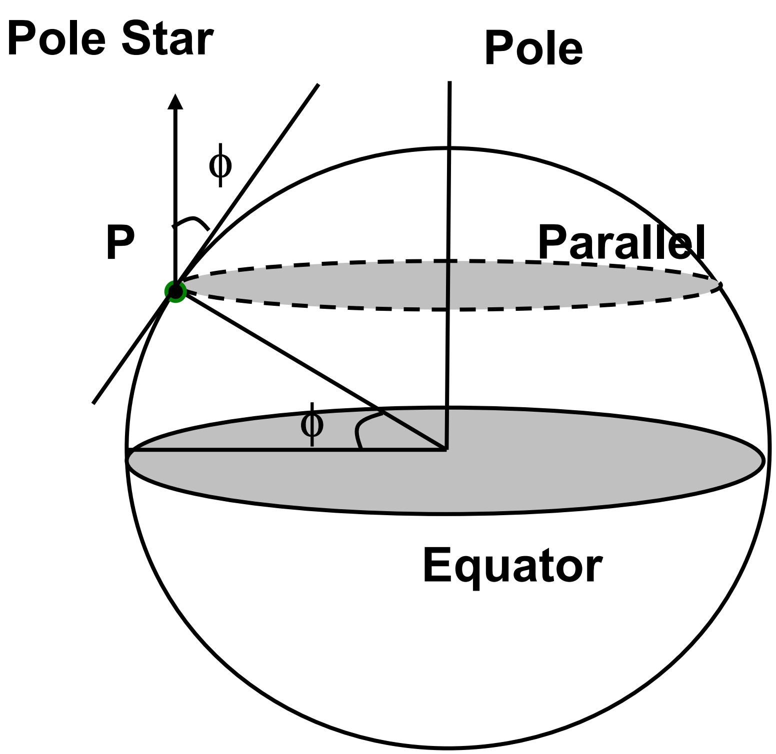
The problem of positioning

- The problem of obtaining a position is always been a matter of:
  - Geodesy
  - Timekeeping
  - Astronomy
- Modern systems also involve
  - Aerospace engineering
  - Telecommunications

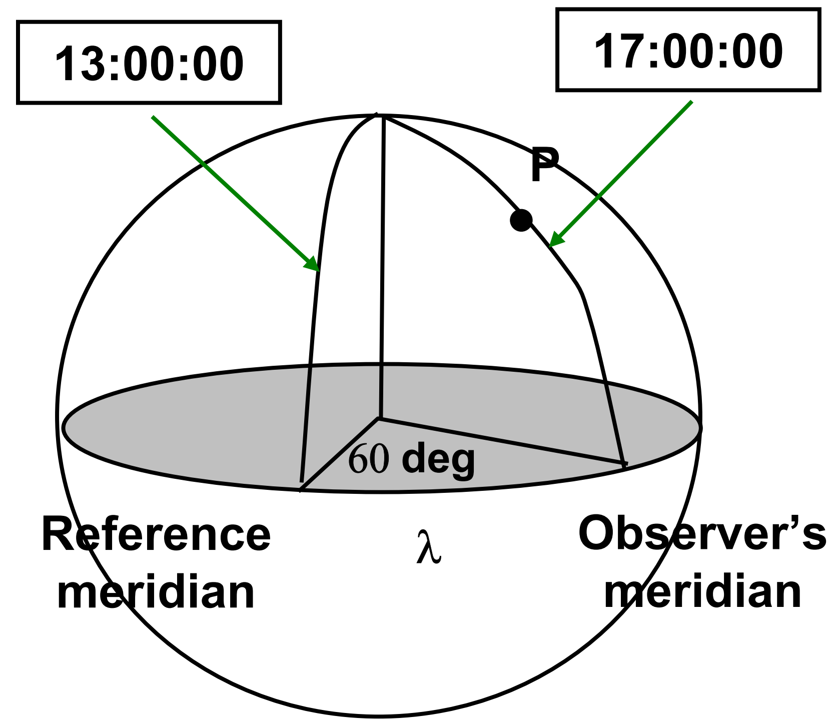
Some Historical notes

- In order to specify the position a reference system is needed:
  - Latitude is not hard to measure from the altitude of the Pole star or the sun at the highest point
  - Longitude calculation is trickier. It has been based on timekeeping: the difference in longitude can be determined if the difference in local times is known

Latitude and longitude



Measurement of latitude



Measurement of longitude

Some History notes

- The comparison of times was perfromed using
- Mechanical clocks: a “precise” clock was transported in order to compare it with local time
  - 4 minutes of error are equal to 1 deg: in 1500 a good clock have an error of 10 min/day
- Astronomical methods: observation of a known celestial event and compare the time of the same event observed at the reference point
  - Galileo(1600): observation of Jupiter’s orbit
  - Newton-Halley(1700): observation of lunar orbits

Long-range navigation system (LORAN)

- Developed during the II world war in the US by the MIT (Loran-A) for ships
- Loran-C developed in late ’50
- Hyperbolic system in LF band 90-110kHz
- Chain of transmitters made of a master and 2-3 secondary stations separated by about 1000 km
- In the US 29 stations (13 chains) covering the coastal regions
- Synchronized transmitters of RF pulses
- The ship measures the time difference between the arrival of master and secondary stations

The TRANSIT system

- First satellite based system (1964)
- 4-7 satellites at 1100 km
- Nearly circular orbit, polar orbit
- Signal at 150 Mhz and 400 Mhz
- Measured Doppler shift from a single station is sufficient if the satellite orbit is known

