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Series III: Computer and Systems Sciences – Vol. 197 ISSN: 1387-6694

Differential Geometry and Topology, Discrete and Computational Geometry

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IOS Press

Amsterdam • Berlin • Oxford • Tokyo • Washington, DC

Published in cooperation with NATO Public Diplomacy Division

Proceedings of the NATO Advanced Study Institute on
Differential Geometry and Topology, Discrete and Computational Geometry
Marrakech, Morocco
17–31 May 2004

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ISBN 1-58603-507-X
Library of Congress Control Number: 2005923356

Publisher
IOS Press
Nieuwe Hemweg 6B
1013 BG Amsterdam
Netherlands
fax: +31 20 620 3419
e-mail: order@iospress.nl

Distributor in the UK and Ireland
IOS Press/Lavis Marketing
73 Lime Walk
Headington
Oxford OX3 7AD
England
fax: +44 1865 750079

Distributor in the USA and Canada
IOS Press, Inc.
4502 Rachael Manor Drive
Fairfax, VA 22032
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e-mail: iosbooks@iospress.com

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PRINTED IN THE NETHERLANDS

Differential Geometry and Topology, Discrete and Computational Geometry
M. Boucetta and J.-M. Morvan (Eds.)
IOS Press, 2005
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Preface

This volume collects papers based on the lectures given at the University of Marrakech, (Morocco), Faculté des Sciences et Techniques de Guéliz, in May 2004, in the connection with a School organized by the Centre International de Mathématiques Pures et Appliquées (C.I.M.P.A.) and promoted by the N.A.T.O. Science Programme, Advanced Study Institute, (N.A.T.O. A.S.I.). We received also a financial support from U.N.E.S.C.O., the I.C.T.P. of Trieste, and I.N.R.I.A. Sophia Antipolis.
The aim of the School was to provide participants with an introduction and an overview to differential topology, differential geometry and computational geometry with an emphasis on some interconnections between these three domains of mathematics. The lecturers gave the background required to begin research in these fields or at their interfaces. They also introduced new research domains, old and new conjectures in these different subjects showing some interactions between other sciences close to mathematics.

Let us give now a description of the content of the lectures given during the School which constitutes also the content of this volume.

- The basis of differential topology and combinatorial topology was introduced by H. Abchir and Ch. Blanchet. A series of four lectures has been devoted to the basis of differential topology and combinatorial topology. H. Abchir developed in particular the notions of isotopy and transversality. Then, he introduced the theory of knots, and the smooth Morse theory, to get the decomposition in handles of smooth manifolds. Ch. Blanchet introduced P.L. manifolds, triangulations, and regular neighborhoods. One interesting point here is that a discrete version of Morse theory exists now since some years. It has been developed by Forman. Ch. Blanchet introduced this new idea, and made the link between smooth and discrete topology. Some questions appeared during the talks : is there a good notion of discrete vector field, and discrete flow? Can a smooth vector field be approximated by a discrete one? In what sense? Then, Ch. Blanchet introduced surgery as a tool for presentation of manifolds of dimention 3 by diagrams, and Kirby theorem which gives the relations between these diagrams. After different questions during the talk, and a discussion with J.D. Boissonnat, new problems appeared : is it possible to find explicit algorithms for each construction mentionned above? What about their implementation?

- The link between differential geometry and topology has been exposed by A. Abouqateb, A. El Kacimi, E. Salhi and H. Hattab. A. Abouqateb and A. Elkacimi gave an elementary introduction to the theory of the geometry and the topology of foliations, with many examples and illustrations about the fundamental notion of transverse structure. Open questions were discussed, in particular the problem of finding a good discrete notion of foliation, which can be of fundamental interest in structural geology, to get a good modelisation of a succession of strata. The goal of E. Salhi (and his co-worker H. Hawete) was to study the dynamical properties of some groups of homeomorphisms of a metric space.

- Riemanian geometry was introduced by M. Boucetta, J.M. Morvan, S. Haesen and A. Ikemakhen. After the basic notions introduced by M. Boucetta, (Levi-Civita connexion, curvature tensor, geodesic, completeness and curvature tensor), the lecturers insisted in the theory of Riemannian submanifolds, and in particular, on curves and surfaces. The Willmore conjecture has been mentionned, with different essays to solve it, using discrete or smooth geometry. A. Ikemakhen extended the frame of Riemannian geometry by introducing Pseudo-Riemannian manifolds. He described the relation between holonomy and curvature, recalled the De Rham-Wu decomposition theorem and the fundamental holonomy principle. He gave in particular some classification and description theorems for symmetric spaces. S. Haesen, in collaboration with L. Verstraelen, gave two lectures on the application of Riemannian geometry in general relativity: he recalled the notion of a semi- symmetric manifold, generalising the locally symmetric spaces defined by E. Cartan (1926) and presented a geometrical interpretation of the semi-symmetry condition in terms of parallel transport of planes. Then, he generalised this condition on the Riemannian tensor to the pseudo-symmetry condition defined by R. Deszcz. Several geometrical characterisations of this class of spaces have been presented, and the author pointed out their relevance in the theory of general relativity.

- The characteristic classes were introduced by two different points of view : by V. Borrelli in a geometrical way and by A. Bouarich in a topological one. They gave five lectures on characteristic classes. The aim of this theory is to associate to every fiber bundle some cohomology classes and, with the help of these classes, to distinguish non isomorphic fiber bundles. In particular, it is easy to distinguish non diffeomorphic differentiable manifolds by comparing the characteristic classes of their respective tangent bundles. V. Borrelli gave an overview on the subject, and A. Bouarich presented three different constructions of the Stiefel-Whitney classes. By way of application - and of motivation - V. Borrelli built the Milnor exotic sphere, that is, a differentiable manifold homeomorphic but not diffeomorphic to a standard sphere. It is rather delicate to get a discrete analogue of these classes. Some constructions have been proposed by J.G. Fu, based on the notion of normal cycle (see the lectures of J.M. Morvan). This last question appeared as a basic but difficult one.

- The link between differential geometry and the geometry of non smooth objects has

been treated by J.M. Morvan. He emphasized Steiner formula for the volume of tubes, which allows to define curvatures on convex (smooth or non smooth) subsets. To generalize these discrete curvatures of non smooth convex bodies, to non convex subsets, he introduced the theory of the normal cycle as a tool to define classical differential invariants as curvatures on discrete or singular objects. He paid a special attention to the case of polyhedra on which Gauss curvature and mean curvature are defined as curvature measures. The main questions arised : what kind of convergence theorem can be proved? If a sequence of polyhedra converges to a smooth surface, what can be said about the sequence of curvatures of the polyhedra? The theory of J. Fu has been presented, and classical examples like the Lantern of Schwarz has been studied. Finally, he mentionned the difficulty to find discrete equivalents of particular smooth quantities like principal curvatures, umbilic points, ridges, etc...

- J.D. Boissonnat introduced computational geometry. He began with the main geometric data structures : convex hull, triangulations, Voronoi diagrams. The explicit construction of a convex hull illustrated the important algorithmic paradigms : incremental and adaptative algorithms, randomisation, the sweep-line algorithms,... The link between Voronoi diagrams and polytopes allowed to obtain bounds on the combinatorial complexity of the diagrams, to define a dual triangulation (Delaunay triangulation), and to obtain efficient algorithms. The last lectures showed the interest of Voronoi diagram in geometric modeling. The notion of sample of a surface was presented, with its relations with the notion of Delaunay triangulation restricted to a surface.

- Finally, some tools developed above allowed to deal with "concrete applications" as structural geology, graphism... J.M. Morvan gave the latest developpements on the relations between differential geometry, discrete geometry and other sciences, as structural geology, graphism, medical imaging, C.A.O., etc... He gave in particular the techniques to draw lines of curvatures on triangulated surfaces. J.D. Boissonnat gave interesting applications of Delaunay triangulations to C.A.O and visualisation of surfaces. He insisted in the different ways to find a "good" triangulation adapted to a particular sample of a smooth surface.

The editors would like to thank all the lecturers for their excellent talks, the C.I.M.P.A. and N.A.T.O. A.S.I. for the promotion of the School and their financial help, the U.N.E.S.C.O., I.C.T.P. and I.N.R.I.A. for their financial help. Finally, Rémi Morvan kindly assured an excellent page make-up of this text, using LaTeX. We would like to thank him warmly.

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Differential Geometry and Topology, Discrete and Computational Geometry
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IOS Press, 2005
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Chapter 1

Differential Topology

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Introduction

The purpose of this chapter is to give a survey of some basic tools needed for studying low dimensional geometric topology.
The first section is devoted to an introduction to differential topology. We recall the regular value theorem. Transversality theorem and Whitney embedding theorem are stated in the large context of manifolds with boundary. We discuss orientation, tubular neighborhoods and collars. The end of this section is devoted to the isotopy relation which will play a central part in what follows.
In the second section, we apply differential topology to the study of knots and links. We define the diagram of a link, state Reidemeister theorem and give some classical knot invariants.
The third section is mainly devoted to Morse theory. We conclude with Heegaard splitting of 3-manifolds and handle decomposition. The next chapter will consider similar notions in the combinatorial context.
Most proofs can be found in the classic literature given in the bibliography.

1.1 Introduction to Differential Topology

1.1.1 Manifolds With Boundary

We use the following notation
$\mathbb{R}^m_+ = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_m \geq 0\}$,
$\partial\mathbb{R}^m_+ = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_m = 0\}$.

Definition 1 *Let X be any subset of $\mathbb{R}^m$. A map $f : X \longrightarrow \mathbb{R}^n$ is said to be differentiable if for any $x \in X$, there exists an open subset $U \subseteq \mathbb{R}^m$, $x \in U$, and a differentiable map $F : U \longrightarrow \mathbb{R}^n$ such that $F_{|U \cap X} = f$. We define the differential of f at x to be the one of F at x.*

Definition 2 *Let $m \geq 0$ be an integer. A differentiable manifold of dimension m (or an m-manifold) is a Hausdorff paracompact space M with an open cover $(U_i)_{i \in I}$ such that*

1. *for each $i \in I$ there is a continuous map $\varphi_i : U_i \longrightarrow \mathbb{R}^m_+$ which maps U_i homeomorphically onto an open subset of $\mathbb{R}^m_+$;*
2. *the coordinate transformation $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$ is a C^∞ diffeomorphism for all i, $j \in I$.*

Definition 3 *Let $x \in M$. We say that x is a boundary point of M if there exists a chart (φ, U) at x such that $x \in \varphi^{-1}(\partial \mathbb{R}^m_+)$. The set of such points is denoted by ∂M and is called the boundary of M.*

Definition 4 *Let M, N be manifolds of respective dimensions m, n, and let $f : M \longrightarrow N$ be a continuous map. We say that f is a differentiable map if for all charts (φ, U) of M and (ψ, V) of N such that $f(U) \subset V$, the map*

$$\psi f \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^m_+ \longrightarrow \psi(V) \subseteq \mathbb{R}^n_+$$

is differentiable.

