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Converse Lyapunov functions for exponentially stable periodic orbits*

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Abstract: In this paper, we present a technique for constructing a class of quadratic Lyapunov functions for exponentially stable periodic orbits. This construction is facilitated by the use of a special set of local coordinates that serve to highlight the tangential and transverse dynamics of the system.

Keywords: Converse Lyapunov functions; periodic orbits, stability; nonlinear systems; tansverse linearization; periodic Lyapunov equation.

1. Introduction

Lyapunov theory provides powerful tools for stability and robustness analysis. For example, given a Lyapunov function that proves the asymptotic stability of an invariant set, one can estimate the domain of attraction of that set and determine a class of disturbances under which the states of system remain bounded. Converse Lyapunov theorems [3, 6, 9] guarantee a rich supply of such functions.

The mere *existence* of Lyapunov functions is not often sufficient – it is desirable to have techniques for *constructing* Lyapunov functions. Quadratic Lyapunov functions can be constructed for uniformly asymptotically stable linear systems by solving a Lyapunov equation. The result is a (uniformly) positive-definite quadratic Lyapunov function with a (uniformly) negative definite (quadratic) derivative. Since an equilibrium point of a smooth autonomous nonlinear system is exponentially stable if and only if the linearization about that point is, the quadratic Lyapunov function derived from the linearization is a valid Lyapunov function for the original nonlinear system.

Although a similar technique can be used to construct nonautonomous Lyapunov functions for exponentially stable *trajectories*, the story is somewhat more subtle for exponentially stable *orbits*. In particular, if we linearize about a specific trajectory on a periodic orbit, we will find out that, although the periodic orbit is exponentially stable, the linearization will not be exponentially stable. This is due to the fact that variations tangent to the orbit do not asymptotically converge to zero since they correspond to initial conditions *on* the orbit. It can be shown, however, that a periodic orbit is exponentially stable if and only if the linearization of the dynamics *transverse* to the orbit are exponentially stable.

In this paper, we use this fact to construct an autonomous Lyapunov function for exponentially stable periodic orbits. First, we provide a new technique for constructing a local coordinate system around the

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periodic orbit that essentially separates the tangential and transverse dynamics of the system. Then, using a periodic Lyapunov function constructed from the linearized transverse dynamics, we construct an autonomous Lyapunov function that proves the exponential stability of the periodic orbit.

1. Local coordinates about a periodic orbit

We are interested in finding a Lyapunov function for systems possessing a locally exponentially stable periodic orbit. Our approach is based on the following fact. The linearization of the system about a trajectory on the periodic orbit satisfies a certain (spectrum) condition if and only if the orbit is exponentially stable. This property is best explored by finding a convenient set of coordinates for a neighborhood of the periodic orbit. Such a set of coordinates was introduced by Urabe [8] and analyzed in detail by Hale [4]. In this section, we provide an alternate method for constructing a set of local coordinates with the desired properties. Roughly speaking, these coordinates highlight the tangential and transverse dynamics of the system.

Consider the smooth dynamical system

$$\dot{x} = f(x) \tag{1}$$

on $\mathbb{R}^n$ and suppose that $\eta \subset \mathbb{R}^n$ is a periodic orbit of (1) with period T.

Using the distance function

$$d(x,\eta) := \min_{y \in \eta} \|x - y\|$$

an ε-neighborhood of η can be specified as

$$B_\varepsilon(\eta) := \{x \in \mathbb{R}^n \colon d(x,\eta) < \varepsilon\}.$$

The orbit η is *stable* if trajectories starting near η stay near η, i.e. for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in B_\delta(\eta) \to \phi_t^f(x) \in B_\varepsilon(\eta)$ for all $t \geq 0$ (ϕ_t^f is the *flow* of the vector field f); η is *asymptotically stable* if it is stable and trajectories starting near η converge to η, i.e. there is a $\delta > 0$ such that $d(\phi_t^f(x), \eta) \to 0$ as $t \to \infty$ for all $x \in B_\delta(\eta)$; η is *exponentially stable* if it is asymptotically stable and the convergence is exponential, i.e. there exist $\delta, M, \lambda > 0$ such that $d(\phi_t^f(x), \eta) \leq d(x,\eta) M e^{-\lambda t}$ for all $x \in B_\delta(\eta)$. This type of stability is often called *orbital* stability to distinguish it from the often used concept of Lyapunov stability [3].

