#LAMIN JAMMEH
#EE597
#HW8

#Pr 11.2 (Cosine series)

restart;
with(*plots*) :
#Step 1 Defien the f as a piecewise function
?piecewise

$$f := piecewise\left(-\frac{\text{Pi}}{2} < x, 1, x < \frac{\text{Pi}}{2}, 1, \quad \frac{\text{Pi}}{2} < x, -1, x < \frac{3\cdot\text{Pi}}{2\cdot\text{Pi}}, -1\right);$$

$$f := \begin{cases} 1 & -\frac{\pi}{2} < x \\ 1 & x < \frac{\pi}{2} \\ -1 & \frac{\pi}{2} < x \\ -1 & x < \frac{3}{2} \end{cases} \tag{1}$$

#Step2 find the Fourier coefficient a0, an, bn from these intervals $\left[-\frac{\text{Pi}}{2} .. \frac{\text{Pi}}{2},\ @f=1\right]$ **to** $\left[\left[\frac{\text{Pi}}{2} .. \frac{3\ \text{Pi}}{2\ \text{Pi}},\ @f=-1\right]\right.$

$$a0 := \frac{1}{\text{Pi}}\cdot int\left(1, x=-\frac{\text{Pi}}{2} .. \frac{\text{Pi}}{2}\right) + \frac{1}{2\cdot\text{Pi}}\cdot int\left(-1,\ x=\frac{\text{Pi}}{2} .. \frac{3\cdot\text{Pi}}{2\cdot\text{Pi}}\right);$$

$$a0 := 1 + \frac{\frac{\pi}{2} - \frac{3}{2}}{2\,\pi} \tag{2}$$

$$an := \frac{2}{\text{Pi}}\cdot int\left(1\cdot\cos(n\cdot x), x=-\frac{\text{Pi}}{2} .. \frac{\text{Pi}}{2}\right) + \frac{1}{\text{Pi}}\cdot int\left(-1\cdot\cos(n\cdot x),\ x=\frac{\text{Pi}}{2} .. \frac{3\cdot\text{Pi}}{2\cdot\text{Pi}}\right);$$

$$an := \frac{4\sin\left(\frac{\pi\,n}{2}\right)}{\pi\,n} - \frac{\sin\left(\frac{3\,n}{2}\right) - \sin\left(\frac{\pi\,n}{2}\right)}{\pi\,n} \tag{3}$$

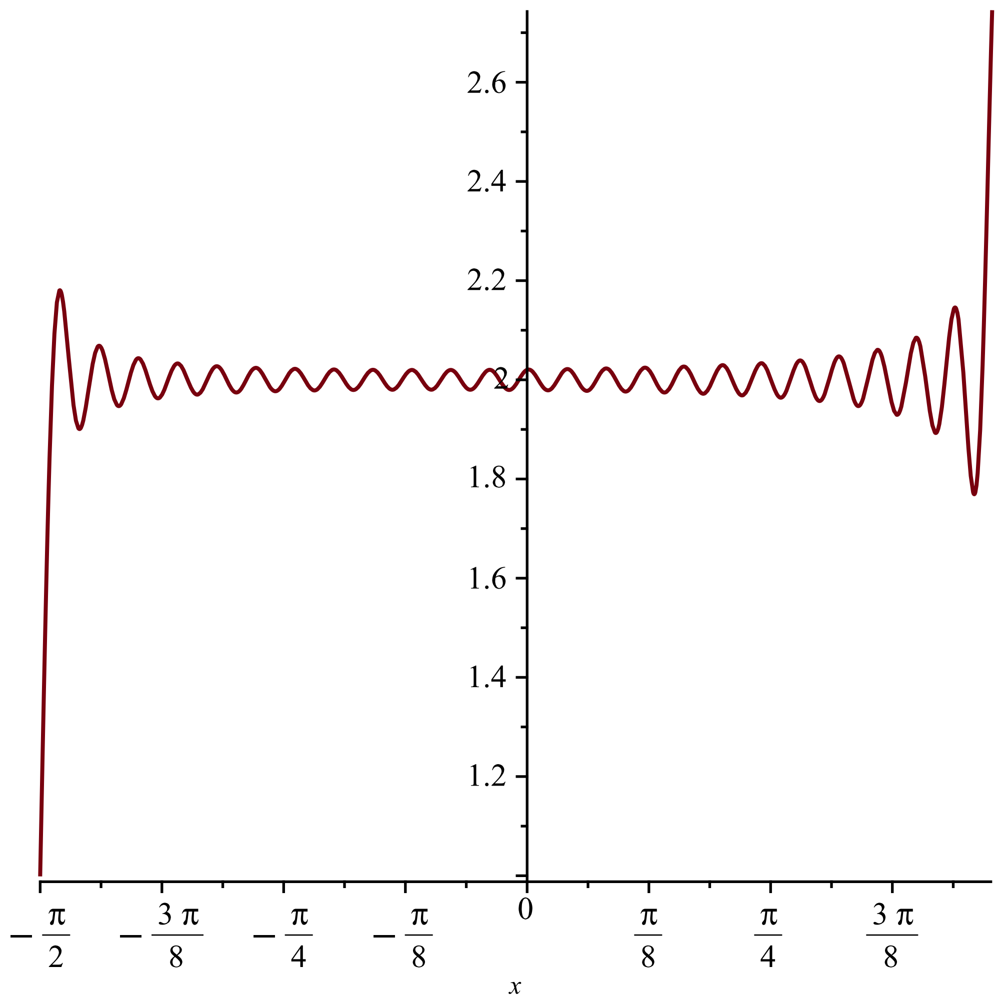
$$bn := \frac{2}{\text{Pi}}\cdot int\left(1\cdot\sin(n\cdot x), x=-\frac{\text{Pi}}{2} .. \frac{\text{Pi}}{2}\right) + \frac{1}{\text{Pi}}\cdot int\left(-1\cdot\sin(n\cdot x),\ x=\frac{\text{Pi}}{2} .. \frac{3\cdot\text{Pi}}{2\cdot\text{Pi}}\right);$$

$$bn := -\frac{-\cos\left(\frac{3\,n}{2}\right) + \cos\left(\frac{\pi\,n}{2}\right)}{\pi\,n} \tag{4}$$