Definition 5 *Let $f : M \longrightarrow N$ be a differentiable map. If f is bijective and f^{-1} is differentiable, we say that f is a diffeomorphism.*

Lemma 1 *Let U and V be two open subsets of $\mathbb{R}^m$ or of $\mathbb{R}^m_+ = \mathbb{R}^{m-1} \times [0, +\infty[$. Let f be a diffeomorphism between U and V. If U is an open subset of $\mathbb{R}^{m-1} \times [0, +\infty[$ and $x \in U \cap \mathbb{R}^{m-1} \times \{0\}$, then V is also an open subset of $\mathbb{R}^{m-1} \times [0, +\infty[$ and $f(x) \in V \cap \mathbb{R}^{m-1} \times \{0\}$.*

Proof. Let U be an open subset of $\mathbb{R}^{m-1} \times [0, +\infty[$ and let x be an element in $U \cap \mathbb{R}^{m-1} \times \{0\}$. By definition f extends to a diffeomorphism F from an open subset U' of $\mathbb{R}^m$ onto another open subset V' of $\mathbb{R}^m$ such that $U \subseteq U'$ and $V \subseteq V'$. Since x is in the closure of $U' \setminus U$, it follows that $f(x)$ is also in the closure of $V' \setminus V$. So V is not a neighborhood of $f(x)$ in $\mathbb{R}^m$, what means that V is not an open subset of $\mathbb{R}^{m-1} \times [0, +\infty[$ and $f(x) \in V \cap \mathbb{R}^{m-1} \times \{0\}$.

□

Corollary 1 *Let M be an m-manifold. The set ∂M is an $(m-1)$-manifold.*

Corollary 2 *let M, N be manifolds and $f : M \longrightarrow N$ be a diffeomorphism. Then we have $\partial N = f(\partial M)$.*

Exercise. Show that any connected 1-manifold with non empty boundary is either diffeomorphic to $[0, 1]$ or to $[0, +\infty[$.

Remark. The notions of tangent vectors and tangent bundle are defined as for manifolds without boundary, as well as immersion, submersion and embedding.

1.1.2 Submanifolds

Definition 6 *A subspace $A \subset \mathbb{R}^m$ is a p-submanifold of $\mathbb{R}^m$ if, for each $x \in A$, there exists a chart (φ, U) of $\mathbb{R}^m$ at x such that*

$$U \cap A = \varphi^{-1}(\mathbb{R}^p_+)$$

Definition 7 *Let M be an m- manifold (∂M can be empty). A subset $A \subset M$ is a p-submanifold of M if for each $x \in A$ there exists a chart (φ, U) of M at x such that $\varphi(U \cap A)$ is a p-submanifold of $\mathbb{R}^m$.*

Remark. The above definition is equivalent to the fact that A is the image of some embedding in M.

Definition 8 *A p-submanifold A of M is said to be neat if $\partial A = A \cap \partial M$ and A is covered by charts (φ, U) of M such that*

$$A \cap U = \varphi^{-1}(\mathbb{R}^p)$$

Definition 9 *An embedding is said to be neat if its image is a neat submanifold.*

Remark. In general, a submanifold A is neat iff $\partial A = A \cap \partial M$ and at any point $x \in \partial A$, we have $T_x A \not\subset T_x \partial M$.

The regular value theorem for manifolds with boundary is the following.

Theorem 1 *Let M be a manifold with boundary (i.e. $\partial M \neq \emptyset$) and let N be a manifold. Let $f : M \longrightarrow N$ be a differentiable map. If $y \in N - \partial N$ is a regular value for both f and $f_{|\partial M}$, then $f^{-1}(y)$ is a neat submanifold of M.*

Proof: This theorem follows from the theorem 3.2. proved in [6].

Exercise. Let M be a manifold and $f : M \longrightarrow \mathbb{R}$ a differentiable map. If a is a regular value for f, then the set $M_a = \{x \in M | f(x) \leq a\} = f^{-1}(]-\infty; a])$ is a manifold which has the dimension of M and whose boundary is $f^{-1}(a)$.

Example. If $M = \mathbb{R}^m$ and $f(x) = 1 - |x|^2$, then the closed unit m-disk is a manifold whose boundary is S^{m-1}.

Theorem 2 *Let M be a compact m-manifold. Then there exists a neat embedding of M into $\mathbb{R}^{2m+1}_+$.*

1.1.3 Orientation

Let E be a real finite dimensional vector space of dimension $n > 0$. Two bases $(e_1, \cdots, e_n)$, $(f_1, \cdots, f_n)$ of E are equivalent if the automorphism $A : E \longrightarrow E$ such that $Ae_i = f_i$ has positive determinant.
An orientation of E is an equivalence class $\omega = [e_1, \cdots, e_n]$ of bases. If $dimE > 0$, then there are two orientations.
An oriented vector space E is a pair (E, ω) where ω is an orientation of E.

Remark. If $dimE = 0$, an orientation of E simply a sign ± 1.
The standard orientation of $\mathbb{R}^n$, $n > 0$, is $\omega^n = [e_1, \cdots, e_n]$ represented by the canonical basis. The standard orientation of $\mathbb{R}^0$ is $+1$.

Let M be a m-manifold. We suppose that $\mathbb{R}^m$ has its standard orientation. For each $x \in M$, we choose an orientation for the tangent space T_xM. We say that those orientations are *coherent* if, for each $x \in M$, there exists a chart (φ, U), such that $T_x\varphi$ preserves the orientation (i.e. the isomorphism $T_x\varphi$ takes the chosen orientation on a base in the class of the standard base of $\mathbb{R}^m$).

Definition 10 *The manifold M is said to be orientable if we can choose a coherent orientation on the tangent spaces T_xM, $x \in M$. An oriented manifold is an orientable manifold with a choice of orientation.*

Remark. If M is a 0-manifold, an orientation of M is the assignment of a sign to each point of M.

Proposition 1 *A manifold M is orientable if and only if there exists charts whose domain cover M, such that any coordinate transformation has positive Jacobian.*

Proposition 2 *A connected orientable manifold has exactly two orientations.*

Remark. If M is an oriented m-manifold with boundary. Then, at each point x of M, the tangent space T_xM is oriented. Let $x \in \partial M$ and (φ, U) a chart of M at x. Then $\varphi(U)$ is an open subset of $\mathbb{R}^{m-1} \times [0, +\infty[$.
Let u be a vector of T_xM such that the last coordinate of $T_x\varphi(u)$ is strictly negative (we say that u is an outside vector). We have the following decomposition

$$T_xM = T_x\partial M \oplus \mathbb{R}u$$

There exists a unique orientation of $T_x\partial M$ such that in the above equality of vector spaces orientations coincide. Here the right hand side is oriented by an oriented basis of $T_x\partial M$ followed by u.
It is easy to see that the orientation of $T_x\partial M$ is independent of the chart and the outside vector and these orientations define an orientation of ∂M.

1.1.4 Transversality

Definition 11 *Let $f : M \longrightarrow N$ be a differentiable map and A a submanifold of N. We say that f is transverse to A if for each $y = f(x) \in A$ we have*

$$T_yA + T_xf(T_xM) = T_yN$$

We denote that phenomenon by $f \pitchfork A$.

Theorem 3 *Let $f : M \longrightarrow N$ be a differentiable map and A a codimension k submanifold of N. If $f \pitchfork A$ then $f^{-1}(A)$ is a codimension k submanifold of M.*

Proof. Let m, n and p be respectively the dimensions of M, N and A.
We remark that it suffices to prove the theorem locally (i.e. for each $x \in f^{-1}(A)$, there exists an open subset U in M, $x \in U$ such that $f^{-1}(A) \cap U$ is a submanifold of M).
Let $a \in A$. There exists a chart (φ, U), $a \in U$ such that $\varphi : U \subseteq N \longrightarrow \varphi(U) \subseteq \mathbb{R}^n$ and

$$U \cap A = \{y \in U \mid \varphi(y) = (\varphi_1(y), \ldots, \varphi_p(y), 0, \ldots, 0)\}$$

Let $\pi : \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^{n-p} \longrightarrow \mathbb{R}^{n-p}$ be the canonical projection.
Let $g = \pi \circ \varphi : U \longrightarrow \mathbb{R}^{n-p}$. Then $g^{-1}(0) = U \cap A$.
We remark that T_ag is surjective.
We consider the map $g \circ f : f^{-1}(U) \longrightarrow \mathbb{R}^{n-p}$. If $a = f(x)$, $x \in f^{-1}(A) \cap f^{-1}(U)$ and $T_x(g \circ f) = T_ag \circ T_xf$. Clearly $T_ag \circ T_xf$ is surjective if and only if $T_ag(T_xf(T_xM)) = \mathbb{R}^{n-p}$.
Since f is transverse to A, then

$$T_aN = T_xf(T_xM) + T_aA$$

Noting that $Ker(T_ag) = T_aA$, we apply T_ag and then we obtain

$$T_ag(T_xf(T_xM)) = T_ag(T_aN) = \mathbb{R}^{n-p}$$

Hence $T_x(g \circ f)$ is surjective. So, we have proved that 0 is a regular value of $g \circ f$ and $(g \circ f)^{-1}(0) = f^{-1}(A \cap U)$ is a submanifold of M by using theorem 1.

□

Remark. If $A = \{y\}$, $y \in N$, then we have the tangent space $T_yA = \{0\}$. So the transversality of f to A is expressed by the formula $T_xf(T_xM) = T_yN$ for each $x \in f^{-1}(y)$, i.e. f is a submersion at any point of $f^{-1}(y)$ and then y is a regular value of f. We then find again the theorem of the regular value announced above.

Particular case:
Let A, B be two submanifolds of a manifold M. Let $i : A \hookrightarrow M$ be the inclusion of A into M. The map i is transverse to B if and only if for each $x = i(x) \in B$ (i.e. $x \in A \cap B$) we have:

$$T_xB + T_xi(T_xA) = T_xM$$

Since T_xi is just the inclusion of T_xA in T_xM, the above relation becomes for any $x \in A \cap B$

$$T_xB + T_xA = T_xM$$

Definition 12 *Two submanifolds A and B of the same manifold M are said to be transverse if for every $x \in A \cap B$*

$$T_xB + T_xA = T_xM$$

We write $A \pitchfork B$.

Proposition 3 *The intersection $A \cap B$ of two transverse submanifolds of a manifold M is also a submanifold such that*

$$dim(A \cap B) = dimA + dimB - dimM$$

Proof. One notes that the inclusion i of A into M satisfies $i \pitchfork B$. By applying the theorem 3, $i^{-1}(B) = A \cap B$ is a submanifold of A and then of M. The relation between dimensions follows.

□

Remark. A necessary condition for the submanifolds A and B of a manifold M to be transverse is that $dimA + dimB \geq dimM$. A generalization of Theorem 3 to manifolds with boundary is:

Theorem 4 *Let M, N be two manifolds and let $A \subset N$ be a submanifold without boundary. If $f : M \longrightarrow N$ is a differentiable map such that f and $f_{|\partial M}$ are both transverse to A, then $f^{-1}(A)$ is a submanifold of M whose boundary is $f^{-1}(A) \cap \partial M$.*

Proof. We suppose that A is contained in the interior of N. In order to prove that $f^{-1}(A)$ is a manifold with boundary, it suffices to check it locally.
Suppose that $x \in f^{-1}(A) \cap \partial M$, then by using small charts we may suppose that M is a half space of $\mathbb{R}^p$, and N is $\mathbb{R}^n$ and A is $\mathbb{R}^q \subset \mathbb{R}^n$. One may also suppose that $x = 0 = f(x)$. By definition f extends to a map g defined on a neighborhood of $M \subset \mathbb{R}^p$.
Let π be the projection of $\mathbb{R}^n$ onto $\mathbb{R}^{n-q}$ and let $h = \pi \circ g$. Since the restriction of f (or of g) to ∂M is transverse at x to A, then the differential of h at x maps surjectively $\mathbb{R}^{p-1}$ on $\mathbb{R}^{n-q}$.
Let F be the kernel of the above differential and ϕ a linear map from $\mathbb{R}^p$ into F which is the identity on F.
Let ψ be the projection of $\mathbb{R}^p$ on $\mathbb{R}$ which sends M on $[0, +\infty[$.
Let Φ be the map from $\mathbb{R}^p$ into $\mathbb{R}^{n-q} \times F \times \mathbb{R}$ which maps y on $(h(y), \varphi(y), \psi(y))$.
The map ϕ is a diffeomorphism which maps M in $\mathbb{R}^{n-q} \times F \times [0, +\infty[$ and takes $f^{-1}(A)$ in $0 \times F \times [0, +\infty[$. Hence $f^{-1}(A)$ is, at the neighborhood of x, a submanifold of M whose boundary is $f^{-1}(A) \cap \partial M$.

