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DEEP LEARNING ON ABSTRACT SIMPLICIAL COMPLEXES

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Anno Accademico 2020/2021

Introduction

Most of the deep learning techniques used today are based on models which learn a partition of the set of smooth functions defined on euclidean domains into human friendly equivalence classes. Although this approach has been successful in modern machine learning, it only deals with a really small set of domains. The goal of geometric deep learning is to extend this method to data defined on manifolds and simplicial complexes.

Convolution on euclidean domains is itself based on the translation invariance of such domains. In fact the convolution of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, with some filter $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is $(f * g)(x) = \langle f, g \circ T^{-1} \rangle_{L^2}$, where $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a translation represented by the vector x . Moreover, one could also consider such a convolution to be defined on the translation group itself, represented by some \mathbb{R}^n . In the same way one can define a convolution on the group $(\mathbb{Z}, +)$ as $(a * b)_n = \sum_{i \in \mathbb{Z}} a_i b_{n-i}$. Such definitions of convolution operators are equivariant with respect to the action of the group they are defined upon. In image recognition translation invariance is necessary, but graphs and simplicial complexes do not in general need this symmetry.

Contents

Introduction	i
1 Preliminaries on topology	1
Simplicial Complexes	1
Simplicial Homology	5
Simplicial Cohomology	6
Laplacian Operators	7
2 Simplicial Neural Networks	9
Convolutional Neural Networks	9
Simplicial Neural Networks	9
A Category Theory	10
B Construction of Simplicial Complexes from Point Clouds	12
Bibliography	13

Chapter 1

Preliminaries on topology

The essential idea in algebraic topology is to convert problems about topological spaces and continuous functions into problems about algebraic objects and their homomorphisms, this way one hopes to end up with an easier problem to solve.

Simplicial Complexes

In this section we shall define structures called simplicial complexes and discuss some of their properties. In order to define these structures we need the definitions of convex hull and affine independence in \mathbb{R}^n . For a deeper insight into simplicial complexes we refer to [1, 2].

Definition 1.1. Let $A \subset \mathbb{R}^n$, we define A to be *convex* if

$$x, y \in A \Rightarrow tx + (1 + t)y \in A$$

for all $t \in [0, 1]$.

Definition 1.2. Let $\sigma := \{x_i\}_{i \in I}$ be a finite subset of \mathbb{R}^n , we define σ to be *affinely independent* if $\{x_0 - x_i\}_{i \in I - \{0\}}$ is linearly independent.

For the definition of affine independence of $\sigma = \{x_i\}_{i \in I} \subset \mathbb{R}^n$ to be well stated we need it to be independent of the choice of x_0 .

Proposition 1.3. Let $\sigma := \{x_i\}_{i \in I}$ be a finite subset of \mathbb{R}^n , let $j \in I$ then, if $\{x_j - x_i\}_{i \in I - \{j\}}$ is linearly independent, also $\{x_0 - x_i\}_{i \in I - \{0\}}$ is.

Proof. If $j = 0$ the statement is trivially true. Let $j \neq 0$ and $\lambda_i \in \mathbb{R}$ for all $i \neq j$, then

$$\sum_{i \in I - \{j\}} \lambda_i (x_j - x_i) = 0 \Rightarrow \lambda_i = 0 \quad \forall i \in I - \{j\}.$$

Let then $\mu_i \in \mathbb{R}$ for all $i \neq 0$, and suppose

$$\sum_{i \in I - \{0\}} \mu_i (x_0 - x_i) = (x_0 - x_j) \sum_{i \in I - \{0\}} \mu_i + \sum_{i \in I - \{0\}} \mu_i (x_j - x_i) = 0.$$

If we define $\mu_0 := -\sum_{i \in I - \{0\}} \mu_i$ we have that

$$0 = \sum_{i \in I} \mu_i (x_j - x_i) = \sum_{i \in I - \{j\}} \mu_i (x_j - x_i) \Rightarrow \mu_i = 0 \quad \forall i \in I - \{j\},$$

therefore the definition of affine independence is well stated. □

Definition 1.4. Let $\sigma := \{x_i\}_{i \in I}$ be a finite subset of \mathbb{R}^n , we define the *convex set generated by* σ to be the smallest convex set containing X according to the inclusion relation.