#Step3 determine the Fourier series F=a0+sum[an·cos(nx)+bn·sin(nx)] from n=1..50

$F := a0 + sum(an\cdot\cos(n\cdot x) + bn\cdot\sin(n\cdot x), n=1..50) :$

$plot\left(F,\quad x=-\frac{\mathrm{Pi}}{2} .. \frac{3\cdot\mathrm{Pi}}{2\cdot\mathrm{Pi}}\right);$



#Pr 11.4 (Half-wave rectifier)

restart;

#define the function f=sin(t)

$f := \sin(t);$

$$f := \sin(t) \tag{5}$$

#Step2 find the Fourier coefficient a0, an, bn from t=0..Pi

$a0 := \frac{1}{\mathrm{Pi}}\cdot int(f, t=0..\mathrm{Pi}\ )$

$$a0 := \frac{2}{\pi} \tag{6}$$

$an := \frac{2}{\mathrm{Pi}} \cdot int(f \cdot \cos(n \cdot t),\ t = 0 .. \mathrm{Pi}\ )$

$$an := -\frac{2\,(\cos(\pi\, n) + 1)}{\pi\,(-1 + n)\,(1 + n)} \tag{7}$$

$bn := \frac{2}{\mathrm{Pi}} \cdot int(f \cdot \sin(n \cdot t),\ t = 0 .. \mathrm{Pi}\ )$

$$bn := -\frac{2\sin(\pi\, n)}{\pi\,(n^2 - 1)} \tag{8}$$

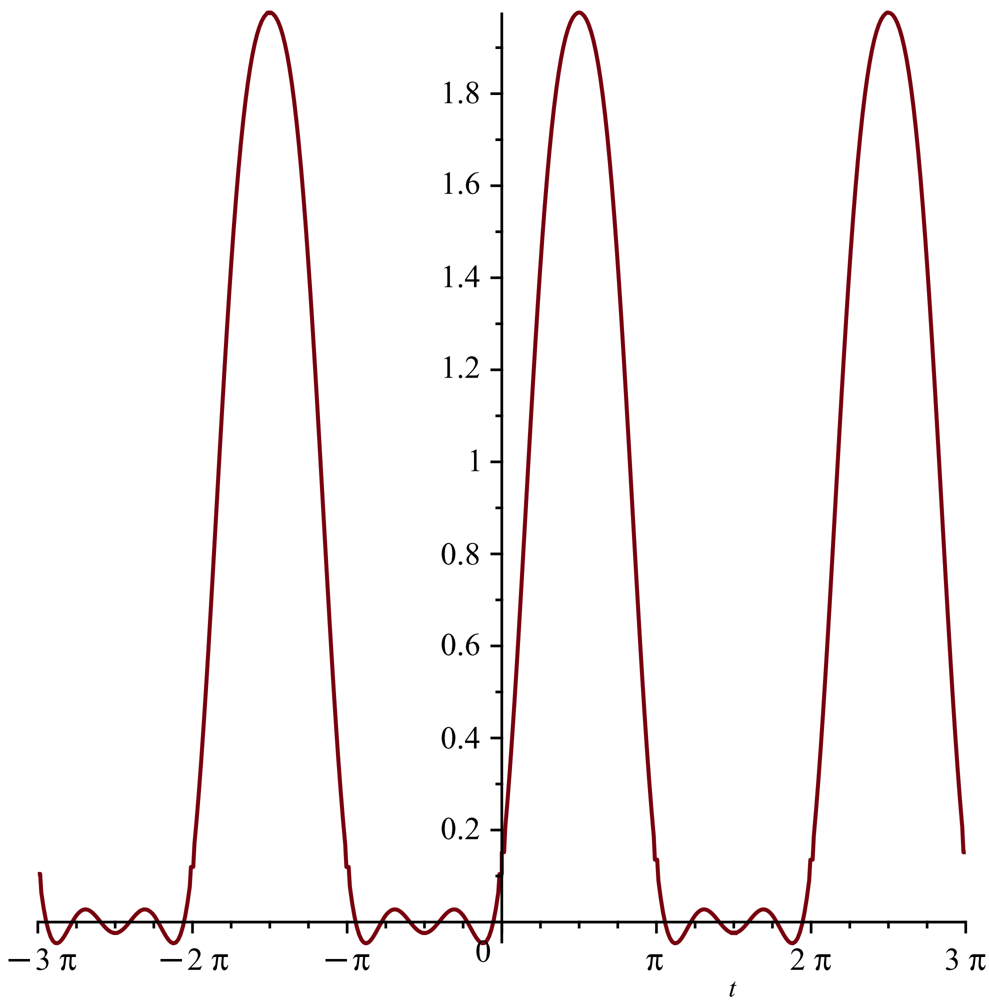
#Step3 determine the Fourier series F=a0+sum[an·cos(nx)+bn·sin(nx)] from n=1..5