1.1.5 Collar and Tubular Neighborhood

Definition 13 *Let M be a manifold with boundary. A collar on M is an embedding*

$$f : \partial M \times [0, \infty[\longrightarrow M$$

such that $f(x, 0) = x$.

Theorem 5 *If M is a manifold with boundary, then ∂M has a collar.*

Proof. Using charts covering ∂M and a partition of unity, one finds a differentiable vector field X on a neighborhood $U \subset M$ of ∂M which is nowhere tangent to ∂M and which points into M (in local coordinates). Let $W \subset \partial M \times [0, \infty[$ be a neighborhood of $\partial M \times 0$ on which the flow η of the vector field is defined. There is a an embedding $h : \partial M \times [0, \infty[\longrightarrow W$ which leaves $\partial M \times 0$ pointwise fixed. The required map F is the composition

$$F : \partial M \times [0, \infty[\xrightarrow{h} W \xrightarrow{\eta} M$$

□

Remark. For more details we refer to section 6 of the chapter 4 in [6].

Definition 14 *Let M be a compact smooth m-manifold embedded in $\mathbb{R}^k$. Then the normal bundle of M in $\mathbb{R}^k$ is $\mathcal{N}(M) = \{\langle x, v\rangle \in M \times \mathbb{R}^k \mid v \perp T_x(M)\}$. We let $\pi : \mathcal{N}(M) \longrightarrow M$ be the projection $\pi\langle x, v\rangle = x$.*

Remark. The normal bundle $\mathcal{N}(M)$ is a manifold.

Proposition 4 *Each point $x \in M$ has a neighborhood U such that $\pi^{-1}(U) \approx U \times \mathbb{R}^{k-m}$ with the projection $\pi : \pi^{-1}(U) \longrightarrow U$ corresponding to the canonical projection $U \times \mathbb{R}^{k-m} \longrightarrow U$.*

Proof. Let $\phi : V \longrightarrow \mathbb{R}^{k-m} \times \mathbb{R}^m = \mathbb{R}^k$ be a chart making M flat, i.e., $U = V \cap M$ corresponds to $\{0\} \times \mathbb{R}^m$. Let $\lambda_1, \cdots, \lambda_{k-m}$ be the first $k - m$ coordinate projections $\mathbb{R}^k \longrightarrow \mathbb{R}$. Then the $\lambda_i \circ \phi$ are constant on U. Thus the vectors $\nabla(\lambda_i \circ \phi)$, at a point of U, give a normal frame to U and provide the splitting $\pi^{-1}(U) \approx U \times \mathbb{R}^{k-m}$. These vectors are independent since they form the first $k - m$ rows of the Jacobian matrix of ϕ.

□

Definition 15 *In the present situation, let $\theta : \mathcal{N}(M) \longrightarrow \mathbb{R}^k$ be given by $\theta\langle x, v\rangle = x + v$. Also let $\mathcal{N}(M, \varepsilon) = \{(x, v) \in \mathcal{N}(M) \mid \|v\| < \varepsilon\}$.*

Theorem 6 *(Tubular Neighborhood Theorem). Let M be a compact m-submanifold of $\mathbb{R}^k$. Then there is an $\varepsilon > 0$ such that $\theta : \mathcal{N}(M, \varepsilon) \longrightarrow \mathbb{R}^k$ is a diffeomorphism onto the neighborhood $\{y \in \mathbb{R}^k \mid d(M, y) < \varepsilon\}$ of M in $\mathbb{R}^k$.*

Proof. First note that there is a canonical splitting

$$T_{(x,0)}(\mathcal{N}(M)) \approx T_x(M) \times N_x(M)$$

where $N_x(M)$ is the normal space to $T_x(M)$ in $\mathbb{R}^k$.
Now let us compute θ_* at $\langle x, 0\rangle \in \mathcal{N}(M)$. Since $\theta\langle x, v\rangle = x + v$ is just a translation for x fixed and v variable, θ_* is the standard inclusion (identity) on $N_x(M) \longrightarrow \mathbb{R}^k$. Also

$\theta_* : T_x(M) \longrightarrow T_x(\mathbb{R}^k)$ is just the differential of the inclusion of $M \subset \mathbb{R}^k$ and so this part of θ_* is just the standard inclusion of $T_x(M)$ in $T_x(\mathbb{R}^k) = \mathbb{R}^k$. Thus

$$\theta_* : \mathbb{R}^k = T_x(\mathbb{R}^k) = T_x(M) \times N_x(M) \longrightarrow \mathbb{R}^k$$

is the identity. Therefore θ_* is an isomorphism at $\langle x, 0\rangle$ for each $x \in M$, and so θ is a diffeomorphism on some neighborhood of $\langle x, 0\rangle$. Consequently, θ_* is an isomorphism at $\langle x, v\rangle$ for any x and for $\|v\|$ small.
By compactness, there is a $\delta > 0$, such that θ_* is an isomorphism at all points of $\mathcal{N}(M, \delta)$. Thus $\theta : \mathcal{N}(M, \delta) \longrightarrow \mathbb{R}^k$ is a local diffeomorphism. We wish to show that θ is one-one on $\mathcal{N}(M, \varepsilon)$ for some $0 < \varepsilon \leq \delta$.
Suppose that θ is not one-one on $\mathcal{N}(M, \varepsilon)$ for any $\varepsilon > 0$. Then there exist sequences $\langle x_i, v_i\rangle \neq \langle y_i, w_i\rangle$ in $\mathcal{N}(M)$ such that $\|v_i\| \to 0$, $\|w_i\| \to 0$, and $\theta\langle x_i, v_i\rangle = \theta\langle y_i, w_i\rangle$. Since M is compact, metrizable, there exists a subsequence such that (by reindexing) $x_i \to x$ and $y_i \to y$. Then $\theta\langle x_i, v_i\rangle \to \theta\langle x, 0\rangle = x$ and $\theta\langle y_i, w_i\rangle \to \theta\langle y, 0\rangle = y$, so that $x = y$. But then, for i large, both $\langle x_i, v_i\rangle$ and $\langle y_i, w_i\rangle$ are close to $\langle x, 0\rangle$. Since θ is one-one locally near $\langle x, 0\rangle$, this is a contradiction, and thus, as claimed, θ must be one-one on some $\mathcal{N}(M, \varepsilon)$.
To finish, we must prove the final contention that $\theta(\mathcal{N}(M, \varepsilon)) = \{y \mid d(y, M) < \varepsilon\}$. The containment $\subset$ is clear, so suppose that y is such that $d(y, M) < \varepsilon$ and let $x \in M$ be such that $d(y, x)$ is a minimum (and hence $< \varepsilon$). Then the vector $y - x$ is a normal vector, at the point x, of length $< \varepsilon$ and so y does lie in $\theta(\mathcal{N}(M, \varepsilon))$.

□

Remarks.

1. Note that the map $r = \pi \circ \theta^{-1} : \theta(\mathcal{N}(M, \varepsilon)) \longrightarrow M$ is a smooth retraction of the tubular neighborhood onto M. It is also clear that r is homotopic to the identity via a smooth homotopy. That is r is a smooth *deformation retraction.* Also every point of M is a regular value of r. We call r the *the normal retraction* of the tubular neighborhood onto M.

2. There are more versions of the Tubular Neighborhood Theorem. Such a theorem can be proved for submanifolds of arbitrary manifolds, and not just euclidian space.

3. Compactness can be removed as a restriction, with a slight modification of the conclusion (the ε must be allowed to vary with the point).

The following theorem is a generalization of 6 and it is proved in the chapter 6 of [6]:

Theorem 7 *Let A be a submanifold of M such that $\partial M = \partial A = \emptyset$. Then A has a tubular neighborhood in M.*

1.1.6 Isotopy

Let $M \subset N$ be a submanifold. It is useful to be able to compare two embeddings of M into N. Furthermore it is interesting to know when a diffeomorphism of M can be extended to a diffeomorphism of N. The isotopy is a powerful tool allowing the study of such problems.

Definition 16 *Let M, N be two manifolds. Two embeddings f_0 and f_1 of M into N are said to be isotopic if there exists a continuous map $F : M \times [0,1] \longrightarrow N$ such that for each $t \in [0,1]$ the map*

$$\begin{array}{rccl} F_t : & M & \longrightarrow & N \\ & x & \mapsto & F_t(x) = F(x,t) \end{array}$$

is an embedding.
Then F is called an isotopy from f_0 to f_1. We also say that $f_0(M)$ is isotopic to $f_1(M)$.

Particular cases:

- If $M \subset N$ is a submanifold and f_0 is inclusion, then we say that F is an isotopy of M into N.
- If $M = N$, every map f_t is a diffeomorphism and $f_0 = id_M$, then we say that F is an ambient isotopy of M or a diffeotopy.

Proposition 5 *An ambient isotopy preserves the orientations.*

Theorem 8 *Let M be a compact submanifold of N. Let $F : M \times [0,1] \longrightarrow N$ be an isotopy of M. If either $F(M \times [0,1]) \subset \partial N$ or $F(M \times [0,1]) \subset N \setminus \partial N$, then F extends to a diffeotopy of N.*

Proof. We consider the spaces $Diff(N)$ and $Emb(M,N)$ where the last one is the space of embeddings of M into N. The restriction induces a map from $Diff(N)$ into $Emb(M,N)$ which will be proved to be a locally trivial bundle map.
Choose an embedding of N into $\mathbb{R}^n$ for a sufficiently large n. One can construct a differentiable map π from a neighborhood of $N \subset \mathbb{R}^n$ to N which is the identity on N. To prove that we have a locally trivial bundle, it suffices to show it is the case where M is a subset of N for a neighborhood of the inclusion of M into N.
There exists a neighborhood U of $M \subset N$ such that every $x \in U$ can be written in a unique way in the form $x = y + u$ with $y \in M$ and $u \in \mathbb{R}^n$ is orthogonal to the tangent space to M at y. We then chose a differentiable map f from U into $[0,1]$ which vanishes in a neighborhood of M and equals to 1 next to the frontier of U.
Let ϕ be an embedding of M into N near the inclusion. Let Φ be the map which is the identity outside of U and which maps $x = y + u \in U$ onto $\pi(u + f(x)y + (1 - f(x))\phi(y))$. When ϕ is near to the inclusion that map is a diffeomorphism of N onto N which extends ϕ.
Then we can associate to a diffeomorphism F of N which is next to the identity of M the couple (ϕ, G) where ϕ is the restriction of F to M and G is the map $F^{-1} \circ \Phi$. That gives a trivialization of the bundle next to the inclusion.
Hence the restriction map is a locally trivial bundle and so the announced theorem follows.