Since the intersection of convex sets is convex, the convex set generated by σ can be equivalently defined as the intersection of all convex sets containing σ .

Theorem 1.5. Let $\sigma := \{x_i\}_{i \in I}$ be a finite subset of \mathbb{R}^n , if X is affinely independent then the convex set generated by X is

$$[\sigma] := \left\{ \sum_{i \in I} \lambda_i x_i : \lambda_i \geq 0, \sum_{i \in I} \lambda_i = 1 \right\}.$$

Furthermore for any point $x \in [\sigma]$ we have that

$$x = \sum_{i \in I} \lambda_i x_i = \sum_{i \in I} \mu_i x_i \Rightarrow \lambda_i = \mu_i \forall i \in I,$$

where $\lambda_i, \mu_i \geq 0$ and $\sum_{i \in I} \lambda_i = \sum_{i \in I} \mu_i = 1$.

Proof. Let $C := \{\bigcap_{\alpha} C_{\alpha} : \sigma \in C_{\alpha}, C_{\alpha} \text{ convex}\}$, we want to show that:

(i) $C \subset [\sigma]$.

This is true if $[\sigma]$ is convex and contains σ . The proof that it contains σ is trivial. In fact for every vertex $x_j = \sum_{i \in I} \delta_{ij} x_i$, and $\sum_{i \in I} \delta_{ij} = 1$.

To prove that it is convex we chose two points $a = \sum_{i \in I} a_i x_i, b = \sum_{i \in I} b_i x_i$ where $a_i, b_i \geq 0 \forall i \in I$ and $\sum_{i \in I} a_i = \sum_{i \in I} b_i = 1$. For $t \in [0, 1]$

$$ta + (1-t)b = t \sum_{i \in I} a_i x_i + (1-t) \sum_{i \in I} b_i x_i = \sum_{i \in I} (ta_i + (1-t)b_i) x_i.$$

It follows that $ta_i + (1-t)b_i \geq 0$ and $\sum_{i \in I} (ta_i + (1-t)b_i) = t \sum_{i \in I} a_i + (1-t) \sum_{i \in I} b_i = 1$ for all $i \in I$, which proves our statement.

(ii) $[\sigma] \subset C$.

If all but one the λ_i are zero certainly $\sum_{i \in I} \lambda_i x_i \in C$, since C contains all the vertexes. The inductive hypothesis, by relabeling, is that if the first $\lambda_0, \dots, \lambda_{n-1}$ are non-zero, hence not even 1, then $\sum_{i \in I} \lambda_i x_i \in C$. We want to show that whenever $\lambda_0, \dots, \lambda_n$ are non-zero then also $\sum_{i \in I} \lambda_i x_i \in C$, since $\lambda_n \neq 0$ we have that

$$\sum_{i \in I} \lambda_i x_i = \sum_{i=0}^n \lambda_i x_i = \lambda_n x_n + \sum_{i=0}^{n-1} \lambda_i x_i = \lambda_n x_n + (1 - \lambda_n) \sum_{i=0}^{n-1} \frac{\lambda_i}{1 - \lambda_n} x_i.$$

Since $\sum_{i=0}^{n-1} \frac{\lambda_i}{1 - \lambda_n} = 1$, for the inductive hypothesis $\sum_{i=0}^{n-1} \frac{\lambda_i}{1 - \lambda_n} x_i \in C$. Also the vertex x_n is contained in C by definition, therefore, being C convex and $\lambda_n \in [0, 1]$, it follows that

$$\lambda_n x_n + (1 - \lambda_n) \sum_{i=0}^{n-1} \frac{\lambda_i}{1 - \lambda_n} x_i \in C.$$

Accordingly $\sum_{i \in I} \lambda_i x_i \in C$, by induction we conclude the proof.

(iii) $\sum_{i \in I} \lambda_i x_i = \sum_{i \in I} \mu_i x_i \Rightarrow \lambda_i = \mu_i \forall i \in I$.

