Biomedical Signal Processing

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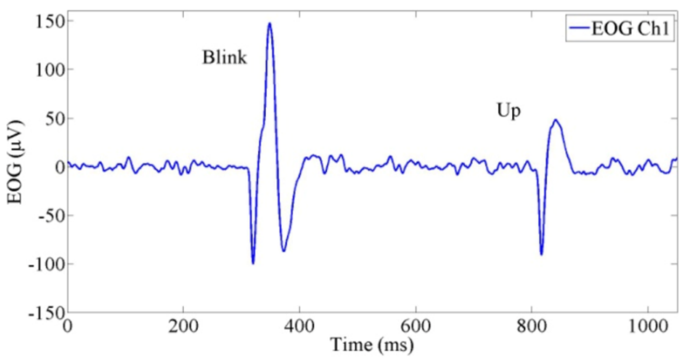
Curriculum

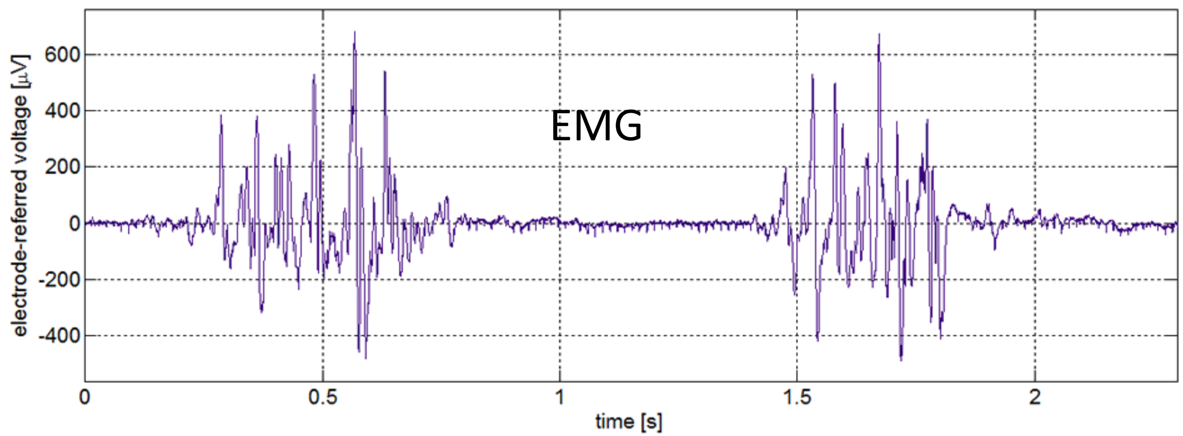
- Ch 1: Signal and Biomedical signal
- Ch 2: Discretizing
- Ch 3: Signal Analysis
- Ch 4: Signal Filtering
- Ch 5: Signal modeling

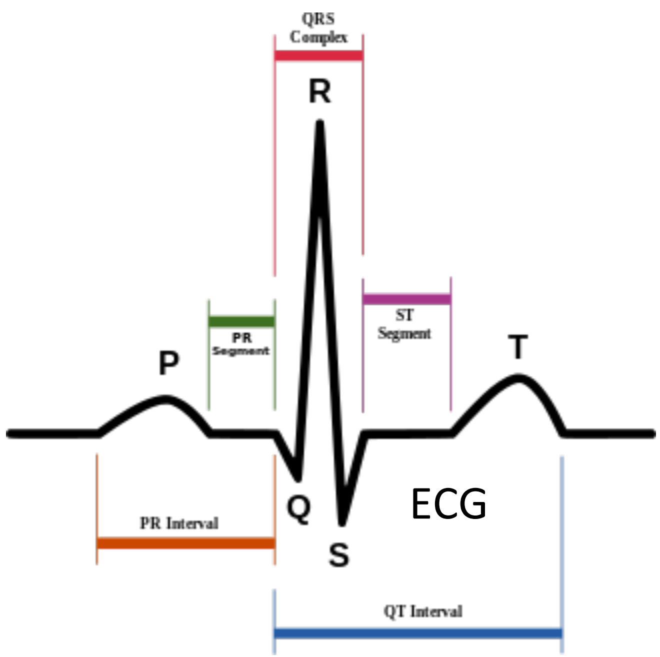
Chapter 1

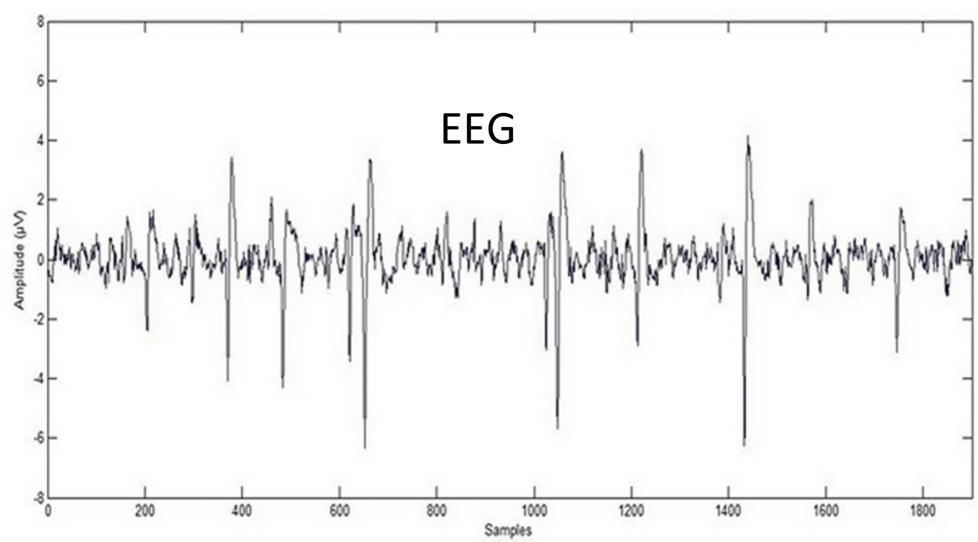
1.1 Bio-electrical Signal

EOG



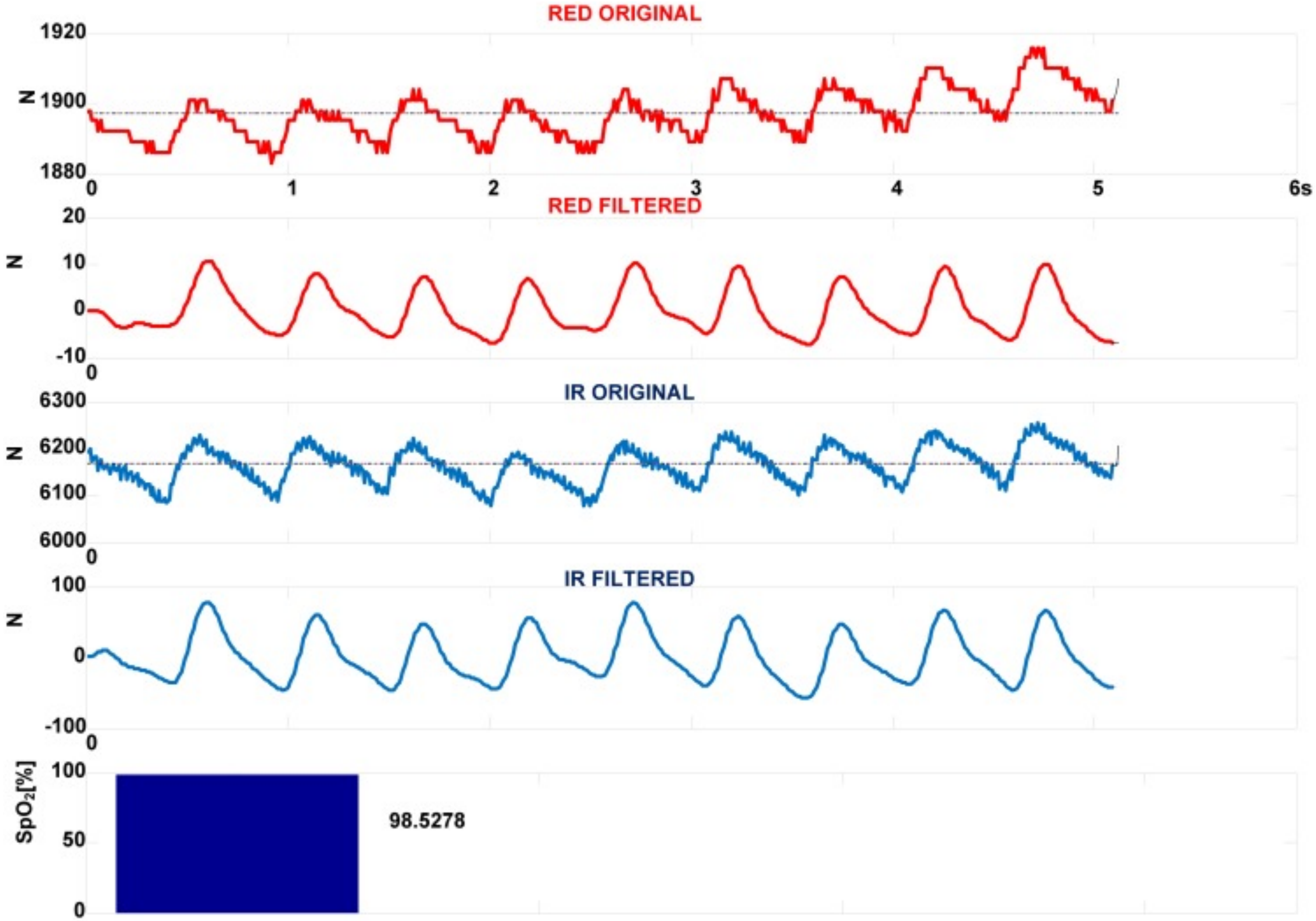




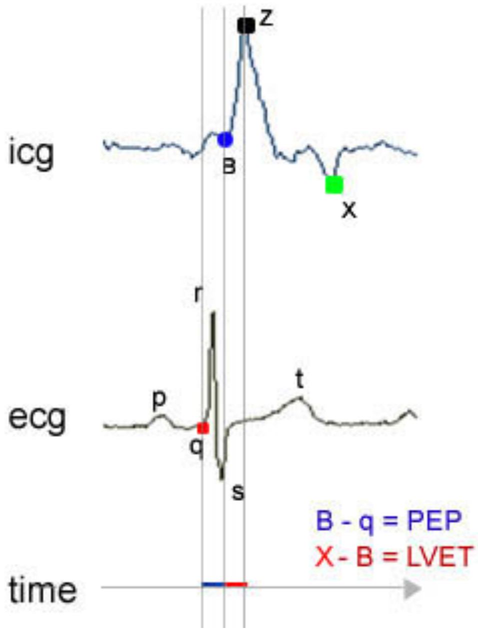


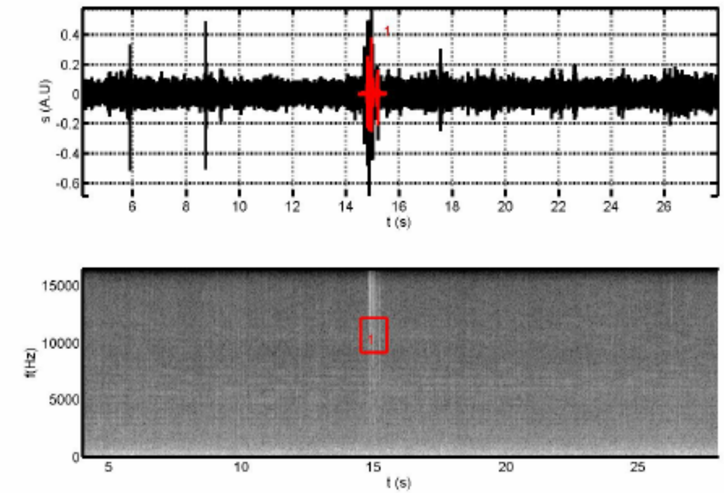
1.2 Bio-optical Signal

SPO2

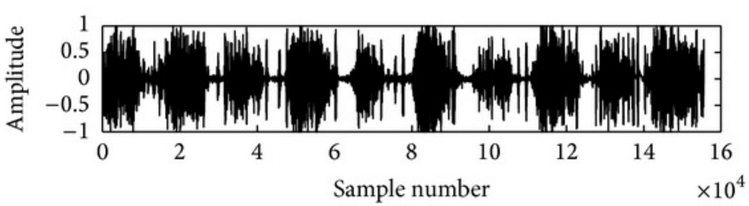


1.3 Bio-acoustic signal

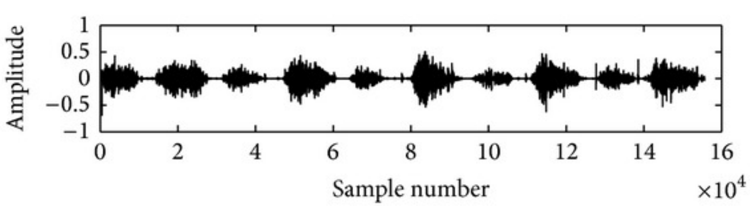




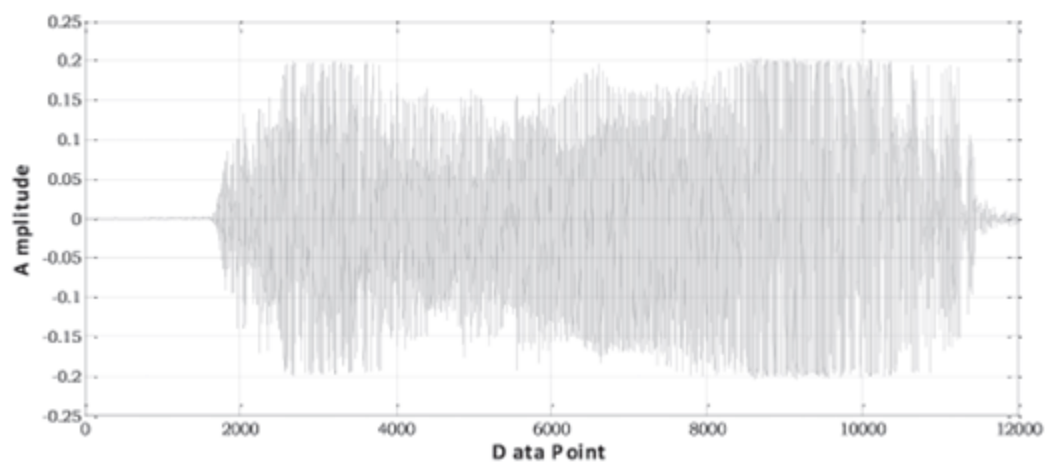
Heart sound



Lung sound (a)

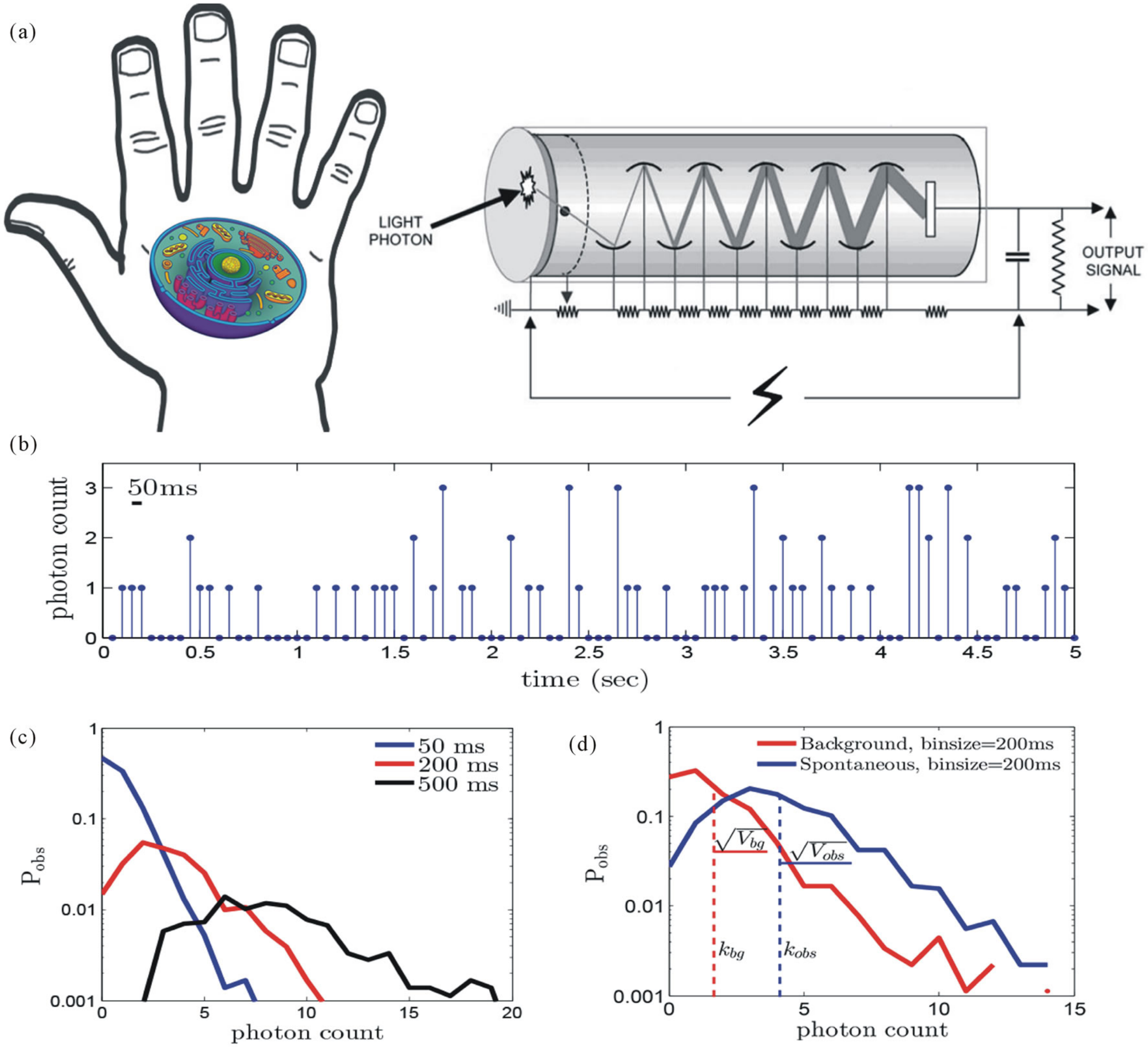


(b)



Abdomen sound

1.4 Bio-photon signal



Bức xạ sinh học

1.5 Basics of biomedical signal

- Definition: The electromagnetic wave including biological information and clinical conditions.