Global Positioning System

- A military system managed by the Department of Defense (DOD) of the United States of America
- Started in the ’80
- Modernized and maintained through new satellite vehicles (SV) constantly added

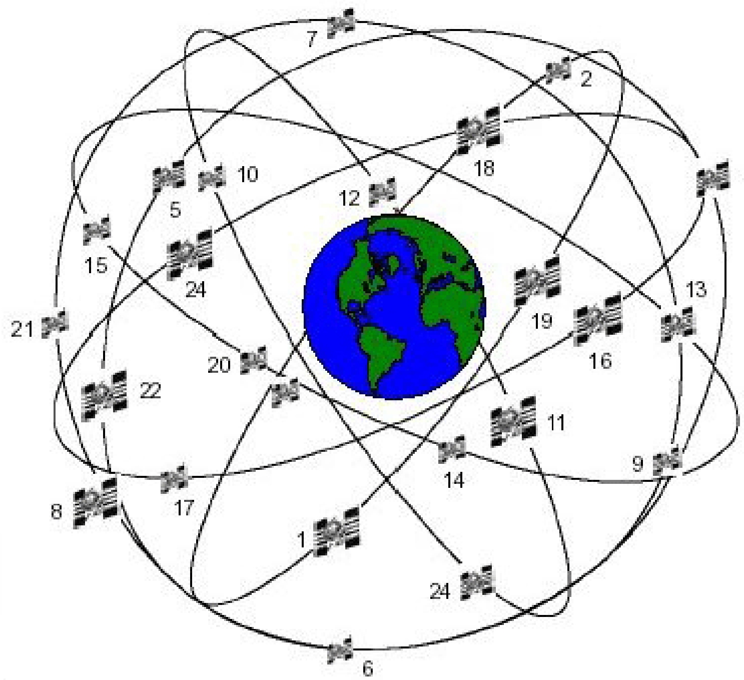
GPS Segments

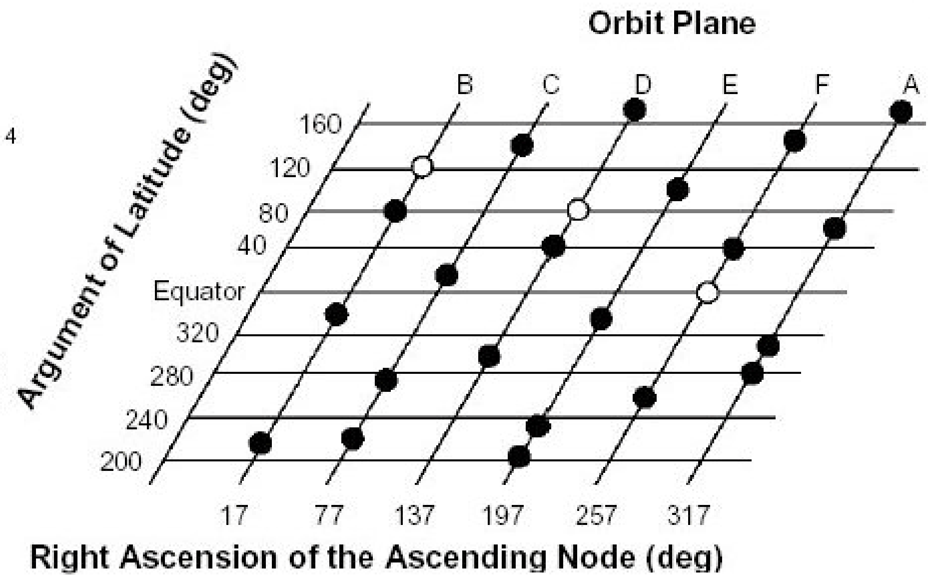
- space
  - 24 satellites constellation
- control
  - tracking stations
    - continuously monitoring the orbital data
  - master station
    - data processing, update orbits and time scale
  - up-loading stations
    - transmit updated data to satellites
- users
  - receivers determining their own position, velocity and time

