Alma Mater Studiorum · Università di Bologna

Scuola di Scienze
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Corso di Laurea in Fisica

GEOMETRIC DEEP LEARNING

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Abstract in italiano...

Abstract in english...

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1 Introduction

1.1 Simplicial complexes

Definition 1.1.1. ***Abstract simplicial complex (finite)***
Let $\mathcal{F}$ be a family of sets we then define an abstract simplicial complex $\mathcal{A}$ to be

$$\mathcal{A} := \{\sigma = \{A_i\}_{i\in I_\sigma} \subset \mathcal{F} : \tau \subset \sigma \Rightarrow \tau \in \mathcal{A}\}$$

where I_σ is a finite set of indexes, we shall call σ abstract simplexes of $\mathcal{A}$.

Definition 1.1.2. ***Dimension of an abstract simplicial complex***
Let $\mathcal{A}$ be an abstract simplicial complex we define its dimension to be

$$dim\mathcal{A} := max_{\sigma\in\mathcal{A}} dim(\sigma),$$

where $dim(\sigma) := |\sigma| - 1$.

Definition 1.1.3. ***Abstract graph***
An abstract graph $\mathcal{G}$ is a 1-dimensional abstract simplicial complex whose vertexes and edges are respectively

$$\mathcal{V} := \{\sigma \in \mathcal{G} : dim(\sigma) = 0\} \text{ and}$$
$$\mathcal{E} := \{\sigma \in \mathcal{G} : dim(\sigma) = 1\} \text{ .}$$

In Definition 1.1.1. we tacitly assumed the definition of the abstract simplex σ invariant with respect to permutations of the indexes I_σ, this assumption establishes the difference between directed and undirected graphs.

Definition 1.1.4. ***Convex envelop of points in*** $\mathbb{R}^n$
Let I be a finite set of indexes, we define the convex envelope of $\{x_i\}_{i\in I} \subset \mathbb{R}^n$ to be

$$\langle x_i\rangle_{i\in I} := \{a = \sum_{i\in I} \lambda_i x_i : \lambda_i \in \mathbb{R},\ \lambda_i > 0,\ \sum_{i\in I} \lambda_i = 1\},$$

which is the smallest convex set containing $\{x_i\}_{i\in I}$.

Definition 1.1.5. ***Affine independency of points in*** $\mathbb{R}^n$
Let $\{x_i\}_{i\in I} \subset \mathbb{R}^n$ we define $\{x_i\}_{i\in I}$ to be affinely independent if and only if

$$\sum_{i\in I} \lambda_i x_i = \sum_{i\in I} \mu_i x_i \quad \Rightarrow \quad \lambda_i = \mu_i \ \forall i \in I,$$

where $\sum_{i\in I} \lambda_i = \sum_{i\in I} \mu_i = 1$.

Definition 1.1.6. ***Geometric k-simplexes***
We define a geometric k-simplex to be a convex envelop $\langle x_i\rangle_{i\in I}$ where $\{x_i\}_{i\in I} \subset \mathbb{R}^n$ are affinely independent and $|I| = k + 1$.

Definition 1.1.7. ***Faces and cofaces of geometric k-simplexes***
Let σ be a geometric k-simplex, we say that another t-simplex τ is a face of σ or equivalently that σ is a coface of τ, by our notiation $\tau \le \sigma$, if and only if $\tau \subset \sigma$, where $t \le k$.

Definition 1.1.8. ***Geometric Simplicial Complex***
We define a geometric simplicial complex $\mathcal{K}$ to be a collection of geometric simplexes such that

(i) $\tau \le \sigma \in \mathcal{K} \Rightarrow \tau \in \mathcal{K}$,

(ii) $\sigma, \tau \in \mathcal{K} \Rightarrow \sigma \cup \tau \in \mathcal{K}$.