$F := a0 + sum(an \cdot \cos(n \cdot t) + bn \cdot \sin(n \cdot t), n = 1 ..5);$

$$F := \frac{2}{\pi} + \sin(t) - \frac{4\cos(2\,t)}{3\,\pi} - \frac{4\cos(4\,t)}{15\,\pi} \tag{9}$$

$with(plots):$

$plot(F,\quad t = -3 \cdot \mathrm{Pi} .. 3 \cdot \mathrm{Pi});$



#*Pr* 11.6 (*Behavior near a jump*)

restart;

#define the function f and period p

$f := \frac{\mathrm{Pi}\cdot x^2}{2}$: $\quad p := 2$:

#Step2 find the Fourier coefficient a0, an, bn from x=-1..1

$a0 := \frac{1}{p}\cdot int(f, x = 0..p);$

$$a0 := \frac{2\,\pi}{3} \tag{10}$$

$an := \frac{2}{p}\cdot int\left(f\cdot\cos\left(\frac{n\cdot x\cdot \mathrm{Pi}}{p}\right), x = 0..p\right)$

$$an := \frac{4\,\left(n^2\,\pi^2\,\sin(n\,\pi) - 2\,\sin(n\,\pi) + 2\,n\,\pi\,\cos(n\,\pi)\right)}{\pi^2\,n^3} \tag{11}$$

$bn := \frac{2}{p}\cdot int\left(f\cdot\sin\left(\frac{n\cdot x\cdot \mathrm{Pi}}{p}\right), x = 0..p\right)$

$$bn := \frac{4\,\left(-n^2\,\pi^2\,\cos(n\,\pi) + 2\,n\,\pi\,\sin(n\,\pi) + 2\,\cos(n\,\pi) - 2\right)}{\pi^2\,n^3} \tag{12}$$

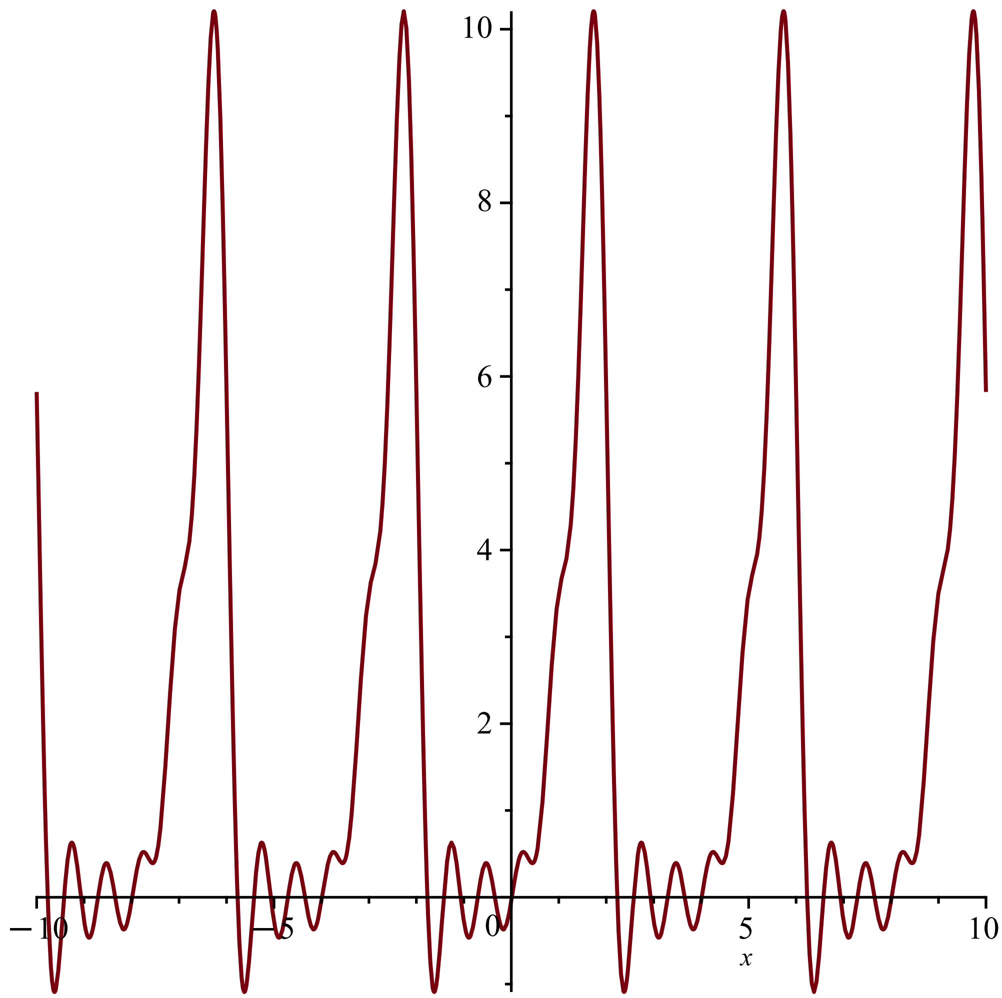
#Step3 determine the Fourier series F=a0+sum[an·cos(nx)+bn·sin(nx)] from n=1..5