□

Remark. M. Hirsch gives another proof in his book Differential Topology [6]

Theorem 9 *Let $A \subset N$ be a compact submanifold. Let f_0, $f_1 : A \hookrightarrow M \backslash \partial M$ be embeddings which are isotopic in $M \setminus \partial M$. If f_0 extends to an embedding $M \longrightarrow N$ then so does f_1.*

Proof. Let $H : A \times [0,1] \longrightarrow N$ be an isotopy from f_0 to f_1. Then we can consider the isotopy

$$\begin{array}{rccc} F: & f_0(A) \times [0,1] & \longrightarrow & N \\ & (y,t) & \longmapsto & F(y,t) = H(f_0^{-1}(y),t) \end{array}$$

from the inclusion to the map $f_1 \circ f_0^{-1}$.
Since $f_0(A)$ is a compact submanifold of N, by using theorem 8 we get that F extends to a diffeotopy $\overline{F}$ of N:

$$\overline{F} : N \times [0,1] \longrightarrow N$$

For each $t \in [0,1]$, $\overline{F}_t$ is a diffeomorphism.
Let g be the extension of f_0 to M, then $\overline{F} \circ g$ is an embedding of M into N which extends f_1.

□

Let M, N be two n dimensional manifolds with boundary. Let $f : \partial M \longrightarrow \partial N$ be a diffeomorphism. We denote by $V = M \cup_f N$ the topological space $M \amalg N$ where we identify ∂M with $f(\partial M)$. We can give the space V a differential structure extending the given ones on M and N.
For notational simplicity we identify M and N with their images in V. Let $\partial M = \partial N = W$. By using collars of W in M and N, we find a homeomorphism from a neighborhood $U \subset V$ of W onto $W \times \mathbb{R}$ taking $x \in W$ to $(x,0)$, and which maps $U \cap M$ and $U \cap N$ diffeomorphically onto $W \times [0,\infty[$ and $W \times]-\infty,0]$, respectively. We give U the differential structure induced by this homeomorphism. The required differential structure on V is obtained by collation from M, N, and U.
In defining this differential structure on V, various choices were made.The uniqueness of gluing, says that the diffeomorphism type of V is independent of these choices.

Theorem 10 *Let M and N be two manifolds. If f and $g : \partial M \longrightarrow \partial N$ are two isotopic diffeomorphisms then the manifolds $M \cup_f N$ and $M \cup_g N$ are diffeomorphic.*

Proof: A collar of M and a collar of N allow one to construct $M \cup_f N$.
Furthermore the collars and the isotopy between f and g allow one to construct a diffeomorphism from $M \cup_f N$ on $M \cup_g N$.

□

Finally we give the following useful result about diffeotopies of the circle.

Theorem 11 *Any diffeomorphism of S^1 is isotopic either to the identity or to complex conjugation. Therefore, every diffeomorphism of S^1 extends to a diffeomorphism of D^2.*

Proof. see chapter 8 in [6].

1.2 Introduction to Knot Theory

1.2.1 Introduction

We can imagine a *knot* as a thin tangled rope in space whose ends are glued together. A *link* can be imagined as several tangled ropes (which may also tangle with each other) each of which has its ends glued together.
The possibility of a mathematical study of knots was probably recognized first by K. F. Gauss. The first attempts to classification of knot types started some years later. Today, this classification hasn't yet been done. However, the advent of algebraic topology in the beginning of the last century had largely contributed to the study of links.
The development of the works on knot theory was marked by several discoveries of some important results which allow to establish some relations with other mathematical objects such manifolds. Nowadays, knot theory has been shown to have applications in various branches of science, for example, physics, molecular biology, chemistry, *et cetera* [14].
Among the most interesting results in the knot theory, one may cite Lickorish's theorem [10] which says that, up to a diffeomorphism, each closed oriented connected 3-manifold can be described by an equivalence class of a link in S^3, the Kirby's theorem which shows that, there is a one-to-one correspondence between links with coefficients, modulo equivalence, and closed, connected, orientable 3-manifolds, up to diffeomorphism [9], and finally the remarkable discovery of the Jones polynomial invariant [7].

1.2.2 Definitions

Links

We suppose that the ambient space is $\mathbb{R}^3$ or S^3.

Definition 17 *Let $S^1 \amalg S^1 \amalg \cdots \amalg S^1$ be the differentiable manifold that consists of n copies of the circle S^1. A n components link L in S^3 is the image of an embedding $f : S^1 \amalg S^1 \amalg \cdots \amalg S^1 \hookrightarrow S^3$. We denote it by $L = K_1 \amalg K_2 \amalg \cdots \amalg K_n$.*
If $n = 1$, the link with one component is called a knot and is denoted by K.

Remark. Since $S^3 = \mathbb{R}^3 \cup \infty$, if a link in S^3 goes through the point at infinity, one can avoid it by a little elementary move. So the theories of links in S^3 and in $\mathbb{R}^3$ are the same. Henceforth, we won't distinguish the two theories.

Examples:

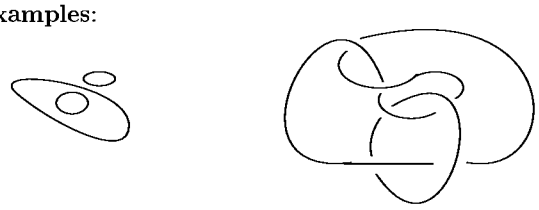


Figure 1.1: A trivial link and the Whitehead link

Tubular Neighborhood

Definition 18 *Let K be a knot in S^3. A neighborhood V of K is said to be tubular if there exists an embedding $h : S^1 \times D^2 \hookrightarrow S^3$ such that $V = h(S^1 \times D^2)$ and $K = h(S^1 \times 0)$.*

Remark. The theorem 6 shows that such neighborhoods exist.

Proposition 6 *The tubular neighborhoods of a knot K in S^3 are all isotopic.*

The Complement of a Knot

Definition 19 *Let K be a knot in S^3. Let V be a tubular neighborhood in K. The space $X = S^3 \setminus \overset{\circ}{V}$ is called the exterior or the complement of K in S^3.*

Remark. We define the exterior of a link in a similar manner.

Proposition 7 *The space X is a differentiable compact connected orientable 3-manifold with boundary.*

The Seifert Surface

Definition 20 *A Seifert surface for an oriented knot K in S^3 is a connected compact oriented surface contained in S^3 which has K as its oriented boundary.*

Theorem 12 *Any oriented knot in S^3 has a Seifert surface.*

Proof. Let V be a tubular neighborhood of a knot K in S^3. We know that V is diffeomorphic to $S^1 \times D^2$ and ∂V is diffeomorphic to $S^1 \times S^1$.
Let $X = S^3 \setminus \overset{\circ}{V}$. We have $\partial X = \partial V$. Let p be the projection of ∂X on the second factor $* \times S^1$ of the space $S^1 \times S^1$. That map extends to $p : X \longrightarrow S^1$ which has a regular value. We conclude by using the theorem of the regular value.

□

Remark. One may construct a Seifert surface for a knot K by destroying in some precise manner all the crossings of a diagram of K. After this operation the considered diagram falls apart into (oriented) closed curves called Seifert circles which will be ranged in the space by placing the inner circle at the top. Then, one attaches the little twisted rectangles removed at the time of the first operation. the result is the required surface.

Examples.

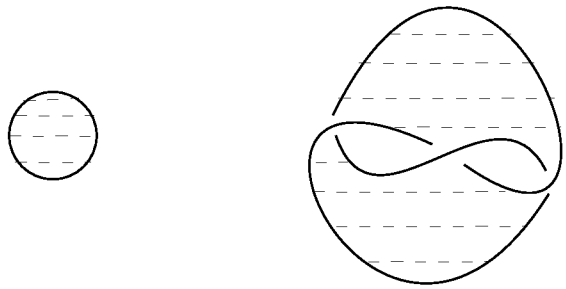


Figure 1.2: Seifert surfaces of the trivial knot and the trefoil

1.2.3 Equivalence of Links

Elementary move in space:
Let L be a link and K a component of L. Let P and Q be two points of K. Let $l \subset K$ be an arc joining P to Q. We consider another arc l' in the space joining P to Q which is isotopic to l such that there exists a disc δ whose boundary is $l \cup l'$ and such that $\delta \cap L = l$. We call an elementary move on L the change of l by l' or the inverse operation. The obtained link L' (after the smoothing of the corners) is said to be obtained from L by an elementary move.

Examples:

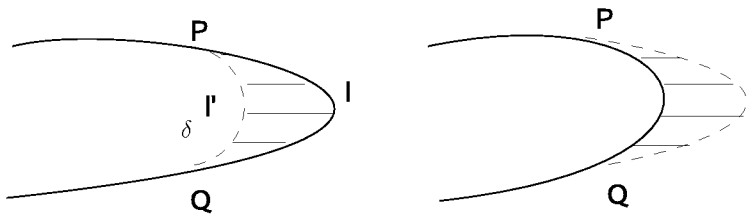


Figure 1.3: Elementary move

Definition 21 *Two links L and L' are said to be equivalent if and only if one can be obtained from the other by a finite sequence of elementary moves.*

Let n be a positive integer, $n \geq 1$, we introduce the following equivalence relation in the set of links with n components:

Definition 22 *Two links, L and L', both with n ordered components are said to be isotopic if there exists an ambient isotopy*

$$\begin{array}{rccl} F: & S^3 \times [0,1] & \longrightarrow & S^3 \\ & (x,t) & \longmapsto & F(x,t) = F_t(x) \end{array}$$

such that $F_0 = id_{S^3}$, $F_1(L) = L'$ *and* F_1 *preserves the order of the components (i.e.* $F_1(K_i) = K'_i$*).*

Theorem 13 *Let* L *and* L' *be two links in* S^3*. The following assertions are equivalent:*

1. L *and* L' *are isotopic,*
2. L *and* L' *are equivalent,*
3. *there exists a diffeomorphism* $f : S^3 \longrightarrow S^3$ *preserving the orientation such that* $f(L) = L'$*.*

1.2.4 Link Diagram

Definitions

It is very difficult to compare links in the space. To avoid those difficulties, one can turn back to plane by using orthogonal projections. The image of a link by such a projection on a plane involves all the informations needed in the restoration of the link in the space. In what follows we show that the comparison between links in the space can be made by comparing their orthogonal projections on a plane.
Let p be an orthogonal projection of the space on the plane $\mathbb{R}^2$. Let L be a link. A point $P \in p(L) \subset \mathbb{R}^2$ whose inverse image $p^{-1}(P)$ contains more than one antecedent is called a multiple point.

Definition 23 *Let* p *be the orthogonal projection of* $\mathbb{R}^3$ *on a plane* $\mathfrak{P}$*. We say that* p *is regular if for each link* $L \subset \mathbb{R}^3$*:*

1. *The tangent lines to the link at all points are projected onto lines on the plane (i.e., the projections on the plane never degenerate into points);*
2. *The double points are the only multiple points. A point on which two points project is said a crossing point;*
3. *The set of crossing points is finite and at each crossing point the projections of the two tangents do not coincide.*

Remark. In particular, the two following situations are forbidden.

Figure 1.4: Two forbidden projections

Definition 24 *A link diagram is the image of a link by a regular projection together with "over and under" information at the crossings.*

Remark. Let $p : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the standard projection. Let L be a link in the space. Then L can easily be changed by elementary moves until the projection $p(L)$ becomes regular.

Equivalence of Diagrams

Let K be a component of a link L. Let P and Q be two points of K and let $l \subset K$ be an arc joining P to Q. Replace l by an other arc l' joining P to Q which is isotopic to l and such that there exists a surface δ with boundary, $\partial\delta = l \cup l'$ satisfying $\delta \cap L = l$. After smoothing the corners, we obtain a new link L_1. We say that L_1 is obtained from L by an elementary move on L.
A finite sequence of such moves and their inverses is called a plane isotopy.

