Dice Battle - formulas and linear program

Chapter 1

Probabilities

$Q(d,k)$: the probability to obtain k points when throwing d dices knowing that we didn't obtain 1
for $d \geq 2$ and $2d \leq k \leq 6d$: $Q(d,k) = \sum_{j=2}^{6} \frac{Q(d-1,k-j)}{5}$

$P(d,k)$: the probability to obtain k points when throwing d dices $(d = 1, ..., D)$

- $P(d,1) = 1 - (\frac{5}{6})^d$
- $P(d,k) = 0$ for $2 \leq k \leq 2d-1$ and for $k > 6d$
- $P(d,k) = (\frac{5}{6})^d Q(d,k)$ for $2d \leq k \leq 6d$

Chapter 2

Sequential game

2.1 Blind strategy

The blind strategy consist in throwing d^* dices such as :

$$d^* = \arg\max_d 4d(\frac{5}{6})^d + 1 - (\frac{5}{6})^d$$

This strategy throws the number of dices that will maximize the expectation of gain, without considering in which situation he is.

2.2 Optimal strategy

We write d^* the optimal number of dices that should be thrown.
The recursive formula to compute the expectation of gain considering a situation (i, j) where the first player has i points and the second j points is :

$$EG(i,j) = \sum_{k=1}^{6d^*} P(d^*, k) * -EG(j, i+k) = \sum_{k=1}^{6d^*} P(d^*, k) * EG(i+k, j)$$

For $i \geq N, \forall j, EG(i,j) = 1$: the first player won.
For $j \geq N, 0 \leq i \leq N, EG(i,j) = -1$: the second player won.

Then we choose :

$$d^* = \max_{1 \leq d \leq D} \sum_{k=1}^{6d} P(d,k) * -EG(j, i+k)$$

Chapter 3

Simultaneous game

3.1 One turn game

The formula to compute the expectation of gain of player 1 when he throws $d1$ dices and his oppenent throws $d2$ dices is :

$$EG_1(d_1,d_2) = 1 * \sum_{k=1}^{6d_1}\sum_{l=1}^{k-1} P(d_1,k) * P(d_2,l) + (-1) * \sum_{k=1}^{6d_2}\sum_{l=1}^{k-1} P(d_2,k) * P(d_1,l)$$

It's the probability that the first player has more points than his oppenent minus the probability that the second player has more points that player 1.

To choose the strategy of player 1 (ie his vector of probability $[p_1,...,p-D]$ where he throws i dices with probabilty p_i) knowing that his opponent will respond optimally, we solve the following linear program :

$$\max_p \min_q p^t EG_1 q$$

$$\sum_{d=1}^{D} p(d) = 1, p(d) \geq 0$$

$$\sum_{d=1}^{D} q(d) = 1, q(d) \geq 0$$

3.2 General game

The expectation of gain of player 1 when he throws $d1$ dices and his oppenent throws $d2$ dices knowing that we are in a situation (i,j) is :

$$EG1_{d_2}^{d_1}(i,j) = \sum_{k=1}^{6d_1}\sum_{l=1}^{6d_2} P(d_1,k) * P(d_2,l) * EG_1(i+k, j+l)$$