$F := a0 + sum\left(an\cdot\cos\left(\frac{n\cdot x\cdot \mathrm{Pi}}{p}\right) + bn\cdot\sin\left(\frac{n\cdot x\cdot \mathrm{Pi}}{p}\right), n = 1..5\right);$

$$\begin{aligned} F := {} & \frac{2\,\pi}{3} - \frac{8\,\cos\left(\frac{\pi\,x}{2}\right)}{\pi} + \frac{4\,\left(\pi^2 - 4\right)\,\sin\left(\frac{\pi\,x}{2}\right)}{\pi^2} + \frac{2\,\cos(\pi\,x)}{\pi} - 2\,\sin(\pi\,x) \\ & - \frac{8\,\cos\left(\frac{3\,\pi\,x}{2}\right)}{9\,\pi} + \frac{4\,\left(9\,\pi^2 - 4\right)\,\sin\left(\frac{3\,\pi\,x}{2}\right)}{27\,\pi^2} + \frac{\cos(2\,\pi\,x)}{2\,\pi} - \sin(2\,\pi\,x) \\ & - \frac{8\,\cos\left(\frac{5\,\pi\,x}{2}\right)}{25\,\pi} + \frac{4\,\left(25\,\pi^2 - 4\right)\,\sin\left(\frac{5\,\pi\,x}{2}\right)}{125\,\pi^2} \end{aligned} \tag{13}$$

with(*plots*) :

plot(*F*, $x = -10..10$);



#Pr 11.8 (Triangular wave)

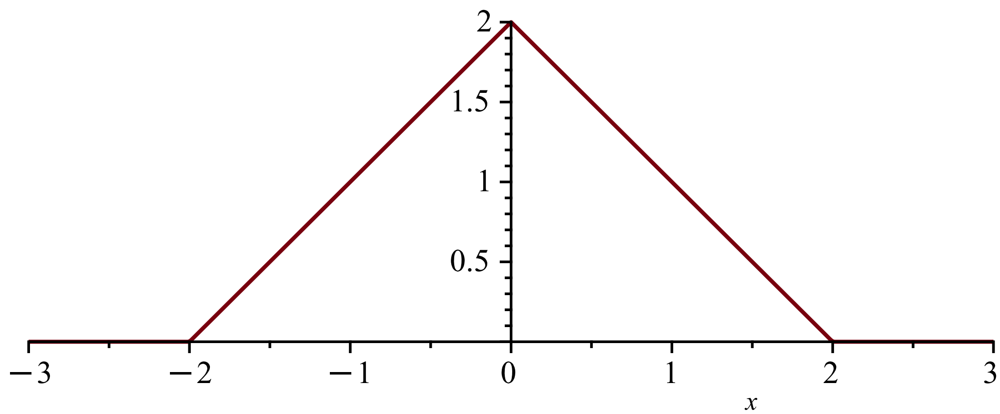
restart;

#define the function f in piecewise

$f := piecewise(-2 < x < 0,\ 2 + x,\ \ 0 < x < 2,\ 2 - x);$

$$f := \begin{cases} 2 + x & -2 < x < 0 \\ 2 - x & 0 < x < 2 \end{cases} \tag{14}$$

$plot(f, x = -3\,..3,\ scaling = constrained);$



#Step2 find the Fourier coefficient a0, an, bn from these intervals $[-2..0,\ @f=2+x]$ **to** $[[0..2,\ @f=2-x]$

$$a0 := \frac{1}{2\cdot \mathrm{Pi}}\cdot int(2+x, x=-2..0) + \frac{1}{2\cdot\mathrm{Pi}}\cdot int(2-x,\ x=0..2);$$

$$a0 := \frac{2}{\pi} \tag{15}$$

$$an := \frac{2}{2\cdot \mathrm{Pi}}\cdot int(2+x\cdot\cos(n\cdot x), x=-2..0) + \frac{2}{2\cdot\mathrm{Pi}}\cdot int(2-x\cdot\cos(n\cdot x),\ x=0..2);$$

$$an := -\frac{2\left(2\, n \sin(2\, n) - 4\, n^2 + \cos(2\, n) - 1\right)}{\pi\, n^2} \tag{16}$$

$$bn := \frac{2}{2\cdot \mathrm{Pi}}\cdot int(2+x\cdot\sin(n\cdot x), x=-2..0) + \frac{2}{2\cdot\mathrm{Pi}}\cdot int(2-x\cdot\sin(n\cdot x),\ x=0..2);$$