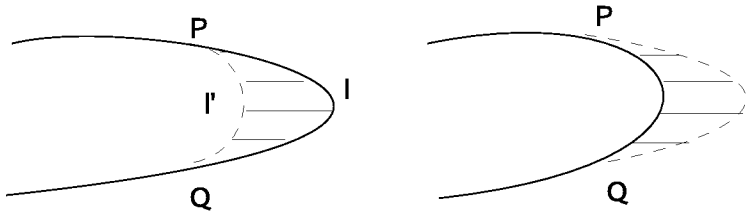


Figure 1.5: Plane isotopy

Remark. During a plane isotopy the following situations can never arise :

1. a new crossing point appears or an old one disappears;
2. the projections of two branches of the link become tangent;
3. the points of three branches of the link are projected on the same point.

These forbidden (by the definition of plane isotopy) situations occur in the ambient isotopies represented in the figure 1.6. The corresponding transformations are called the *first*, *second*, *third* Reidemeister moves and are denoted by Ω_1, Ω_2, Ω_3

Definition 25 *Two link diagrams are said to be equivalent if one can be obtained from the other by a finite sequence of Reidemeister moves and plane isotopies.*

1.2.5 Reidemeister Theorem

The Reidemeister moves allow, in certain sense, to reduce ambient isotopy of links to equivalence of diagrams in plane as the following theorem shows.

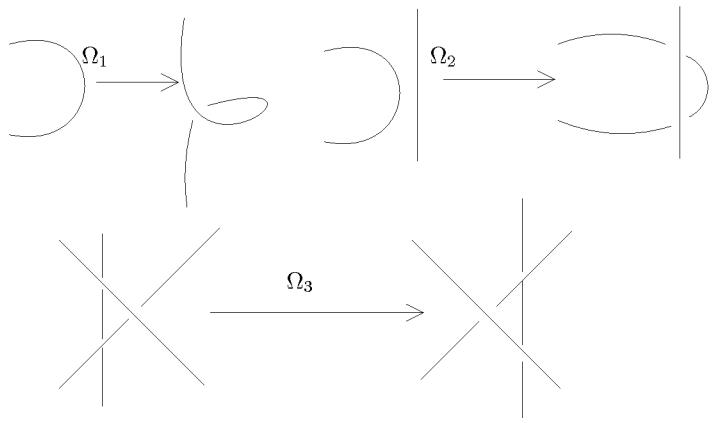


Figure 1.6: Reidemeister moves

Theorem 14 *(Reidemeister theorem) Two links are isotopic if and only if they have equivalent diagrams.*

Examples. In the next figure there are:

1. Two equivalent diagrams of the unknot.
2. Two equivalent diagrams of the trefoil.

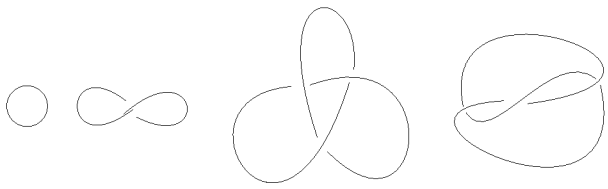


Figure 1.7: Equivalent diagrams of the trivial knot and the trefoil

Remark. We may consider oriented links, then in Reidemeister theorem all orientations have to be considered.

1.2.6 Particular knots

Alternating knot

Let us now take an arbitrary point P on a regular diagram D of a knot K and move it once round D. If P, at the crossing points of D, is shown to, alternatively, move between

Figure 1.8: An alternating knot

a segment that passes over and a segment that passes under, then we say that the regular diagram is an *alternating* (regular) diagram.

Definition 26 *A knot that possesses (at least one) alternating diagram is called an alternating knot.*

Example. The trivial knot is an alternating knot.

Figure 1.9: Trivial knot

The first non-alternating knot on the table of prime knots is the 8_{19} shown on the figure 1.10.

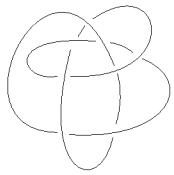


Figure 1.10: 8_{19}-knot

Prime Knots

Let K_1 and K_2 be two oriented knots, which are regarded as being in distinct copies of S^3. Remove from each S^3 a (small) ball that meets the given knot in an unknotted spanning arc, and then identify together the resulting boundary spheres, and their intersections with the knots, so that all orientations match up. The obtained knot is embedded in S^3 and is denoted by $K_1 + K_2$ or $K_1 \sharp K_2$.

Proposition 8 *The addition of oriented knots is commutative, associative. The unknot is a zero for this addition, but no knot other than the unknot has an additive inverse.*

Definition 27 *A knot K is a prime knot if it is not the unknot, and $K = K_1 + K_2$ implies that K_1 or K_2 is the unknot.*

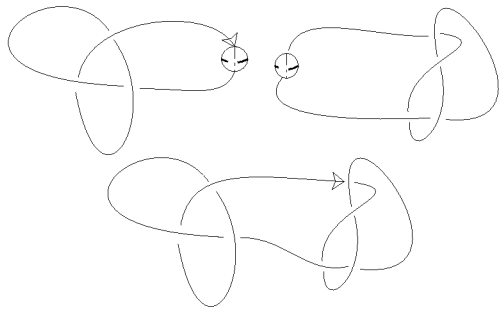


Figure 1.11: $K_1 + K_2$

The proof of the following theorem can be found in [10].

Theorem 15 *Up to ordering of summands, there is a unique expression for a knot K as a finite sum of prime knots.*

Torus knot

We consider the standard unknotted torus $T = S^1 \times S^1$ in S^3. One can see T as the boundary of the tubular neighborhood of the trivial knot. We know that $\pi_1(T) \simeq \mathbb{Z} \oplus \mathbb{Z}$. Let $[m]$ and $[l]$ be the generators of $\pi_1(T)$ which are respectively the homotopy classes of the meridian and the parallel of T. So the homotopy (or homology) class of any simple closed curve γ on T writes

$$[\gamma] = a[m] + b[l]$$

where a and b are integers.

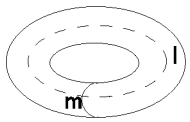


Figure 1.12: The meridian and parallel curves

Definition 28 *A torus knot of type (a, b) is a simple closed curve K on T such that its homotopy class can be written*

$$[K] = a[m] + b[l]$$

where a and b are relatively prime integers.

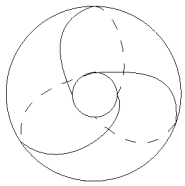


Figure 1.13: The torus knot $(2,3)$

1.2.7 Some classical knot invariants

The knot complement

Let K be a knot in S^3 and let V be a tubular neighborhood of K.

Theorem 16 *[4] Two knots K and K' are isotopic if and only if they have diffeomorphic complements.*

Remark. The last theorem fails for links with more than one component as shown in the following example.

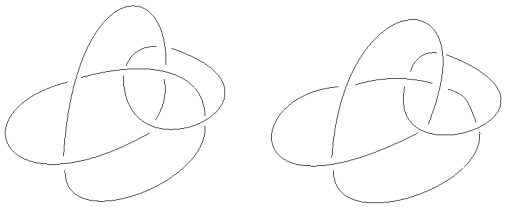


Figure 1.14: Non-isotopic links with diffeomorphic complements

The minimum number of crossing points

Let L be a link. Let D be a diagram of L. We denote by $\mathcal{D}$ the set of diagrams of L. Let $c(D)$ the number of crossing points of D. We set

$$c(L) = inf\{c(D) \mid D \in \mathcal{D}\}$$

Proposition 9 *Let L and L' be two isotopic links then $c(L) = c(L')$.*

Remarks.

1. The trivial knot is the unique knot K such that $c(K) \leq 2$.
2. If K is a (either right or left-handed) trefoil knot, then $c(K) = 3$. Furthermore, among all knots and links the trefoil knots are the only ones with $c(K) = 3$.

Conjecture. If K_1 and K_2, are two alternating knots, then

$$c(K_1 \# K_2) = c(K_1) + c(K_2)$$

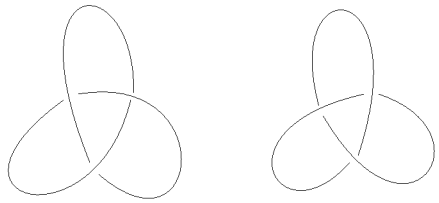


Figure 1.15: Non-isotopic knots

The Bridge Number

Let D be a diagram of a knot K. An overpass is a subarc of D which passes over the crossing points it meets. The number of maximal overpasses is called *the bridge number* of the diagram. The *bridge index* $b(K)$ of K is the least bridge number of all diagrams of K. Then, if we denote by $\mathcal{D}$ the set of diagrams of K, we have

$$b(L) = inf\{b(D) \mid D \in \mathcal{D}\}$$

By convention the bridge index of the unknot is taken to be 1.

Examples.

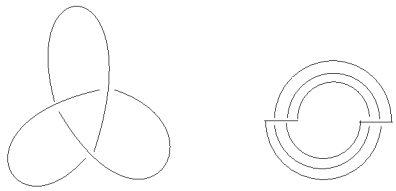


Figure 1.16: The trefoil is a 2-bridge knot

The last figure shows two diagrams of the trefoil, the first with bridge number 3, the second with bridge number 2. It is shown that the trefoil is a 2-bridge knot.

Remark. The unknot is the unique knot whose bridge index is 1.

Proposition 10 *If K_1 and K_2 are two isotopic knots then $b(K) = b(K')$.*

Proposition 11 *If K_1 and K_2 are two arbitrary knots then*

$$b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$$

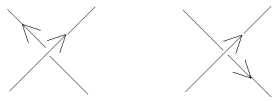
Conjecture. If K is a knot, then

$$c(K) \geq 3(b(K) - 1)$$

where equality holds only when K is the trivial knot, a trefoil knot, or the (connected) sum of trefoil knots.
Remark. Any 2-bridge knot is prime (not a connected sum of two nontrivial knots).

The Linking Number

In this subparagraph we consider oriented links.
Let D be a diagram of a link L with n components, $L = \{K_1, K_2, \cdots, K_n\}$.
Convention. To each crossing c of D which is the intersection of the projections of two distinct components K_i and K_j of L we assign the integer $\varepsilon(c) = \pm 1$ according to the following rule



$\varepsilon(c) = +1$ $\varepsilon(c) = -1$

Figure 1.17: The assignment convention of a sign to a crossing

Let $\mathcal{C}$ be the set of all crossings of the components K_i and K_j. The *linking number* of the two components K_i and K_j is the integer

$$lk(K_i, K_j) = \frac{1}{2} \sum_{c \in \mathcal{C}} \varepsilon(c)$$

Example. In figure 1.21 we have $lk(K_1, K_2) = -2$ and $lk(K_1', K_2') = 0$

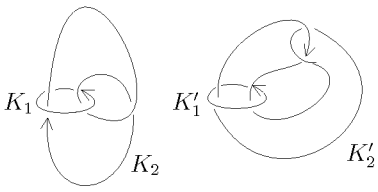


Figure 1.18:

Remarks.