GPS constellation

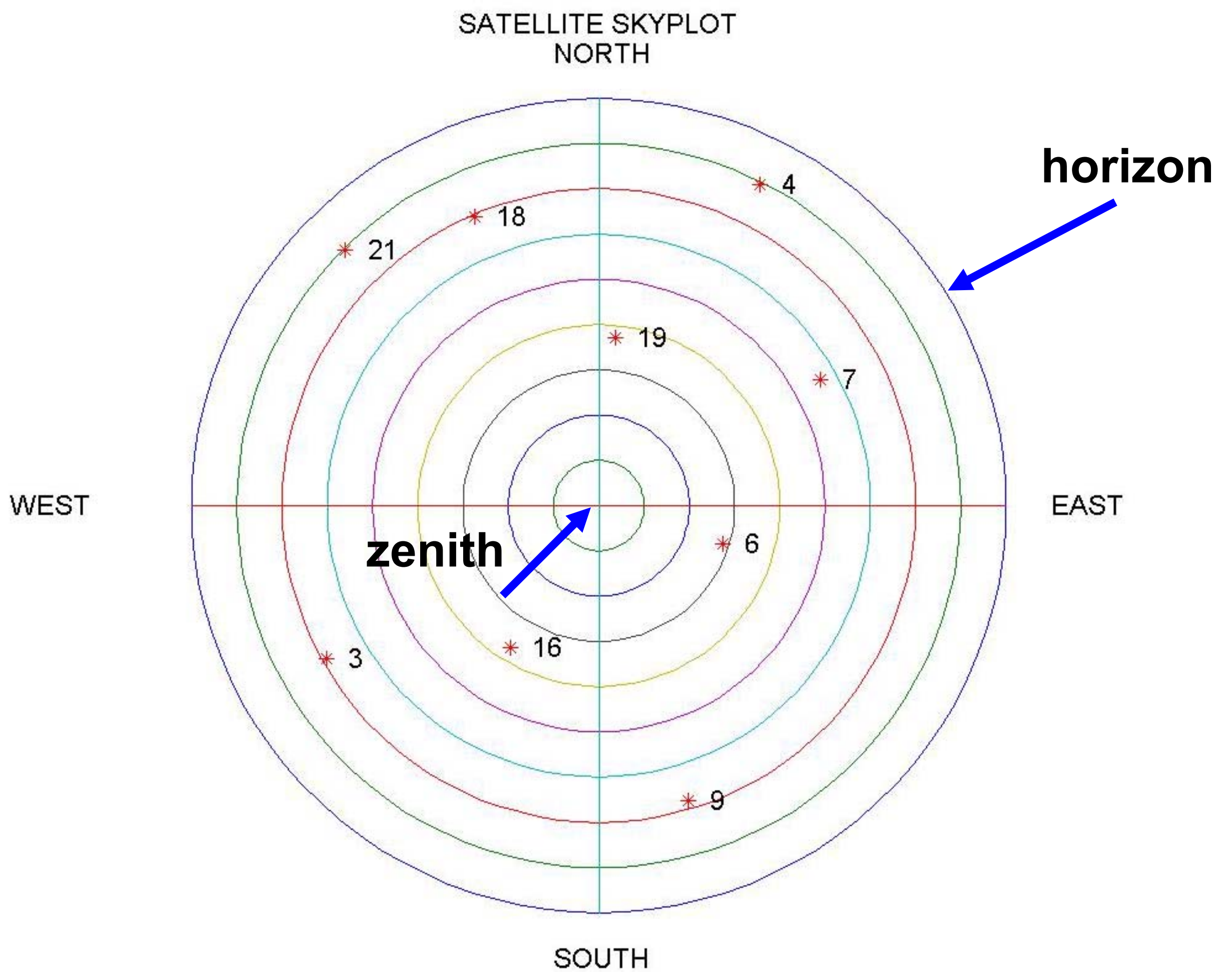
- The baseline nominal constellation
  - Semi-major axis: a=26560 km
  - Altitude 20.200 km
  - Eccentricity <0.01 (about circular orbit)
  - Period: about 12 h.
  - Six orbital planes (A-F) with inclination at 55 degrees. The RAAN of the six orbital planes are separated by 60 deg in the equatorial plane
  - Four satellites per plane distributed unevenly (to minimize the effect of a single satellite failure)

