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GEOMETRIC DEEP LEARNING

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Abstract in italiano...

Abstract in english...

Contents

1	Introduction	1
1.1	Simplicial complexes	1
1.2	Forms and integration on abstract simplicial complexes	2
1.3	Smooth real manifolds and abstract graphs	4
1.4	Convolutional neural networks on euclidean domains	4

1 Introduction

1.1 Simplicial complexes

Definition 1.1.1. *Abstract simplicial complex (finite)*

Let \mathcal{F} be a family of sets we then define an abstract simplicial complex \mathcal{A} to be

$$\mathcal{A} := \{\sigma = \{A_i\}_{i \in I_\sigma} \subset \mathcal{F} : \tau \subset \sigma \Rightarrow \tau \in \mathcal{A}\}$$

where I_σ is a finite set of indexes, we shall call σ abstract simplexes of \mathcal{A} .

Definition 1.1.2. *Dimension of an abstract simplicial complex*

Let \mathcal{A} be an abstract simplicial complex we define its dimension to be

$$\dim \mathcal{A} := \max_{\sigma \in \mathcal{A}} \dim(\sigma),$$

where $\dim(\sigma) := |\sigma| - 1$.

Definition 1.1.3. *Abstract graph*

An abstract graph \mathcal{G} is a 1-dimensional abstract simplicial complex whose vertexes and edges are respectively

$$\mathcal{V} := \{\sigma \in \mathcal{G} : \dim(\sigma) = 0\} \text{ and}$$

$$\mathcal{E} := \{\sigma \in \mathcal{G} : \dim(\sigma) = 1\}.$$

In Definition 1.1.1. we tacitly assumed the definition of the abstract simplex σ invariant with respect to permutations of the indexes I_σ , this assumption establishes the difference between directed and undirected graphs.

Definition 1.1.4. *Convex envelop of points in \mathbb{R}^n*

Let I be a finite set of indexes, we define the convex envelope of $\{x_i\}_{i \in I} \subset \mathbb{R}^n$ to be

$$\langle x_i \rangle_{i \in I} := \{a = \sum_{i \in I} \lambda_i x_i : \lambda_i \in \mathbb{R}, \lambda_i > 0, \sum_{i \in I} \lambda_i = 1\},$$

which is the smallest convex set containing $\{x_i\}_{i \in I}$.

Definition 1.1.5. *Affine independency of points in \mathbb{R}^n*

Let $\{x_i\}_{i \in I} \subset \mathbb{R}^n$ we define $\{x_i\}_{i \in I}$ to be affinely independent if and only if

$$\sum_{i \in I} \lambda_i x_i = \sum_{i \in I} \mu_i x_i \quad \Rightarrow \quad \lambda_i = \mu_i \quad \forall i \in I,$$

where $\sum_{i \in I} \lambda_i = \sum_{i \in I} \mu_i = 1$.

Definition 1.1.6. *Geometric k -simplexes*

We define a geometric k -simplex to be a convex envelop $\langle x_i \rangle_{i \in I}$ where $\{x_i\}_{i \in I} \subset \mathbb{R}^n$ are affinely independent and $|I| = k + 1$.

Definition 1.1.7. *Faces and cofaces of geometric k -simplexes*

Let σ be a geometric k -simplex, we say that another t -simplex τ is a face of σ or equivalently that σ is a coface of τ , by our notation $\tau \leq \sigma$, if and only if $\tau \subset \sigma$, where $t \leq k$.

Definition 1.1.8. *Geometric Simplicial Complex*

We define a geometric simplicial complex \mathcal{K} to be a collection of geometric simplexes such that

$$(i) \quad \tau \leq \sigma \in \mathcal{K} \Rightarrow \tau \in \mathcal{K},$$

$$(ii) \quad \sigma, \tau \in \mathcal{K} \Rightarrow \sigma \cup \tau \in \mathcal{K}.$$

Definition 1.1.9. Geometric realization of an abstract simplicial complex

Let \mathcal{K} be a geometric simplicial complex, and let $\text{Vert}(\mathcal{K}) := \{\sigma \in \mathcal{K} : \dim(\sigma) = 0\}$, we call the abstract simplicial complex $\mathcal{A} := \{\{x_i\}_{i \in I} \subset \text{Vert}(\mathcal{K}) : \langle x_i \rangle_{i \in I} \in \mathcal{K}\}$ a vertex scheme for \mathcal{K} or equivalently we might say that \mathcal{K} is a geometric realization of \mathcal{A} .

