Are Graphs Manifolds?

1 Graphs

Definition 1.1. *Let $\mathscr{G}$ be a connected simplicial complex such that $dim\mathscr{G} = 1$ and let $\mathscr{V} := Vert\mathscr{G}$ and $\mathscr{E} := \mathscr{G}^{(i)}$, let $F, G : \mathscr{V} \to \mathscr{E}$ and $f, g : \mathscr{V} \to \mathbb{R}$ we want to define a scalar product on $L^2(\mathscr{E})$ and on $L^2(\mathscr{V})$:*
$< f, g >_{\mathscr{V}} = \sum_{i\in\mathscr{V}} a_i f_i g_i$
$< F, G >_{\mathscr{E}} = \sum_{\mathscr{E}} w_{ij} F_{ij} G_{ij}$ *DO NOT COUNT THE EDGES TWICE!**

Definition 1.2. $grad : L^2(\mathscr{V} \to L^2(\mathscr{E})$ *such that* $f \mapsto f_i - f_j$
$div : L^2(\mathscr{E}) \to L^2(\mathscr{V})$ *such that* $F \mapsto \frac{1}{a_i} \sum_{j:(i,j)\in\mathscr{E}} w_{ij} F_{ij}$

Proposition 1.1. $< divF, f >_{\mathscr{V}} = < F, gradf >_{\mathscr{E}}$ *if* $F_{ij} = -F_{ji}$

Proof. $\sum_{i\in\mathscr{V}} a_i f_i divF_i = \sum_{\mathscr{E}} w_{ij} F_{ij}(f_i - f_j) = * \sum_{i,j:(i,j)\in\mathscr{E}} w_{ij} F_{ij} f_i$
then $a_i divF_i = \sum_{j:(i,j)\in\mathscr{E}} w_{ij} F_{ij}$ □

Definition 1.3. $\partial\mathscr{A} \subset \mathscr{V} := \{(i,j) \in \mathscr{E} : i \in \mathscr{A}, j \notin \mathscr{A}\}$ *the second condition is equivalent to outwards oriented surface*

Proposition 1.2. $\sum_{i\in\mathscr{A}} divF_i = \sum_{i,j:(i,j)\in\partial\mathscr{A}} F_{ij}$ *assuming* $a_i = w_{ij} = 1$

Proof. $\sum_{i,j:i\in\mathscr{A},(i,j)\in\mathscr{E}} = \sum_{i,j:i\in\mathscr{A},(i,j)\in\mathscr{E},j\notin\mathscr{A}} + \sum_{i,j:i\in\mathscr{A},(i,j)\in\mathscr{E},j\in\mathscr{A}}$
In the second sum the simmetric adjacency matrix kills the antisimmetric F_{ij}, so only the first one remains □