- Derivation of the active biomedical signal:

  - Bio-electrical signal -> from metabolism in living cells.

  - Bio-acoustic signal -> from movement of living organs.

  - Bio-photon signal -> from oxidized bio-chemical reaction.

- Derivation of the inactive biomedical signal:

  - Response from body under excitation of a physic energy (Electrics, Photon, Magnet, acoustic, heat)

  - Response from electromagnetic wave: X-ray, CT, MRI, PET, SPECT

  - Response from acoustic wave: Ultrasound

1.6 Approaches

- Profits of biomedical signal:

  - Biomedical signal includes the significant biological information to observe the clinical conditions of living body-> clinical diagnosis.

  - Response to excitation provide information of living organs, tissues, then observe the clinical conditions -> Clinical diagnosis and therapy.

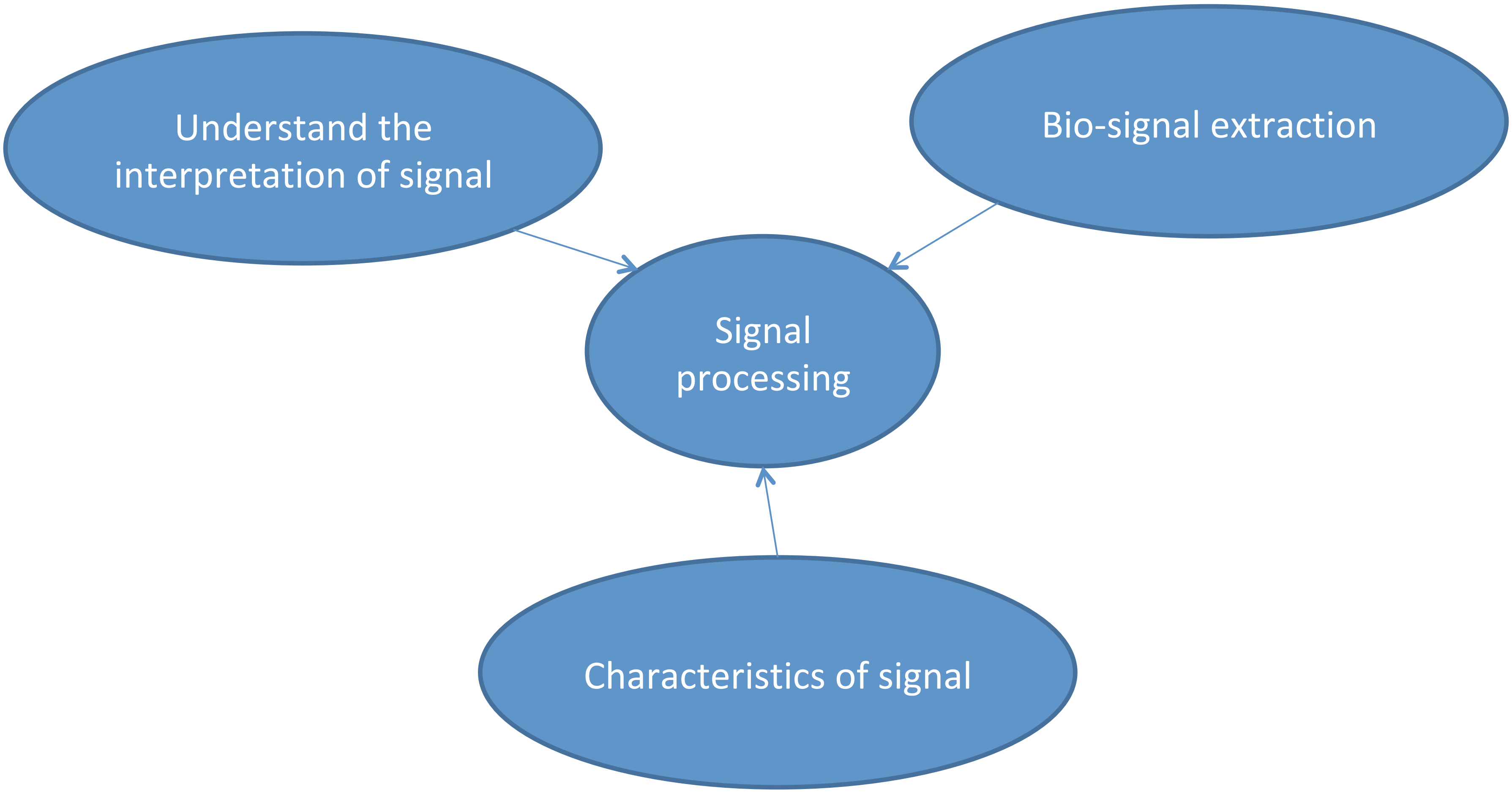
- The roles of inactive biomedical signal

  - Observe the clinical condition

  - Observe the biological info.

Biomedical Imaging

1.7 Problems of biomedical diagnosis



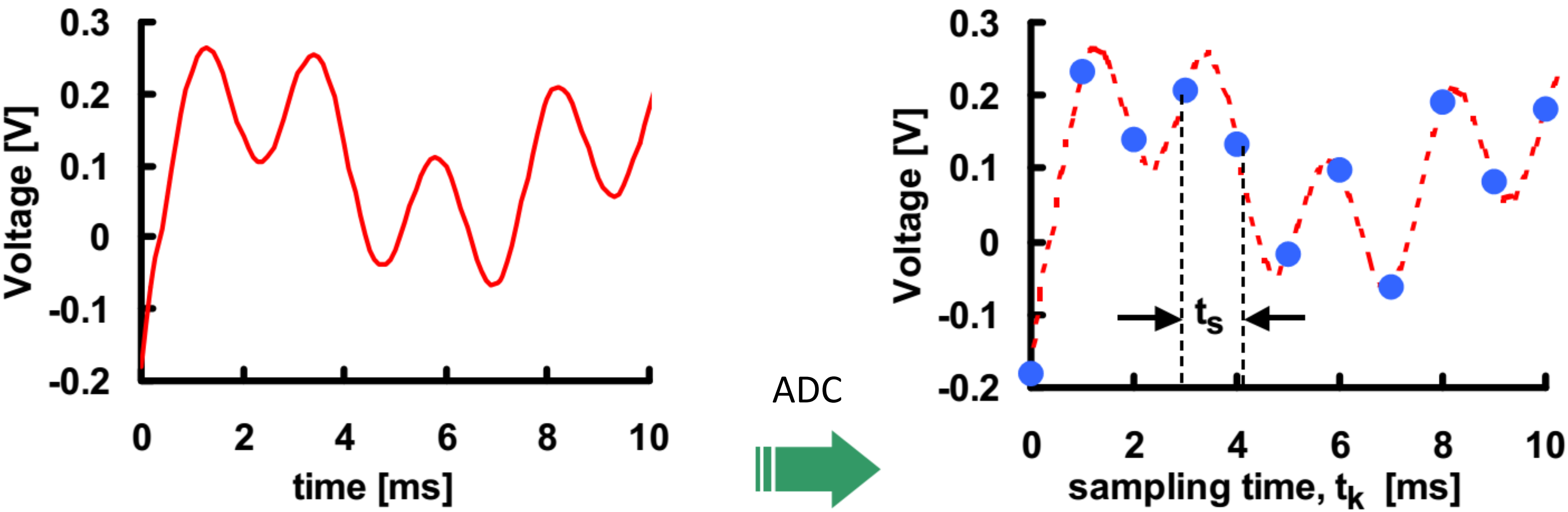
1.8 Why biomedical signal processing?

- The sophistication of biomedical signal -> multi wave forms, multi frequencies.
- The dynamic state of biomedical signal -> stochastic, multi parameters.
- Measured signal contaminated from environment noises.

1.9 Biomedical Signal Processing Methods

- Separating signal to individual specified signal
- Convert the signal from a domain to other domain
- Analyzing signal from distribution of specified signal
- Filter design to reduce the noises
- Check the efficiency of the filters through analysis of signal before and after filtered

Chapter 2: Discretizing



Analog signal

Digital signal

2.1 Definition of signal format

- Analogue: Electromagnetic wave varying in time continuously.

  - Properties:

  + Time resolution -> infinity

  + Value resolution -> infinity
- Digital: Electromagnetic wave varying in time with each period time.

  - Properties:

  + Time resolution -> limited

  + Value resolution -> limited

2.2: Pros and Cons of Analog Signal

Pros

- Continuity -> continuous information (infinity)

- Allow to observe the variation in small time -> The time length resolution is infinity.

- Cons

  - Difficulties to noise removal

  - Difficulties in complicated math functions implement

  - Difficulties in design signal analysis based on electronic circuit.

2.3: Pros and Cons of Digital Signal

- Pros:

  - Easy to process the signal based on math function

  - Small effected from noise (or no noise) from other sources after discretized.

  - Easy to analysis, separate the signal based on math function.

- Cons:

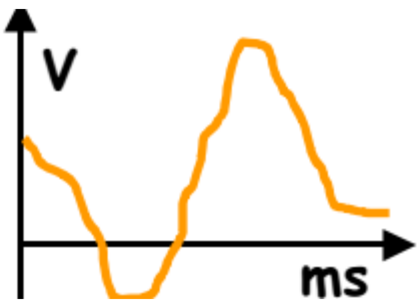
  - The discrete values (discontinuity) in both value and time. The limitation of the resolution

  - Limitation of the accuracy.

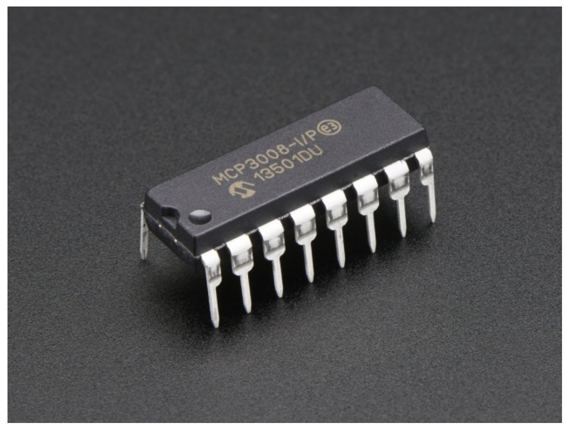
  - Real-time observation is limited

2.3 Conversion tools

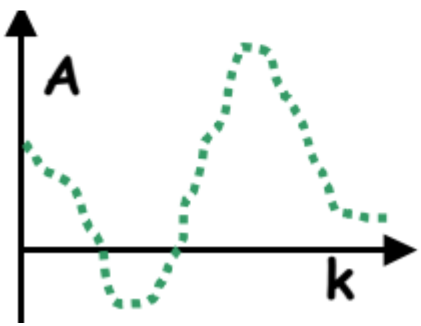
ADC (Analog to digital converter)



Analog

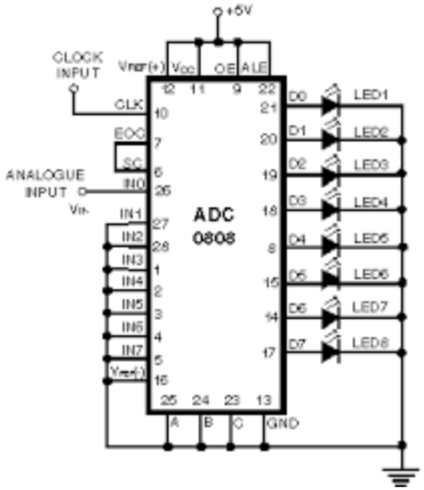


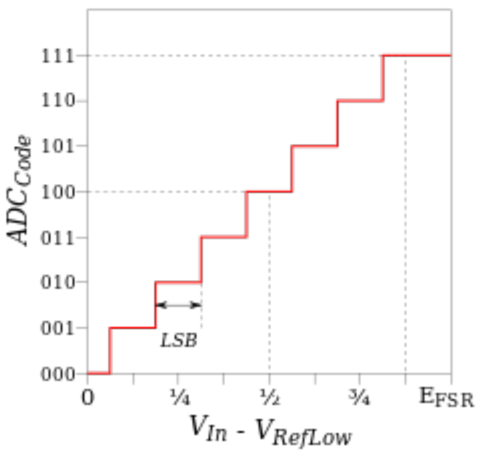
Conversion



Digital

2.4 parallel ADC

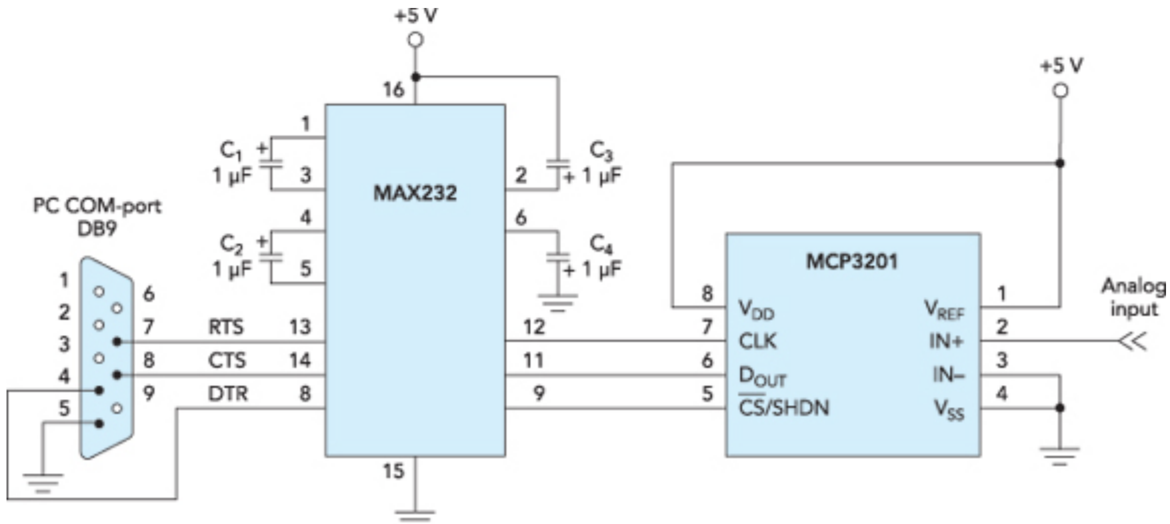




-Bit data of ADC provided parallelly
-The value of ADC is integer, corresponding to physic value
-Convert the value of ADC to physic value depending on ref+ và ref- voltage to define the measurement range

$$V_{\text{vật lý}} = \frac{N_{ADC}}{2^n}\left[V_{ref+} - V_{ref-}\right]$$

2.5 Serial ADC



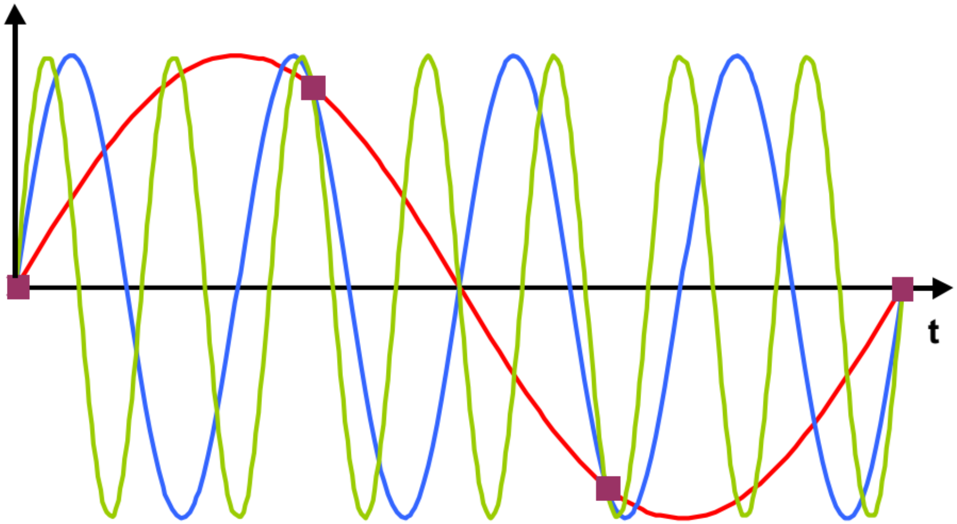
-Data bit is arranged as serial type in a frame.
-Number of bit from the data frame usually high (24 bit to 32 bit)
-The data bit stored in shifted register to transfer from parallel to serial.

2.6 Sampling time

When the wheel turning fast, the direction movement of the wheel reversing?

