Geometric Deep Learning
Beyond Euclidean Domains

1 Geometric Priors

Definition 1.1. *Our compact euclidean domain* Ω
$\Omega := \prod_{i \in I}[0,1]$.

Definition 1.2. *Classification*
Let $x \in L^2 := L^2(\Omega)$ *then* $f : L^2 \to \mathscr{C}$ *surjective is said to be a classification of* L^2 *on the set* $\mathscr{C}$.

Definition 1.3. *Training set*
Let f *be a classification of* L^2 *on* $\mathscr{C}$ *and* $\{x_i\}_{i \in I} \subset L^2$ *then the set* $\{(x_i, f(x_i))\}_{i \in I}$ *is called a training set for* f.

Proposition 1.1. *The classification* f *is not injective*
Let f *be a classification of* L^2 *on* $\mathscr{C}$ *then, given the inevitable noise acting on data, there exists a real positive* ε *such that* $\forall (x, x_\varepsilon) \in L^2 \times L^2 : \int_\Omega |x - x_\varepsilon|^2 < \varepsilon$ *we have that* $f(x) = f(x_\varepsilon)$.

Given ideal data classification we can define two functions f-equivalent if and only if their images via the classification f are equal according to an equivalence on $\mathscr{C}$ which so far can be any set.

Proposition 1.2. *The relation* $\simeq$ *is an equivalence relation*
Let $x, y, z \in L^2$ *we define* $x \simeq y \iff f(x) = f(y)$ *where* f *is a classification of* L^2 *on* $\mathscr{C}$, *then:*
(i) $x \simeq x$
(ii) $x \simeq y \iff y \simeq x$
(iii) $x \simeq y, y \simeq z \implies x \simeq z$

Proof. (i),(ii) and (iii) follow from the the equivalence on $\mathscr{C}$ by which they are defined. □

Definition 1.4. *Translation operator*
Let $x \in L^2$ *and* $v \in \Omega$ *then* $T_v : L^2 \to L^2$ *such that* $x(\xi) \mapsto x(\xi - v)$ *is said to be a translation operator.*

Definition 1.5. *Local deformation operator*
Let $x \in L^2$ *and* $\tau \in C^\infty(\Omega, \Omega)$ *then* $L_\tau : L^2 \to L^2$ *such that* $x(\xi) \mapsto x(\xi - \tau(\xi))$ *is said to be a local deformation operator according to the smooth vector field* τ.

Definition 1.6. *Invariance*
A classification f *of* L^2 *on* $\mathscr{C}$ *is said to be* A*-invariant, where* $A : L^2 \to L^2$, *if and only if* $f(A(x)) = f(x)\ \forall x \in L^2$.

Definition 1.7. *Equivariance*
A classification f *of* L^2 *on* $\mathscr{C}$ *is said to be* A*-equivariant, where* $A : L^2 \to L^2$, *if and only if* $f(A(x)) = A(f(x))\ \forall x \in L^2$.
Problem: A is not defined on $\mathscr{C}$.