Theorem 1.1.1. *Let \mathcal{A} be a d -dimensional abstract simplicial complex, it admits a geometric realization in \mathbb{R}^{2d+1} .*

Kuratowski theorem proves the previous statement to be also sharp.

1.2 Forms and integration on abstract simplicial complexes**Definition 1.2.1. Linear space of simplicial p -chains**

Let \mathcal{A} be an abstract simplicial complex, and let $\mathcal{A}_p := \{\sigma \in \mathcal{A} : \dim(\sigma) = p\}$, we define the linear space $C_p = C_p(\mathcal{A})$ of simplicial p -chain on \mathcal{A} to be

$$C_p = \left\{ \sum_{\sigma \in \mathcal{A}_p} \lambda^\sigma \sigma, \quad \lambda^\sigma \in \mathbb{Z}_2 \right\},$$

where the formal operations of the linear space are given by the definition itself.
(Possible extension from \mathbb{Z}_2 to \mathbb{R} , naming C_p by the dual notation Λ_p)

The set \mathcal{A}^p is a canonical base of p -simplexes for C_p .

Definition 1.2.2. Boundary operator on C_{p+1}

Let σ be an element of the canonical base of C_{p+1} we define $\partial : C_{p+1} \rightarrow C_p$ such that

$$\partial \sigma = \sum_{i=0}^{p+1} (-1)^i \sigma_i,$$

where if $\sigma = \{x_0, \dots, x_{p+1}\} \in C_{p+1}$ we define $\sigma_i := \{x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{p+1}\} \in C_p$.
Furthermore we extend this operator linearly on the whole space C_{p+1}

$$\partial \left(\sum_{\sigma \in \mathcal{A}_p} \lambda^\sigma \sigma \right) = \sum_{\sigma \in \mathcal{A}_p} \lambda^\sigma \partial \sigma.$$

Lemma 1.2.1. *Let $\sigma \in \mathcal{A}_{p+2}$, $p \geq 0$ then $\partial^2 \sigma = 0$.*

Proof. We have

$$\begin{aligned} (\partial^2 \sigma)_x &= \sum_{i=0}^{p+2} (-1)^i (\partial \sigma)_i \\ &= \sum_{i=0}^{p+2} (-1)^i \left[\sum_{j=0}^{i-1} (-1)^j \sigma_{ij} + \sum_{j=i+1}^{p+2} (-1)^{j-1} \sigma_{ij} \right] \\ &= \sum_{i=0}^{p+2} \sum_{j=0}^{i-1} (-1)^{i+j} \sigma_{ij} - \sum_{i=0}^{p+2} \sum_{j=i+1}^{p+2} (-1)^{i+j} \sigma_{ij} = 0. \end{aligned}$$

□

Definition 1.2.3. p -forms on abstract simplicial complexes

Let \mathcal{A} be an abstract simplicial complex we define the linear space of p -forms $\Lambda^p = \Lambda^p(\mathcal{A})$ to be

$$\Lambda^p := \{\omega : C_p \rightarrow \mathbb{R}\}, \text{ such that}$$

$$\omega \left(\sum_{\sigma \in \mathcal{A}_p} \lambda^\sigma \sigma \right) = \sum_{\sigma \in \mathcal{A}_p} \lambda^\sigma \omega(\sigma) \quad \forall \omega \in \Lambda^p, \lambda_\sigma \in \mathbb{Z}_2,$$

with linear space operations defined as

$$+ : \Lambda^p \times \Lambda^p \rightarrow \Lambda^p \quad (\omega + \eta)(\sigma) = \omega(\sigma) + \eta(\sigma) \quad \sigma \in C_p, \omega, \eta \in \Lambda^p,$$

$$\cdot : \mathbb{R} \times \Lambda^p \rightarrow \Lambda^p \quad (\lambda\omega)(\sigma) = \lambda\omega(\sigma) \quad \sigma \in C_p, \omega \in \Lambda^p, \lambda \in \mathbb{R}.$$