The errors of the information depends on sampling time

2.7 Sampling frequency



— $s_1(t) = \sin(8\pi f_0 t)$

— $s_2(t) = \sin(14\pi f_0 t)$

— $s(t) = \sin(2\pi f_0 t)$

■ $s(t)$ @ f_S

$f_0 = 1$ Hz, $f_S = 3$ Hz

$s_k(t) = \sin(2\pi (f_0 + k f_S) t)$, $|k| \in \mathbb{N}$

2.8 Sampling theories

- A signal can rebuilt if and only if it is sampled with the sampling frequency at least of double its own frequency. Nyquist theorem

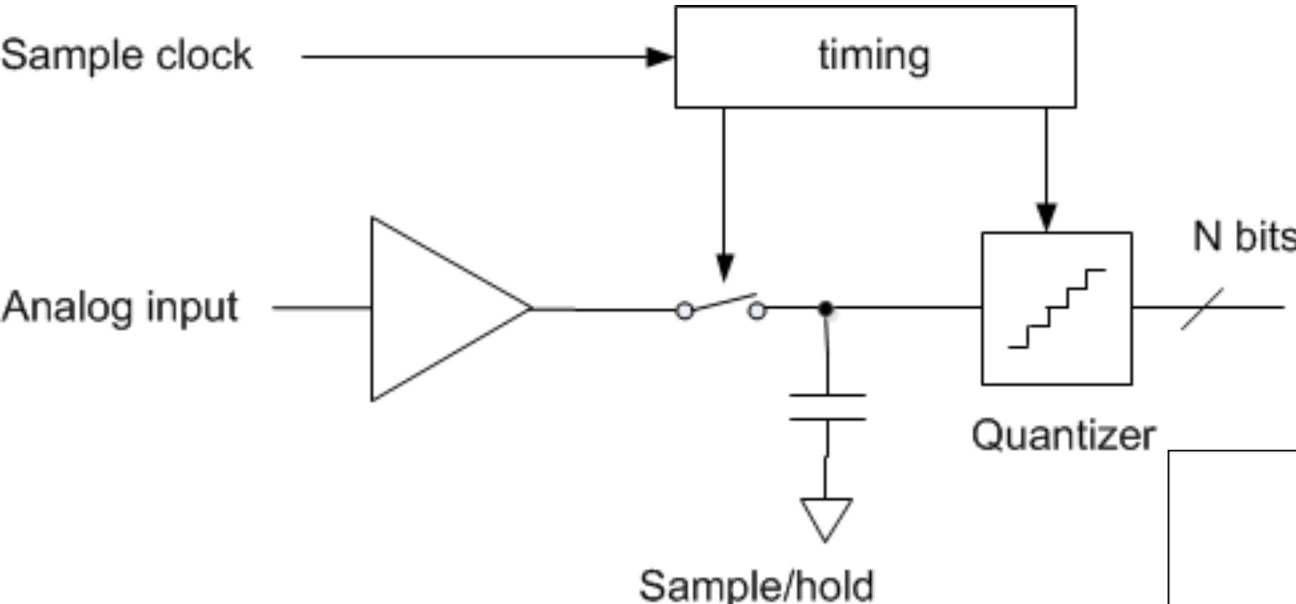
Nyquist frequency (rate) $f_N = 2\, f_{MAX}$ *or* f_{MAX} *or* $f_{S,MIN}$ *or* $f_{S,MIN}/2$

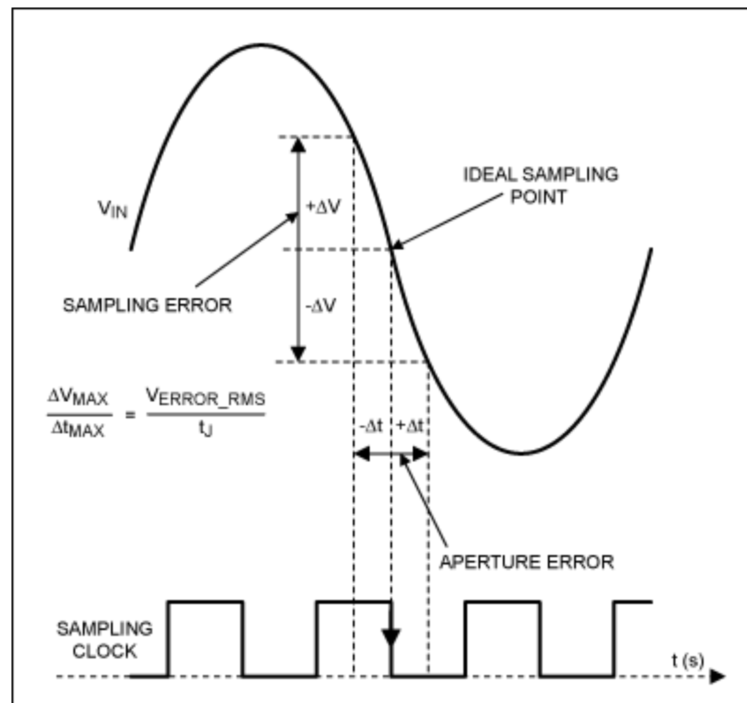
$$s(t) = 3 \cdot \underbrace{\cos(50\pi t)}_{F_1} + 10 \cdot \underbrace{\sin(300\pi t)}_{F_2} - \underbrace{\cos(100\pi t)}_{F_3}$$

$F_1 = 25$ Hz, $F_2 = 150$ Hz, $F_3 = 50$ Hz $\longrightarrow$ $\boxed{f_S > 300 \text{ Hz}}$

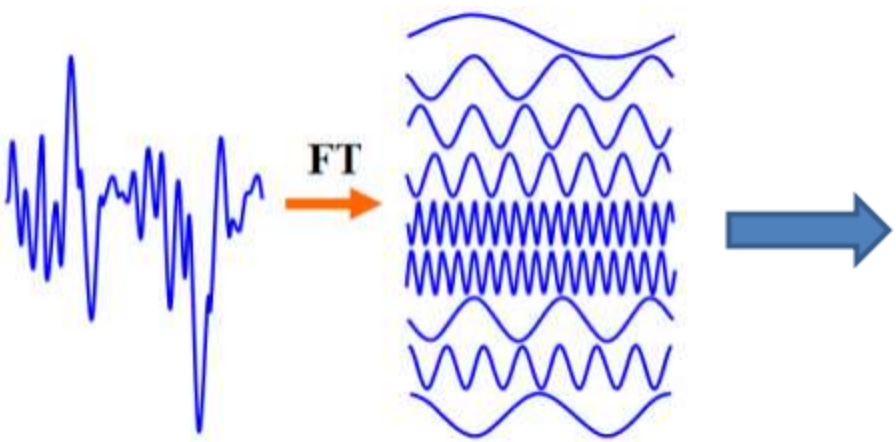
f_{MAX}

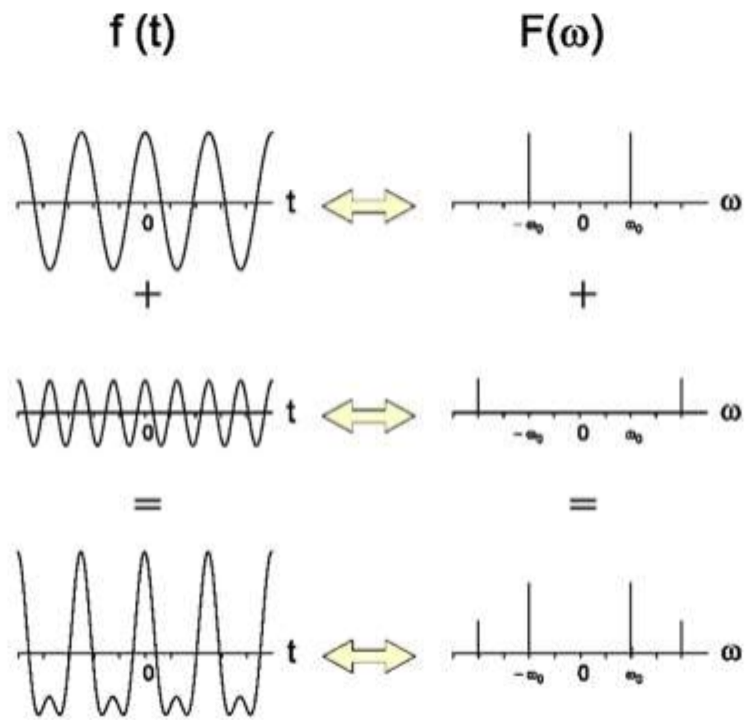
2.9 How to sampling





Ch. 3: Signal Analysis





Time Domain

Frequency Domain

3.1: Advantage of Signal Conversion in Frequency Domain

- Separation a complicated wave to be multi component waves.
- Each component wave and number of component wave contained in a signal wave make its characteristics.
- Component basic wave is considered as sine wave (Fourier) and included information
- Distribution of component waves from a signal called spectrum. Spectrum of biomedical signal contains biological information.

3.2 Difficulties of numerical analysis

- Analog signal analysis based on electronic circuit to perform complicated math functions.
- Fourier transform.
- Fourier transform needed to be performed in discrete domain.
- The transform from analog to discrete domain called z transfrom

3.2.1 Signal Components

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \qquad \text{for all } 0 < x < \ell \text{ and } t > 0$$

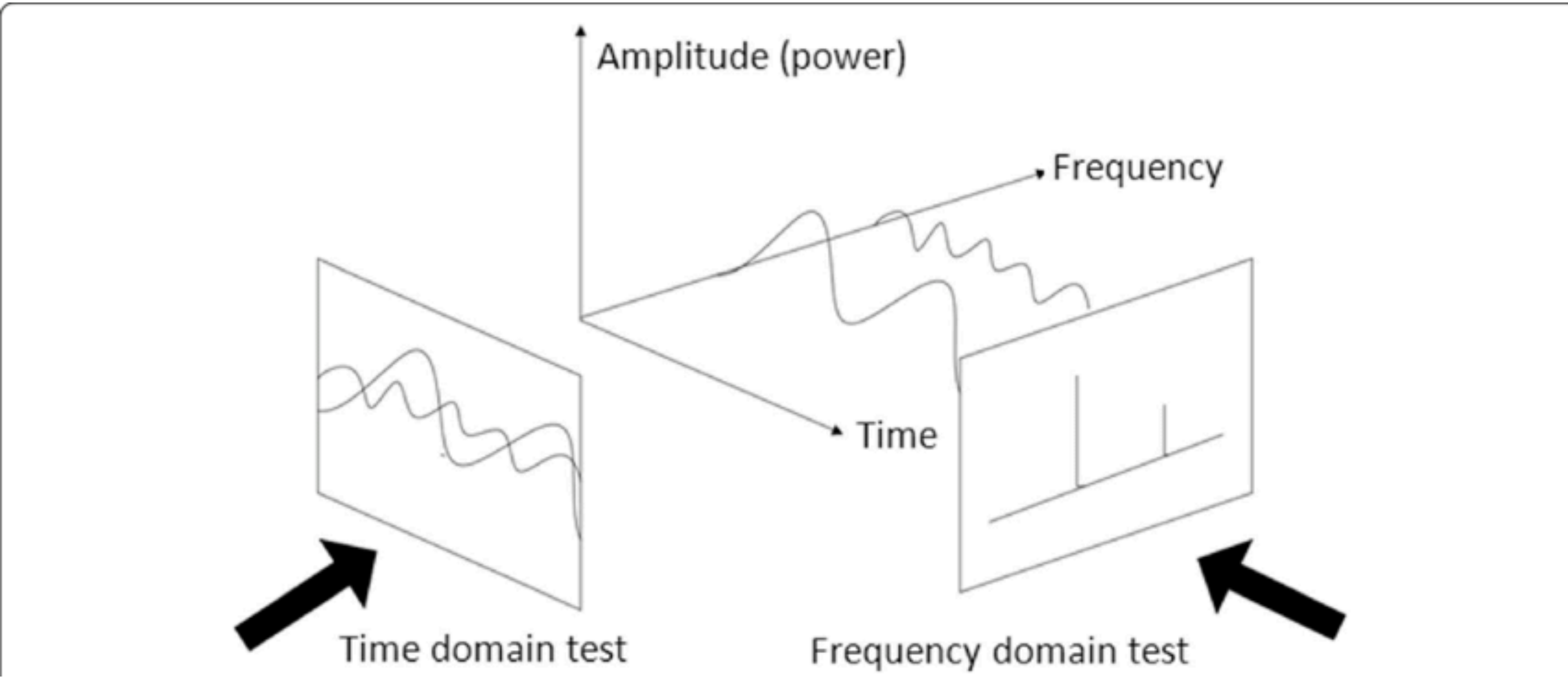
Solution

$$u(x,t) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{\ell}x\right)\left[\alpha_k \cos\left(\frac{ck\pi}{\ell}t\right) + \beta_k \sin\left(\frac{ck\pi}{\ell}t\right)\right]$$

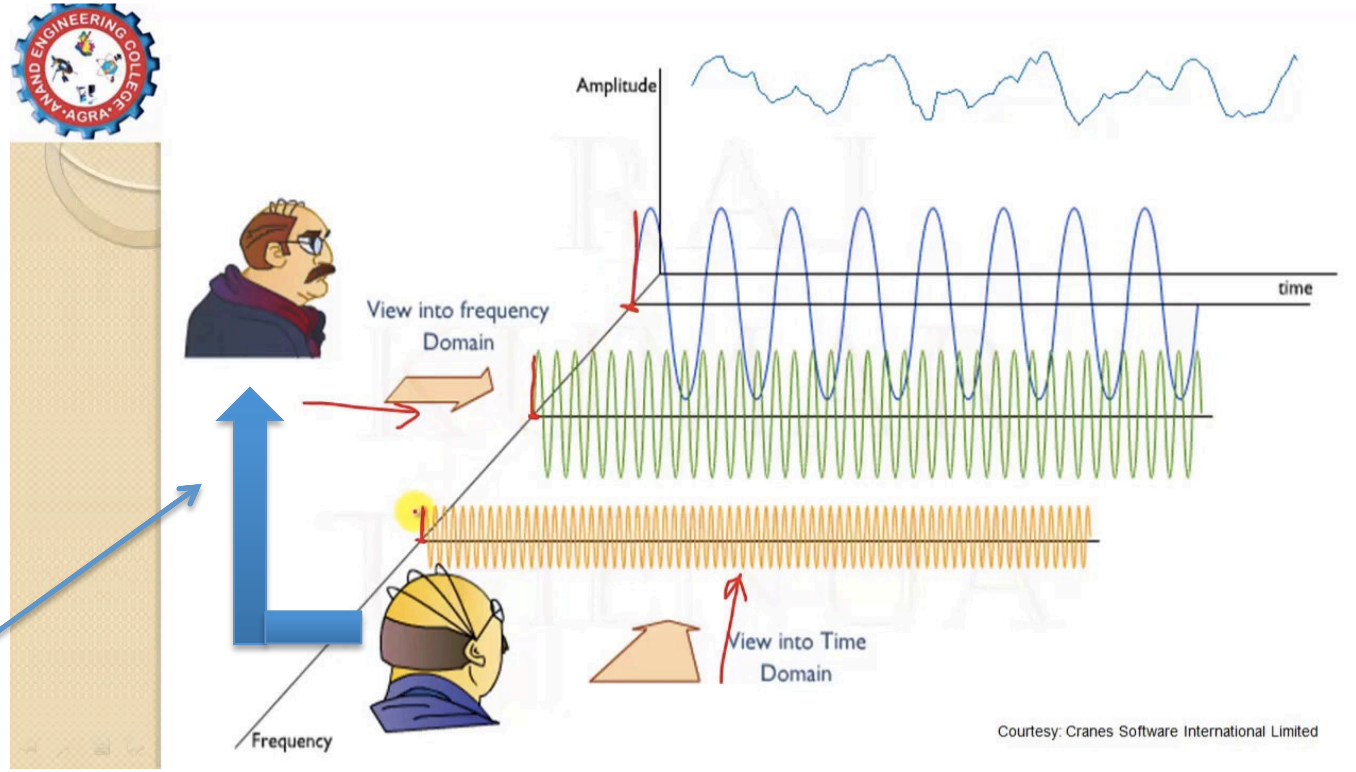
If the waveform includes many frequencies???

Complex wave

3.2.2 Domain Observation



3.2.3 How to change?



Conversion
(Transform)

F(t) -> F(f)

3.2.3 Terms of signal

- Time-domain signal -> waveform
- Frequency-domain signal -> spectrum
- Signal: the range of frequency should be observed
- Noise: the range of frequency should be eliminated
- Signal to noise ratio (SNR): Quality of measurement SNR=20*log(Signal Level/Noise Level)

3.3 Z Transform

Laplace transform of *f*(*t*):

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

Z transform of a side of *x*(*n*):

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$$

Z transform of both side of *x*(*n*):

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

3.4 Discrete Fourier Transform

$$S(f) = \int_{-\infty}^{+\infty} s(t) \cdot e^{-j2\pi f t} dt \quad \longrightarrow \quad S(f) = \sum_{n=-\infty}^{+\infty} s[n] \cdot e^{-j2\pi f n}$$

Fourier transform of analog signal — Fourier transform of digital signal

$$X(re^{i\omega}) = \sum_{n=-\infty}^{\infty} x(n)(re^{i\omega})^{-n}, \; or$$

$$X(re^{i\omega}) = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-i\omega n}, \; and \; if \; r = 1,$$

$$X(e^{i\omega}) = X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-i\omega n}$$

3.5 Digital Fourier Transform & Fast Fourier Transform FFT

$$S(f) = \int_{-\infty}^{+\infty} s(t)\cdot e^{-j2\pi f t}\,dt \xrightarrow{\text{FFT}} c_k = \frac{1}{T}\cdot\int_{0}^{T} s(t)\cdot e^{-jk\omega t}\,dt$$

Fourier Transform

Fast Fourier Transform

$$S(f) = \sum_{n=-\infty}^{+\infty} s[n]\cdot e^{-j2\pi f n} \xrightarrow{\text{DFT}} \tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} s[n]\cdot e^{-j\frac{2\pi k n}{N}}$$

Digital Fourier Transform

Fast Fourier Transform

3.6 Fourier Series

Signal contains many components. Fourier transform implements to a wave to extract that wave to be components called Fourier series.

synthesis

$$s(t) = a_0 + \sum_{k=1}^{+\infty} [a_k \cdot \cos(k\omega t) - b_k \cdot \sin(k\omega t)]$$

For all t but discontinuities

a_0, a_k, b_k : Fourier coefficients.

k: harmonic number,

T: period, $\omega = 2\pi/T$

analysis

$$a_0 = \frac{1}{T} \cdot \int_0^T s(t)dt$$

(signal average over a period, i.e. DC term & zero-frequency component.)

$$a_k = \frac{2}{T} \cdot \int_0^T s(t) \cdot \cos(k\omega t)dt$$

$$-b_k = \frac{2}{T} \cdot \int_0^T s(t) \cdot \sin(k\omega t)dt$$

Note: $\{\cos(k\omega t), \sin(k\omega t)\}_k$ form orthogonal base of function space.

