The University of Hong Kong
Department of Statistics and Actuarial Science

STAT 6015/6115/7615 Advanced Quantitative Risk Management and Finance

Solution for Assignment 2

1. Let $a = var(X)$, $b = var(W)$ and $c = cov(X, W)$. Then

$$f(\alpha) := var(\alpha X + (1-\alpha)W) = \alpha^2 a + (1-\alpha)^2 b + 2\alpha(1-\alpha)c$$

and by setting $f'(\alpha) = 0$, $\alpha = \frac{b-c}{a+b-2c}$. It can be checked that $f''(\alpha) \geq 0$. The minimum variance is $\frac{ab-c^2}{a+b-2c}$. When X and W are independent, the minimum variance is $\frac{ab}{a+b}$ and $\alpha = \frac{b}{a+b}$.

2. Denote $S_t = S_0 e^{Y_t}$, where $Y_t = (r - \sigma^2/2)t + \sigma W_t$. So we have

$$\left(\prod_{i=1}^{k} S_{\frac{iT}{k}}\right)^{1/k} = S_0 \exp\left(\frac{1}{k}\sum_{i=1}^{k} Y_{\frac{iT}{k}}\right) = S_0 e^{\bar{Y}},$$

where $\bar{Y} = \frac{1}{k}\sum_{i=1}^{k} Y_{\frac{iT}{k}}$ and $Y_{\frac{iT}{k}} = \left(r - \frac{\sigma^2}{2}\right)\frac{iT}{k} + \sigma W_{\frac{iT}{k}}$. By letting $Y_0 = 0$, $\bar{Y}$ can be written as

$$\begin{aligned}
\bar{Y} = \frac{1}{k}\sum_{i=1}^{k} Y_{\frac{iT}{k}} &= \frac{1}{k}\sum_{i=1}^{k}(k-i+1)\left(Y_{\frac{iT}{k}} - Y_{\frac{(i-1)T}{k}}\right) \\
&= \frac{1}{k}\sum_{i=1}^{k}(k-i+1)\left[\left(r - \frac{\sigma^2}{2}\right)\frac{T}{k} + \sigma\left(W_{\frac{iT}{k}} - W_{\frac{(i-1)T}{k}}\right)\right] \\
&= \frac{1}{k}\sum_{i=1}^{k}(k-i+1)\left[\left(r - \frac{\sigma^2}{2}\right)\frac{T}{k} + \sigma\sqrt{\frac{T}{k}}Z_i\right],
\end{aligned}$$

where $Z_i = W_i - W_{i-1} \overset{\text{i.i.d.}}{\sim} N(0, 1)$. Therefore, $\bar{Y}$ is normally distributed with mean

$$\begin{aligned}
E(\bar{Y}) &= \frac{1}{k}\sum_{i=1}^{k}(k-i+1)\left[\left(r - \frac{\sigma^2}{2}\right)\frac{T}{k} + \sigma\sqrt{\frac{T}{k}}E(Z_i)\right] \\
&= \frac{1}{k}\sum_{i=1}^{k}(k-i+1)\left[\left(r - \frac{\sigma^2}{2}\right)\frac{T}{k}\right] \\
&= \left(r - \frac{\sigma^2}{2}\right)\frac{T}{k}\frac{1}{k}\sum_{i=1}^{k} i \\
&= \left(r - \frac{\sigma^2}{2}\right)\frac{T}{k}\frac{1}{k}\left[\frac{1}{2}k(k+1)\right] \\
&= \frac{1}{2}\left(r - \frac{\sigma^2}{2}\right)\left(1 + \frac{1}{k}\right)T
\end{aligned}$$

and variance

$$
\begin{aligned}
\operatorname{Var}(\bar{Y}) &= \frac{1}{k^2}\sum_{i=1}^{k}(k-i+1)^2\,\sigma^2\frac{T}{k} \\
&= \sigma^2\frac{T}{k}\frac{1}{k^2}\sum_{i=1}^{k} i^2 \\
&= \sigma^2\frac{T}{k}\frac{1}{k^2}\left[\frac{1}{3}k\left(k+\frac{1}{2}\right)(k+1)\right] \\
&= \frac{\sigma^2}{3}\left(1+\frac{1}{k}\right)\left(1+\frac{1}{2k}\right)T.
\end{aligned}
$$