Definition 1.1.9. ***Geometric realization of an abstract simplicial complex***
Let $\mathcal{K}$ be a geometric simplicial complex, and let $Vert(\mathcal{K}) := \{\sigma \in \mathcal{K} : dim(\sigma) = 0\}$, we call the abstract simplicial complex $\mathcal{A} := \{\{x_i\}_{i\in I} \subset Vert(\mathcal{K}) : \langle x_i \rangle_{i\in I} \in \mathcal{K}\}$ a vertex scheme for $\mathcal{K}$ or equivalently we might say that $\mathcal{K}$ is a geometric realization of $\mathcal{A}$.

Theorem 1.1.1. *Let $\mathcal{A}$ be a d-dimentional abstract simplicial complex, it admits a geometric realization in $\mathbb{R}^{2d+1}$.*

Kuratowski theorem proves the prevuois statement to be also sharp.

1.2 Forms on abstract simplicial complexes

Definition 1.2.1. ***p-forms on abstract simplicial complexes***
Let $\mathcal{A}$ be an abstract simplical complex we define the linear space of p-forms $\Lambda^p = \Lambda^p(\mathcal{A})$ to be

$$\Lambda^p := \{\omega : Vert(\mathcal{A})^{p+1} \to \mathbb{R}\},$$
$$+ : \Lambda^p \times \Lambda^p \to \Lambda^p$$
$$(\omega + \eta)(x) = \omega(x) + \eta(x) \quad x \in Vert(\mathcal{A})^{p+1},\ \omega, \eta \in \Lambda^p,$$
$$\cdot : \mathbb{R} \times \Lambda^p \to \Lambda^p$$
$$(\lambda\omega)(x) = \lambda\omega(x) \quad x \in Vert(\mathcal{A})^{p+1},\ \omega \in \Lambda^p,\ \lambda \in \mathbb{R},$$

where $dim(\Lambda^p) = |Vert(\mathcal{A})|^{p+1}$.

Although this definition seems to stand apart from the concept of simplicial complex we might notice that if we take the subset $\{x \in Vert(\mathcal{A})^{p+1} : \{x\} \in \mathcal{A}\}$ as the domain of our forms we are defining p-forms on p-simplexes of $\mathcal{A}$. For a lighter notation we shall define $\mathcal{V} := Vert(\mathcal{A})$.

Proposition 1.2.1. *A canonical base of elementary forms for Λ^p is*

$$\{e^x \in \Lambda^p : x \in \mathcal{V}^{p+1},\ e^x(y) = \delta_{xy} \quad y \in \mathcal{V}^{p+1}\},$$

therefore giving us an expression for every other p-form

$$\omega = \sum_{x\in\mathcal{V}^{p+1}} \omega_x e^x, \quad \omega_x \in \mathbb{R}.$$

This basis, according to the restriction $\{x \in Vert(\mathcal{A})^{p+1} : \{x\} \in \mathcal{A}\}$, is the dual basis of the basis of the linear space of simplicial p-chains.

Definition 1.2.2. ***Exterior derivative of a p-form***
Let $\omega \in \Lambda^p$ we define $d : \Lambda^p \to \Lambda^{p+1}$ such that

$$(d\omega)_x = \sum_{i=0}^{p+1} (-1)^i \omega_{\hat{x}_i}\ ,$$

where if $x = (x_0, ..., x_{p+1}) \in \mathcal{V}^{p+1}$ we define $\hat{x}_i := (x_0, ..., x_{i-1}, x_{i+1}, ..., x_{p+1}) \in \mathcal{V}^p$.

Lemma 1.2.1. *Let $\omega \in \Lambda^p$, $p \geq 0$ then $d^2\omega = 0$.*

Proof. We have for $x \in \Lambda^{p+2}$

$$(d^2\omega)_x = \sum_{i=0}^{p+2} (-1)^i (d\omega)_{\hat{x}_i}$$
$$= \sum_{i=0}^{p+2} (-1)^i \sum_{j=0}^{i-1} (-1)^j \omega_{\hat{x}_{ij}} + \sum_{j=i+1}^{p+2} (-1)^{j-1} \omega_{\hat{x}_{ij}}$$

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