* see next slide

3.7 Example with square wave

FS of odd* function: square wave.

$$\boxed{T = 2\pi \quad \Rightarrow \omega = 1}$$

$$a_0 = \frac{1}{2\pi}\cdot\left\{\int_0^{\pi} dt + \int_{\pi}^{2\pi}(-1)dt\right\} = 0$$

(zero average)

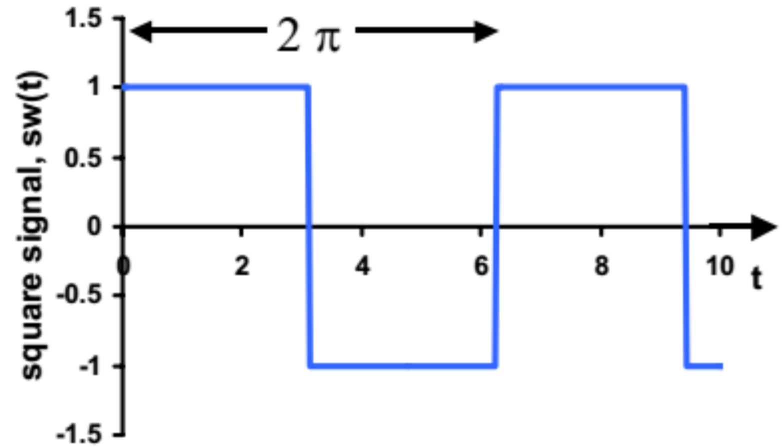
$$a_k = \frac{1}{\pi}\cdot\left\{\int_0^{\pi} \cos kt\, dt - \int_{\pi}^{2\pi} \cos kt\, dt\right\} = 0$$

(odd function)

$$-b_k = \frac{1}{\pi}\cdot\left\{\int_0^{\pi} \sin kt\, dt - \int_{\pi}^{2\pi} \sin kt\, dt\right\} = \ldots = \frac{2}{k\cdot\pi}\cdot\{1 - \cos k\pi\} =$$

$$= \begin{cases} \dfrac{4}{k\cdot\pi} & , \text{k odd} \\ 0 & , \text{k even} \end{cases}$$

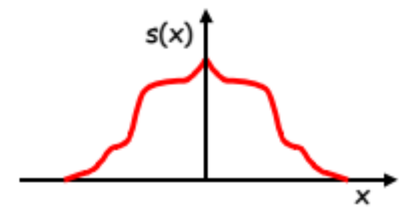
$$\boxed{sw(t) = \frac{4}{\pi}\cdot\sin t + \frac{4}{3\cdot\pi}\cdot\sin 3\cdot t + \frac{4}{5\cdot\pi}\cdot\sin 5\cdot t + \ldots}$$



* Even & Odd functions

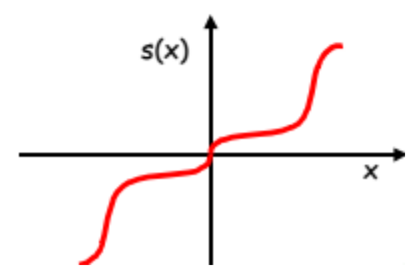
Even :

$s(-x) = s(x)$



Odd :

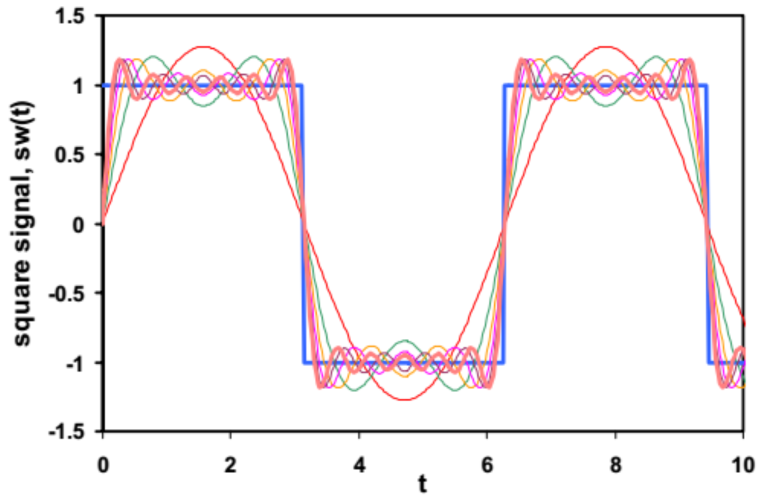
$s(-x) = -s(x)$



3.8 Square wave synthesis

$$sw_9(t) = \sum_{k=1}^{9}\left[-b_k \cdot \sin(kt)\right]$$

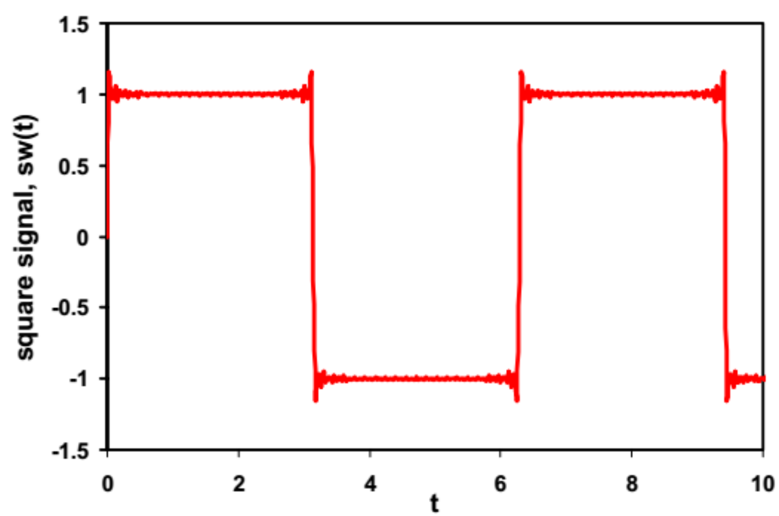
Tổng hợp xung vuông từ sóng sin



Nếu số lượng sóng sin hữu hạn -> hiện tượng Gibbs

Overshoot exist @ each discontinuity

$$sw_{79}(t) = \sum_{k=1}^{79}\left[-b_k \cdot \sin(kt)\right]$$



Hiện tượng Gibbs

3.9 Phase analysis

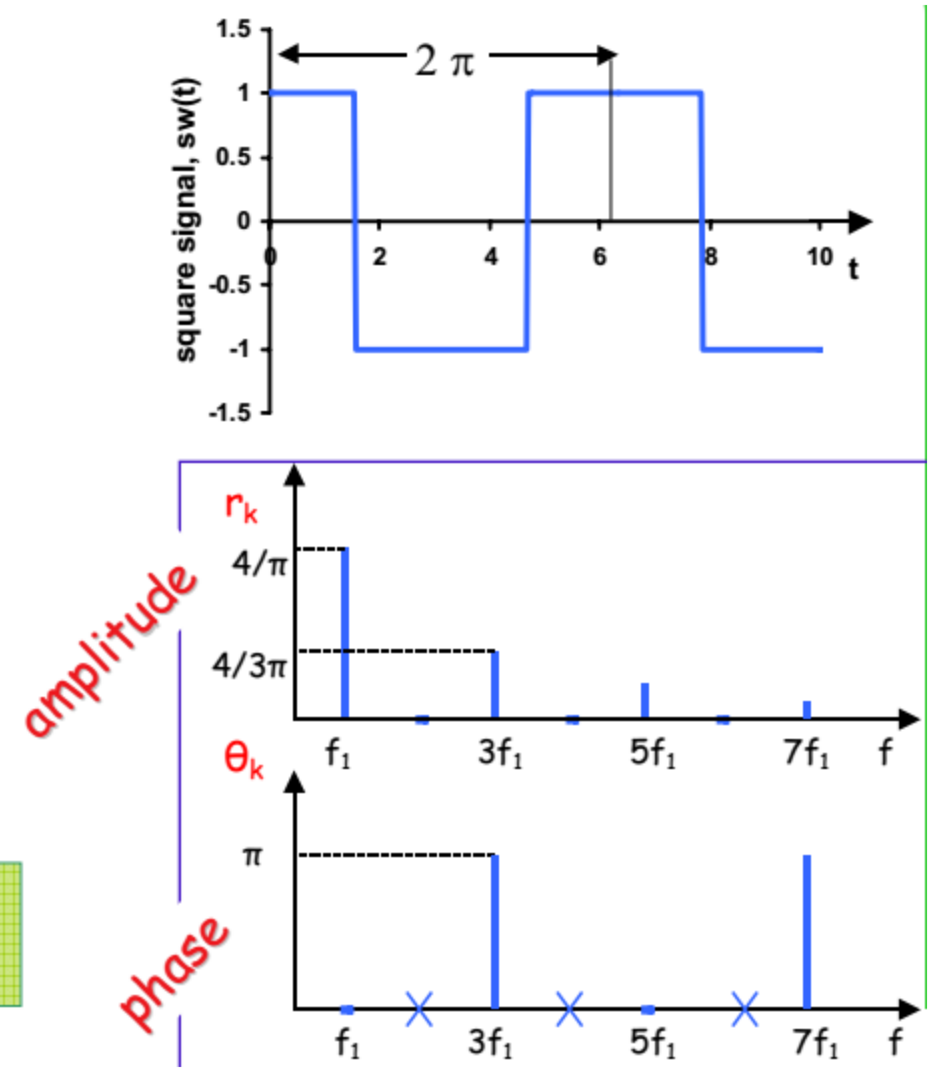
FS of even function:
$\pi/2$-advanced square-wave

$a_0 = 0$ *(zero average)*

$$a_k = \begin{cases} \dfrac{4}{k\cdot\pi}, & k \text{ odd}, \quad k = 1, 5, 9\ldots \\ -\dfrac{4}{k\cdot\pi}, & k \text{ odd}, \quad k = 3, 7, 11\ldots \\ 0, & k \text{ even.} \end{cases}$$

$-b_k = 0$ *(even function)*

Note: amplitudes unchanged ***BUT*** phases advance by $k\cdot\pi/2$.



3.10 Z Operation

	Time	Frequency
Homogeneity	$a \cdot s(t)$	$a \cdot S(k)$
Additivity	$s(t) + u(t)$	$S(k)+U(k)$
Linearity	$a \cdot s(t) + b \cdot u(t)$	$a \cdot S(k)+b \cdot U(k)$
Time reversal	$s(-t)$	$S(-k)$
Multiplication *	$s(t) \cdot u(t)$	$\sum_{m=-\infty}^{\infty} S(k-m)U(m)$
Convolution *	$\frac{1}{T} \cdot \int_0^T s(t-\bar{t}) \cdot u(\bar{t})\, d\bar{t}$	$S(k) \cdot U(k)$
Time shifting	$s(t-\bar{t})$	$e^{-j\frac{2\pi k \cdot \bar{t}}{T}} \cdot S(k)$
Frequency shifting	$e^{+j\frac{2\pi m t}{T}} \cdot s(t)$	$S(k - m)$

3.11 Spectral Power Density

Average power W : $W = \frac{1}{T}\int_{0}^{T} |s(t)|^2 \, dt \equiv s(t) \otimes s(t)$

Parseval's Theorem

$$\boxed{W = \sum_{k=-\infty}^{\infty} |c_k|^2 = a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}\left(a_k^2 + b_k^2\right)}$$

- FS convergence ~1/k

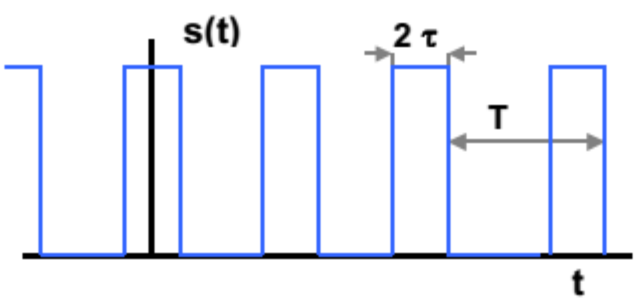
  ⇒ *lower frequency terms*

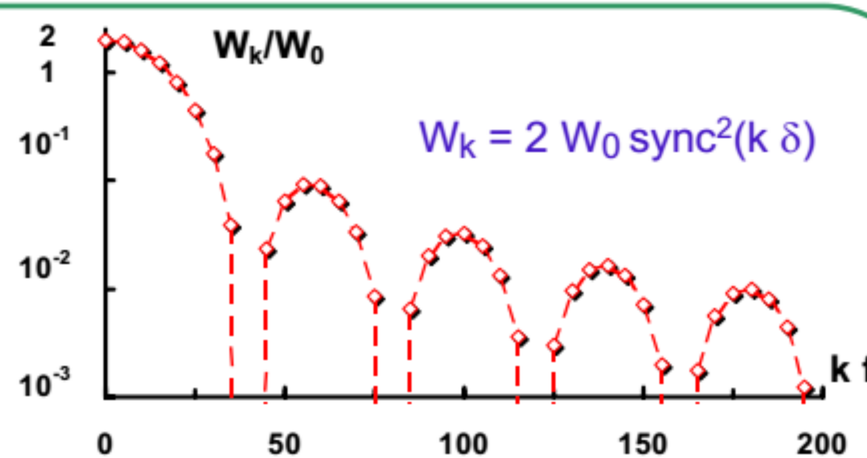
  $W_k = |c_k|^2$ carry most power.

- W_k vs. ω_k: Power density spectrum.

Example

Pulse train, duty cycle $\delta = 2\tau / T$





$W_k = 2\, W_0 \,\mathrm{sync}^2(k\,\delta)$

$b_k = 0 \quad a_0 = \delta\, s_{MAX}$

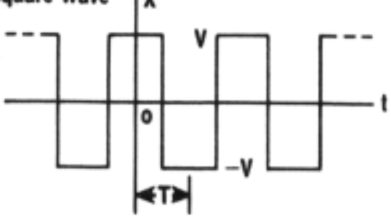
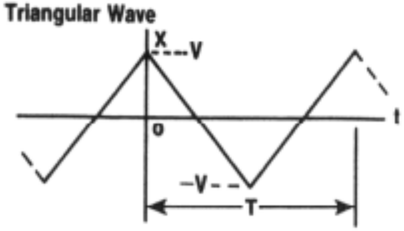
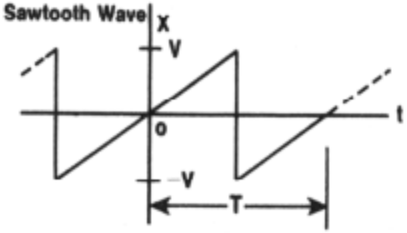
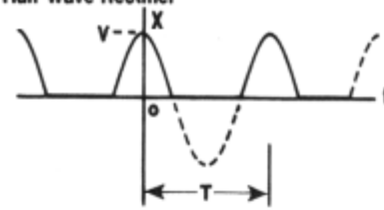
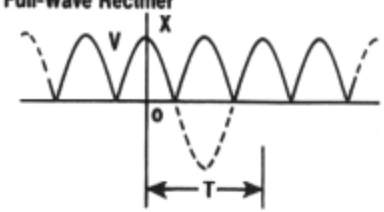
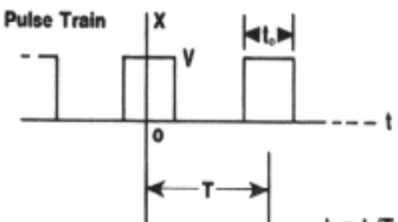
$a_k = 2\delta s_{MAX}\, \mathrm{sync}(k\,\delta)$

$W_0 = (\delta\, s_{MAX})^2$

$\mathrm{sync}(u) = \sin(\pi u)/(\pi u)$

$$W = W_0 \cdot \left\{1 + \sum_{k=1}^{\infty} \frac{W_k}{W_0}\right\}$$

3.12: DFT basic function

Wave Shape	Fourier Series -- $\omega_o = 2\pi/T$
Square Wave 	$x(t)=\frac{4V}{\pi}(\cos \omega_o t-\frac{1}{3}\cos 3\omega_o t+\frac{1}{5}\cos 5\omega_o t-\frac{1}{7}\cos 7\omega_o t+\dots)$
Triangular Wave 	$x(t)=\frac{8V}{\pi^2}(\cos \omega_o t+\frac{1}{9}\cos 3\omega_o t+\frac{1}{25}\cos 5\omega_o t+\dots)$
Sawtooth Wave 	$x(t)=\frac{2V}{\pi}(\sin \omega_o t-\frac{1}{2}\sin 2\omega_o t+\frac{1}{3}\sin 3\omega_o t-\frac{1}{4}\sin 4\omega_o t+\dots)$
Half-Wave Rectifier 	$x(t)=\frac{V}{\pi}(1+\frac{\pi}{2}\cos \omega_o t+\frac{2}{3}\cos 2\omega_o t-\frac{2}{15}\cos 4\omega_o t+\frac{2}{35}\cos 6\omega_o t-\dots (-1)^{n/2+1}\frac{2}{n^2-1}\cos n\omega_o t\dots)$ n even
Full-Wave Rectifier 	$x(t)=\frac{2V}{\pi}(1+\frac{2}{3}\cos 2\omega_o t-\frac{2}{15}\cos 4\omega_o t+\frac{2}{35}\cos 6\omega_o t-\dots (-1)^{n/2+1}\frac{2}{n^2-1}\cos n\omega_o t\dots)$ n even
Pulse Train  1754-16 $k = t_o/T$	$x(t)=V[k+\frac{2}{\pi}(\sin k\pi \cos \omega_o t+\frac{1}{2}\sin 2k\pi \cos 2\omega_o t+\dots+\frac{1}{n}\sin nk\pi \cos n\omega_o t+\dots)]$ $k = t_o/T$

3.13: Digital Fourier Transform

Band-limited signal s[n], period = N.