Let $x(\cdot)$ be a trajectory on the periodic orbit (i.e. $\bar{x}(t) \in \eta$ for all $t \geq 0$) and consider the (periodic) variational linear system

$$\dot{\delta x} = F(t)\delta x, \tag{2}$$

where

$$F(t) := \frac{\partial f}{\partial x}(\bar{x}(t)).$$

Let $\Phi_F(t,\tau)$ be the state transition matrix for (2). We have the following theorem.

Theorem 1.1. *The orbit η is an exponentially stable orbit of* (1) *if and only if $n - 1$ of the eigenvalues of the* monodromy matrix $\Phi_F(T,0)$ *belong to the open unit disk.*

Proof. The sufficiency of the spectrum condition is well known (see e.g. [5, 4]). To show necessity, suppose that η is exponentially stable and consider the Poincaré map $G(\cdot)$ for an arbitrary section S transverse to the flow. Since any trajectory near η converges to η exponentially fast and the first return time is a continuous map, it is clear that the Poincaré map exhibits exponential convergence, i.e. there exist δ, $M > 0$, $0 < \mu < 1$ such that $\|G^k(x) - y\| \leq \|x - y\| M\mu^k$ for $k = 0, 1, \ldots$ for all $x \in B_\delta(\eta) \cap S$, where $y \in \eta \cap S$ is the fixed point of G. This can only happen if the $n-1$ eigenvalues of the linearization of G belong to the open unit disk. Necessity follows since the spectrum of the Poincaré map linearization is a subset of the spectrum of the monodromy matrix. □

Note that the eigenvalue of $\Phi_F(T,0)$ corresponding to variations tangent to η will always be one. Furthermore, the restriction of $\Phi_F(T,0)$ to the subspace orthogonal to the tangent to η at time $t=0$ is exactly the differential of a Poincaré map (with suitably chosen section) of the system.

It is clear that the behavior we are interested in is determined by the $n-1$-dimensional *transverse* dynamics. To highlight these dynamics, we construct a set of local coordinates $(\theta,\rho)\in S^1\times\mathbb{R}^{n-1}$ as follows.

First, parameterizing S^1 by $\theta\in[0,T)$ (i.e. the closed segment $[0,T]$ with the two endpoints identified), define a mapping $\theta\mapsto y(\theta)$ by solving

$$\frac{\mathrm{d}y(\theta)}{\mathrm{d}\theta}=f(y(\theta)),$$

with $y(0)\in\eta$ arbitrary (but fixed). Then the θ coordinate of a point x in a neighborhood of η is defined by

$$\theta=\psi_1(x):=\arg\min_{\theta\in[0,T)}\|y(\theta)-x\|^2. \tag{3}$$

Note that the function $\psi_1(\cdot)$ is smooth since $f(\cdot)$ is and the minimizer is unique on a small neighborhood of η. Next, letting $\psi_i(\cdot)$, $i=2,\dots,n$, be a set of functions that are independent and vanish on η, the ρ coordinate of x is given by

$$\rho_{i-1}=\psi_i(x),\quad i=2,\dots,n. \tag{4}$$

The following proposition shows that it is always possible to find such functions.

Proposition 1.2. *Suppose that η is a periodic orbit of $\dot{x}=f(x)$. Then there exist $n-1$ independent functions $\psi_i(\cdot)$, $i=2,\dots,n$ defined on a neighborhood of η that vanish on η.*

Proof. It is well known that a periodic orbit of a smooth dynamical system is diffeomorphic to the circle S^1 (i.e. it is a smooth Jordan curve). Let $\xi=\varphi(x)$ be a diffeomorphism defined on a neighborhood of η that takes η to the unit circle defined by

$$\{\xi\in\mathbb{R}^n:\ \tilde{\psi}_2(\xi)=\tilde{\psi}_3(\xi)=\cdots=\tilde{\psi}_n(\xi)=0\},$$

where $\tilde{\psi}_2(\xi)=\xi_1^2+\xi_2^2-1$ and $\tilde{\psi}_i(\xi)=\xi_i$, $i=3,\dots,n$. Clearly, the functions $\psi_i(x):=\tilde{\psi}_i(\varphi(x))$ vanish on η and are independent as desired. □

We now show that (θ,ρ) is a well-defined set of coordinates in a neighborhood of η.