GPS constellation

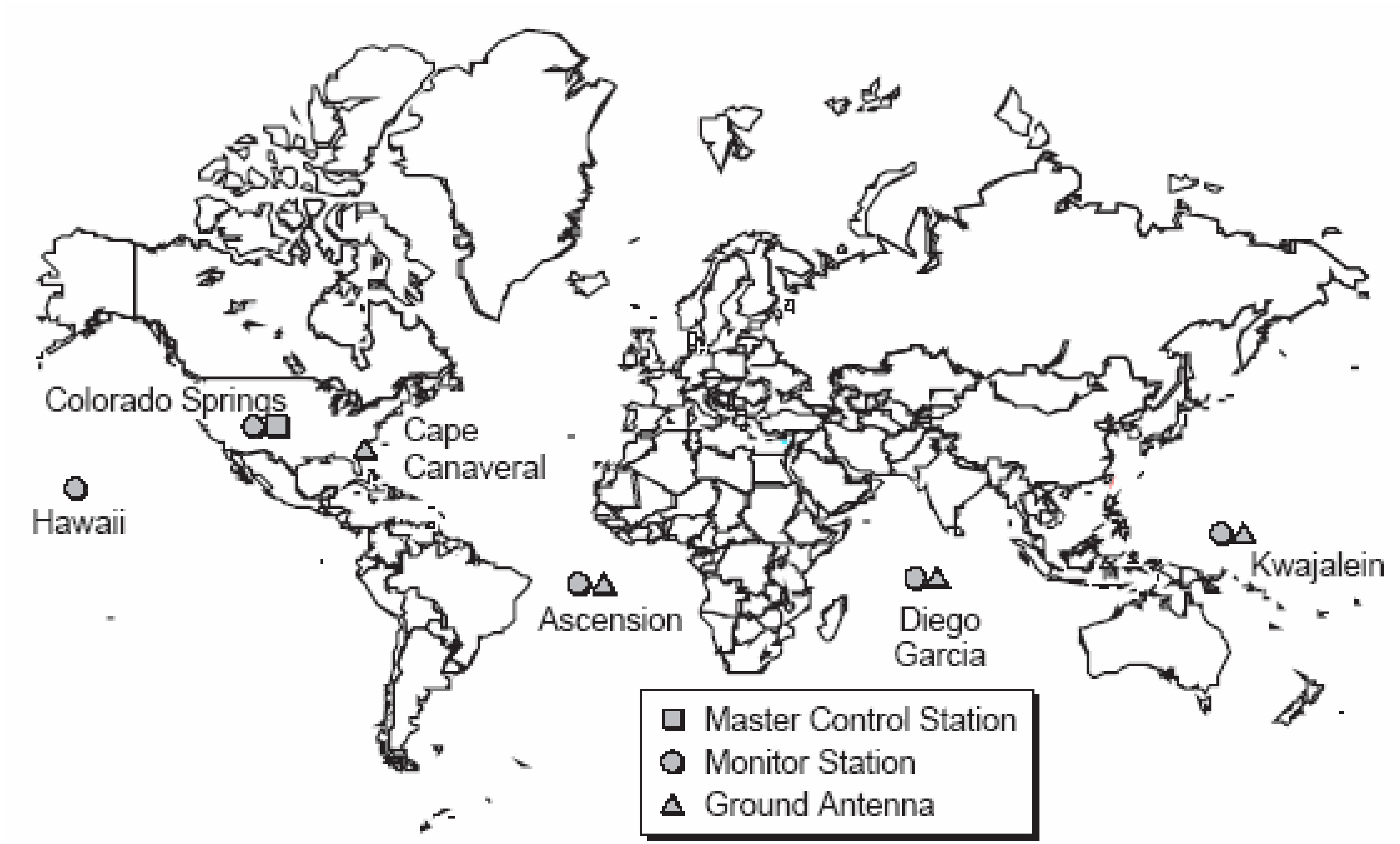




GPS Constellation from the ground



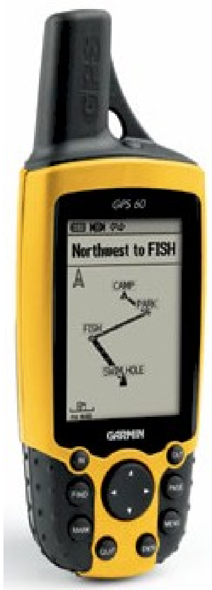
The Control Segment

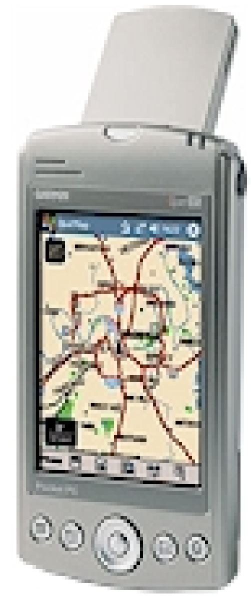


User Segment

- The user segment is made of a wide range of different receivers, with different performance levels
- The receiver estimates the position of the user on the basis of the signals transmitted by the satellites
- The functionalities common to any kind of receiver can be summarized as
  - Identification of the satellites in view
  - Estimation of the distance user-satellite
  - Triangulation
- Additional functionalities aim at
  - easing and/or improving the position estimation (augmentations)
  - improve the user output interface
  - added value services (e.g. route calculation, integration with communication systems)

User segment





- See as an example:
- http://www.tramsoft.ch/gps/

User segment

Functional basics (1)

- If a satellites transmit a pulse at t_0, it is received at time $t_0+\tau$ and the distance between TX and RX can be estimated as:

$$R = c \cdot \tau$$

where **c** is the speed of light

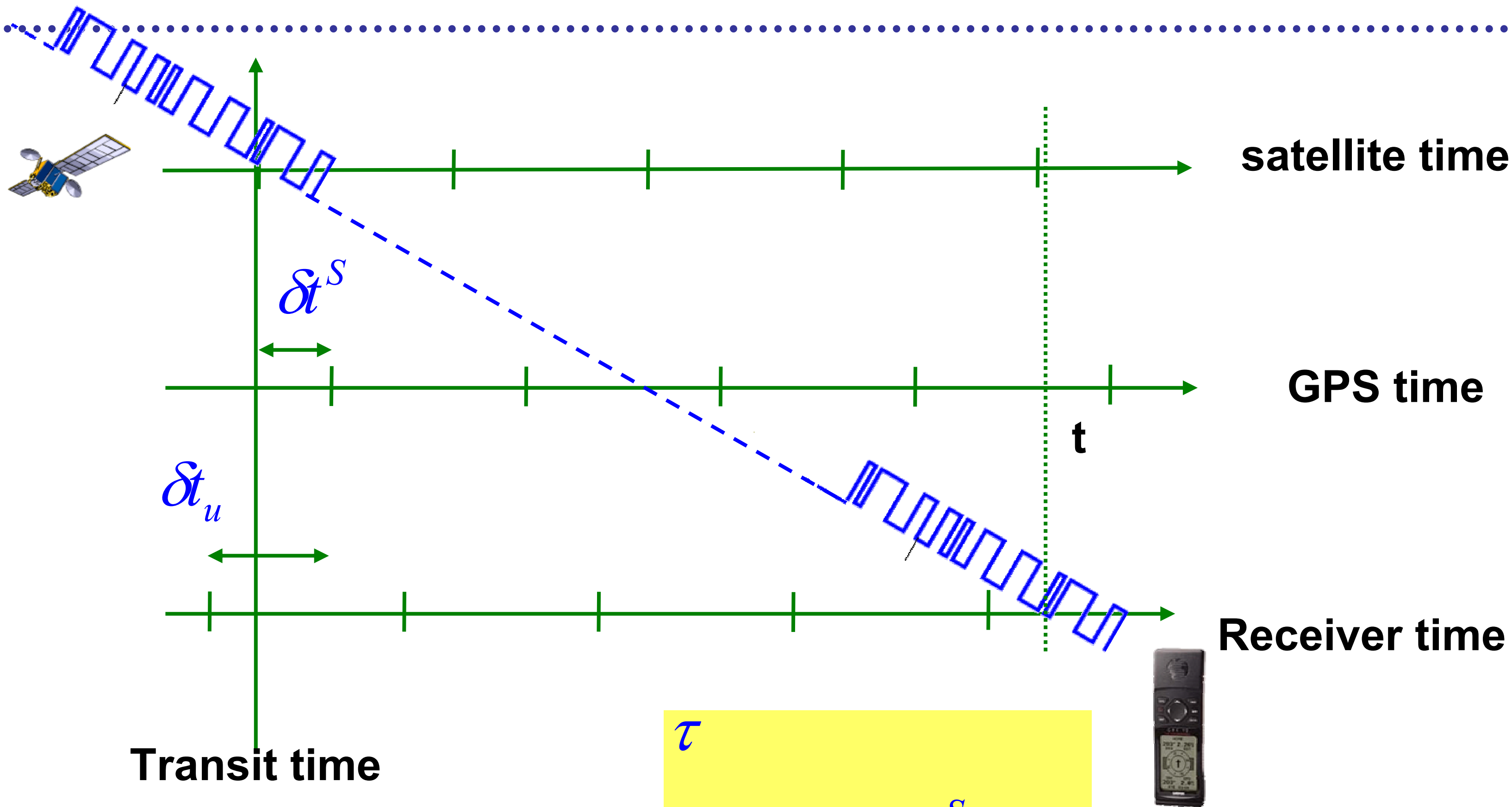
- If both the oscillators are perfect the measure of $t_0+\tau$ allows for **R** determination