DFS defined as:

analysis

$$\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} s[n]\cdot e^{-j\frac{2\pi k n}{N}}$$

Note: $\tilde{c}_{k+N} = \tilde{c}_k \Leftrightarrow$ same period N

i.e. time periodicity propagates to frequencies!

synthesis

$$s[n] = \sum_{k=0}^{N-1} \tilde{c}_k \cdot e^{j\frac{2\pi k n}{N}}$$

Synthesis: finite sum $\Leftarrow$ band-limited s[n]

DFS generate periodic c_k with same signal period

Orthogonality in DFS:

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi\, n(k-m)}{N}} = \delta_{k,m}$$

Kronecker's delta

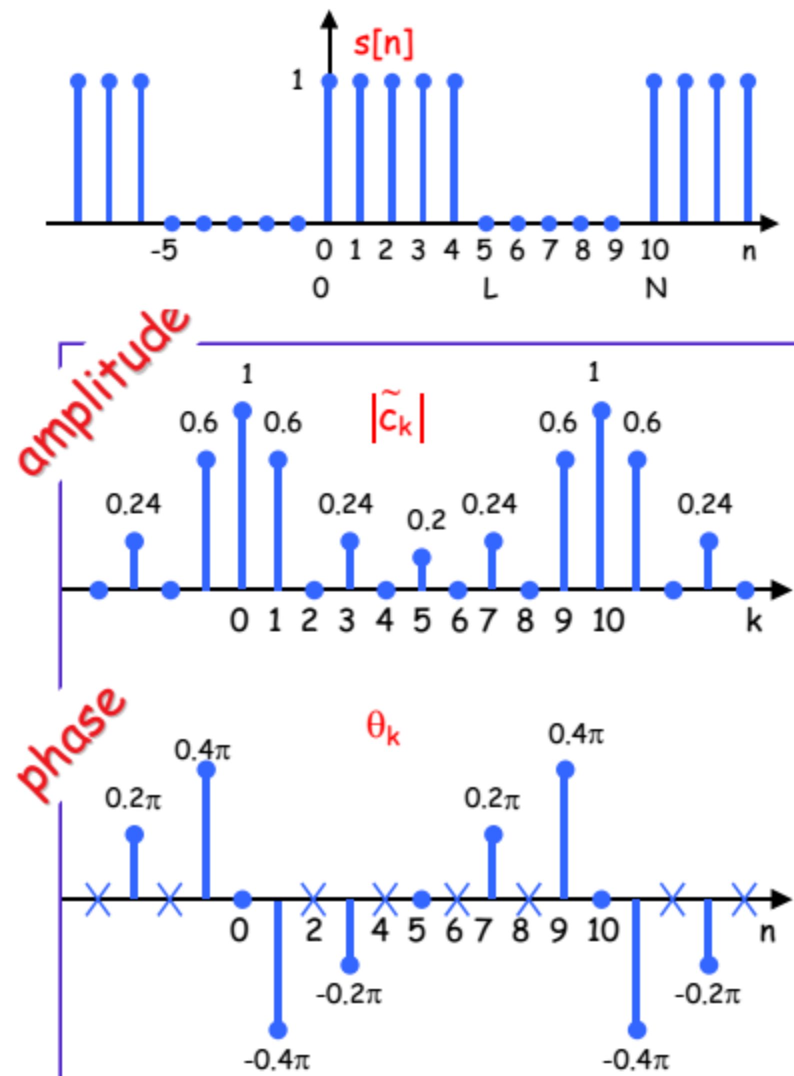
N consecutive samples of s[n] completely describe s in time or frequency domains.

3.14: Digital Fourier Integral

DFS of periodic discrete 1-Volt square-wave

s[n]: period N, duty factor L/N

$$\tilde{c}_k = \begin{cases} \dfrac{L}{N} \quad , & k = 0, +N, \pm 2N, \dots \\[2ex] \dfrac{e^{-j\frac{\pi k (L-1)}{N}}}{N} \cdot \dfrac{\sin\left(\frac{\pi kL}{N}\right)}{\sin\left(\frac{\pi k}{N}\right)} \quad , & \text{otherwise} \end{cases}$$



Discrete signals ⇒ periodic frequency spectra.
Compare to continuous rectangular function (slide # 10, "FS analysis - 1")

3.15: Basic function of Digital Fourier

	Time	Frequency
Homogeneity	$a\cdot s[n]$	$a\cdot S(k)$
Additivity	$s[n] + u[n]$	$S(k)+U(k)$
Linearity	$a\cdot s[n] + b\cdot u[n]$	$a\cdot S(k)+b\cdot U(k)$
Multiplication *	$s[n]\cdot u[n]$	$\frac{1}{N}\cdot\sum_{h=0}^{N-1} S(h)U(k-h)$
Convolution *	$\sum_{m=0}^{N-1} s[m]\cdot u[n-m]$	$S(k)\cdot U(k)$
Time shifting	$s[n - m]$	$e^{-j\frac{2\pi k\cdot m}{T}}\cdot S(k)$
Frequency shifting	$e^{+j\frac{2\pi h t}{T}}\cdot s[n]$	$S(k - h)$

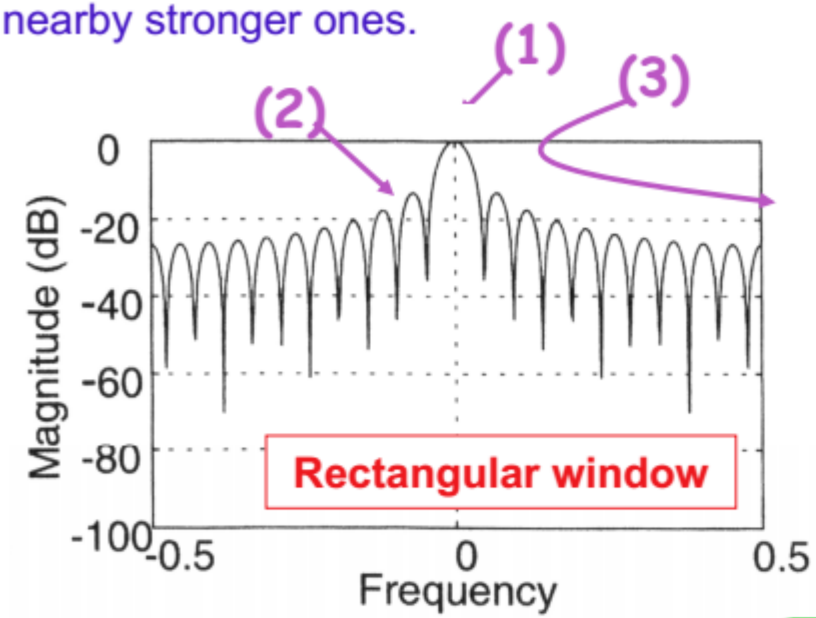
3.16: Spectral Windows

- Finite discrete sequence ⇒ spectrum convoluted with rectangular window spectrum.
- Leakage amount depends on chosen window & on how signal fits into the window.

(1) **Resolution**: capability to distinguish different tones. Inversely proportional to main-lobe width. *Wish: as high as possible.*

(2) **Peak-sidelobe level**: maximum response outside the main lobe. Determines if small signals are hidden by nearby stronger ones. *Wish: as low as possible.*

(3) **Sidelobe roll-off**: sidelobe decay per decade. Trade-off with (2).

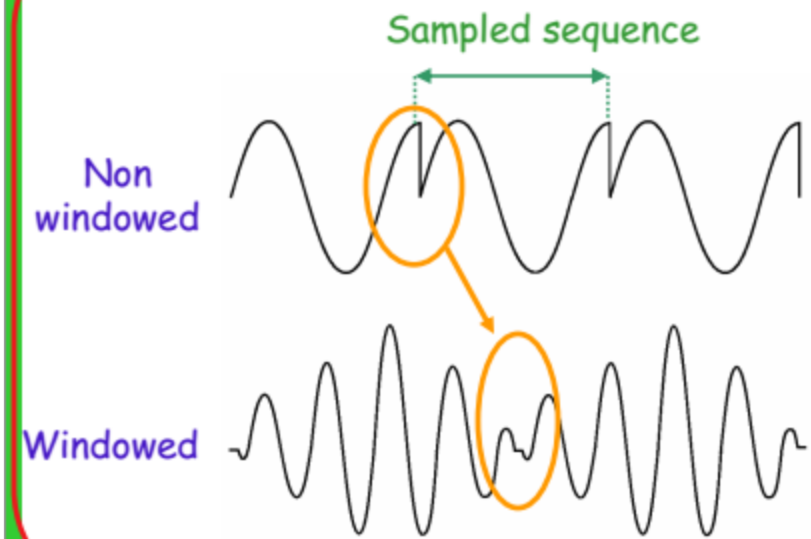


Several windows used (**application-dependent**): Hamming, Hanning, Blackman, Kaiser ...

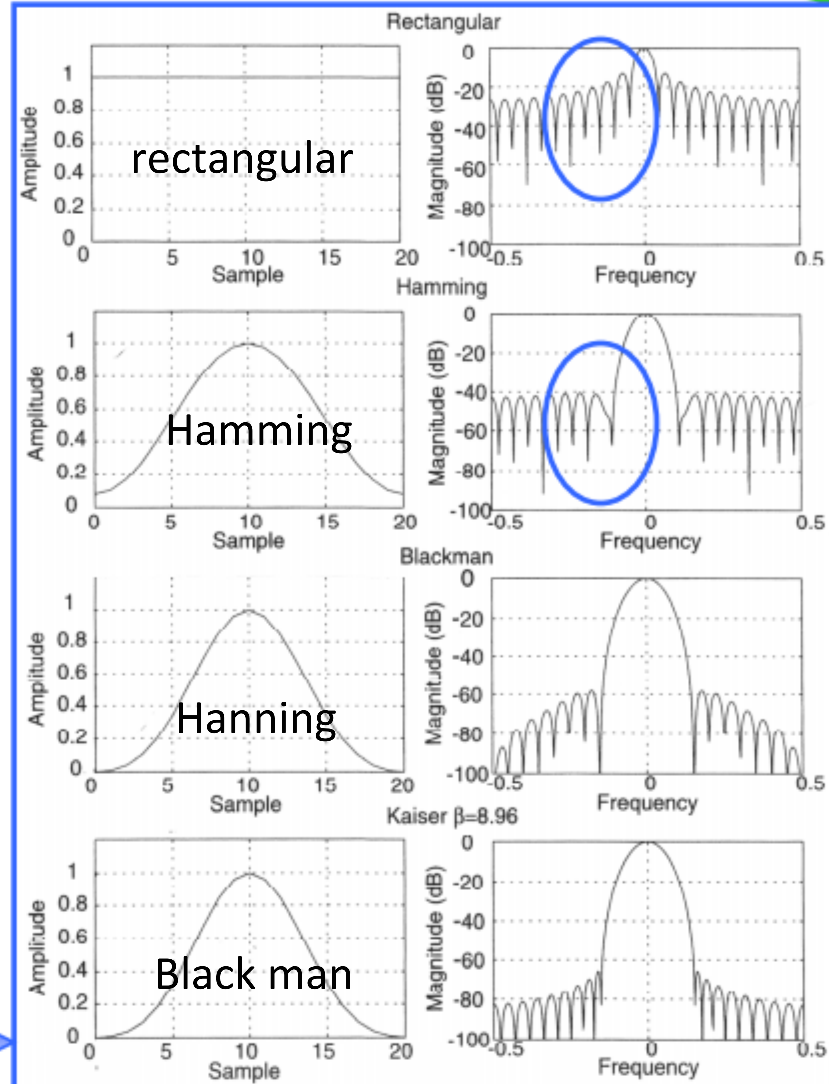
3.16: Popular Windows

Windowing reduces leakage by minimising sidelobes magnitude.

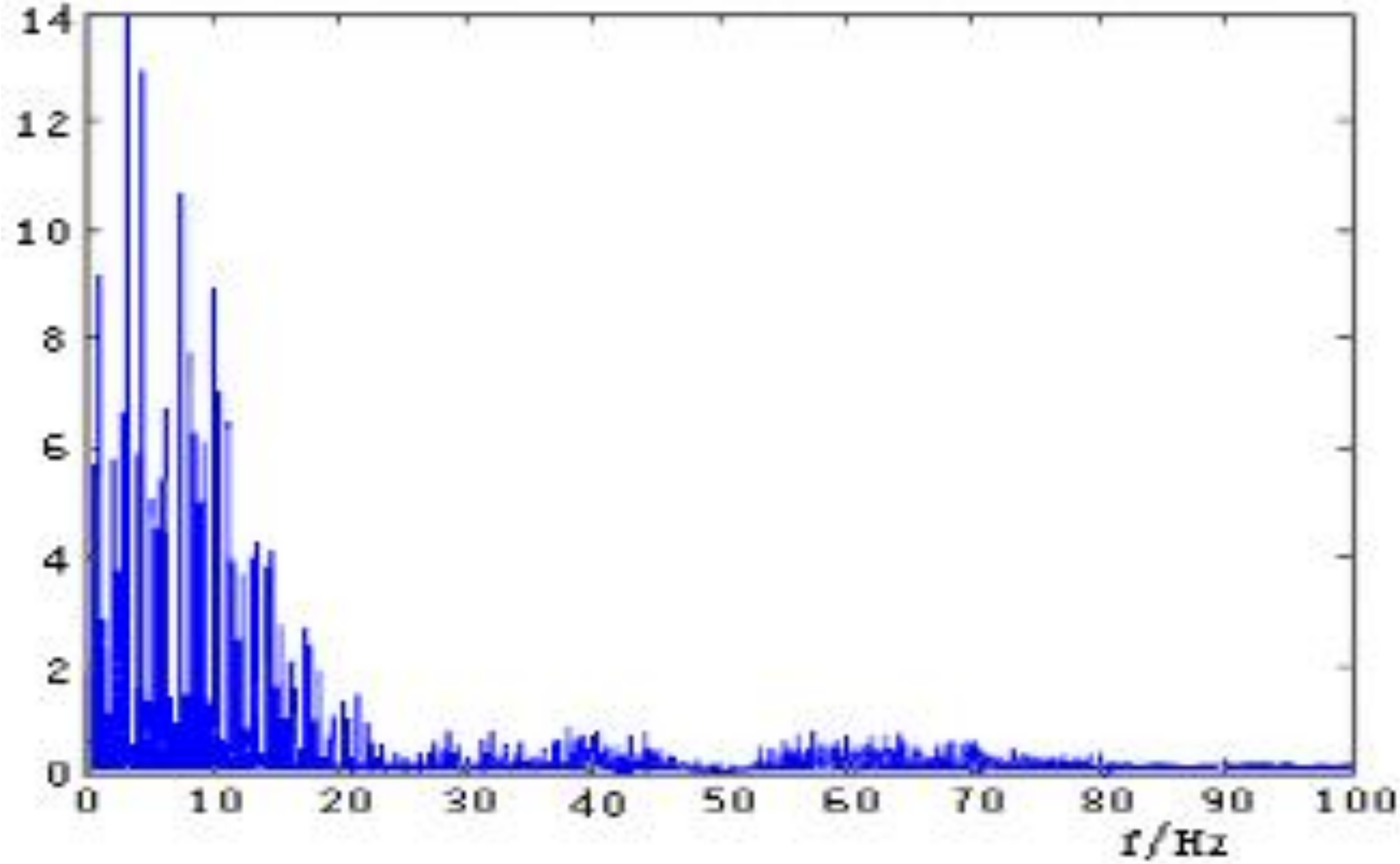
In time it reduces end-points discontinuities.