1. The linking number is an integer.
2. We have $lk(K_1, K_2) = lk(K_2, K_1)$.
3. Let K be an oriented knot. If $-K$ is the knot K provided with reverse orientation, then we have the following relation

$$lk(K_1, -K_2) = -lk(K_1, K_2)$$

Theorem 17 *Let $L = \{K_1, K_2\}$ be a link, then $lk(K_1, K_2)$ is an invariant, i.e., it does not depend on the choice of the diagram of the link.*

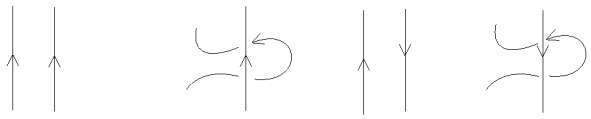


Figure 1.19:

Proof.
According to Reidemeister theorem, it suffices to prove that $lk(K_1, K_2)$ does not change under Reidemeister moves. The moves Ω_1 and Ω_3 never change the set of values of ε, while the move Ω_2 either does nothing, or adds (or destroys) two numbers ε of opposite signs
In the two cases we see that the considered move generates two crossings c_1 and c_2 such that $\varepsilon(c_1) + \varepsilon(c_2) = 0$, therefore the linking number is unchanged.

□

Two closed curves J and K in S^3 are called *unlinked* if there exists an isotopy that takes them to two curves J' and K' lying in two non-intersecting 3-balls. In the converse case, they are called *linked*. The linking number is a very simple invariant that allows one to prove (in many cases) that two curves are linked.

Theorem 18 *If $lk(K, J) \neq 0$ then J and K are linked.*

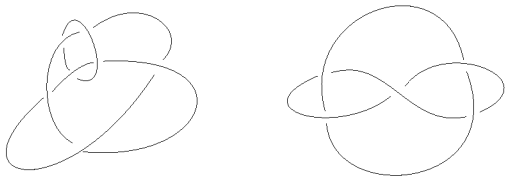


Figure 1.20: Linked curves with zero linking numbers

Definition 29 *Let $L = \{K_1, K_2, \cdots, K_n\}$ be a link with n components. We call the total linking number of L the following sum:*

$$lk(L) = \sum_{1 \leq i < j \leq n} lk(K_i, K_j)$$

Proposition 12 *The total linking number is an invariant of the link.*

1.2.8 On the Tubular Neighborhood of a Knot

For more details about the following definitions and results we refer to [15].

Definition 30 *Let V be a tubular neighborhood of a knot K. Let μ be a closed simple curve which does not bound a disk in ∂V. We say that μ is a meridian of V if there exists an embedding $h : S^1 \times D^2 \hookrightarrow S^3$ such that $V = h(S^1 \times D^2)$ and $\mu = h(* \times \partial D^2)$*

Proposition 13 *Any two meridians of V are equivalent by an ambient isotopy of V.*

Definition 31 *Let V be a tubular neighborhood of a knot K. Let λ be a closed simple curve in ∂V. We say that λ is a parallel of V if there exists an embedding $h : S^1 \times D^2 \hookrightarrow S^3$ such that $V = h(S^1 \times D^2)$ and $\lambda = h(S^1 \times *)$.*

Proposition 14 *Let K be a knot in S^3. Let V be a tubular neighborhood of K. If F is a Seifert surface of K then $F \cap \partial V$ is a parallel of V.*

Proposition 15 *There are an infinitely many ambient isotopy classes of parallels in a tubular neighborhood V of a knot.*

Proposition 16 *A simple closed curve λ on ∂V is a parallel if and only if it intersects a meridian transversally at only one point.*

Theorem 19 *Let K be a knot in S^3. Let V be a tubular neighborhood of K. Up to ambient isotopy of V, there is a unique parallel λ, which bounds in $X = S^3 \setminus \mathring{V}$ (i.e. homologically trivial in X). It is called the preferred framing of a solid torus V.*

Proposition 17 *Let K be a knot in S^3. Let V be a tubular neighborhood of K. A diffeomorphism $f : \partial V \longrightarrow \partial V$ extends to a self-diffeomorphism of V if and only if f takes a meridian to a meridian.*

1.3 Morse Theory

In the first three subsections we analyse the level sets $f^{-1}(y)$ of a function $f : M \longrightarrow \mathbb{R}$ having only simplest possible critical points. Such a function is called a "Morse function". The decomposition of M into these level sets contains an amazing amount of information about the topology of M. We have presented only the beginning of Morse theory as done in Milnor's book [12].
In the last subsection we state the important theorems of Heegaard splitting and handle decomposition. For more details about these notions the reader should consult [13].

1.3.1 Introduction

Let T be a torus embedded in $\mathbb{R}^3$ as depicted in figure 1.21, Let $f : T \longrightarrow \mathbb{R}$ be the height function (third coordinate).
Recall that if $a \in \mathbb{R}$ is a regular value of f, then

$$T_a = \{x \in T | f(x) \leq a\}$$

is a manifold such that $\partial T_a = f^{-1}(a)$.

1. If $a < 0 = f(P)$, then $T_a = \emptyset$.

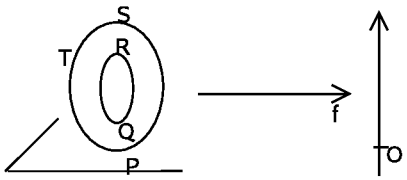


Figure 1.21: The height function on the torus T

2. If $0 = f(P) < a < f(Q)$, then T_a is diffeomorphic to a 2-cell.
3. If $f(Q) < a < f(R)$, then T_a is diffeomorphic to a cylinder.
4. If $f(R) < a < f(S)$, then T_a is diffeomorphic to a 2-compact manifold of genus 1 having a circle as boundary.
5. If $f(S) < a$, then $T_a = T$

In order to describe how T_a is changes when a passes through one of the points $f(P)$, $f(Q)$, $f(R)$, $f(S)$, it is convenient to consider homotopy type rather than diffeomorphism type.

So,in terms of homotopy type:

- the change from the state 1 to the state 2 is the operation of attaching a 0-cell.

Figure 1.22: Attaching a 0-cell

- the change from the state 2 to the state 3 is the operation of attaching a 1-cell.

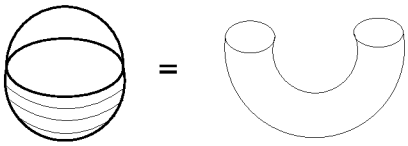


Figure 1.23: Attaching a 1-cell

- the change from the state 3 to the state 4 is the operation of attaching an other 1-cell.
- the change from the state 4 to the state 5 is the operation of attaching an other 2-cell.

One notes that the homotopy type of T_a changes as a passes through one of the points $f(P)$, $f(Q)$, $f(R)$, $f(S)$.

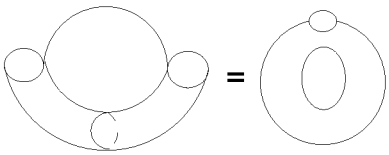


Figure 1.24: Attaching a 1-cell

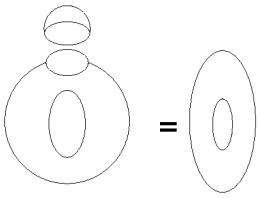


Figure 1.25: Attaching a 2-cell

1.3.2 Morse Lemma

Let M be an n manifold and let $f : M \longrightarrow \mathbb{R}$ be a differentiable map.

Definition 32 *A point p in M is called a critical point of f if the induced map $T_pf : T_pM \longrightarrow T_{f(p)}\mathbb{R} = \mathbb{R}$ is zero. If we choose a local coordinate system $(x = (x_1, \ldots, x_m), U)$ of M at p this means that for each i, $1 \leq i \leq m$,*

$$\frac{\partial f}{\partial x_i}(p) = 0$$

We say that $f(p)$ is a critical value of f.

Definition 33 *A critical point p is called non-degenerate if and only if there exists a chart $(x = (x_1, \ldots, x_n), U)$ of M at p such that the matrix*

$$(\partial^2 f/\partial x_i \partial x_j(p))_{1 \leq i,j \leq n}$$

is non-singular i.e. $det(\partial^2 f/\partial x_i \partial x_j(p))_{1 \leq i,j \leq n} \neq 0$

Remark. The non-degeneracy does not depend on the chart.

Definition 34 *We say that the differentiable map $f : M \longrightarrow \mathbb{R}$ is a Morse function if all of its critical points are non-degenerate.*

Proposition 18 *Let M be a compact manifold, then there exists a Morse function on M.*

Remarks.

1. The set of critical points of a Morse function is closed discrete in M. So, it is finite if M is compact.

2. If $p \in M$ is a critical point, the matrix $(\partial^2 f/\partial x_i \partial x_j(p))_{1\leq i,j\leq n}$ is symmetric.

Definition 35 *Let $p \in M$ be a critical point of f. The index of f at p is the number of strictly negative eigenvalues of the matrix*
$(\partial^2 f/\partial x_i \partial x_j(p))_{1\leq i,j\leq n}$.

Lemma 2 *Let $p \in M$ be a non degenerate critical point for f. Then there is a chart $(x = (x_1, \ldots, x_n), U)$ of M at p with $x(p) = 0$ and such that the identity*

$$f = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$$

holds throughout U, where k is the index of f at p.

1.3.3 Homotopy Type in Terms of Critical Values

We recall that if f is a smooth real valued function on an n-manifold M, then

$$M_a = f^{-1}(]-\infty, a]) = \{p \in M | f(p) \leq a\}$$

is an n-manifold such that $\partial M_a = f^{-1}(a)$.

Theorem 20 *Let f be a smooth real valued function on a manifold M. Let $a < b$ and suppose that the set $f^{-1}([a,b]) = \{p \in M | a \leq f(p) \leq b\}$ is compact, and contains no critical points of f. Then M_a is diffeomorphic to M_b. Furthermore, M_a is a deformation retract of M_b, so that the inclusion map $M_a \hookrightarrow M_b$ is a homotopy equivalence.*

Proof: Choose a Riemannian metric on M; and let $\langle u, v\rangle$ denote the inner product of two tangent vectors, as determined by this metric.
We consider the vector field $grad\, f$ on M characterized by the identity:

$$\langle X, grad\, f\rangle = X(f) = T_X f$$

for any vector field X.
This vector field $grad\, f$ vanishes precisely at the critical points of f.
We note that if $\gamma : \mathbb{R} \longrightarrow M$ is a curve in M, we have:

$$\langle \frac{d\gamma}{dt}, grad\, f\rangle = \frac{d(f \circ \gamma)}{dt}$$

Let $g : M \longrightarrow \mathbb{R}$ be a smooth function which is equal to $1/\langle grad\, f, grad\, f\rangle$ throughout the compact set $f^{-1}([a,b])$; and which vanishes outside of a compact neighborhood of this set, $f^{-1}([a,b]) \subset K$.
Then the vector field, defined by

$$X(q) = g(q) grad\, f(q)$$

satisfies the condition of the following lemma

Lemma 3 *A smooth vector field on M which vanishes outside of a compact set $K \subset M$ generates a unique 1-parameter group of diffeomorphisms of M such that for each point $q \in M$ we have*

$$\frac{d\varphi_t(q)}{dt} = X_{\varphi_t(q)}, \quad \text{et } \varphi_0(q) = q$$

Hence by using the last lemma the introduced vector field X generates a unique 1-parameter group of diffeomorphisms of M

$$\varphi_t : M \longrightarrow M$$

For fixed $q \in M$ consider the function

$$\begin{array}{ccccc} M & \xrightarrow{\varphi_t} & M & \xrightarrow{f} & \mathbb{R} \\ q & \longrightarrow & \varphi_t(q) & \longrightarrow & f(\varphi_t(q)) \end{array}$$

If $\varphi_t(q) \in f^{-1}([a,b])$, then

$$\frac{df(\varphi_t(q))}{dt} = \langle \frac{d(\varphi_t(q))}{dt}, grad\, f \rangle = \langle X, grad\, f \rangle = +1$$

Thus the correspondence $t :\longrightarrow f(\varphi_t(q))$ is linear with derivative $+1$ as long as $f(\varphi_t(q)) \in [a,b]$.
Now consider the diffeomorphism $\varphi_{b-a} : M \longrightarrow M$. Clearly this carries M_a diffeomorphically onto M_b. This proves the first half of the theorem.
Define a 1-parameter family of maps $r_t : M_b \longrightarrow M_b$, $t \in \mathbb{R}$ by

$$r_t(q) = \begin{cases} q & if \quad f(q) \leq a \\ \varphi_{t(a-f(q))}(q) & if \quad a \leq f(q) \leq b \end{cases}$$

Then $r_0 = id$, and r_1 is a retraction from M_b to M_a.