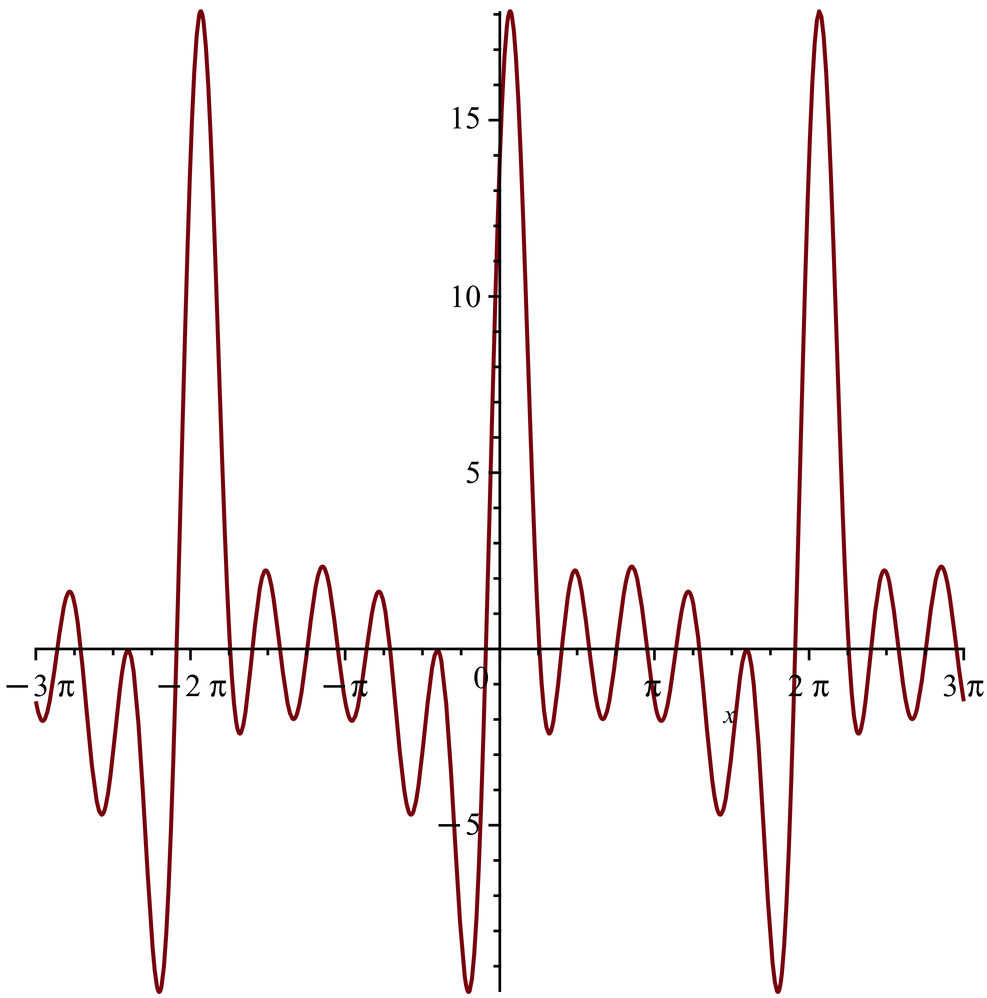
$$bn := -\frac{2\, n\cos(2\, n) - 4\, n^2 - \sin(2\, n)}{\pi\, n^2} + \frac{2\, n\cos(2\, n) + 4\, n^2 - \sin(2\, n)}{\pi\, n^2} \tag{17}$$

#Step3 determine the Fourier series F=a0+sum[an·cos(nx)+bn·sin(nx)] from n=1..5

$F := a0 + sum(an\cdot\cos(n\cdot x) + bn\cdot\sin(n\cdot x), n=1..5)$:

$with(plots)$:

$plot(F,\quad x=-3\cdot\mathrm{Pi}..3\cdot\mathrm{Pi});$



#Pr 11.10 (Herringbone wave)

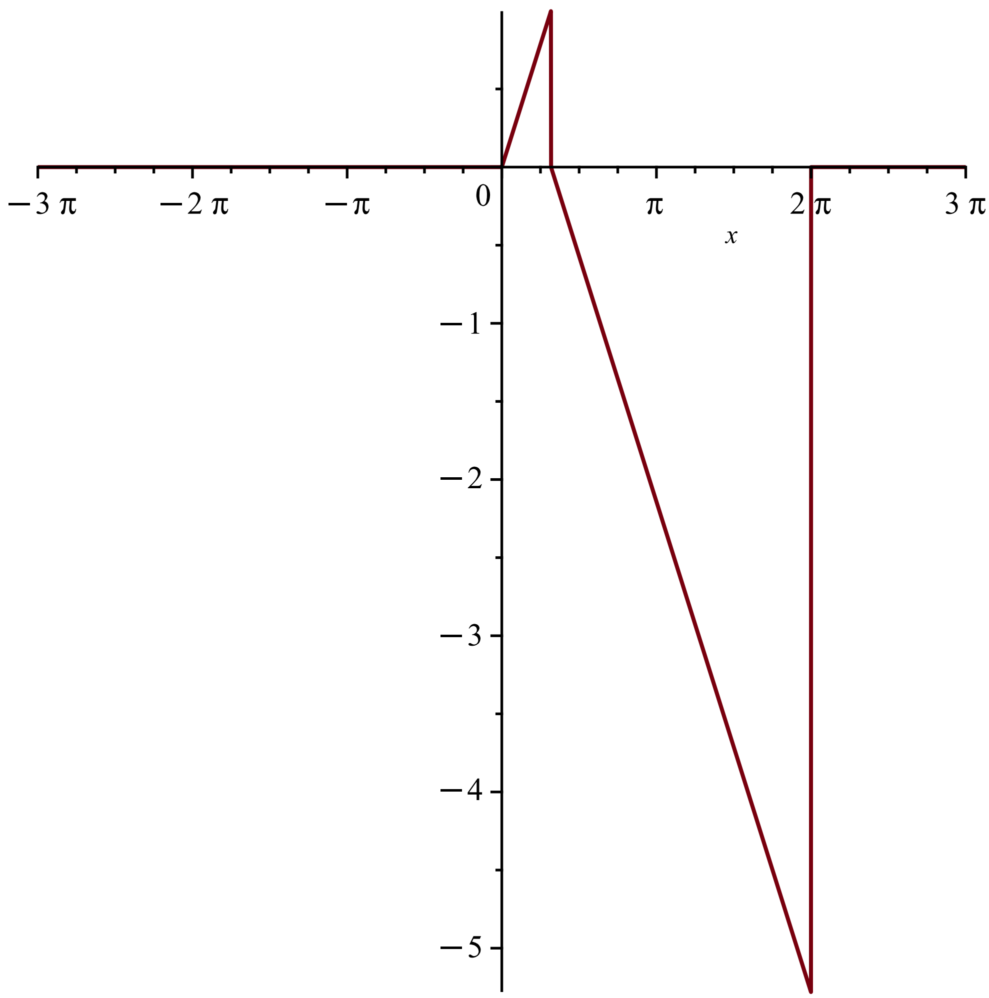
restart;