Proposition 1.3. *The mapping $x\mapsto(\theta,\rho)$ given by $(\theta,\rho)=\Psi(x)$ in (3) and (4) is a diffeomorphism on a neighborhood of η.*

Proof. Using the fact that

$$\langle y(\theta)-x, f(y(\theta))\rangle=0$$

for x in a neighborhood of η, we see that

$$\mathrm{d}\psi_1(x)=\frac{f(y(\theta))^{\mathrm{T}}}{\langle f(y(\theta)),f(y(\theta))\rangle+\langle y(\theta)-x,\mathrm{D}f(y(\theta))f(y(\theta))\rangle}.$$

Since $f(x)$ is orthogonal to each $\mathrm{d}\psi_i(x)$, $i=2,\dots,n$, for each $x\in\eta$, the differentials $\mathrm{d}\psi_i$, $i=1,\dots,n$ are linearly independent on η. □

Roughly speaking, since the ρ coordinates are *transverse* to the periodic orbit at each point of η, the stability of the orbit is largely determined by the behavior of the ρ coordinates. This is seen more clearly by examining the system dynamics in the (θ, ρ) coordinates.

Proposition 1.4. *The dynamics of the nonlinear system* (1) *in a neighborhood of the periodic orbit η have the form*

$$\begin{aligned} \dot{\theta} &= 1 + f_1(\theta, \rho), \\ \dot{\rho} &= A(\theta)\rho + f_2(\theta, \rho), \end{aligned} \tag{5}$$

where $f_1(\cdot,\cdot)$ and $f_2(\cdot,\cdot)$ satisfy

$$f_1(\theta, 0) = 0, \qquad f_2(\theta, 0) = 0, \quad \textit{and} \quad \frac{\partial f_2(\theta, 0)}{\partial \rho} = 0.$$

Proof. To avoid confusion, we will use $f_j(\cdot)$ to refer to the components of $f(\cdot)$ from (1) and $\bar{f}_1(\cdot,\cdot)$ and $\bar{f}_2(\cdot,\cdot)$ to refer to the nonlinear terms in (5). We begin by calculating

$$\dot{\theta} = \frac{\langle f(y(\theta)), f(x)\rangle}{\langle f(y(\theta)), f(y(\theta))\rangle + \langle y(\theta) - x, \mathrm{D}f(y(\theta)) f(y(\theta))\rangle}. \tag{6}$$

Evaluating $\dot{\theta} - 1$ at $\rho = 0$ (i.e. $x = y(\theta)$), we see that $\bar{f}_1(\theta, 0) = 0$. Let $\Gamma = \Psi^{-1}$, i.e. $\Psi(\Gamma(\theta, \rho)) = (\theta, \rho)$. Then we have

$$\begin{aligned} x &= \Gamma(\theta, \rho) \\ &= \Gamma(\theta, 0) + \frac{\partial \Gamma}{\partial \rho}(\theta, 0)\rho + r(\theta, \rho) \\ &= y(\theta) + Z(\theta)\rho + r(\theta, \rho), \end{aligned}$$

where

$$Z(\theta) := \frac{\partial \Gamma}{\partial \rho}(\theta, 0)$$

and $r(\cdot,\cdot)$ is a remainder term that is higher order in ρ. Straightforward calculations show that $Z(\cdot)$ satisfies

$$\begin{bmatrix} \mathrm{d}\psi_2(y(\theta)) \\ \vdots \\ \mathrm{d}\psi_n(y(\theta)) \end{bmatrix} Z(\theta) = I_{n-1}$$