□

Theorem 21 *Let $f : M \longrightarrow \mathbb{R}$ be a smooth function, and let p be a non-degenerate critical point with index k. Setting $f(p) = c$, suppose that $f^{-1}([c-\varepsilon, c+\varepsilon])$ i compact, and contains no critical points of f other than p, for some $\varepsilon > 0$. Then, for all sufficiently small $\varepsilon > 0$, the set $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with a k-cell attached.*

Proof. By using the Morse Lemma , there exists a chart $(x = (x_1, \ldots, x_m), U)$ of M at p such that

$$f = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$$

inside U and $x(p) = 0$.
Choose $\varepsilon > 0$ sufficiently small so that

1. The region $f^{-1}([c-\varepsilon, c+\varepsilon])$ is compact and contains no critical points other than p.

2. The image of U by x contains the closed ball

$$\left\{(x_1,\ldots,x_m)|\sum(x_i)^2 \leq 2\varepsilon\right\}$$

Let

$$e_k = \left\{q \in U|x_1^2(q)+\ldots x_k^2(q) \leq \varepsilon \quad et \quad x_{k+1}(q)=\cdots=x_m(q)=0\right\}$$

The resulting situation is illustrated schematically in the diagram of the figure 1.26.

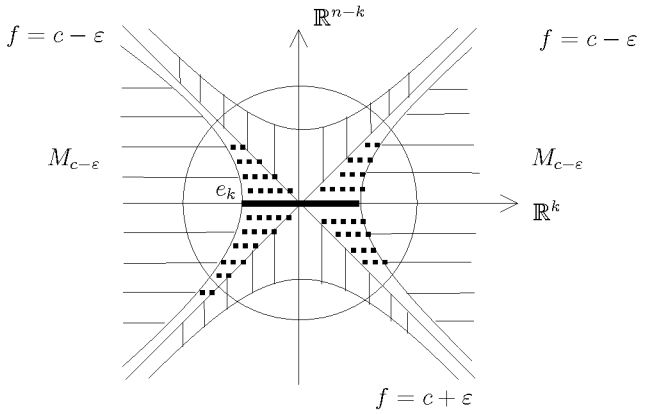


Figure 1.26:

To show that $M^{c-\varepsilon} \cup e_k$ is a deformation retract of $M^{c+\varepsilon}$ we use an adequate other Morse function F. The coordinate lines represent the planes $x_{k+1}(q)=\cdots=x_m(q)=0$ and $x_1(q)=\cdots=x_k(q)=0$ respectively; the circle represents the boundary of the ball of radius $\sqrt{2\varepsilon}$; and the hyperbolas represent the hypersurfaces $f^{-1}(c-\varepsilon)$ and $f^{-1}(c+\varepsilon)$. The region $M_{c-\varepsilon}$ is horizontally shaded; the region $f^{-1}([c-\varepsilon,c])$ is lightly dotted; and the region $f^{-1}([c,c+\varepsilon])$ is vertically shaded. The horizontal thick dark line through p represents the cell e_k.

□

Remark. In general, if we set that $p_1,p_2,\ldots,p_r$ are some non-degenerate critical points of f respectively with the indexes $k_1,k_2,\ldots,k_r$ in $f^{-1}(c)$. Then a same proof as that of the previous theorem shows that $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon} \cup e^{k_1} \cup \cdots \cup e^{k_r}$.

Theorem 22 *If f is a differentiable function on a manifold M with no degenerate critical points, and if each M_a is compact, then M has the homotopy type of a CW-complex, with one cell of dimension k for each critical point of index k.*

1.3.4 Applications: Heegaard splitting and handle decomposition

An n-dimensional k-handle, $0 \leq k \leq n$, is an n-manifold diffeomorphic to $D^k \times D^{n-k}$.
The core of the handle: $D^k \times 0$.
The attaching region: $\partial D^k \times D^{n-k}$.
The attaching sphere: $\partial D^k \times 0$.

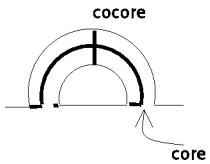


Figure 1.27: Anatomy of a handle

Definition 36 *Let M be an n-manifold with boundary. The attaching of the k-handle $D^k \times D^{n-k}$ to M along $\partial D^k \times D^{n-k}$ is the gluing of $D^k \times D^{n-k}$ to ∂M by an embedding $\varphi : \partial D^k \times D^{n-k} \hookrightarrow \partial M$. The obtained manifold is denoted by*

$$M \cup_\varphi D^k \times D^{n-k}$$

Remark. There exists a deformation retraction of $M \cup_\varphi D^k \times D^{n-k}$ to $M \cup \varphi_{|\partial D^k \times 0} D^k \times 0$. So, up to homotopy, attaching a k-handle in dimension n is the same as the attaching of a k-cell.

Definition 37 *A handlebody of genus g is a 3-manifold diffeomorphic to the 3-ball D^3 to which are attached g 1-handles, $D^1 \times D^2$.*

Figure 1.28: Handlebody of genus 3

The theorems 20, 21 can be used to prove the following result:

Theorem 23 *Any closed connected oriented 3-manifold is diffeomorphic to a gluing of two handlebodies H_1, H_2 along their boundaries and which have the same genus g:*

$$M \simeq H_1 \cup_h H_2$$

where $h : \partial H_1 \longrightarrow \partial H_2$ is a diffeomorphism.

The last presentation of the 3-manifold is called a Heegaard splitting.
We close this subsection with:

Theorem 24 *Any compact manifold admits a handle decomposition.*

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Differential Geometry and Topology, Discrete and Computational Geometry
M. Boucetta and J.-M. Morvan (Eds.)
IOS Press, 2005
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Chapter 2

Combinatorial Topology and Discrete Morse Theory

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Motivations

In the first draft of the program for this Summer School in Marrakech, we planned to expose some standard material in differential topology, and some recent developments in low dimensional topology. We had then a discussion with the organizers, and it appeared that a unifying view on the program of this School was to establish bridges between differential matter on one hand, and discrete or computational geometry on the other hand. So we decided to focus on combinatorial topology and include an introduction to Robin Forman *discrete Morse theory*. In this chapter, we start with some classical combinatorial topology, including piecewise linear manifolds. Then we define discrete Morse functions on CW-complexes and show that we get the usual theorems of Morse theory. Everything here should sound rather familiar for people knowing classical theory. However the relation between differentiable objects, such that gradient vector fields, and their discrete counterpart has still to be explored. The generic questions below could give starting points for interesting research.

- What is a discrete counterpart of a given differentiable notion ?
- What is the interplay between the differentiable and discrete notions for a given

manifold ? In particular in which way the discrete notion could approach the smooth one ?

- Is it computable ?

The subject of *Computational Topology* is the interrelationship between topology and algorithmic problems. Motivations come either from algorithmic questions involving topological matter, or from combinatorial problems supporting topological methods. Studying discrete Morse theory could be a first step towards this new domain. We will give at the end of this exposition some applications of classical Morse theory in low dimensional topology. Most of them can be obtained easily via the discrete theory. Some deeper ones, for example those using Cerf theory, are not so obvious. That would be interesting to revisit them using discrete methods and hopefully to go further towards new interesting results.

2.1 Introduction to Combinatorial Topology

The purpose of this section is to define basic objects in combinatorial topology such as simplicial complexes, triangulations, piecewise linear manifolds and CW-complexes.

2.1.1 Simplicial Complexes

Definition 38 *A n-dimensional simplex is the convex hull of $n+1$ points (the vertices) affinely independent in an real affine space.*

Points in the simplex $\sigma^n = (\alpha_0, \ldots, \alpha_n)$ are

$$x = \sum_{j=0}^{n} x_j \alpha_j \ , \ \forall j \ x_j \geq 0 \ , \ \sum_{j=0}^{n} x_j = 1 \ .$$

A face of σ^n is a simplex obtained with a subset of vertices. In particular, the $(n-1)$-dimensional faces (called facets) are the simplexes

$$\sigma_i^{n-1} = (\alpha_0, \ldots, \hat{\alpha}_i, \ldots, \alpha_n) = \{x = \sum_{j \neq i} x_j \alpha_j \ , \ \forall j \ x_j \geq 0 \ , \ \sum_{j \neq i} x_j = 1\} \ .$$

Definition 39 *A (locally finite) simplicial complex in a real affine space is a set F of simplices such that*

(i) all faces of simplices in F are in F,
(ii) F is locally finite i.e. each point in some simplex of F has a neighborhood meeting only a finite number of simplices in F,
(iii) intersection of 2 simplices in F is either a common face or empty-set.

If S is the set of vertices (0-dimensional faces), we will use the notation $K = (S, F)$. The polyhedron of K is the union of all its simplices; we denote it by $|K|$.

Definition 40 *A triangulation of a topological space is the data given by a simplicial complex K, and an homeomorphism between $|K|$ and X.*

Definition 41 *A simplicial complex $K' = (S', F')$ is a subdivision of $K = (S, F)$ if and only if each simplex in F' is contained in a simplex in F, and each simplex in F is a union of simplices in F'.*

The barycentric subdivision of $K = (S, F)$ is obtained from the set of vertices S' formed by adding to S the barycenter of each face in F. Faces are obtained recursively on dimension. For each n dimensional simplex $\sigma \in F$, we add in F' cones from the barycenter of σ to all simplices in F' which are contained in the faces of σ.

Definition 42 *Let $K = (S, F)$ be a simplicial complex. A map from the polyhedron $|K|$ to a real affine space is said to be simplicial if and only if the restriction to each simplex is linear.*

Definition 43 *Two simplicial complexes $K = (S, F)$ and $K' = (S', F')$ are isomorphic if and only if there exists a bijection between S and S' inducing a bijection between F and F' (such a map induces a simplicial homeomorphism between $|K|$ and $|K'|$).*

Definition 44 *Two simplicial complexes are combinatorially equivalent if and only if they have isomorphic subdivisions. The triangulations $t : |K| \to X$, $t' : |K'| \to X$ are combinatorially equivalent if and only if there exist subdivisions L and L' such that $t'^{-1} \circ t$ defines a simplicial isomorphism between L and L'.*

A topological space together with a combinatorial equivalence class of triangulations is called a polyhedron, or a PL space. In the next section we will consider the case of PL manifolds.

2.1.2 PL manifolds

Definition 45 *A simplicial complex $K = (S, F)$ is a piecewise linear manifold if and only if each point in $|K|$ has a neighborhood combinatorially equivalent to a n dimensional simplex.*

Note that the above property only depends on the combinatorial equivalence class of K.

Definition 46 *A PL manifold structure on a topological space X is a combinatorial equivalence class of triangulations by simplicial complexes which are piecewise linear manifolds.*

A PL manifold is a topological manifold with boundary.

Definition 47 *Let M, M' be PL manifolds. A map $f : M \to M'$ is PL if and only if there exist triangulations*

$$h : |K| \to M, \text{ et } h' : |K'| \to M'$$

such that $h'^{-1} \circ f \circ h$ is a simplicial map.