Let $\sum_{i \in I} \lambda_i x_i = \sum_{i \in I} \mu_i x_i$, then also $x_0 \sum_{i \in I} \lambda_i + \sum_{i \in I} \lambda_i (x_i - x_0) = x_0 \sum_{i \in I} \mu_i + \sum_{i \in I} \mu_i (x_i - x_0)$, and since both λ_i and μ_i are normalised we have that

$$\sum_{i \in I} (\lambda_i - \mu_i)(x_0 - x_i) = \sum_{i \in I - \{0\}} (\lambda_i - \mu_i)(x_0 - x_i) = 0 \Rightarrow \lambda_i = \mu_i \quad \forall i \in I - \{0\},$$

because of the affine independence. □

A proof of theorem 1.5 can be found in [1]. The set $[\sigma]$ is often called *convex hull* of σ .

Definition 1.6. We define a *p-simplex* $[\sigma]$ to be the convex hull of an affinely independent set $\sigma := \{x_i\}_{i \in I} \subset \mathbb{R}^n$, where $p = |I| - 1$ is called dimension of the p-simplex.

Theorem 1.5 gives us the possibility to represent a point in a simplex $[\sigma]$ via a finite set of real parameters defined in the range $[0, 1]$ and satisfying the normalisation condition $\sum_{i \in I} \lambda_i = 1$. Such parameters are called *baricentric coordinates* of $[\sigma]$.

The points in σ are called *vertexes* of the simplex $[\sigma]$, accordingly we define the vertex set of a simplex $[\sigma]$ to be $Vert([\sigma]) = \sigma$.

Definition 1.7. Let $[\sigma]$ be a p-simplex and $p, t \in \mathbb{N}$, we say that another t-simplex $[\tau]$ is a *face* of $[\sigma]$ or equivalently that $[\sigma]$ is a *coface* of $[\tau]$, and we write $[\tau] \leq [\sigma]$, if $\tau \subset \sigma$, where $t \leq p$.

Now we are ready for our main definitions.

Definition 1.8. We define a *simplicial complex* \mathcal{G} to be a collection of simplexes such that

- (i) if any simplex $[\tau] \leq [\sigma], [\sigma] \in \mathcal{G}$ then $[\tau] \in \mathcal{G}$,
- (ii) if $[\sigma], [\tau] \in \mathcal{G}$ then $[\sigma] \cap [\tau] \in \mathcal{G}$.

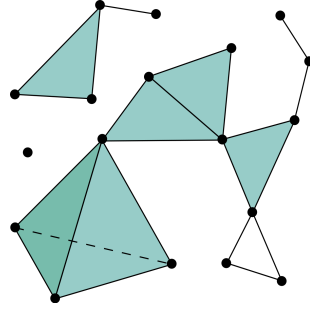


Fig. 1.1: Example of simplicial complex.

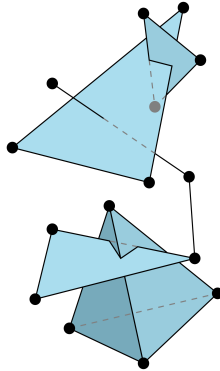


Fig. 1.2: Set of simplexes which is not a simplicial complex.

We can see in Fig. 1.2 that the intersection property of simplicial complexes is not satisfied.

In this first part we analysed simplicial complexes as peculiar subsets of \mathbb{R}^n , namely *geometric simplicial complexes*. Although this approach provides simplicial complexes with the topology inherited from the metric space, it hides the power of simplicial complexes to describe those networks and interactions which would happily exist without that topology. To make this distinction clear enough we will treat simplicial complexes as a realization of more abstract objects called *abstract simplicial complexes*. The discussion of abstract simplicial complexes can be found in [2, 3].

Definition 1.9. Let \mathcal{V} be a finite set, we define an *abstract simplicial complex* \mathcal{A} to be a family of non empty subsets of \mathcal{V} such that:

- (i) if $v \in \mathcal{V}$, then $\{v\} \in \mathcal{A}$,
- (ii) if $\sigma \in \mathcal{A}$ and $\tau \subset \sigma$, then $\tau \in \mathcal{A}$.

We call the member of this family *abstract simplexes*.

One calls \mathcal{V} the *vertex set* of \mathcal{A} and denotes it by $Vert(\mathcal{A})$; since the vertex set is finite we expect every abstract simplex to be also finite, therefore we might use the notation $\sigma = \{v_i\}_{i \in I_\sigma}$, which so far we consider invariant under arbitrary permutations of the finite index set I_σ .