Functional basics (2)

- The satellite payload hosts synchronous satellites
- It is not possible to have user clocks aligned with the satellite time scale at low cost and complexity
- Being δt_u the user clock misalignment the measured distance is the pseudorange

$$\rho = c \cdot \tau + c \cdot \delta t_u = R + c \cdot \varepsilon$$

Functional basics (3)



Transit time	τ
Pseudo transit-time	$\tau + (\delta t_u - \delta t^S)$
Pseudorange	$c \cdot [\tau + (\delta t_u - \delta t^S)]$

Functional basics (4)

- Let consider in the following $\delta t^S=0$

□ δt_u can be determined using a fourth pseudorange

- The user measuring four pseudoranges rj with respect to 4 satellites with known coordinates (x_{sj},y_{sj},z_{sj}) determines its postion (x_u,y_u,z_u) and the correction δt_u to apply to its own clock

$$\begin{cases} \rho_1 = \sqrt{(x_{s1}-x_u)^2+(y_{s1}-y_u)^2+(z_{s1}-z_u)^2}+c\cdot\delta t_u \\ \rho_2 = \sqrt{(x_{s2}-x_u)^2+(y_{s2}-y_u)^2+(z_{s2}-z_u)^2}+c\cdot\delta t_u \\ \rho_3 = \sqrt{(x_{s3}-x_u)^2+(y_{s3}-y_u)^2+(z_{s3}-z_u)^2}+c\cdot\delta t_u \\ \rho_4 = \sqrt{(x_{s4}-x_u)^2+(y_{s4}-y_u)^2+(z_{s4}-z_u)^2}+c\cdot\delta t_u \end{cases}$$

Functional basics (5)

Remarks

- In order to estimate its position a receiver must have at least four satellites in view
- The satellite must be in Line-of-sight
- If a larger number of satellites is in view a better estimation is possible. In the past the combination of four satellites giving the best performance was chosen
- Modern receivers use up to 12 channels in order to perform the position estimation

Satellites Signal-In-Space

- The propagation time is estimated processing a signal transmitted by each satellite
- As an example in GPS:
  - each satellite transmits over two carriers (L1, L2) modulated with a binary pseudonoise sequence
  - each satellite uses the same frequencies
  - each satellite is identified by a different PN sequence (CDMA scheme)

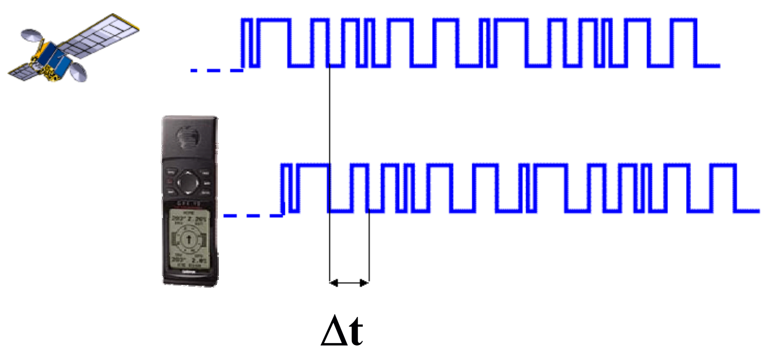
GPS receiver measurements

- Code phase measurements: the propagation time between SV and user is estimated measuring the Δt between a local replica of the C/A code and the received SIS

- Carrier phase measurements: the phase difference between a local carrier and the received one is evaluated; through proper techniques the number N of integer cycles is estimated

