Graphs

1 Graph Operators

Definition 1.1. *Let $\mathcal{G}$ be a graph where $\mathcal{V}$ are its vertexes and $\mathcal{E}$ are its edges, let $f, g : L^2(\mathcal{V})$ and $F, G \in L^2(\mathcal{E})$ be real valued functions, we define $\langle f, g\rangle_{L^2(\mathcal{V})} := \sum_{\mathcal{V}} a_i f_i g_i$, $a_i \in \mathbb{R}$ and $\langle F, G\rangle_{L^2(\mathcal{E})} := \sum_{\mathcal{E}} w_{ij} F_{ij} G_{ij}$, $w_{ij} \in \mathbb{R}$.*

Definition 1.2. ***Graph gradient and divergence***
Let $f \in L^2(\mathcal{V})$ and $F \in L^2(\mathcal{E})$ we define $grad : L^2(\mathcal{V}) \to L^2(\mathcal{E})$ and $div : L^2(\mathcal{E}) \to L^2(\mathcal{V})$, such that $(grad f)_{ij} = f_i - f_j$ and $(divF)_i = \frac{1}{a_i} \sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} w_{ij} F_{ij}$.

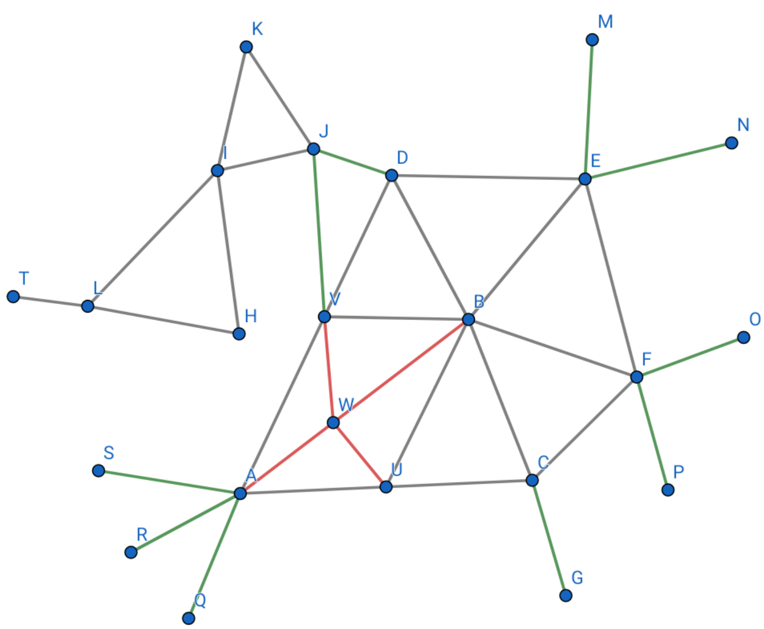
Proposition 1.1. *Let $f \in L^2(\mathcal{V})$ and $F \in L^2(\mathcal{E}) : F_{ij} = -F_{ji}$ then $\langle f, divF\rangle_{L^2(\mathcal{V})} = \langle grad f, F\rangle_{L^2(\mathcal{E})}$, i.e. $divF^\dagger = grad$.*

Proof. $\sum_{\mathcal{V}} a_i f_i (divF)_i = \sum_{\mathcal{E}} w_{ij} F_{ij} (f_i - f_j) = \sum_{i\in\mathcal{V}} \sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} w_{ij} F_{ij} f_i$ thus $a_i (divF)_i = \sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} w_{ij} F_{ij}$. □

Theorem 1.2. ***Fake Gauss theorem***
Let $F \in L^2(\mathcal{E}) : F_{ij} = -F_{ji}$, let $\mathcal{A} \subset \mathcal{V}$ then if $a_i = w_{ij} = 1$ we have $\sum_{\mathcal{A}} (divF)_i = \sum_{\partial^0_+\mathcal{A}} F_{ij}$.

Proof. First of all we recall $\partial^0_+\mathcal{A} = \{(i,j) \in \mathcal{E}, i \in \mathcal{A}, j \in \mathcal{V} \setminus \mathcal{A}\}$, then we see that $\sum_{\mathcal{A}} (divF)_i = \sum_{i\in\mathcal{A}} \sum_{j\in\mathcal{V}:(i,j)\in\mathcal{E}} F_{ij} = \sum_{i\in\mathcal{A}} \sum_{j\in\mathcal{V}\setminus\mathcal{A}:(i,j)\in\mathcal{E}} F_{ij} + \sum_{i\in\mathcal{A}} \sum_{j\in\mathcal{A}:(i,j)\in\mathcal{E}} F_{ij} = \sum_{\partial^0_+\mathcal{A}} F_{ij} + \sum_{(i,j)\in\mathcal{A}^2} adj(\mathcal{A})_{ij} F_{ij}$ where since $adj(\mathcal{A})_{ij} = adj(\mathcal{A})_{ji}$ we have by renaming dummy indexes $adj(\mathcal{A})_{ij} F_{ij} = -adj(\mathcal{A})_{ij} F_{ij} = 0$. □

Figure 1: Coboundary operator applied to A+B+C+D+E+F+U+V+W(green) and to A+B+C+D+E+F+U+V(green and red)



Proposition 1.3. *The use of the coboundary operator makes sense only with antisimmetric functions on the edges, the antisimmetry of those function is somehow related to the orientation of surfaces.*

Proof. $\sum_{\partial^0 \sum_{i\in\mathscr{A}} i} F_{ij} = \sum_{\sum_{i\in\mathscr{A}} \partial^0 i} F_{ij}$ if an edge is in the coboundary of two different vertexes of $\mathscr{A}$ it will be count twice, that means zero times in $\mathbb{Z}_2$, similarly for that same edge we would sum $F_{ij} + F_{ji} = 0$. $\square$

Definition 1.3. ***Graph laplacian***
Let $f \in L^2(\mathscr{V})$ we have that $\langle gradf, gradf \rangle = \langle div(gradf), f \rangle =: \langle \Delta f, f \rangle = \langle f, \Delta f \rangle$, where $\Delta : L^2(\mathscr{V}) \to L^2(\mathscr{V})$ is the Laplacian.

Possibili approfondimenti interessanti:
(i) Studio di equazioni differenziali sui grafi con gli operatori sopra definiti
-Equazione del Calore
$\frac{d(f_i)}{dt} = -c(\Delta f)_i$
-Equazione di Schrödinger
$\frac{d(f_i)}{dt} = -c(\Delta f)_i + U_i f_i$
-Equazione di Navier-Stokes
-Equazione di continuità
-altre
(ii) Vincolare l'apprendimento di funzioni a divergenza nulla delle edge tramite H^0, eventuale applicazione a flussi incomprimibili
(iii) Definire il rotore per 2-simplessi e fare l'analogo con H_1, eventuale applicazione a reti elettriche tridimensionali