Definition 1.10. Let \mathcal{A} be an abstract simplicial complex and \mathcal{G} a geometric simplicial complex, if for all $\{x_i\}_{i \in I} \in \mathcal{A}$ also $[x_i]_{i \in I} \in \mathcal{G}$ we say that \mathcal{G} is a *geometric realization* of \mathcal{A} .

While every geometric simplicial complex can be thought as a geometric realization of an abstract simplicial complex, the existence of a geometric realization for an arbitrary abstract simplicial complex is not trivial at all.

Theorem 1.11. *Let \mathcal{A} be an n -dimensional abstract simplicial complex, then it admits a geometric realization in \mathbb{R}^{2n+1} .*

A proof of this theorem can be found in [3].

Both for abstract and geometric simplicial complexes one can define maps called *simplicial maps* in order to obtain a category (A) whose equivalences are called isomorphisms. The geometric realisation is unluckily not a functor on the whole category of abstract simplicial complexes due to the limitation imposed by theorem 1.11. Nevertheless, when they exist, the geometric realizations of isomorphic abstract simplicial complexes are themselves isomorphic.

In the following sections we shall use abstract simplicial complexes, which can be always thought geometrically in the appropriate euclidean space.

Simplicial Homology

An important field in algebraic topology is homology theory. We shall discuss homology theory to the extent that allows us to define the laplacian operator on simplicial complexes, for further readings see [4]. First we want to equip our simplicial complexes with an orientation.

Definition 1.12. An *oriented* simplicial complex \mathcal{A} is a simplicial complex and a partial order on $Vert(\mathcal{A})$ whose restriction to the vertices of any simplex in \mathcal{A} is a linear order.

Then we define on the simplicial complex a vector space structure.

Definition 1.13. Let \mathcal{A} be an oriented simplicial complex, on \mathcal{A} we define a formal sum in order to obtain a vector space on the real numbers, that is

$$C_p(\mathcal{A}) := \left\{ \sum_i \lambda_i \sigma_i^p \mid \lambda_i \in \mathbb{R} \right\},$$

where σ_i^p are oriented p-simplexes of \mathcal{A} .

All $\sigma_i^p = [v_0, \dots, v_p]$ can have two possible orientations that satisfy $[v_0, \dots, v_p] = \text{sgn}(\pi)[v_{\pi 0}, \dots, v_{\pi p}]$, where π is a permutation of $\{0, \dots, p\}$. We shall call C_p the space of *simplicial p-chains*.

In the study of chains' spaces a special role is played by a particular linear operator called boundary operator.

Definition 1.14. We define the *boundary* operator $\partial_{p+1} : C_{p+1} \rightarrow C_p$ by setting

$$\partial_{p+1}([v_0, \dots, v_p]) = \sum_{i=0}^p (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_p]$$

(where \hat{v}_i means delete the vertex v_i) and extending by linearity.

The set of all chains' spaces with their respective boundary operators is a special category that we call a *chain complex*, the property that defines a chain complex is the following.

Theorem 1.15. $\partial^2 = 0$.

Proof. Let $\partial_{p+1}([v_0, \dots, v_{p+1}]) = \sum_{i=0}^{p+1} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_{p+1}]$ then

$$\partial_p(\partial_{p+1}([v_0, \dots, v_{p+1}])) = \sum_{j=0, j \neq i}^{p+1} \sum_{i=0}^{p+1} (-1)^{i+j} [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_{p+1}] = 0.$$

□

We are interested in boundary operators because they allow us to define the p-holes, which mathematically can be thought as the p-cycles, p-chains that have no boundary, that are not boundary of any higher dimensional simplex. It is possible to show that the p-holes form a vector space and if two simplicial complex are isomorphic also their p-holes' spaces are. This, given what we said in the previous section, allows us to decide if two topological spaces are not homeomorphic by looking at the p-holes' spaces of their triangulations.

Definition 1.16. We define the *p-homology group* to be

$$H_p := \frac{\ker \partial_p}{\text{im} \partial_{p+1}},$$

where $\text{im} \partial_{p+1}$ is the group of simplicial p-cycles and $\ker \partial_p$ is the group of simplicial p-boundaries.

