Graphs

1 Graph Operators

Definition 1.1. *Let $\mathscr{G}$ be a graph where $\mathscr{V}$ are its vertexes and $\mathscr{E}$ are its edges, let $f,g : L^2(\mathscr{V})$ and $F,G \in L^2(\mathscr{E})$ be real valued functions, we define $\langle f,g\rangle_{L^2(\mathscr{V})} := \sum_{\mathscr{V}} a_i f_i g_i$, $a_i \in \mathbb{R}$ and $\langle F,G\rangle_{L^2(\mathscr{E})} := \sum_{\mathscr{E}} w_{ij} F_{ij} G_{ij}$, $w_{ij} \in \mathbb{R}$.*

Definition 1.2. ***Graph gradient and divergence***
Let $f \in L^2(\mathscr{V})$ and $F \in L^2(\mathscr{E})$ we define $grad : L^2(\mathscr{V}) \to L^2(\mathscr{E})$ and $div : L^2(\mathscr{E}) \to L^2(\mathscr{V})$, such that $(grad f)_{ij} = f_i - f_j$ and $(divF)_i = \frac{1}{a_i}\sum_{j\in\mathscr{V}:(i,j)\in\mathscr{E}} w_{ij}F_{ij}$.

Proposition 1.1. *Let $f \in L^2(\mathscr{V})$ and $F \in L^2(\mathscr{E}) : F_{ij} = -F_{ji}$ then $\langle f, divF\rangle_{L^2(\mathscr{V})} = \langle grad f, F\rangle_{L^2(\mathscr{E})}$, i.e. $divF^\dagger = grad$.*

Proof. $\sum_{\mathscr{V}} a_i f_i (divF)_i = \sum_{\mathscr{E}} w_{ij}F_{ij}(f_i - f_j) = \sum_{i\in\mathscr{V}}\sum_{j\in\mathscr{V}:(i,j)\in\mathscr{E}} w_{ij}F_{ij}f_i$ thus $a_i(divF)_i = \sum_{j\in\mathscr{V}:(i,j)\in\mathscr{E}} w_{ij}F_{ij}$. □

Theorem 1.2. ***Gauss theorem***
Let $F \in L^2(\mathscr{E}) : F_{ij} = -F_{ji}$, let $\mathscr{A} \subset \mathscr{V}$ then if $a_i = w_{ij} = 1$ we have $\sum_{\mathscr{A}}(divF)_i = \sum_{\partial^0_+\mathscr{A}} F_{ij}$.

Proof. First of all we recall $\partial^0_+\mathscr{A} = \{(i,j) \in \mathscr{E}, i \in \mathscr{A}, j \in \mathscr{V}\setminus\mathscr{A}\}$, then we see that $\sum_{\mathscr{A}}(divF)_i = \sum_{i\in\mathscr{A}}\sum_{j\in\mathscr{V}:(i,j)\in\mathscr{E}} F_{ij} = \sum_{i\in\mathscr{A}}\sum_{j\in\mathscr{V}\setminus\mathscr{A}:(i,j)\in\mathscr{E}} F_{ij} + \sum_{i\in\mathscr{A}}\sum_{j\in\mathscr{A}:(i,j)\in\mathscr{E}} F_{ij} = \sum_{\partial^0_+\mathscr{A}} F_{ij} + \sum_{(i,j)\in\mathscr{A}^2} adj(\mathscr{A})_{ij}F_{ij}$ where since $adj(\mathscr{A})_{ij} = adj(\mathscr{A})_{ji}$ we have by renaming dummy indexes $adj(\mathscr{A})_{ij}F_{ij} = -adj(\mathscr{A})_{ij}F_{ij} = 0$. □

Definition 1.3. ***Graph laplacian***
Let $f \in L^2(\mathscr{V})$ we have that $\langle grad f, grad f\rangle = \langle div(grad f), f\rangle =: \langle \Delta f, f\rangle = \langle f, \Delta f\rangle$, where $\Delta : L^2(\mathscr{V}) \to L^2(\mathscr{V})$ is the Laplacian.