Proposition 1.2.1. A canonical base of elementary forms for Λ^p is

$$\{e^\sigma \in \Lambda^p : \sigma \in \mathcal{A}_p, e^\sigma(\tau) = \delta_{\sigma\tau} \quad \tau \in \mathcal{A}_p\},$$

therefore giving us an expression for every other p -form

$$\omega = \sum_{\sigma \in \mathcal{A}_p} \omega_\sigma e^\sigma, \quad \omega_\sigma \in \mathbb{R}.$$

Definition 1.2.4. Exterior derivative of a p -form

Let $\omega \in \Lambda^p$ we define $d : \Lambda^p \rightarrow \Lambda^{p+1}$ on its coordinates to be

$$(d\omega)_\sigma = \sum_{i=0}^{p+2} (-1)^i \omega_{\sigma_i}.$$

Lemma 1.2.2. Let $\omega \in \Lambda^p, p \geq 0$ then $d^2\omega = 0$.

Proof. We have for $\sigma \in \mathcal{A}_{p+2}$

$$\begin{aligned} (d^2\omega)_\sigma &= \sum_{i=0}^{p+2} (-1)^i (d\omega)_{\sigma_i} \\ &= \sum_{i=0}^{p+2} (-1)^i \left[\sum_{j=0}^{i-1} (-1)^j \omega_{\sigma_{ij}} + \sum_{j=i+1}^{p+2} (-1)^{j-1} \omega_{\sigma_{ij}} \right] \\ &= \sum_{i=0}^{p+2} \sum_{j=0}^{i-1} (-1)^{i+j} \omega_{\sigma_{ij}} - \sum_{i=0}^{p+2} \sum_{j=i+1}^{p+2} (-1)^{i+j} \omega_{\sigma_{ij}} = 0. \end{aligned}$$

□

Definition 1.2.5. Integration of p -forms on p -chains

Let $\omega \in \Lambda^p$ and $\tau \in C_p$ we define the integral of ω on τ to be a bilinear form $\Lambda^p \times C_p \rightarrow \mathbb{R}$

$$(\omega, \tau)_p := \sum_{\sigma \in \mathcal{A}_p} \omega_\sigma \tau^\sigma,$$

where $\omega = \sum_{\sigma \in \mathcal{A}_p} \omega_\sigma e^\sigma$ and $\tau = \sum_{\sigma \in \mathcal{A}_p} \tau^\sigma \sigma$.

(This might be extended by adding a non trivial permutation invariant measure on \mathcal{A}_p)

Theorem 1.2.1. *Let $\omega = \sum_{\sigma \in \mathcal{A}_p} \omega_\sigma e^\sigma$ and $\tau = \sum_{\sigma \in \mathcal{A}_{p+1}} \tau^\sigma \sigma$ the following identity holds*

$$(d\omega, \tau)_{p+1} = (\omega, \partial\tau)_p,$$

i.e. the operators $d : \Lambda^p \rightarrow \Lambda^{p+1}$ and $\partial : C_{p+1} \rightarrow C_p$ are dual.

Proof. We have

$$(d\omega, \tau)_{p+1} = \sum_{\sigma \in \mathcal{A}_{p+1}} \tau^\sigma (d\omega, \sigma)_{p+1} \quad , \quad (d\omega, \sigma)_{p+1} = (d\omega)_\sigma = \sum_{i=0}^{p+1} (-1)^i \omega_{\sigma_i},$$

while

$$(\omega, \partial\tau)_p = \sum_{\sigma \in \mathcal{A}_{p+1}} \tau^\sigma (\omega, \partial\sigma)_p \quad , \quad (\omega, \partial\sigma)_p = \left(\omega, \sum_{i=0}^{p+1} (-1)^i \sigma_i \right)_p = \sum_{i=0}^{p+1} (-1)^i \omega_{\sigma_i} \quad .$$

□

This theorem can be seen as the generalized Stokes' theorem on abstract simplicial complexes.

1.3 Smooth real manifolds and abstract graphs

1.4 Convolutional neural networks on euclidean domains