The homology group is therefore the space of cycles that are not boundaries.

Simplicial Cohomology

In order to define data on simplicial complexes we are interested in studying the dual of the simplicial chain complex which we shall call a simplicial cochain complex.

Definition 1.17. Let C_p be the space of simplicial p-chains, we define the space of *simplicial cochains* to be $C^p := Hom(C_p, \mathbb{R})$, i.e. the dual space of C_p .

Definition 1.18. The dual of the boundary operator which we shall call *coboundary operator*, is defined to be

$$d_{p+1} : C^p \rightarrow C^{p+1} \quad d_p(\sigma^*) = \sigma^* \circ \partial_{p+1}.$$

One could also show that the property $d^2 = 0$ is verified in the so defined cochain complex. The homology group could in general be define for any chain complex, in particulare for the simplicial cochain complex it takes the name of cohomology group.

Definition 1.19. We define the *p-cohomology group* to be

$$H^p := \frac{kerd_{p+1}}{imd_p},$$

where imd_{p-1} is the group of simplicial p-cocycles and $kerd_p$ is the group of simplicial p-coboundaries.

To have a mirrored intuition of what the coboundary operator actually does we equipe our simplicial chain spaces with an inner product. This finite dimensional Hilbert space structure shall allow us to represent the action of a cochain σ^* on the chain τ as the scalar product $\langle \sigma, \tau \rangle$.

Definition 1.20. We define the scalar product called *integration* $\langle, \rangle : C_p \times C_p \rightarrow \mathbb{R}$ on the canonical basis of C_p to be

$$\langle i, j \rangle = \delta_{ij},$$

where i, j are any two p-simplexes in the canonical basis of C_p and δ_{ij} is the Kronecker Delta.

It is convenient, from now on, to write cochains σ^* as bra vectors $\langle \sigma |$, chains σ as ket vectors $|\sigma \rangle$, and the scalar product $\langle i, j \rangle$ as the bra-ket product $\langle i | j \rangle$.

The same Hilbert space structure allows us to represent the coboundary operator d in the space of chains with its integral kernel ∂^\dagger .

Definition 1.21. Let $d_{p+1} : C^p \rightarrow C^{p+1}$ be the coboundary operator, then for any $\sigma^* \in C^p, \tau \in C_{p+1}$ we can define its dual representation ∂_{p+1}^\dagger by

$$(d_{p+1}\sigma^*)(\tau) = (d_{p+1}\langle \sigma |)|\tau \rangle = \langle \partial_{p+1}^\dagger \sigma | \tau \rangle.$$

It is easy to notice that our definitions lead to the restatement of the equivalent of the generalized Stokes' theorem on simplicial complexes according to the integration previously defined, i.e. $(d\langle \sigma |)|\tau \rangle = \langle \sigma | \partial \tau \rangle$. One can therefore think of the coboundary operator as a discrete exterior derivative acting on cochains.

Laplacian Operators

An important role in the definition of a convolution on simplicial complexes is played by the Laplacian operator, especially by its eigenfunctions and spectrum. A more thorough discussion of this operator can be found in [5].

Definition 1.22. We define the p -Laplacian operator to be

$$\Delta_p = \partial_{p+1} \partial_{p+1}^\dagger + \partial_p^\dagger \partial_p =: \Delta_p^+ + \Delta_p^-.$$

The Laplacian operator is defined to be self-adjoint and positive definite.

Proposition 1.23. $\Delta_p^\dagger = \Delta_p$.

Proof. Let $|\sigma\rangle, |\tau\rangle \in C_p$

$$\begin{aligned} \langle \sigma | \Delta_p \tau \rangle &= \langle \sigma | (\partial_{p+1} \partial_{p+1}^\dagger + \partial_p^\dagger \partial_p) \tau \rangle = \\ &= \langle (\partial_{p+1} \partial_{p+1}^\dagger + \partial_p^\dagger \partial_p)^\dagger \sigma | \tau \rangle = \\ &= \langle (\partial_{p+1} \partial_{p+1}^\dagger + \partial_p^\dagger \partial_p) \sigma | \tau \rangle = \langle \Delta_p \sigma | \tau \rangle. \end{aligned}$$

□

According to the spectral theorem there exists a basis of eigenchains of the Laplacian, and since all $\Delta_p, \Delta_p^+, \Delta_p^-$ are self-adjoint we can say that they all admit a basis of eigenchains.