and

$$f(y(\theta))^{\mathrm{T}} Z(\theta) = 0.$$

Using the definition of ρ, we have, for $i = 2, \ldots, n$,

$$\begin{aligned} \dot{\rho}_{i-1} &= \frac{\partial \psi_i}{\partial x_j}(x) f_j(x) \\ &= \frac{\partial \psi_i}{\partial x_j}(y(\theta) + Z(\theta)\rho + r(\theta, \rho)) f_j(y(\theta) + Z(\theta)\rho + r(\theta, \rho)) \\ &= \left(\frac{\partial \psi_i}{\partial x_j} + \frac{\partial^2 \psi_i}{\partial x_j \partial x_k} Z_{kl}\rho_l + \mathrm{O}(\|\rho\|^2)\right)\left(f_j + \frac{\partial f_j}{\partial x_k} Z_{kl}\rho_l + \mathrm{O}(\|\rho\|^2)\right) \\ &= \frac{\partial \psi_i}{\partial x_j} f_j + \left(\frac{\partial \psi_i}{\partial x_j}\frac{\partial f_j}{\partial x_k} + \frac{\partial^2 \psi_i}{\partial x_j \partial x_k} f_j\right) Z_{kl}\rho_l + r_2(\theta, \rho), \end{aligned}$$

where we have used the Einstein summation convention and the missing arguments are $y(\theta)$ and θ and $r_2(\theta, \rho)$ is higher order in ρ. Since

$$\left(\frac{\partial \psi_i}{\partial x_j} f_j\right) \circ y(\theta) = 0,$$

we see that $\dot{\rho}$ has the desired form with $\bar{f}_2 = r_2$ and

$$A_{il}(\theta) = \left[\left(\frac{\partial \psi_i}{\partial x_j}\frac{\partial f_j}{\partial x_k} + \frac{\partial^2 \psi_i}{\partial x_j \partial x_k} f_j\right) \circ y(\theta)\right] Z_{kl}(\theta). \qquad \square$$

Remark. Equation (5) has exactly the same form as that given by Hale [4]. In that development, a moving orthonormal coordinate system must be constructed. Here we have shown that the family of coordinate systems leading to the dynamics (5) is more general. Clearly, every coordinate change $(\theta, \rho) = \Psi(x)$ taking (1)–(5) defines $n-1$ independent functions $\psi_2(\cdot), \ldots, \psi_n(\cdot)$ that vanish on the orbit. We have shown that the converse is also true: every set of $n-1$ independent functions that vanish on the orbit can be used to place the dynamics of the system into the canonical form (5). Additionally, we have provided explicit formulas for important quantities such as $A(\theta)$.

The canonical form (5) provides a useful characterization of a periodic orbit.

Proposition 1.5. *The orbit η is an exponentially stable orbit of* (1) *if and only if the* transverse linearization

$$\frac{d\rho}{d\theta} = A(\theta)\rho \tag{7}$$

is asymptotically stable.

Proof. This proposition is essentially a corollary to Theorem 1.1. The variational linear system about the periodic trajectory $(\bar{\theta}(t), \bar{\rho}(t)) = (t, 0)$ is given by

$$\begin{bmatrix} \delta\dot{\theta} \\ \delta\dot{\rho} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{\partial f_1}{\partial \rho}(t, 0) \\ 0 & A(t) \end{bmatrix} \begin{bmatrix} \delta\theta \\ \delta\rho \end{bmatrix},$$

with state transition matrix

$$\Phi(t, \tau) = \begin{bmatrix} 1 & * \\ 0 & \Phi_A(t, \tau) \end{bmatrix},$$

where * indicates a quantity of no interest to us. Clearly, the eigenvalues of $\Phi_A(T, 0)$ belong to the open unit disk if and only if the periodic linear system (7) is asymptotically (hence, exponentially) stable. $\square$

2. A family of converse Lyapunov functions

The linearization of a nonlinear system about an exponentially stable equilibrium point is commonly used in the construction of quadratic Lyapunov functions for the nonlinear system. Such functions are constructed by solving a Lyapunov equation. In this section, we show that this simple approach can be extended to the case of exponentially stable periodic orbits. Roughly speaking, the *transverse linearization* takes the place of the equilibrium point linearization and a *periodic Lyapunov equation* takes the place of the Lyapunov equation.

Associated with the periodic linear system (7) is the *periodic Lyapunov equation*

$$P'(\theta) + A(\theta)^{\mathrm{T}} P(\theta) + P(\theta) A(\theta) + Q(\theta) = 0, \tag{8}$$

where

$$P'(\theta) = \frac{\mathrm{d}P}{\mathrm{d}\theta}(\theta)$$

and $Q(\theta)$ is a continuous, periodic matrix. The following result is well known (see e.g. [7, 1]).