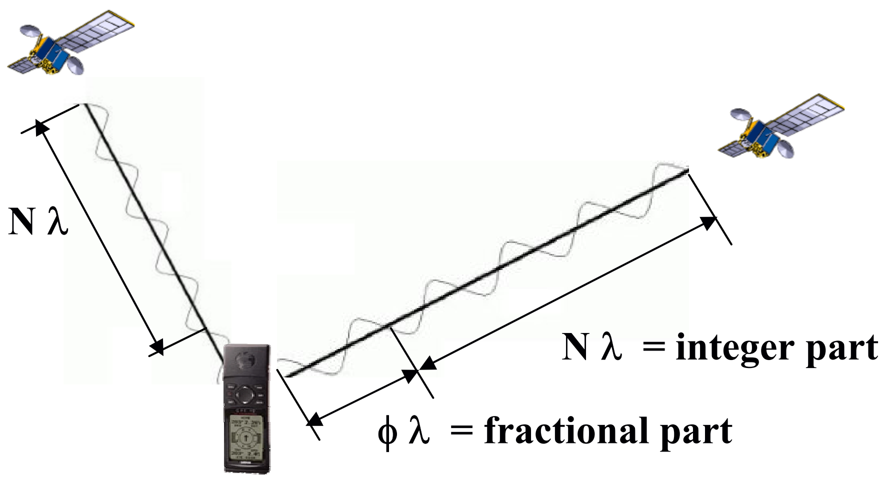
GPS receiver measurements

Code phase measurements



$$\rho = c\,\Delta t$$

Carrier Phase measurements



$$\rho = (N + \phi)\,\lambda$$

The navigation solution

- The generic pseudorange

$$\rho_j = \sqrt{(x_{sj} - x_u)^2 + (y_{sj} - y_u)^2 + (z_{sj} - z_u)^2} - c \cdot \delta t_u$$

- can be approximated through the Taylor expansion around a known location

$$\hat{\rho}_j = \sqrt{(x_{sj} - \hat{x}_u)^2 + (y_{sj} - \hat{y}_u)^2 + (z_{sj} - \hat{z}_u)^2} - c \cdot \delta\hat{t}_u$$

The navigation solution

- at a first order approximation

$$\Delta\rho_j = \rho_j - \hat{\rho}_j$$

$$\Delta\rho_j = a_{xj}\Delta x_u + a_{yj}\Delta y_u + a_{zj}\Delta z_u - c\Delta t_u$$

$$a_{xj} = \frac{x_j - \hat{x}_u}{\hat{r}_j}, a_{yj} = \frac{y_j - \hat{y}_u}{\hat{r}_j}, a_{zj} = \frac{z_j - \hat{z}_u}{\hat{r}_j}$$

- where $\mathbf{a}_j=(a_{xy},a,_{yj},a_{zj})$ are unitary vectors steering from the approximation point towards the j-th satellite

The navigation solution

$$\begin{cases} \Delta\rho_1 = a_{x1}\Delta x_u + a_{y1}\Delta y_u + a_{z1}\Delta z_u - c\Delta t_u \\ \Delta\rho_2 = a_{x2}\Delta x_u + a_{y2}\Delta y_u + a_{z2}\Delta z_u - c\Delta t_u \\ \Delta\rho_3 = a_{x3}\Delta x_u + a_{y3}\Delta y_u + a_{z3}\Delta z_u - c\Delta t_u \\ \Delta\rho_4 = a_{x4}\Delta x_u + a_{y4}\Delta y_u + a_{z4}\Delta z_u - c\Delta t_u \end{cases}$$

$$\boldsymbol{\Delta\rho} = \begin{bmatrix} \Delta\rho_1 \\ \Delta\rho_2 \\ \Delta\rho_3 \\ \Delta\rho_4 \end{bmatrix} \qquad \mathbf{H} = \begin{bmatrix} a_{x1} & a_{y1} & a_{z1} & 1 \\ a_{x2} & a_{y2} & a_{z2} & 1 \\ a_{x3} & a_{y3} & a_{z3} & 1 \\ a_{x4} & a_{y4} & a_{z4} & 1 \end{bmatrix} \qquad \mathbf{\Delta x} = \begin{bmatrix} \Delta x_u \\ \Delta y_u \\ \Delta z_u \\ -c\Delta t_u \end{bmatrix}$$

$$\boldsymbol{\Delta\rho} = \mathbf{H}\mathbf{\Delta x}$$

The navigation solution

- In case four satellite are used

$$\Delta \mathbf{x} = \mathbf{H}^{-1} \Delta \boldsymbol{\rho}$$

- If a larger number of satellite is used

$$\mathbf{H} = \begin{bmatrix} a_{x1} & a_{y1} & a_{z1} & 1 \\ a_{x2} & a_{y2} & a_{z2} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{xn} & a_{yn} & a_{zn} & 1 \end{bmatrix}$$

The navigation solution

- For n>4 a least square solution must be used
- The solution is given by the value of Δx that minimizes the square of the residual:

$$R_{SE}(\mathbf{\Delta x}) = (\mathbf{H}\mathbf{\Delta}x - \mathbf{\Delta\rho})^2$$

- The solution can be obtained differentiating with respect to $\boldsymbol{\Delta x}$ to obtain the gradient of R_{SE}.

$$\nabla R_{SE} = 2(\mathbf{\Delta x})^T \mathbf{H}^T \mathbf{H} - 2(\mathbf{\Delta\rho})^T \mathbf{H}$$

The navigation solution

- The gradient is set to zero and solved for Δx to seek a minimum value
- Taking the transpose and setting it to zero:

$$2\mathbf{H}^T\mathbf{H}(\Delta\mathbf{x}) - 2\mathbf{H}^T(\Delta\boldsymbol{\rho}) = 0$$

- Provided that $\mathbf{H}^T\mathbf{H}$ is non-singular, the equation solution is :

$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\boldsymbol{\rho}$$

The navigation solution

- Such a solution is theoretical and does not take into account:
  - errors in the pseudorange measurements
  - geometrical factors

The navigation solution

- With *n* satellites a Least Square solution can be obtained

$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\boldsymbol{\rho}$$

- Such a solution is theoretical and does not take into account:
  - errors in the pseudorange measurements
  - geometrical factors

Positioning errors

- The pseudorange measurement is error affected

$$\rho_j = \sqrt{(x_{sj}-x_u)^2+(y_{sj}-y_u)^2+(z_{sj}-z_u)^2} - c\cdot t_u + c\cdot t_a + E_j + \eta$$