Proposition 1.24. Let $\Delta_p |\sigma\rangle = \lambda_\sigma |\sigma\rangle$ then $\lambda_\sigma \geq 0$.

Proof. Let $\Delta_p^+ |\sigma\rangle = \lambda_\sigma^+ |\sigma\rangle$, we see that

$$\begin{aligned} \langle \sigma | \Delta_p^+ \sigma \rangle &= \langle \partial_{p+1}^\dagger \sigma | \partial_{p+1} \sigma \rangle \geq 0 \\ \langle \sigma | \Delta_p^+ \sigma \rangle &= \lambda_\sigma^+ \langle \sigma | \sigma \rangle \geq 0 \implies \lambda_\sigma^+ \geq 0. \end{aligned}$$

Let then $\Delta_p^- |\sigma\rangle = \lambda_\sigma^- |\sigma\rangle$, we see that

$$\begin{aligned} \langle \sigma | \Delta_p^- \sigma \rangle &= \langle \partial_p \sigma | \partial_p \sigma \rangle \geq 0 \\ \langle \sigma | \Delta_p^- \sigma \rangle &= \lambda_\sigma^- \langle \sigma | \sigma \rangle \geq 0 \implies \lambda_\sigma^- \geq 0. \end{aligned}$$

Furthermore, since $\Delta_p^+ \Delta_p^- = \Delta_p^- \Delta_p^+ = 0$ we have that $[\Delta_p^+, \Delta_p^-] = 0$, thence $[\Delta_p, \Delta_p^\pm] = 0$, therefore $\Delta_p, \Delta_p^+, \Delta_p^-$ share a basis of eigenchains. Let $|\sigma\rangle$ be in that common basis then $\Delta_p |\sigma\rangle = \lambda_\sigma |\sigma\rangle$, where $\lambda_\sigma = \lambda_\sigma^+ + \lambda_\sigma^- \geq 0$. □

Another really interesting property that was first proven by Beno Eckmann in 1944, is that the kernel of the p-Laplacian is isomorphic to the p-homology group.

Theorem 1.25. $\ker \Delta_p \simeq H_p$.

Proof. We have

$$\Delta_p = \partial_{p+1} \partial_{p+1}^\dagger + \partial_p^\dagger \partial_p =: \Delta_p^+ + \Delta_p^-.$$

It is true that

$$\Delta_p^+ \Delta_p^- = \Delta_p^- \Delta_p^+ = 0 \implies \ker \Delta_p^\pm \subset \ker \Delta_p^\mp.$$

It is trivial to see that since $\ker \Delta_p = \ker \Delta_p^+ \cap \ker \Delta_p^-$

$$\ker \partial_{p+1}^\dagger \cap \ker \partial_p \subset \ker \Delta_p,$$

less trivial is the opposite inclusion:

$$\partial_{p+1}^\dagger \sigma \in \text{im} \partial_{p+1}^\dagger, \sigma \in \ker \Delta_p \implies \partial_{p+1}^\dagger \sigma \in \ker \partial_{p+1} = (\text{im} \partial_{p+1}^\dagger)^\perp,$$

$$\partial_p \sigma \in \text{im} \partial_p, \sigma \in \ker \Delta_p \implies \partial_p \sigma \in \ker \partial_p^\dagger = (\text{im} \partial_p)^\perp,$$

$$\partial_p \sigma = \partial_{p+1}^\dagger \sigma \in \text{im} \partial_{p+1}^\dagger \cap (\text{im} \partial_{p+1}^\dagger)^\perp = \text{im} \partial_p \cap (\text{im} \partial_p)^\perp = 0,$$

therefore $\ker \partial_{p+1}^\dagger \cap \ker \partial_p \subset \ker \Delta_p$. Finally we notice that

$$\ker \Delta_p = \ker \partial_{p+1}^\dagger \cap \ker \partial_p = (\text{im} \partial_{p+1}^\dagger)^\perp \cap \ker \partial_p \simeq H_p.$$

□

Chapter 2

Simplicial Neural Networks

Convolutional Neural Networks

The goal of machine learning is teaching machines to approximate expert observers. Thinking of the space of possible machines as possibly convex and compact, one is able by varying a finite number of parameters to find a unique best approximation of any expert observer. The representation of a random initialized machine is

Definition 2.1 (relu).