Theorem 2.1. *Suppose that* (7) *is asymptotically stable and that $Q(\theta)$ is positive definite for all θ. Then* (8) *has a unique periodic solution $P(\theta)$ that is positive definite for all θ.*

We are now ready to construct quadratic Lyapunov functions that prove the exponential stability of the orbit η. Suppose that we have chosen a set of functions $\psi_i(\cdot)$, $i = 2, \ldots, n$, and calculated the transverse linearization (7) and the unique steady-state solution $P(\cdot)$ to (8) for a suitably chosen $Q(\cdot)$.

Theorem 2.2. *Suppose that η is an exponentially stable periodic orbit of* (1). *The Lyapunov function*

$$V(x) = \rho^{\mathrm{T}} P(\theta) \rho$$

proves the exponential stability of η. That is, on a neighborhood of η, $V(\cdot)$ satisfies

$$k_1 \|x\|_\eta^2 \le V(x) \le k_2 \|x\|_\eta^2 \tag{9}$$

and

$$\left\| \frac{\partial V}{\partial x} \right\| \le k_3 \|x\|_\eta \tag{10}$$

and

$$\dot{V}(x) \le -k_4 \|x\|_\eta^2 \tag{11}$$

for some positive constants k_1, k_2, k_3, k_4 and

$$\|x\|_\eta := \inf_{y \in \eta} \|x - y\|.$$

Proof. Clearly, $V(\cdot)$ is quadratic in $\|x\|_\eta$ since $P(\cdot)$ is positive definite and continuous on S^1 and, on a sufficiently small compact neighborhood of η,

$$l_1 \|x\|_\eta \le \|\rho\| \le l_2 \|x\|_\eta$$

for some $l_1, l_2 > 0$. Differentiating $V(x)$, we have

$$\dot{V}(x) = -\rho^{\mathrm{T}} Q(\theta) \rho + 2\rho^{\mathrm{T}} P(\theta) f_2(\theta, \rho) + \rho^{\mathrm{T}} P'(\theta) f_1(\theta, \rho) \rho.$$

Since the last two terms are cubic in ρ, it is clear that $\dot{V}(\cdot)$ is locally negative definite and quadratic in $\|x\|_\eta$. □

Our approach provides a rich family of Lyapunov functions for an exponentially stable periodic orbit. This family is parameterized by the choice of periodic, positive definite $Q(\cdot)$ as well as the choice of independent functions $\psi_i(\cdot)$ that vanish on η.

The ability to *construct* Lyapunov functions for exponentially stable orbit makes it easy to provide (conservative) estimates of, for example, the domain of attraction of an orbit and the ability of the system to maintain the desired behavior in the face of disturbances.

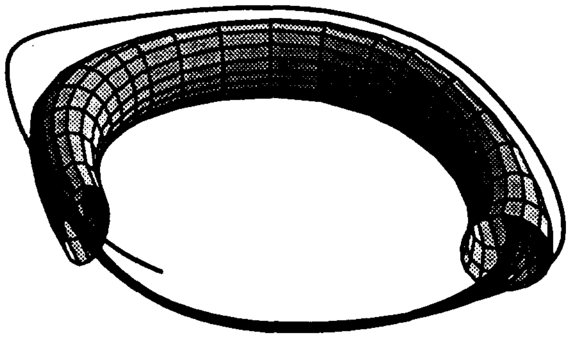

Fig. 1. The level set $V(x) = 1$.

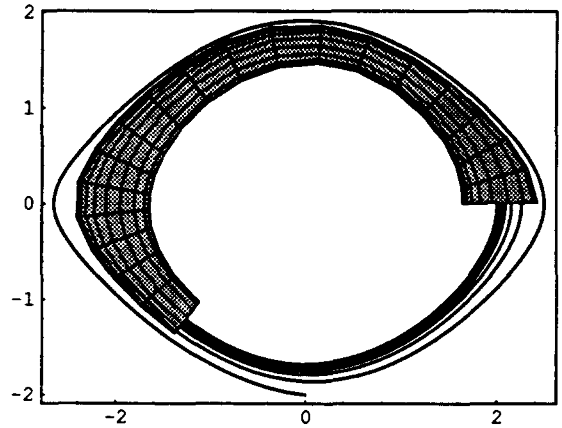

Fig. 2. Top view of the level set $V(x) = 1$.