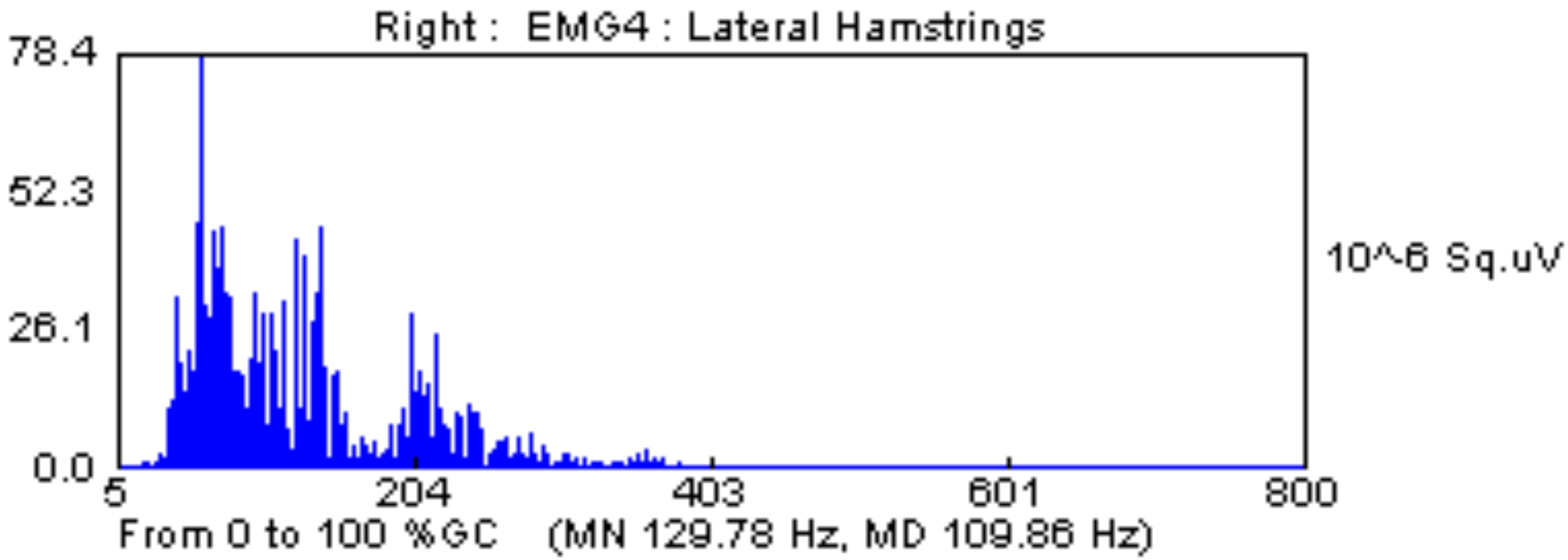
Some window functions



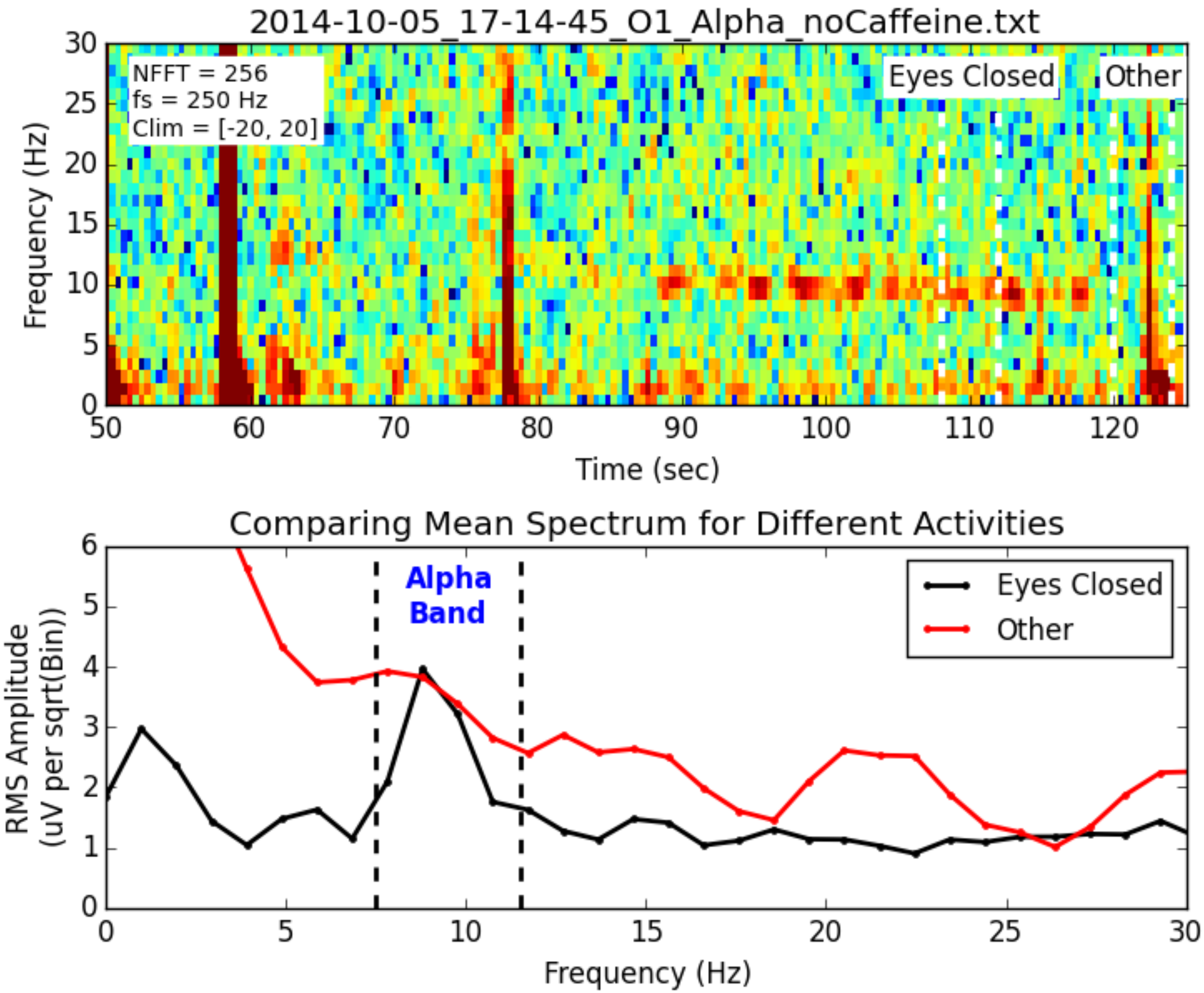
3.17 ECG Spectral



3.18 EMG Spectral



3.19: EEG Spectral



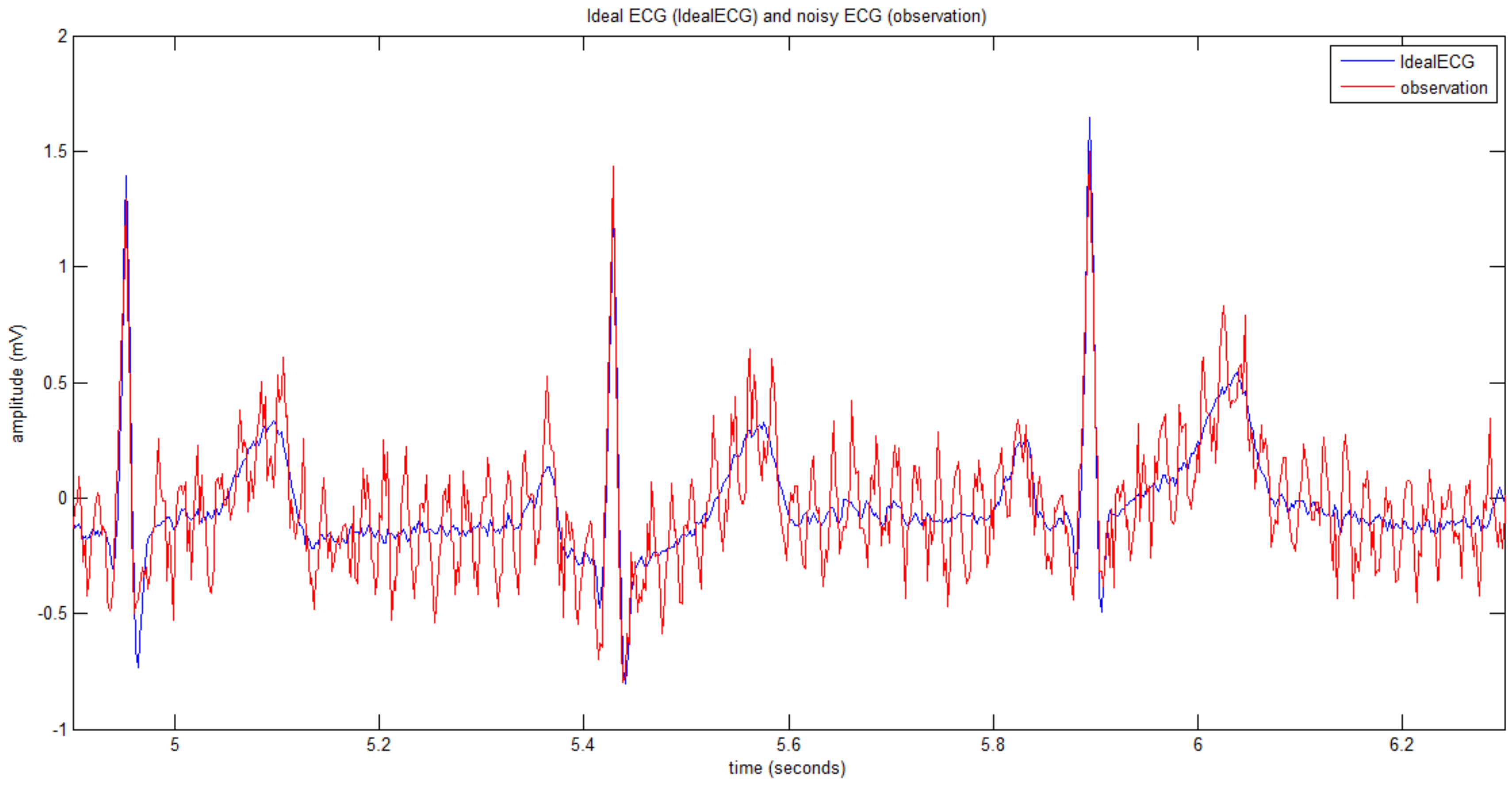
3.20: Tools for spectrum analysis

NATIONAL INSTRUMENTS
LabVIEW

MATLAB

Ch 4: Signal Filtering

- Why filtering?



4.1 Filter Concept

- Each signal has frequencies range (spectrum)
- Noises is other frequencies and/or same frequencies range but emitted from other sources
- The noises from other range (spectrum) can be eliminated (reduced) with electric circuit and/or math function.
- Electric/Electronic circuit possible absorb the noise frequencies
- The math function built from model of electric/ electronic circuit called digital (numerical) filters

4.1 Physic Filters

- Low-pass filters

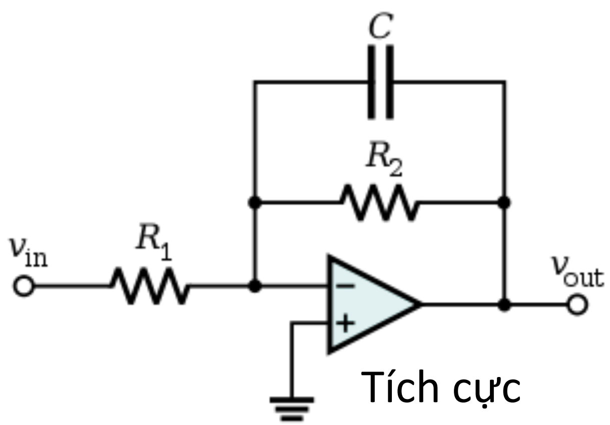
  Allow low frequencies go through
- High pass filters

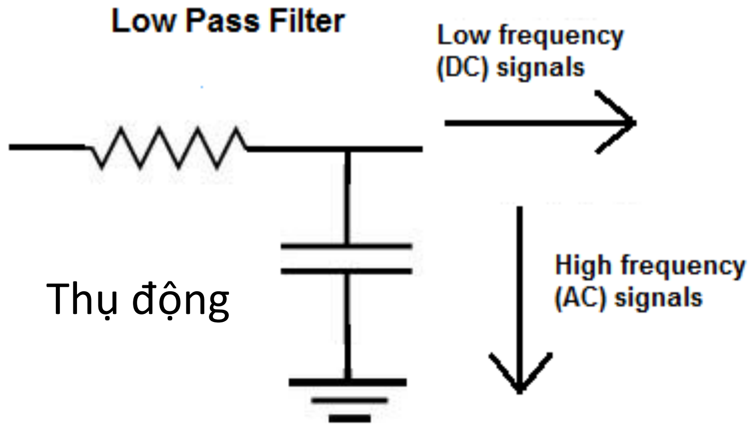
  Allow high frequencies go through
- Band-pass filters

  Allow frequencies in band (range) go through
- Band-stop (Notch) filters

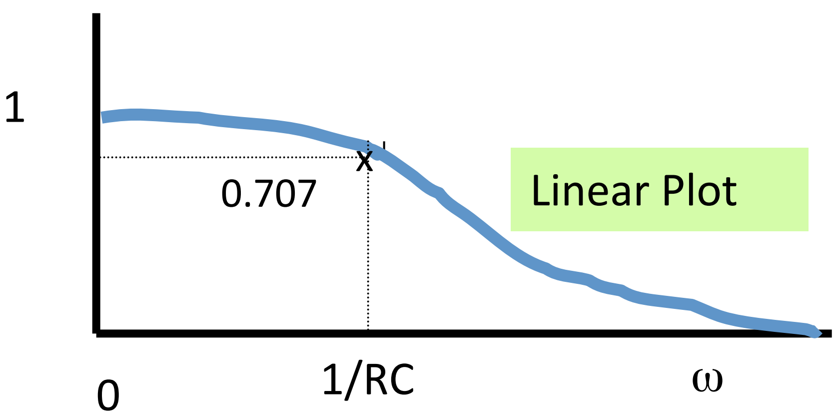
  Does not allow frequencies in band (range) go through

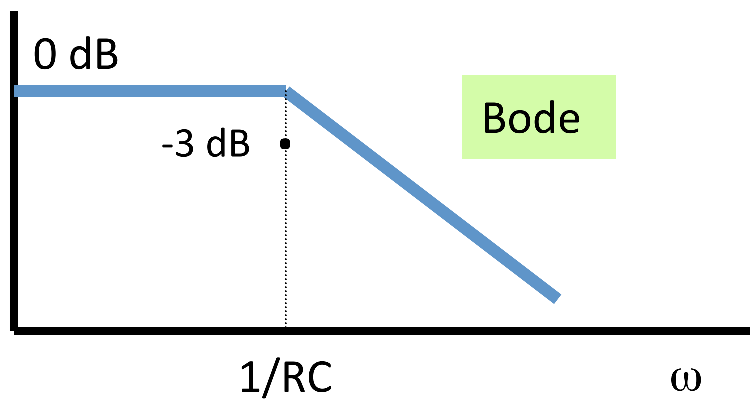
4.2: Low-pass filters



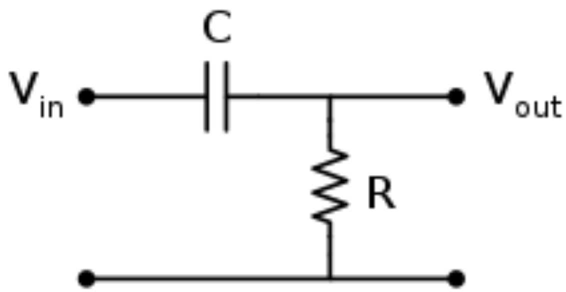


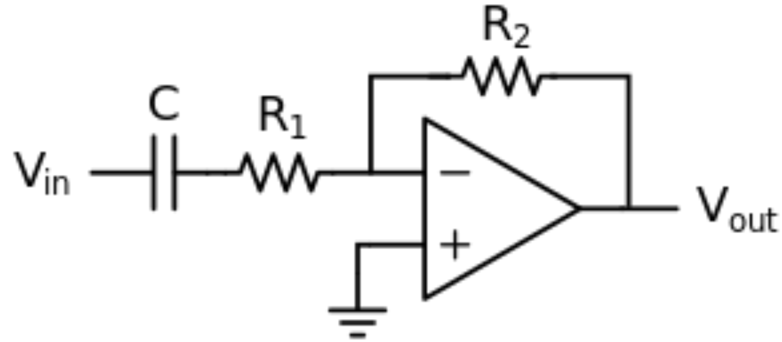
$$\frac{V_o(jw)}{V_i(jw)} = \frac{\frac{1}{jwC}}{R + \frac{1}{jwC}} = \frac{1}{1 + jwRC}$$