Definition 2.2 (convolutional layer).

Definition 2.3 (pooling layer).

Definition 2.4 (drop out).

Definition 2.5 (CNN).

Definition 2.6 (gradient descent).

Simplicial Neural Networks

Applications of simplicial complexes

Definition 2.7 (convolution on a simplicial complex).

Definition 2.8 (heat equation on a simplicial complex).

Definition 2.9 (pooling on a simplicial complex).

Definition 2.10 (pooling on a graph).

Definition 2.11 (simplicialNN).

Appendix A

Category Theory

The essential idea in algebraic topology is to convert problems about topological spaces and continuous functions into problems about algebraic objects and their homomorphisms, this way one hopes to end up with an easier problem to solve. The language of category theory shall be our main tool to formally describe this conversion.

Definition A.1. A *category* \mathbf{C} consists of three ingredients:

a class of *objects* $Obj(\mathbf{C})$; sets of *morphisms* $Hom(A, B)$ for every ordered pair $(A, B) \in Obj(\mathbf{C}) \times Obj(\mathbf{C})$; a composition $Hom(A, B) \times Hom(B, C) \rightarrow Hom(A, C)$, denoted by $(f, g) \mapsto f \circ g$ for every $A, B, C \in Obj(\mathbf{C})$, satisfying the following axioms:

- (i) the family of $Hom(A, B)$ is pairwise disjoint,
- (ii) the composition, when defined, is associative,
- (iii) for each $A \in Obj(\mathbf{C})$ there exists an *identity* $1_A \in Hom(A, A)$ such that for $f \in Hom(A, B)$ and $g \in Hom(C, A)$ we have that $1_A \circ f = f$ and $g \circ 1_A = g$.

Instead of writing $f \in Hom(A, B)$, we usually write $f : A \rightarrow B$.

Definition A.2. Let \mathbf{A} and \mathbf{C} be categories, a *functor* $T : \mathbf{A} \rightarrow \mathbf{C}$ is a function, that is,

- (i) $A \in Obj(\mathbf{A}) \implies TA \in Obj(\mathbf{C})$,
- (ii) $f : A \rightarrow A' \implies Tf : TA \rightarrow TA' \quad A, A' \in Obj(\mathbf{A})$,
- (iii) if f, g are morphisms in \mathbf{A} for which $g \circ f$ is defined, then $T(g \circ f) = (Tg) \circ (Tf)$,
- (iv) $T(1_A) = 1_{TA} \quad \forall A \in \mathbf{A}$.

The property (iii) of the previous definition actually defines what we shall call *covariant functors*. If instead we require $T(g \circ f) = (Tf) \circ (Tg)$, we are defining a so called *contravariant functor*.

Definition A.3. An *equivalence* in a category \mathbf{C} is a morphism $f : A \rightarrow B$ for which there exists a morphism $g : B \rightarrow A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$.

Theorem A.4. If \mathbf{A} and \mathbf{C} are categories and $T : \mathbf{A} \rightarrow \mathbf{C}$ is a functor of either variance, then whenever f is an equivalence on \mathbf{A} then Tf is an equivalence on \mathbf{C} .

Proof. We apply T to the equations $f \circ g = 1_B$ and $g \circ f = 1_A$, that for a covariant functor leads to $(Tf) \circ (Tg) = T(1_B) = 1_{TB}$ and $(Tg) \circ (Tf) = T(1_A) = 1_{TA}$. \square

A category that will be used in the following section is the category of topological spaces and continuous functions.

Theorem A.5. *Topological spaces and continuous functions are a category **Top**, whose equivalences are called homeomorphisms.*

Other examples of categories can be found in [2].

Appendix B

Construction of Simplicial Complexes from Point Clouds

Cech, Alpha and Delauney complexes. Nerve Theorem.

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