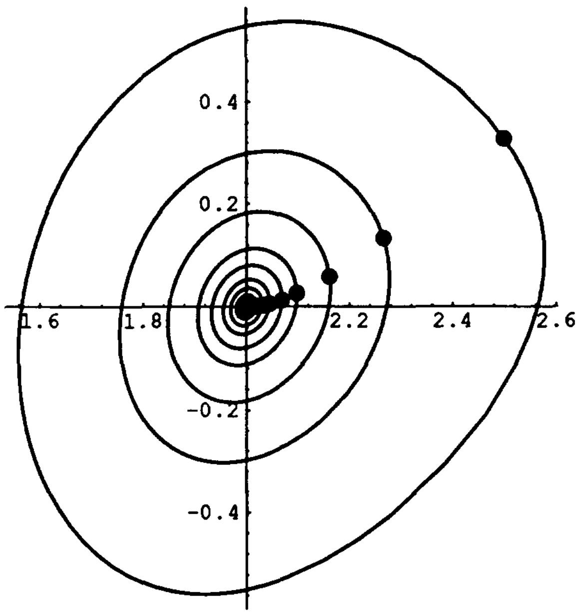

Fig. 3. Illustrating a Poincaré map with a section based at $(2, 0, 0) \in \eta$. Level sets are shown for $V(x) = 1.55, 0.46, 0.17, 0.06, 0.03, 0.014, 0.007, 0.003$.

3. Example

Consider the control system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\sin x_1 - \cos x_1 u, \\ \dot{x}_3 &= u \end{aligned} \tag{12}$$

in $\mathbb{R}_3$. This is a simplified version of the cart and pendulum system studied in [2]. The two-dimensional cart dynamics have been replaced by a one-dimensional dynamics for easy visualization. Clearly, when the control u is zero, the system (12) possesses a family of periodic orbits (corresponding to the undriven pendulum). Define

$$\rho_1 := \psi_2(x) := x_2^2/2 + 1 - \cos x_1 - \hat{H},$$

$$\rho_2 := \psi_3(x) := x_3,$$

where $0 < \hat{H} < 2$ is a desired energy level for the pendulum and suppose we are interested in stabilizing the periodic orbit given by

$$\eta := \{x \in \mathbb{R}^3 \colon \psi_2(x) = \psi_3(x) = 0\}.$$

When $\hat{H} = 1 - \cos 2$, the control law

$$u = 0.1 x_2 \cos x_1 \psi_2(x) - 0.1 x_3$$

stabilizes η (cf. [2]). This can be verified by showing that the linearized transverse dynamics given by

$$\begin{bmatrix} \rho_1' \\ \rho_2' \end{bmatrix} = \begin{bmatrix} -0.1\beta^2(\theta) & 0.1\beta(\theta) \\ 0.1\beta(\theta) & -0.1 \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} =: A(\theta)\rho$$

with $\beta(\theta) = y_2(\theta)\cos y_1(\theta)$ ($y(\theta)$ parameterizes η) is exponentially stable. Alternatively, we can check that periodic solution of

$$P'(\theta) + A(\theta)^{\mathrm{T}} P(\theta) + P(\theta) A(\theta) + I = 0$$

is positive definite for all θ. Figures 1 and 2 show two views of the level set $V(x) = \rho^{\mathrm{T}} P(\theta)\rho = 1$ along with a trajectory $(x(0) = (0, -2, 0.7))$ converging to η. Figure 3 shows a Poincaré section for the system that intersects the orbit at the point $(2, 0, 0) \in \eta$. The large dots show the passage of the given trajectory. For each passage, the corresponding level set of $V(x)$ (on the section) is shown. Note that, for larger values of $V(x)$, the closed curves are significantly distorted from the (linear) ellipse shape.

4. Conclusion

In this paper, we have presented a technique for constructing a class of quadratic Lyapunov functions for exponentially stable periodic orbits. This construction is facilitated by the use of a special set of local coordinates that essentially separates the tangential and transverse dynamics of the system.

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