- where
  - t_a is the atmosferical error due to the propagation in the ionosphere and troposphere (t_{iono}+t_{tropo})
  - E_j is the ephemeris error for the j-th satellite
  - η *represent* other error sources (multipath, receiver noise,)

Positioning errors

- The set of equations to solve is then

$$\boldsymbol{\Delta\rho} + \boldsymbol{\delta\rho} = \mathbf{H}\left(\boldsymbol{\Delta}\mathbf{x} + \boldsymbol{\delta}\mathbf{x}\right)$$

- where $\boldsymbol{\delta}\mathbf{x}$ represents the error in the postion and time estimation

$$\boldsymbol{\delta}\mathbf{x} = \left[\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\right]\boldsymbol{\delta\rho}$$

- The LS solution is valid under the hypotesis of linearly independent equations

Remarks

- There are two different contribution to the error

$$\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T$$

  - depends only on the satellite geometry

$$\boldsymbol{\delta\rho}$$

  - depends on the error in the pseudorange estimation

The geometric factor

- The pseudorange errors can be modeled as random variables
- The elements of the error vector $\delta\rho$ can be considered as random variables
  - gaussian with zero mean
  - identically distributed
  - independent
  - with variance σ^2_{UERE}

The geometric factor

$$\operatorname{cov}(\boldsymbol{\delta}\mathbf{x}) = E\left\{\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\boldsymbol{\delta\rho}\boldsymbol{\delta\rho}^T\mathbf{H}\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\right\}$$

$$\operatorname{cov}(\boldsymbol{\delta}\mathbf{x}) = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\operatorname{cov}(\boldsymbol{\delta\rho})\mathbf{H}\left(\mathbf{H}^T\mathbf{H}\right)^{-1}$$

$$\operatorname{cov}(\boldsymbol{\delta\rho}) = \mathbf{I}_{n\times n}\sigma_{UERE}^2$$

$$\operatorname{cov}(\boldsymbol{\delta}\mathbf{x}) = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\sigma_{UERE}^2$$

The geometric factor

- Let's define

$$\operatorname{cov}(\boldsymbol{\delta}\mathbf{x}) = \mathbf{G}\sigma_{UERE}^2$$

- where

$$\mathbf{G} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix}$$

The geometric factor

- It is then possible to observe the relation of the error for each dimension

$$\sigma_x^2 = g_{11}\sigma_{UERE}^2$$

$$\sigma_y^2 = g_{22}\sigma_{UERE}^2$$

$$\sigma_z^2 = g_{33}\sigma_{UERE}^2$$

$$\sigma_{c\Delta t}^2 = g_{44}\sigma_{UERE}^2$$

Geometric Dilution Of Precision

- The GDOP factor is defined as

$$GDOP = \sqrt{tr\left\{\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\right\}}$$