That is, $\bar{Y} \sim N(\widetilde{\mu}T,\, \widetilde{\sigma}^2 T)$, where

$$
\widetilde{\mu} = \frac{1}{2}\left(r-\frac{\sigma^2}{2}\right)\left(1+\frac{1}{k}\right), \quad \widetilde{\sigma}^2 = \frac{\sigma^2}{3}\left(1+\frac{1}{k}\right)\left(1+\frac{1}{2k}\right).
$$

3. Note that $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. Then the tilted density is

$$
\begin{aligned}
f_t(x) &= (\int e^{ty} f(y)dy)^{-1} e^{tx} f(x) \\
&= e^{-\frac{1}{2\sigma^2}(-2\mu\sigma^2 t+\sigma^4 t^2)} e^{tx}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\
&= \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu-\sigma^2 t)^2}{2\sigma^2}} \sim N(\mu+\sigma^2 t, \sigma^2).
\end{aligned}
$$

4. Let h be a bounded measurable function, then

$$
\begin{aligned}
E_{\mathbb{Q}}(h(X)) &= \int_\Omega h(X(\omega))d\mathbb{Q}(\omega) = \int_\Omega h(X(\omega))\frac{d\mathbb{Q}}{d\mathbb{P}}(\omega)d\mathbb{P}(\omega) \\
&= \int_\Omega h(X(\omega))\frac{g(X(\omega))}{f(X(\omega))}d\mathbb{P}(\omega) = E_{\mathbb{P}}(h(X)\frac{g(X)}{f(X)}) \\
&= \int_{\mathbb{R}} h(x)\frac{g(x)}{f(x)}f(x)dx = \int_{\mathbb{R}} h(x)g(x)dx.
\end{aligned}
$$

5. For the crude Monte Carlo estimator, the variance is

$$
\begin{aligned}
\sigma_c^2 &= \frac{1}{2n}\operatorname{Var}\left(\frac{e^{U_i}-1}{e-1}\right) = \frac{1}{2n(e-1)^2}\operatorname{Var}(e^{U_i}) = \frac{1}{2n(e-1)^2}\left\{E(e^{2U_i}) - [E(e^{U_i})]^2\right\} \\
&= \frac{1}{2n(e-1)^2}\left[\frac{e^2-1}{2} - (e-1)^2\right] = \frac{3-e}{4n(e-1)} = \frac{0.04099}{n}.
\end{aligned}
$$

For the antithetic variable method, the Monte Carlo estimator is

$$
\widehat{\theta}_a = \frac{1}{2n}\sum_{i=1}^{n}\left(\frac{e^{U_i}-1}{e-1} + \frac{e^{1-U_i}-1}{e-1}\right).
$$

The variance of this estimator is

$$
\begin{aligned}
\sigma_a^2 &= \frac{1}{n}\mathrm{Var}\left(\frac{e^{U_i}-1}{2(e-1)} + \frac{e^{1-U_i}-1}{2(e-1)}\right) = \frac{1}{4n(e-1)^2}\mathrm{Var}(e^{U_i}+e^{1-U_i}) \\
&= \frac{1}{4n(e-1)^2}[\mathrm{Var}(e^{U_i}) + \mathrm{Var}(e^{1-U_i}) + 2\mathrm{Cov}(e^{U_i},\, e^{1-U_i})] \\
&= \frac{1}{4n(e-1)^2}\{2\mathrm{Var}(e^{U_i}) + 2[E(e^{U_i}e^{1-U_i}) - E(e^{U_i})E(e^{1-U_i})]\} \\
&= \frac{1}{2n(e-1)^2}\left[\frac{(e-1)(3-e)}{2} + e - (e-1)^2\right] = \frac{0.001325}{n}.
\end{aligned}
$$

So the efficiency of this estimator relative to the crude Monte Carlo estimator is

$$\frac{\sigma_c^2}{\sigma_a^2} = \frac{0.04099/n}{0.001325/n} = 30.9311.$$