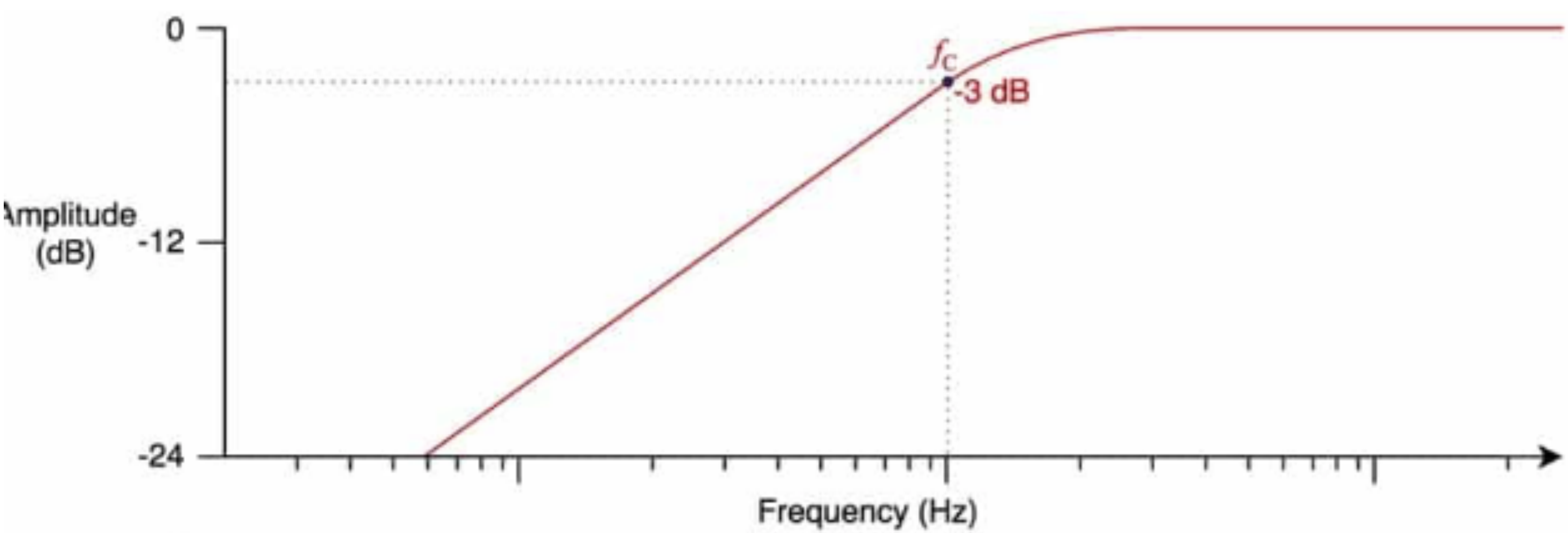


4.3: High-pass filters

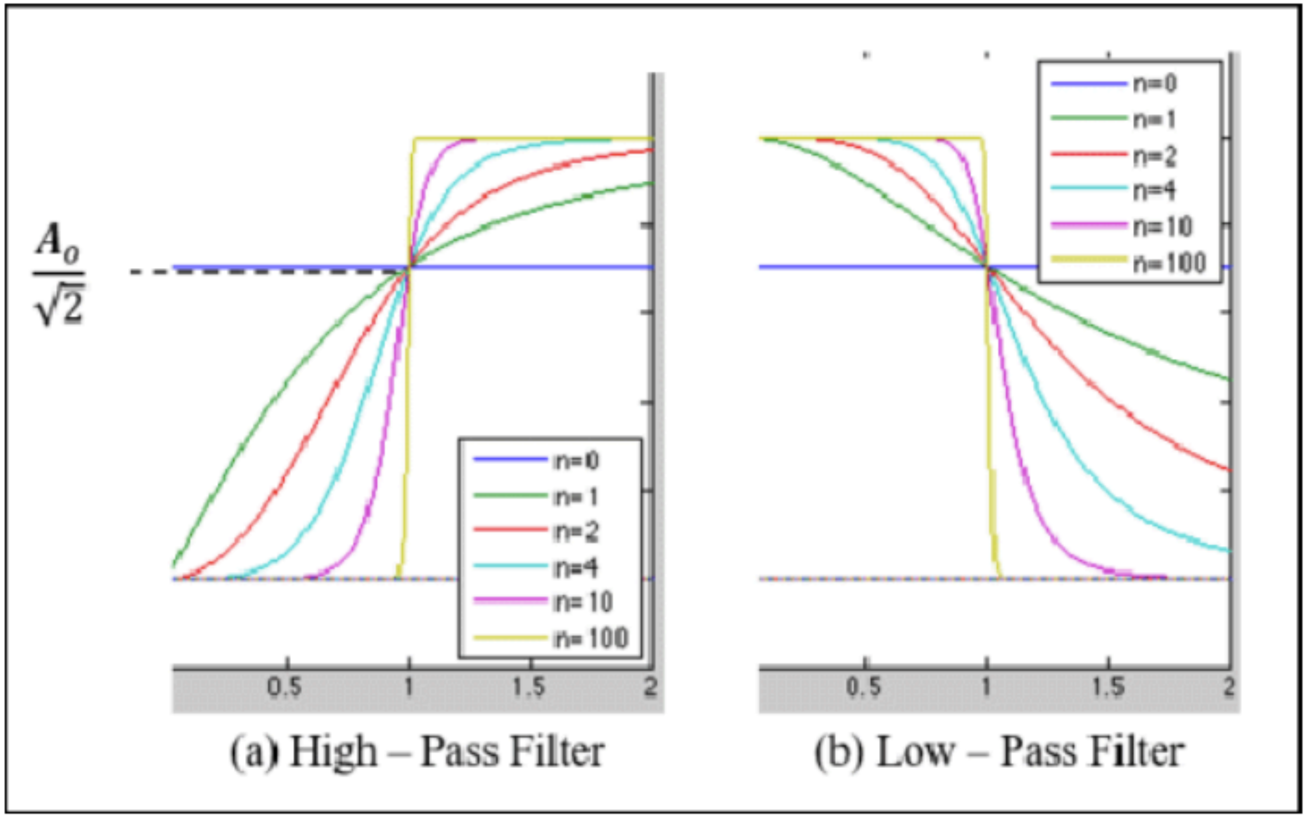




$$\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{sRC}{1 + sRC}.$$



4.4 Order of Filter



(a) High-Pass (b) Low-Pass Butterworth Filters of different orders For example, a cut-off frequency of 1 Hz on the 1st order is specified in a high-pass Butterworth Band-pass filter, shown in Figure 4 (a). One can observe that a parabolic shape is formed, represented by the green line; and, any information inside the established boundary is allowed to pass through. Extending the example, EEG signals on 1 Hz are introduced into the filter. The Butterworth filter transforms the input to its normalized form with Equation 1, consequently it holds: $|H(\omega)| = Ao$ $\sqrt{2}$

Hilbert Problems and Signal Analysis Relationship

Hilbert - Riemann Problems Original

$$M_+(z) = u(z) + iv(z)$$

$$a(z)u(z) - b(z)v(z) = c(z)$$???

Definition

$$M_-(z) = \overline{M_+\left(\bar{z}^{-1}\right)}.$$

$$M_-(z) = \overline{M_+(z)}, \quad z \in \Sigma.$$

$$\frac{a(z) + ib(z)}{2} M_+(z) + \frac{a(z) - ib(z)}{2} M_-(z) = c(z),$$

Hilbert Transform

$$\mathrm{H}(u)(t) = \frac{1}{\pi}\, \mathrm{p.\,v.} \int_{-\infty}^{+\infty} \frac{u(\tau)}{t-\tau}\, \mathrm{d}\tau, \qquad h(t) = \frac{1}{\pi t},$$

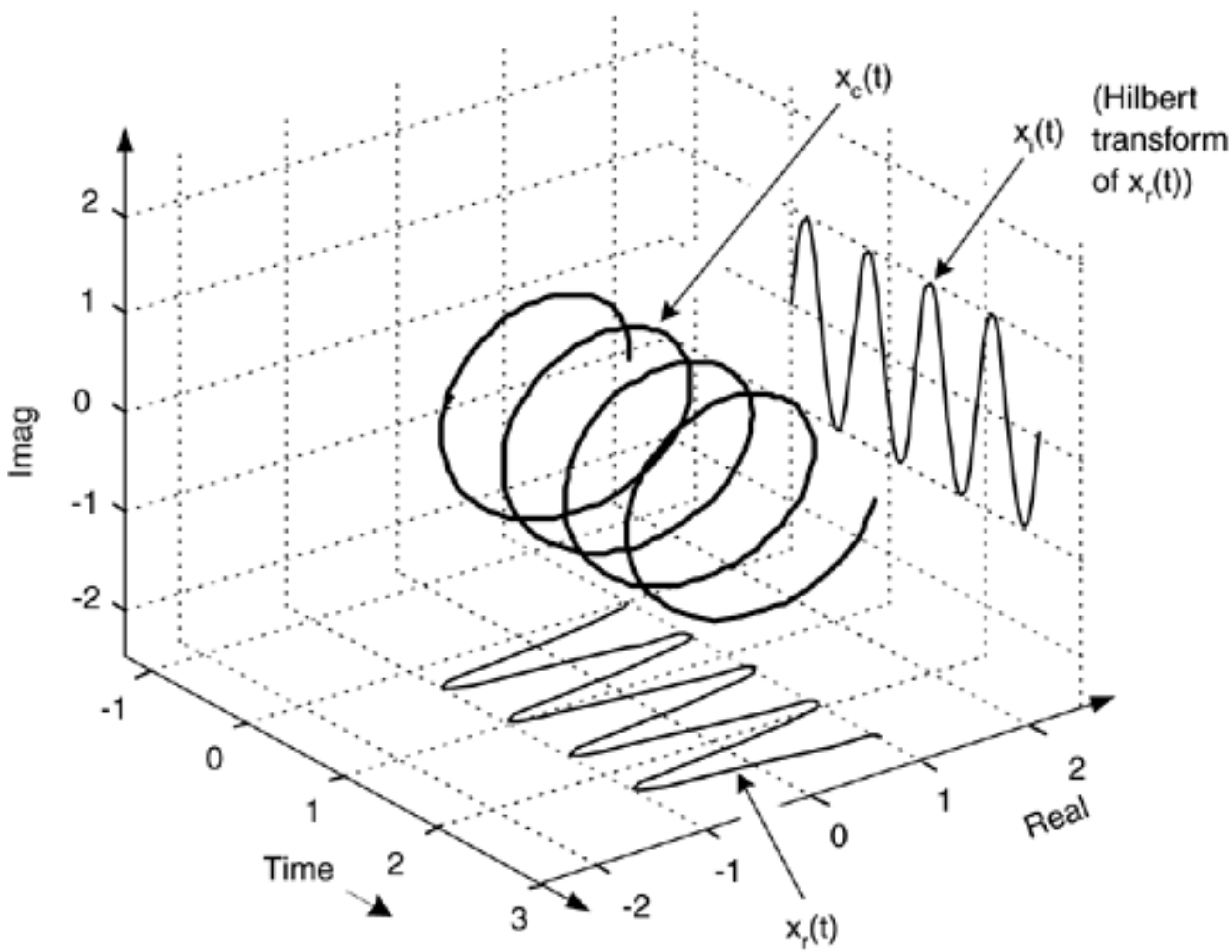
Cauchy principal value

Why Cauchy principle value? t=0? Integral(t) = Infinity

$$\mathrm{H}(u)(t) = \frac{2}{\pi} \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} \frac{u(t-\tau) - u(t+\tau)}{2\tau}\, \mathrm{d}\tau\,.$$

$$\mathrm{H}\big(\mathrm{H}(u)\big)(t) = -u(t),$$

Physic phenomenon



Hilbert transform basic functions

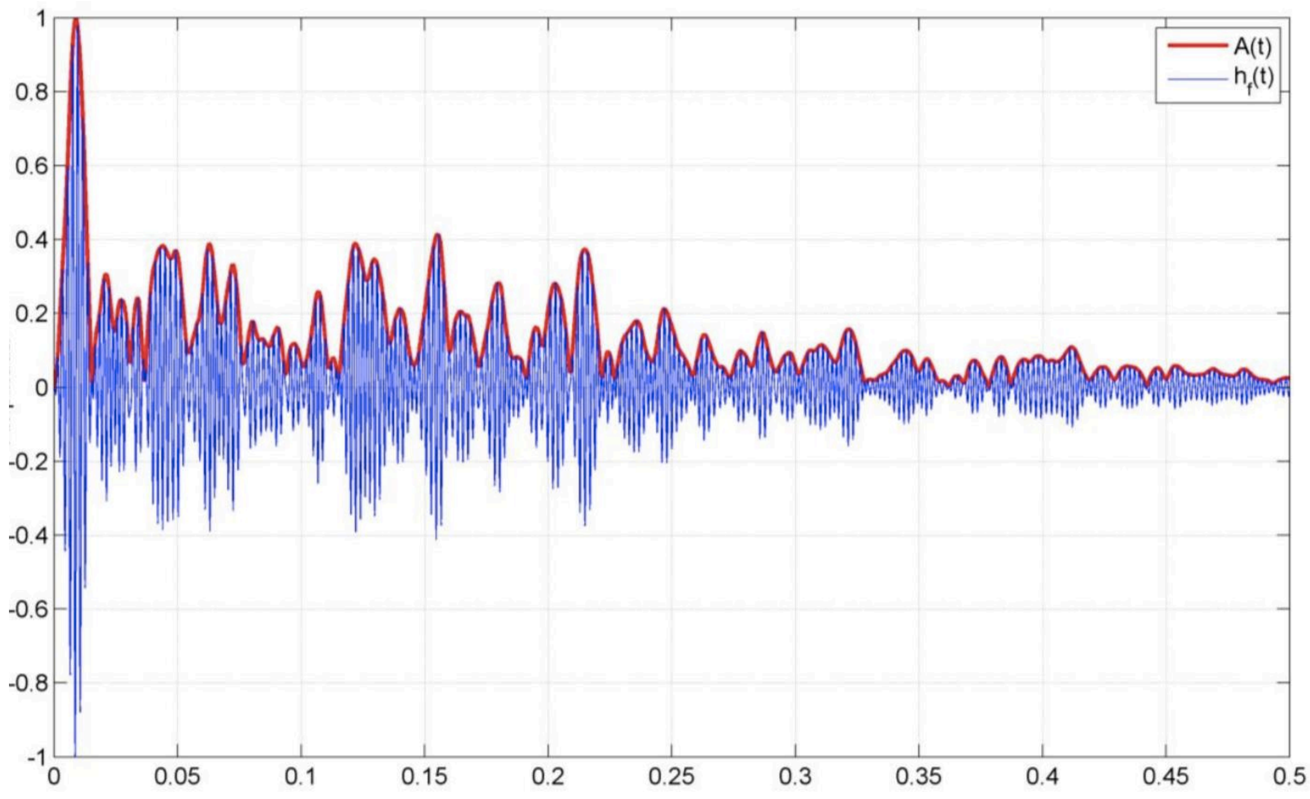
Signal $u(t)$	Hilbert transform[fn 1] $\operatorname{H}(u)(t)$
$\sin(\omega t)$ [fn 2]	$\sin\left(\omega t - \frac{\pi}{2}\right), \quad \omega > 0$ $\sin\left(\omega t + \frac{\pi}{2}\right), \quad \omega < 0$
$\cos(\omega t)$ [fn 2]	$\cos\left(\omega t - \frac{\pi}{2}\right), \quad \omega > 0$ $\cos\left(\omega t + \frac{\pi}{2}\right), \quad \omega < 0$
$e^{i\omega t}$	$e^{i\left(\omega t - \frac{\pi}{2}\right)}, \quad \omega > 0$ $e^{i\left(\omega t + \frac{\pi}{2}\right)}, \quad \omega < 0$
$e^{-i\omega t}$	$e^{-i\left(\omega t - \frac{\pi}{2}\right)}, \quad \omega > 0$ $e^{-i\left(\omega t + \frac{\pi}{2}\right)}, \quad \omega < 0$
$\frac{1}{t^2+1}$	$\frac{t}{t^2+1}$
e^{-t^2}	$\frac{2}{\sqrt{\pi}}F(t)$ (see Dawson function)
Sinc function $\frac{\sin(t)}{t}$	$\frac{1-\cos(t)}{t}$
Dirac delta function $\delta(t)$	$\frac{1}{\pi t}$
Characteristic Function $\chi_{[a,b]}(t)$	$\frac{1}{\pi}\ln\left\lvert\frac{t-a}{t-b}\right\rvert$

Applications

$$s_A(t) = s(t) + j\hat{s}(t)$$

$$\downarrow$$

$$s_A(t) = A(t)e^{j\psi(t)}$$



The topic in Biomedical Engineering?

- EMG analysis
- EEG analysis
- Auscultation analysis
- Ultrasonic imaging
- Voice therapy diagnosis

Limitations of Fourier Transform

- Limitation of the Fourier Analysis

  - For linear input and output

  $ax1(t)+bx2(t)+cx3(t)=dy1(t)+gy2(t)+hy3(t)$

  - Stationary and Periodic
- What is the main problems

  - Non-stationary

  - Local nonlinearity

  - Sinusoidal waveform familiar

  - Does not match with special waveform delta pulse like

Non-stationary wave processing method

- Spectrogram
- Wavelet analysis
- Wigner-Ville distribution
- Evolutionary spectrum
- Empirical orthogonal function expansion
- Smooth moving average
- Trend least square estimation

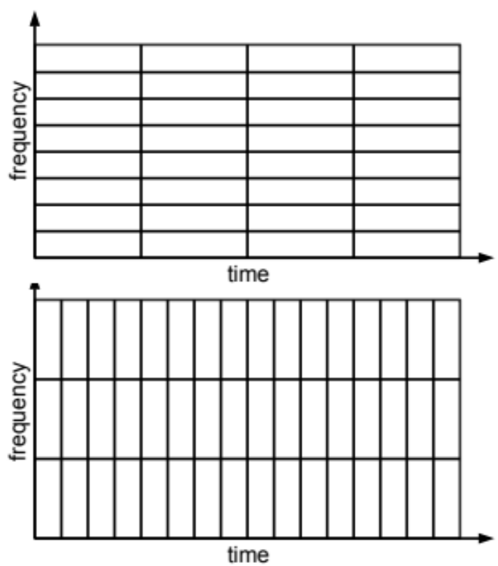
Gabor Transform

$$X(t,f) = \int_{-\infty}^{\infty} x(t_1) w^*(t_1 - t) e^{-j2\pi f t_1} dt_1 \quad + \quad w(t) = e^{-\alpha t^2}$$

Short-time Fourier Transform

Gabor Transform

Optimum resolution for frequency and time



2D-Resolution

Wavelet Transform

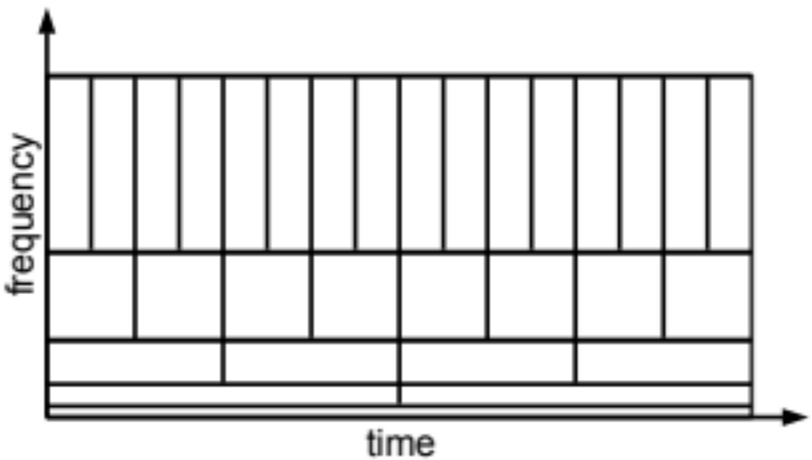
$$CWT(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} x(t)\Psi^* \left(\frac{t-b}{a}\right) dt$$