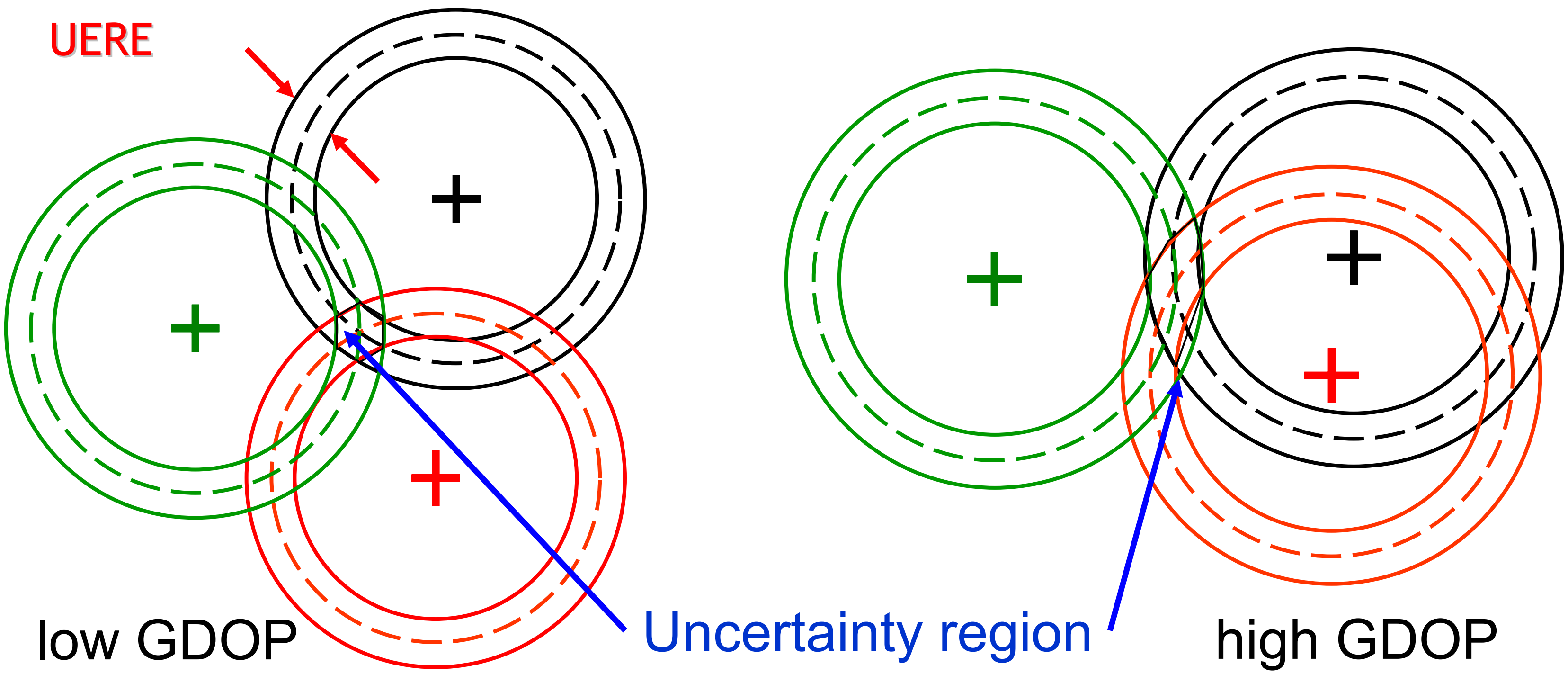
- obtaining

$$\sqrt{tr\{\mathrm{cov}(\boldsymbol{\delta}\mathbf{x})\}} = \sqrt{\sigma_{x_u}^2 + \sigma_{y_u}^2 + \sigma_{z_u}^2 + \sigma_{c\Delta t}^2} =$$

$$= GDOP \times \sigma_{UERE}$$

The geometrical problem

- The impact of the pseudorange error on the final estimated position depends on the displacement of the satellites (reference points)



Dilution of precision

- Partial factors can be defined:
- Position Dilution of Precision

$$\mathrm{PDOP} = \sqrt{g_{11} + g_{22} + g_{33}}$$

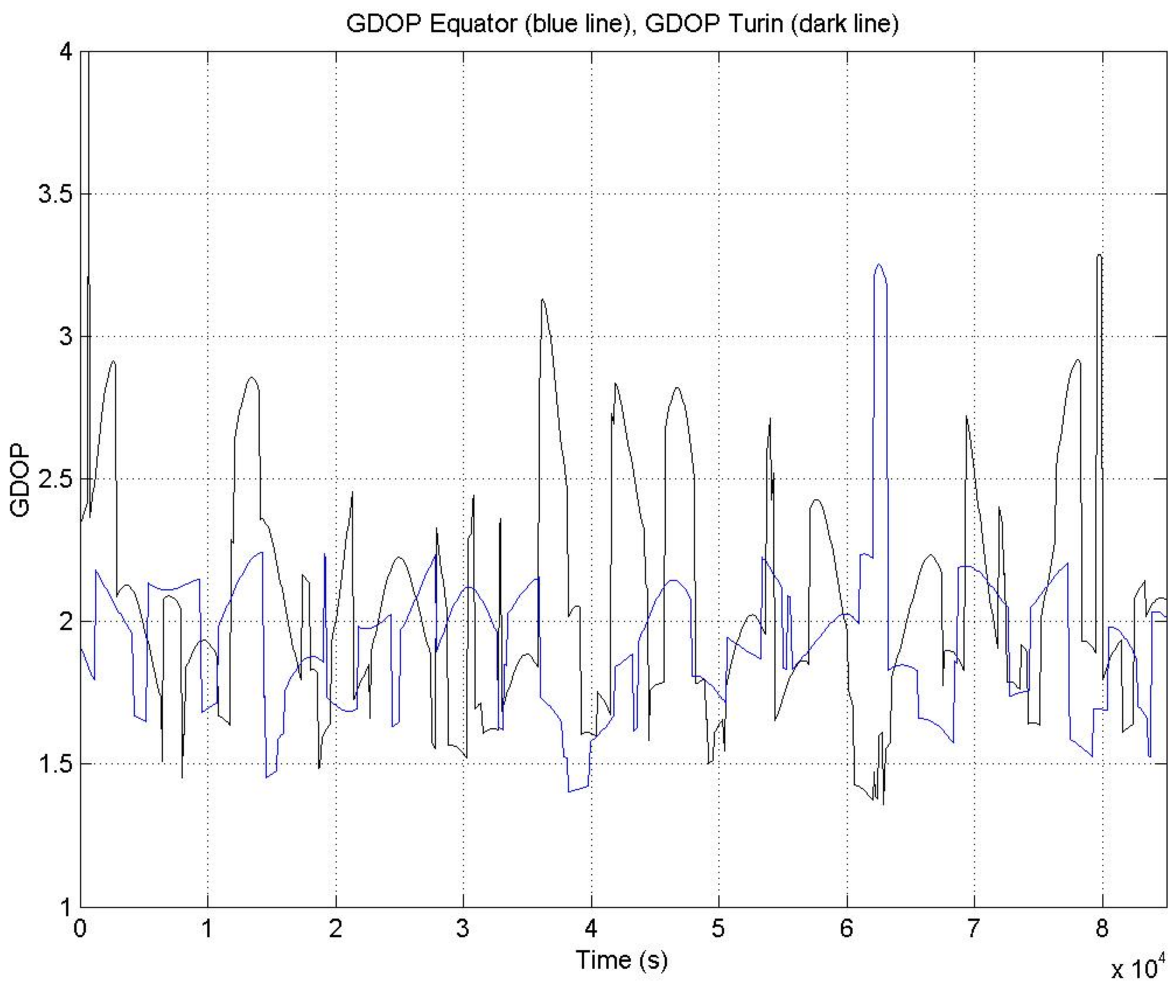
- Time Dilution of Precision

$$\mathrm{TDOP} = \sqrt{g_{44}}$$

- Horizontal Dilution of Precision

$$\mathrm{HDOP} = \sqrt{g_{11} + g_{22}}$$

GDOP example



End of Part II