#define the function f in piecewise

$f := piecewise(0 < x < 1,\ x,\ \ 1 < x < 2\cdot\mathrm{Pi},\ 1 - x);$

$$f := \begin{cases} x & 0 < x < 1 \\ 1 - x & 1 < x < 2\,\pi \end{cases} \tag{18}$$

$plot(f,\ x = -3\cdot\mathrm{Pi}\,..3\cdot\mathrm{Pi});$



#Step2 find the Fourier coefficient a0, an, bn from these intervals $[0..1,\ @f = 2 + x]$ **to** $[[1..2\,\text{Pi},\ @f = 2 - x]$

$$a0 := \frac{1}{2\cdot \text{Pi}} \cdot int(2 + x,\ x = 0..1) + \frac{1}{2\cdot \text{Pi}} \cdot int(2 - x,\ x = 1..2\cdot \text{Pi});$$

$$a0 := \frac{5}{4\,\pi} + \frac{-\frac{3}{2} + 4\,\pi - 2\,\pi^2}{2\,\pi} \tag{19}$$

$$an := \frac{2}{2\cdot \text{Pi}} \cdot int((2 + x)\cdot\cos(n\cdot x), x = 0..1) + \frac{2}{2\cdot \text{Pi}} \cdot int((2 - x)\cdot\cos(n\cdot x),\ x = 1..2\cdot \text{Pi});$$

$$an := \frac{3\,n\sin(n) + \cos(n) - 1}{\pi\,n^2} - \frac{2\,n\sin(2\,n\,\pi)\,\pi - 2\,n\sin(2\,n\,\pi) + n\sin(n) + \cos(2\,n\,\pi) - \cos(n)}{\pi\,n^2} \tag{20}$$

$$bn := \frac{2}{2\cdot \text{Pi}} \cdot int((2 + x)\cdot\sin(n\cdot x), x = 0..1) + \frac{2}{2\cdot \text{Pi}} \cdot int((2 - x)\cdot\sin(n\cdot x),\ x = 1..2\cdot \text{Pi});$$

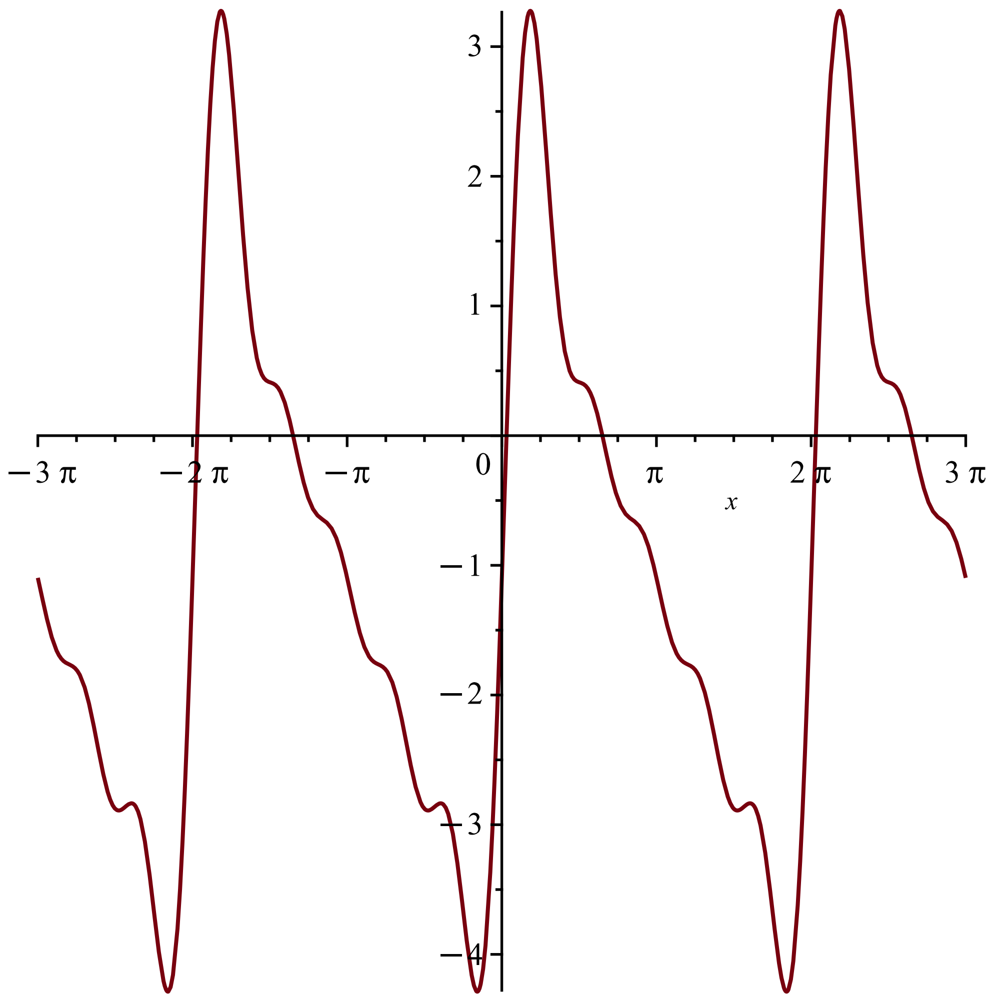
$$bn := \frac{-3\,n\cos(n) + \sin(n) + 2\,n}{\pi\,n^2} \tag{21}$$