+

$$\Psi_{Morlet}(t) = e^{-\alpha t^2} e^{j2\pi f_c t}$$

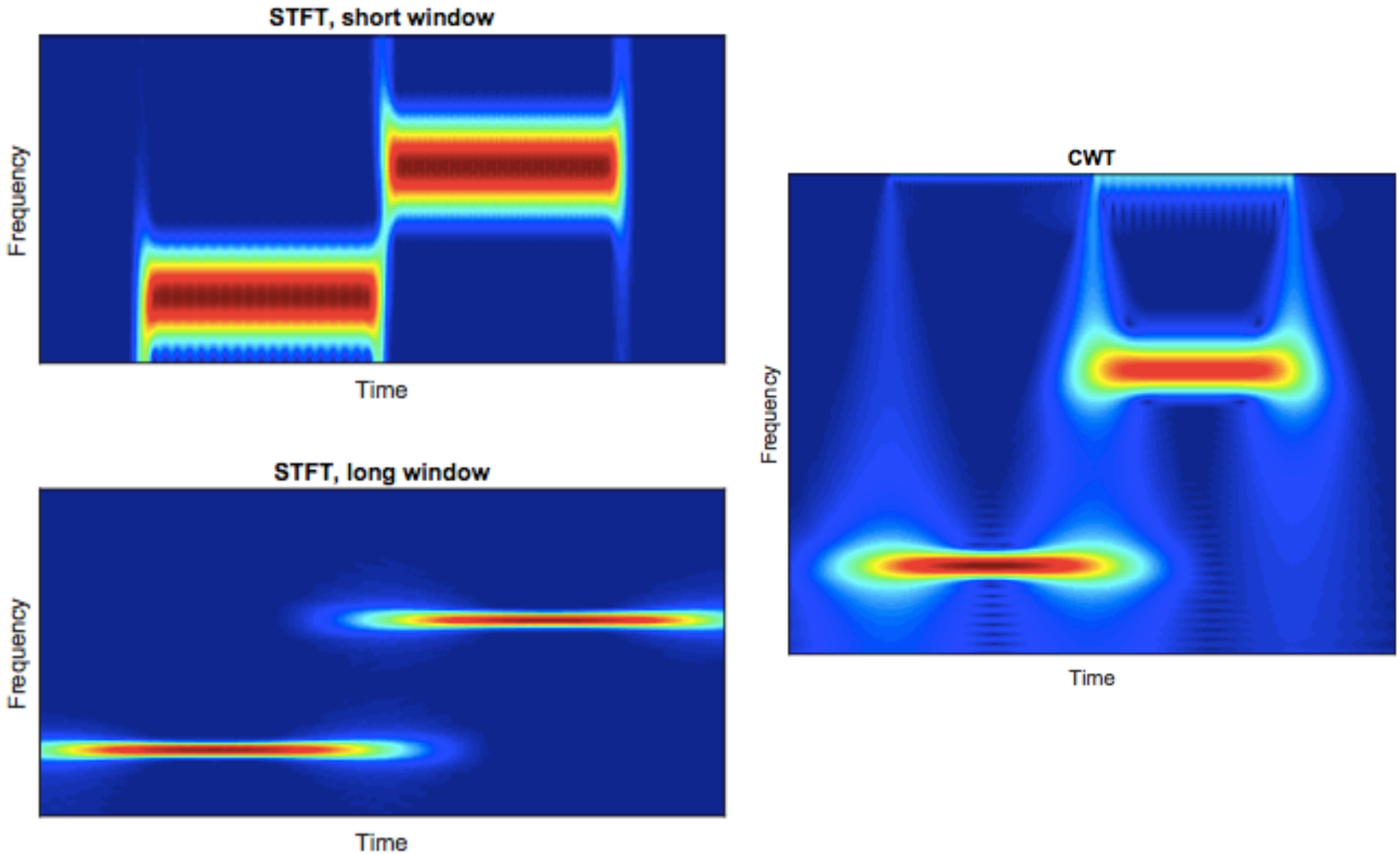
Wavelet Transform

Morlet also called Gabor wavelet



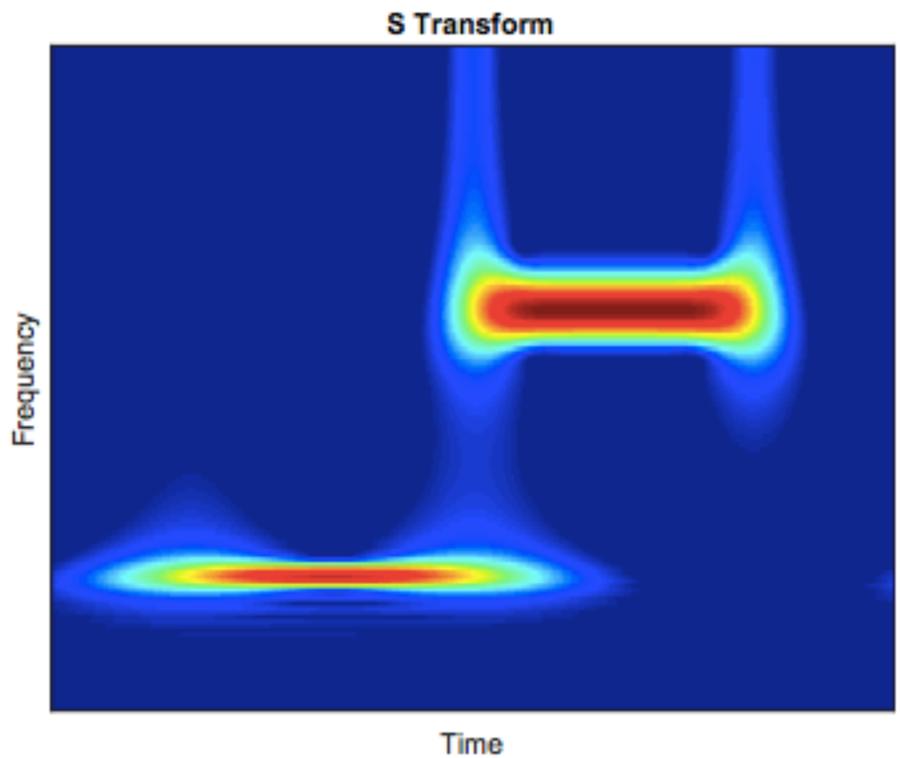
Optimum Resolution Distribution

Fourier vs Wavelet



Stockwell Transform (S-Transform)

$$ST(t,f) = \int_{-\infty}^{\infty} x(t_1)\frac{|f|}{\sqrt{2\pi}} e^{\frac{-f^2(t_1-t)^2}{2}} e^{-j2\pi f t_1} dt_1$$



Limitations

- Fourier provide uniformly resolution in vertical and/or horizon but conflict between frequency and time resolution.
- Wavelet provide partial resolution depending on the order of the decomposition. However the distribution of the resolution in both time frequency are not uniform
- Stockwell is special case of Fourier or Wavelet buts the drawback is the distribution of resolution.

Wigner Ville Distribution

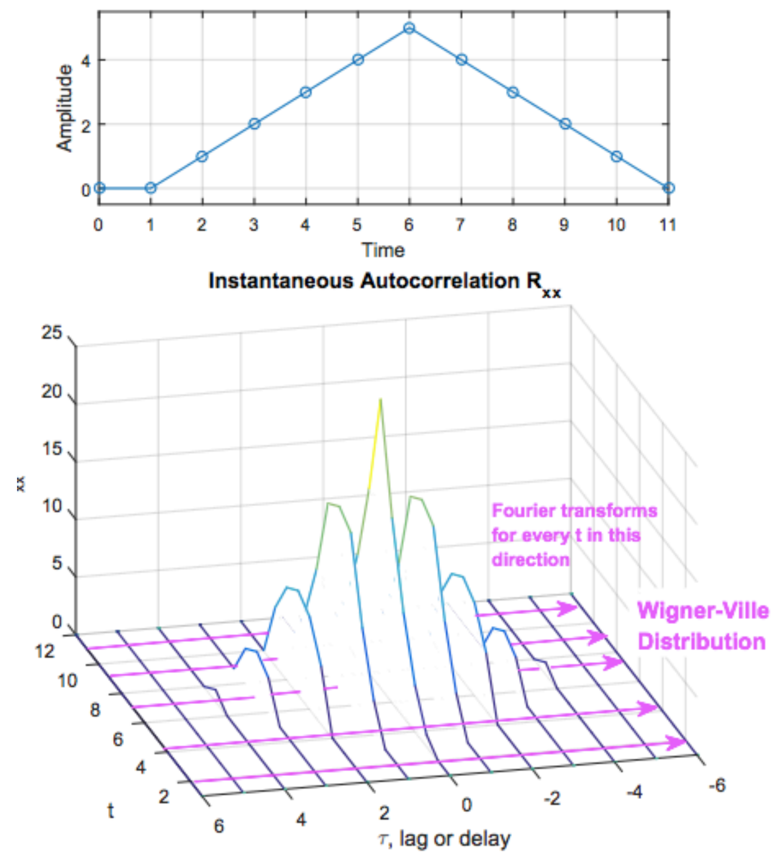
- Idea: Overcome the limitation of resolution in both Fourier and Wavelet

  - Because of the window

Function the transforms is discretizing and delayed

  - To remove the delay,

Apply the autocorrelation function



Autocorrelation

$$r_k = \frac{\sum_{t=k+1}^{T} (y_t - \bar{y})(y_{t-k} - \bar{y})}{\sum_{t=1}^{T} (y_t - \bar{y})^2}, \rightarrow R_{ff}(\tau) = \int_{-\infty}^{\infty} f(t+\tau)\overline{f(t)}\,\mathrm{d}t = \int_{-\infty}^{\infty} f(t)\overline{f(t-\tau)}\,\mathrm{d}t$$

Definition

Wiener–Khinchin theorem

$$\mathrm{R}_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(f)e^{i2\pi f\tau}\,\mathrm{d}f$$

$$S_{XX}(f) = \int_{-\infty}^{\infty} \mathrm{R}_{XX}(\tau)e^{-i2\pi f\tau}\,\mathrm{d}\tau.$$

From Autocorrelation, the distribution of frequency can be built

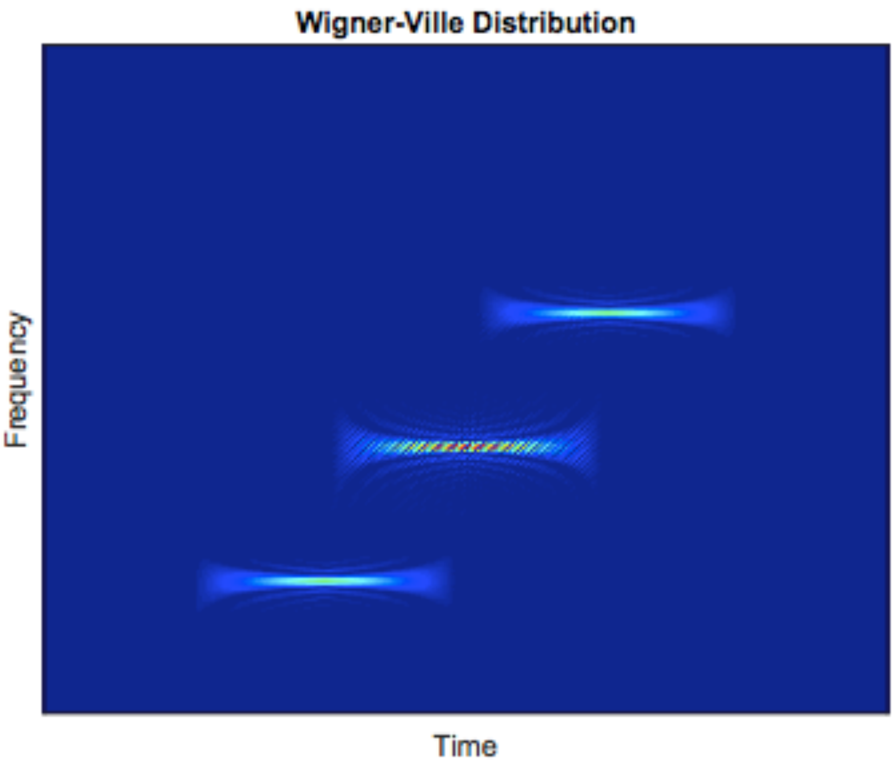
Wigner Ville

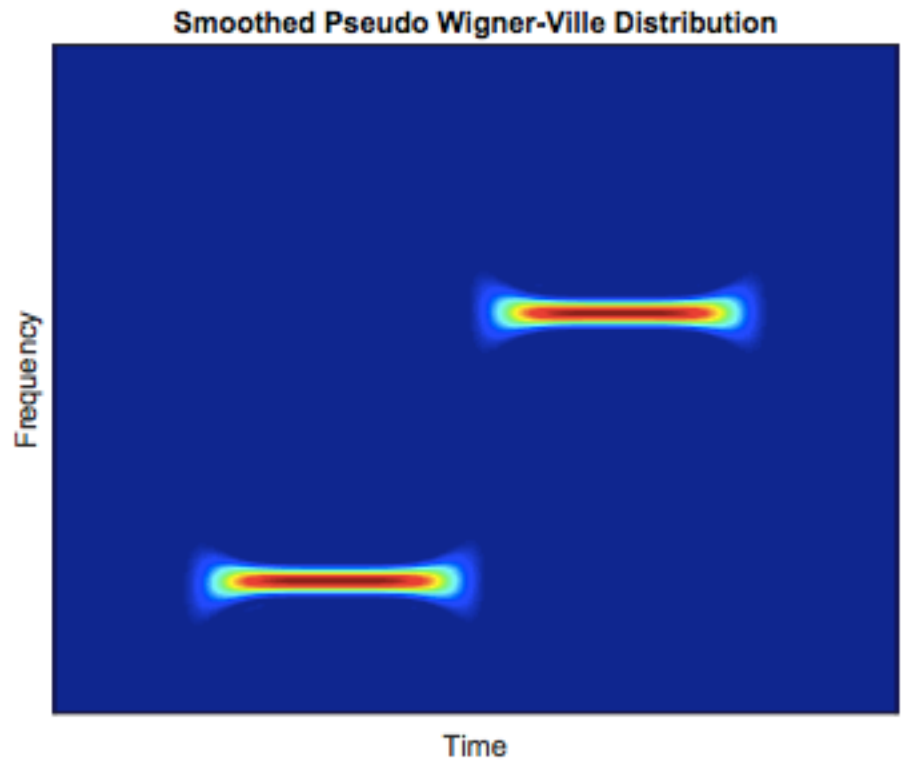
$$r_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t+\tau)dt \xrightarrow{\text{Shift to } \tau/2} R_{xx}(t,\tau) = x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2})$$

$$\begin{aligned} W(t,f) &= \int_{-\infty}^{\infty} R_{xx}(t,\tau)e^{-j2\pi f\tau}d\tau \\ &= \int_{-\infty}^{\infty} x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2})e^{-j2\pi f\tau}d\tau \end{aligned}$$

Resolution go to infinity in continuous domain and limitation in sampling frequency in discrete domain

Wigner-Ville Example



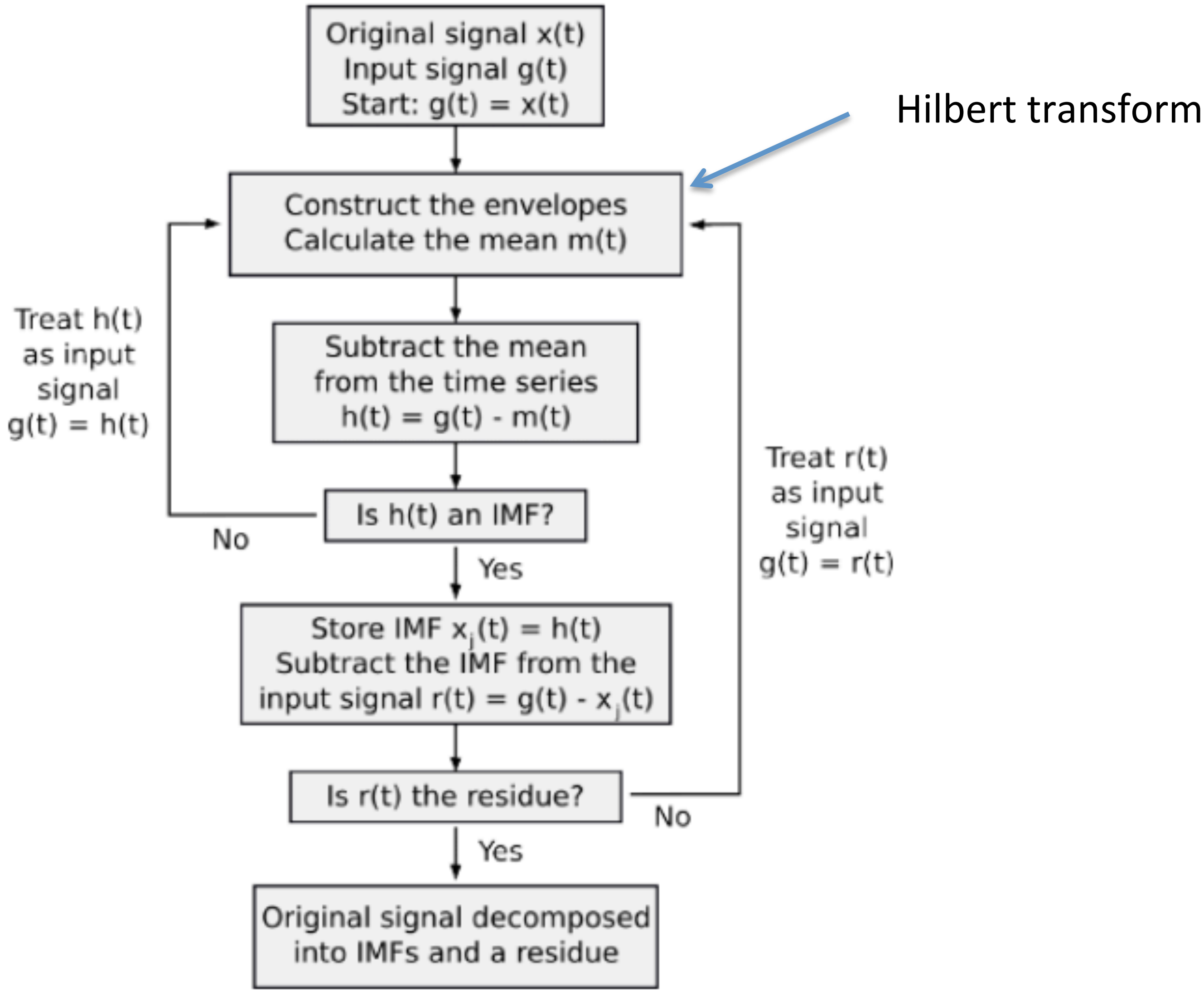


Hilbert-Huang Transform (HHT)

- Wigner-Ville is the universal utility for signal processing. However, it need identified the object to apply.
- Huang based on Hilbert Transform (a special case of Wigner-Ville) with Empirical Mode Decomposition (EMD), extracts the Intrinsic Mode Function (IMF) in definition of:

1) An IMF has only one extremum between two subsequent zero crossings, i.e. the number of local minima and maxima differs at most by one.
2) An IMF has a mean value of zero.

The EMD Algorithm



HHT Example