$$-\frac{-2 n \cos(2 n \pi) \pi + 2 n \cos(2 n \pi) - n \cos(n) + \sin(2 n \pi) - \sin(n)}{\pi n^2}$$

#Step3 determine the Fourier series F=a0+sum[an·cos(nx)+bn·sin(nx)] from n=1..5

$F := a0 + sum(an \cdot \cos(n \cdot x) + bn \cdot \sin(n \cdot x), n = 1..5)$:

$plot(F,\quad x = -3 \cdot \mathrm{Pi}..3 \cdot \mathrm{Pi});$



#Pr 12.2 (Animation)

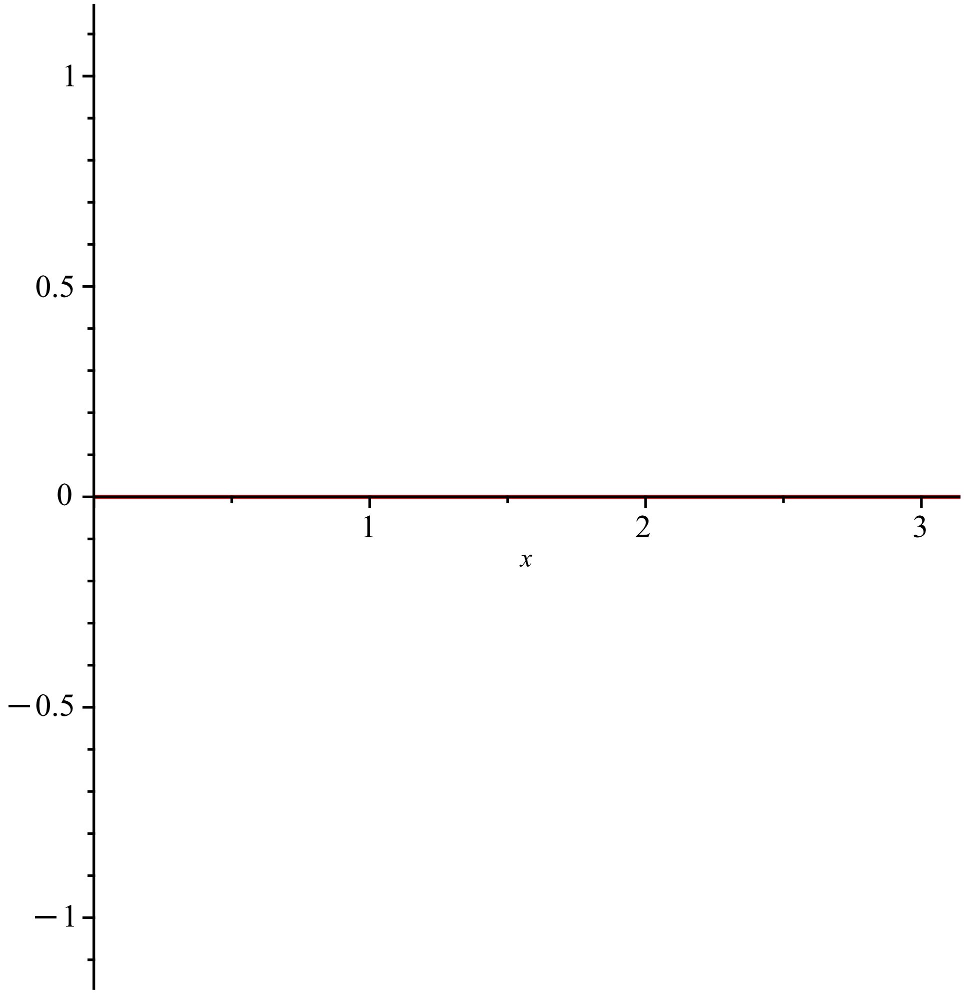
restart;

$u(x, t) := \sin(x) \cdot \cos(t) - \frac{1}{9} \cdot \sin(3 \cdot x) \cdot \cos(3 \cdot t) + \frac{1}{25} \cdot \sin(5 \cdot x) \cdot \cos(5 \cdot t) - \frac{1}{49} \cdot \sin(7 \cdot x) \cdot \cos(7 \cdot t);$

$$u := (x, t) \mapsto \sin(x) \cdot \cos(t) - \frac{\sin(3 \cdot x) \cdot \cos(3 \cdot t)}{9} + \frac{\sin(5 \cdot x) \cdot \cos(5 \cdot t)}{25} - \frac{\sin(7 \cdot x) \cdot \cos(7 \cdot t)}{49} \quad \textbf{(22)}$$

with(*plots*) :

$animate\left(u(x, t), x = 0..\mathrm{Pi},\ t = \frac{\mathrm{Pi}}{2}..10 \cdot \mathrm{Pi},\ frames = 1000\right);$



#Pr 12.4 (Separation of variables)

restart;
with(*PDETools*) :
$U(x) = U(y);$

$$U(x) = U(y) \tag{23}$$

$pde := diff(U(x), x, x) = diff(U(y), y, y);$

$$pde := \frac{d^2}{dx^2} U(x) = \frac{d^2}{dy^2} U(y) \tag{24}$$

#when the separation constant p > 0

#Setting rhs and lhs of pde to be -p and divide by U(y) and U(x) respectively

$sol1 := dsolve\left(\frac{lhs(pde)}{U(x)} = -p\right);$

$$sol1 := U(x) = c_1 \sin\left(\sqrt{p}\, x\right) + c_2 \cos\left(\sqrt{p}\, x\right) \tag{25}$$

$sol2 := dsolve\left(\frac{rhs(pde)}{U(y)} = -p\right);$

$$sol2 := U(y) = c_1 \sin\left(\sqrt{p}\, y\right) + c_2 \cos\left(\sqrt{p}\, y\right) \tag{26}$$

 # General Solution U(x,y)= sol1·sol1

Gen := *sol1*·*sol2*

$$Gen := U(x)\, U(y) = \left(c_1 \sin\left(\sqrt{p}\, x\right) + c_2 \cos\left(\sqrt{p}\, x\right)\right) \left(c_1 \sin\left(\sqrt{p}\, y\right) + c_2 \cos\left(\sqrt{p}\, y\right)\right) \tag{27}$$

#when the separation constant p=0

#Setting rhs and lhs of pde to be equal to zero

sol3 := *dsolve*(*lhs*(*pde*) = 0);

$$sol3 := U(x) = c_1 x + c_2 \tag{28}$$

sol4 := *dsolve*(*rhs*(*pde*) = 0);

$$sol4 := U(y) = c_1 y + c_2 \tag{29}$$

 # General Solution U(x,y)= sol3·sol4

Gen1 := *sol3*·*sol4*;

$$Gen1 := U(x)\, U(y) = \left(c_1 x + c_2\right) \left(c_1 y + c_2\right) \tag{30}$$

#when the separation constant p <0

#Setting lhs(pde)-pU(x)=0 and rhs(pde)+pU(y)=0

sol5 := *dsolve*(*lhs*(*pde*) − *p*·*U*(*x*) = 0);

$$sol5 := U(x) = c_1 e^{\sqrt{p}\, x} + c_2 e^{-\sqrt{p}\, x} \tag{31}$$

sol6 := *dsolve*(*rhs*(*pde*) + *p*·*U*(*y*) = 0);

$$sol6 := U(y) = c_1 \sin\left(\sqrt{p}\, y\right) + c_2 \cos\left(\sqrt{p}\, y\right) \tag{32}$$

 # General Solution U(x,y)= sol5·sol6

Gen2 := *sol5*·*sol6*;

$$Gen2 := U(x)\, U(y) = \left(c_1 e^{\sqrt{p}\, x} + c_2 e^{-\sqrt{p}\, x}\right) \left(c_1 \sin\left(\sqrt{p}\, y\right) + c_2 \cos\left(\sqrt{p}\, y\right)\right) \tag{33}$$

#Example 12.8 One-Dimentional Heat Equation

restart;

$f := x\cdot(5 - x);$

$$f := x\,(5 - x) \tag{34}$$

$Bn := simplify\left(\frac{2}{5}\cdot int\left(f\cdot\sin\left(\frac{n\cdot \mathrm{Pi}}{5}\cdot x\right),\ x = 0..5\right)\right);$

$$Bn := \frac{-50\,\pi\, n \sin(\pi n) - 100\cos(\pi n) + 100}{\pi^3 n^3} \tag{35}$$

$S := sum\left(Bn\cdot\sin\left(\frac{n\cdot \mathrm{Pi}}{5}\cdot x\right)\cdot\exp\left(-\left(\frac{1\cdot \mathrm{Pi}\cdot n}{5}\right)^2\cdot t\right),\ n = 1..13\right);$

$$S := \frac{200\sin\left(\frac{\pi x}{5}\right) e^{-\frac{\pi^2 t}{25}}}{\pi^3} + \frac{200\sin\left(\frac{3\pi x}{5}\right) e^{-\frac{9\pi^2 t}{25}}}{27\,\pi^3} + \frac{8\sin(\pi x)\, e^{-\pi^2 t}}{5\,\pi^3} \tag{36}$$

$$+\frac{200\sin\left(\frac{7\pi x}{5}\right)e^{-\frac{49\pi^2 t}{25}}}{343\pi^3}+\frac{200\sin\left(\frac{9\pi x}{5}\right)e^{-\frac{81\pi^2 t}{25}}}{729\pi^3}+\frac{200\sin\left(\frac{11\pi x}{5}\right)e^{-\frac{121\pi^2 t}{25}}}{1331\pi^3}$$

$$+\frac{200\sin\left(\frac{13\pi x}{5}\right)e^{-\frac{169\pi^2 t}{25}}}{2197\pi^3}$$

$S0 := eval(subs(t=0, S)):$ $S1 := eval(subs(t=0.1, S)):$ $S2 := eval(subs(t=0.2, S)):$ $S10 := eval(subs(t=1.0, S)):$

$S20 := eval(subs(t=2.0, S)):$ $S30 := eval(subs(t=3.0, S)):$

$plot([S0, S1, S2, S10, S20, S30, f],\ x=0..5);$