Remark. The definition of a PL map makes sense if M or M' is a polyhedron.

Collar Neighborhood

Definition 48 *Let $i : N \to M$ be a PL embedding between PL manifolds. A collar is a PL embedding $j : [0,1] \times N \to M$ such that*

$\forall x \in N,\ j(0,x) = i(x),$
$j([0,1[\times N)$ *is an open neighborhood of* $i(N)$.

A local collar at $x \in N$ is a collar for the restriction of i to some neighborhood of x. The theorem below is proved in [15, chapitre 2]. The statement holds for polyhedra and can be extended to the non compact case.

Theorem 25 (Collar neighborhood) *If N is compact and $i : N \to M$ admits local collars, then there exists a collar.*

Proof. We suppose that M is a piecewise linear manifold in $\mathbf{R}^n$, and we consider $M^+ = [0,1] \times N \cup 1 \times N \subset \mathbf{R} \times \mathbf{R}^n$. If we have a local collar at x

$$j_x : [0,1] \times N_x \to M$$

where N_x is a neighborhood of x which is a combinatorial ball. Then we define a PL homeomorphism of M^+, which is identity outside $[0,1] \times N_x \cup j_x([0,1] \times N_x)$, moves $(1,x)$ to $(\frac{1}{2},x)$ and is defined by extending linearly on the square $[0,1] \times N_x \cup j_x([0,1] \times N_x) \approx [0,2] \times N_x$. Note that the projection $\mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ induces a PL projection $j_x(N) \to N$.

There exists a finite number of such local collars

$$j_{x_\nu} : [0,1] \times N_{x_\nu} \to M\ ,\ 1 \leq \nu \leq m$$

such that the $Int(N_{x_\nu})$ cover N. We consider the composition

$$h = h_{x_m} \circ \cdots \circ h_{x_1}\ .$$

Let $M' = h(M)$. The image $N' = h(N)$ is included in $]0,1[\times N$, and the projection $p : N' \to N$ is a PL map. In $M' \approx M$, the intersection $M' \cap [0,1] \times N$ is a collar for N'. □

Corollary 3 *If M is a compact PL manifold, then its boundary admits a collar.*

Regular Neighborhood

Definition 49 *Let P be a polyhedron, and $Q \subset P$ a sub-polyhedron.*
a) We say that Q is an elementary collapse of P if and only if the closure $\sigma = \overline{P - Q}$ is a simplex with exactly one facet not included in Q.
b) We say that P collapses onto Q if and only ifXS there exits a sequence of sub-polyhedra $Q = Q_0 \subset Q_1 \cdots \subset Q_r = P$ such that Q_{i-1} is an elementary collapse of Q_i, for $1 \leq i \leq r$.

Definition 50 *Let M be a PL manifold, and $N \subset M \setminus \partial M$ be a PL embedded compact polyhedron. A regular neighborhood of N is a neighborhood $V \subset M \setminus \partial M$ such that,*

(i) V is a compact PL sub-manifold,
(ii) V collapse onto N.

Definition 51 *A PL ambient isotopy (resp. rel. N) is a PL homeomorphism $H : [0,1] \times M \to [0,1] \times M$ preserving the level and such that*

$$\forall x \ H(0,x) = (0,x) \quad (resp. \forall t, \ \forall x \in N, \ H(t,x) = (t,x) \ ) \ .$$

The following theorem is proved in [15, Chapter 3]

Theorem 26 *a) For any compact polyhedron N in a PL manifold M, there exists a regular neighborhood.*
b) Two regular neighborhoods V et V' are PL ambient isotopic rel. N.

In particular two regular neighborhoods are PL homeomorphic.

Corollary 4 *Let M be a PL n-manifold with boundary and v a point in $M \setminus \partial M$ such that M collapse onto v, then M is a PL n-simplex.*

Proof. In the PL n-manifold M, the point v has a regular neighborhood which is a PL $n - simplex$. Another regular neighborhood is M minus an open collar of the boundary. They are PL homeomorphic. □

Ambient Isotopy and Isotopy by Moves for Knots

For PL embeddings, there exists a couple of different notions of isotopy that will not be developed here (see for example [9]). For knot theory one can show results stating *equivalence of equivalences.* We give below an example of such result.

Definition 52 *a) A polygonal knot is an PL embedding of S^1 in $\mathbb{R}^3$.*
b) Two knots are (PL) isotopic if and only if they correspond by an ambient (PL) isotopy.
c) Two knots are equivalent by moves if and only if they correspond by a sequence of moves consisting in replacing in a knot K a segment AB by two segments BC and CA in such a way that the triangle ABC intersects K along $ABC \cap K = AB$, or the converse.

Theorem 27 *For two polygonal knots, the statements below are equivalent.*
a) K and K' corresponds by a PL homeomorphism.
b) K and K' are PL ambient isotopic.
c) K and K' are equivalent by moves.

A proof can be found in [2].

2.1.3 Triangulation of Manifolds

Triangulation of C^1 manifolds has been established by Cairns in the years 1930. Then Whitehead has proved that a C^1 manifold has a canonical PL structure, which is represented by a C^1 triangulation.

In dimension lower than 3, any topological manifold supports a smooth structure, and C^1 manifolds are diffeomorphic if and only if they are homeomorphic (resp. PL homeomorphic). One says that in these dimensions Top, PL and Diff classifications coincide.

In dimension 4, PL and Diff classifications coincide, but there exist compact topological manifolds without PL structure, and homeomorphic C^1 manifolds which are not diffeomorphic.

A list of results and unsolved problems can be found in [1]. Moise's book [13] gives a detailed exposition in dimension 2 and 3. Triangulation of smooth manifolds is exposed in [14, chapitre II]; one can find here the following theorems.

Definition 53 *Let M be a C^r manifolds, $r \geq 1$. A triangulation $f : |K| \to M$ is C^r if and only if the restriction of f to each simplex in K is C^r and has maximal rank.*

Theorem 28 *Let $r \geq 1$, $f : |K| \to M$, $f' : |K'| \to M$ be two C^r triangulations of the C^r manifold M. Then K and K' have isomorphic subdivisions.*

Theorem 29 *Any C^r manifold M, $r \geq 1$, support a C^r triangulation. Moreover any C^r triangulation of the boundary ∂M can be extended.*

2.2 CW-complexes

We give here basic definitions and properties of CW-complexes, with a specific view on the regular case. All the notions on CW-complexes exposed below can be found in [11].

2.2.1 Combinatorial Cell Complexes

A *closed Euclidean n-cell* E^n is a homeomorphic image of the Euclidean n-cube $\mathbf{I}^n$, the Cartesian product of n copies of the closed interval $\mathbf{I} = [0, 1]$. We denote by $\dot{E}^n$ the boundary of E^n.

Definition 54 *Let X be a set. A* cell structure *on X is a pair (X, Φ) where Φ is a collection of maps of closed Euclidean cells into X satisfying the following conditions.*

1. *If $\phi \in \Phi$ and ϕ has domain E^n, then ϕ is injective on $E^n - \dot{E}^n$.*
2. *The images $\{\phi(E^n - \dot{E}^n) | \phi \in \Phi\}$ form a partition of X.*
3. *If $\phi \in \Phi$ has domain E^n, then*

$$\phi(\dot{E}^n) \subset \bigcup_{\substack{\psi \in \Phi \text{ has domain } E^k \\ k \leq (n-1)}} \psi(E^k) \ .$$

If $\phi \in \Phi$ and ϕ has domain E^n, the image $\phi(E^n) = \overline{\sigma}^{(n)}$ is called a *closed n-cell* of (X, Φ). $\phi(\dot{E}^n) = \dot{\sigma}^{(n)})$ is called the *boundary* of $\sigma^{(n)}$, and $\phi(E^n - \dot{E}^n)$ is called its *interior*. If $n > 0$, $\phi(E^n - \dot{E}^n)$ is called an *open n-cell* and we denote it by $\sigma^{(n)}$.
We call ϕ a *characteristic map* for the n-cell $\sigma^{(n)}$. Then Φ is the set of characteristic maps for the cells of (X, Φ). Unless otherwise specified, a *cell* in X will mean an *open cell*.

The union

$$X^{n-1} = \bigcup_{\substack{\psi \in \Phi \text{ has domain } E^k \\ k \leq (n-1)}} \psi(E^k)$$

which appears in condition *3* is called the $(n-1)$-*skeleton* of the cell structure. Thus $\forall n \geq 1,\ \dot{\sigma}^{(n)} = \phi(\dot{E}^n) \subset X^{n-1}$.

We say that two cell structures (X, Φ) and (X, Φ') are *strictly equivalent* if there is a one-to-one correspondence between Φ and Φ' such that a characteristic function with domain E^n corresponds to a characteristic function with domain E^n, and corresponding functions differ only by a reparametrization of their domain. That is, if ϕ and ϕ' are corresponding functions of Φ and Φ', respectively, then $\phi' = \phi \circ h$, where $h : (E^n, \dot{E}^n) \to (E^n, \dot{E}^n)$ is an homeomorphism of pairs. One can check that it is an equivalence relation on the collection of cell structures on the set X.
If (X, Φ) is a cell structure, let S_Φ consists of all pairs $(\sigma^{(n)}, [\phi])$, where $\sigma^{(n)} = \phi(E^n - \dot{E}^n)$ and $[\phi]$ is a strict equivalence class of $\phi \in \Phi$.

Definition 55 *A cell complex structure on a set X (or a* cellular decomposition *of a set X) is an equivalence class of cell structures (X, Φ) under strict equivalence. We will denote by (X, S) the set X equipped with a cell complex structure, where $S = S_\Phi$ for some representative cell structure (X, Φ). The set S is called the set of (open) cells of (X, S).*

Definition 56 *A* sub-complex *(A, T) of a cell complex (X, S), which we denote by $(A, T) \subset (X, S)$, is a cell complex such that $A \subset X$ et $T \subset S$.*

Subsequently, we'll need the following proposition which characterizes the cells of a sub-complex.

Proposition 19 *Let $(A, T) \subset (X, S)$ and σ be a cell of S. Then σ is a cell of T if and only if $\sigma \cap A \neq \emptyset$.*

Proof. If $\sigma \in T$ then $\sigma \subset A$ and so we have $\sigma \cap A \neq \emptyset$.
Conversely, suppose that $\sigma \cap A \neq \emptyset$. Choose characteristic maps Φ for (X, S) and Ψ for (A, T) such that $\Psi \subset \Phi$. If we have $y \in \sigma \cap A$, then $y \in \phi_\sigma(E_\sigma - \dot{E}_\sigma)$ for $\phi_\sigma \in \Phi$. Since open cells in A form a partition, the condition $y \in A$ implies $\phi_\sigma \in \Psi$. Thus σ belongs to T. □

Definition 57 *A cell complex (X, S) is* finite *or* countable *if S is a finite or countable set. A cell is called a* regular cell *if it has one (and hence every equivalent) characteristic map bijective on all E^n.*

A cell complex is closure finite *if each n-cell meets only a finite number of open cells $\sigma^{(p)}$ with $p < n$.*
A cell complex is regular *if all its cells are regular.*
A cell complex is normal *if, for each closed cell $\overline{\sigma}$, the subset $\overline{\sigma}$ carries the structure of a sub-complex.*
The cell complex (X, S) has dimension *n if it has no cells $\sigma^{(p)}$ of dimension p greater than n, and has at least one cell of dimension n.*

2.2.2 CW Complex Structure

A CW-complex structure is a cell structure on a set X whose topology is given by the characteristic maps.

Definition 58 *A Hausdorff topological space is a* CW-complex *with respect to a family of cells S provided:*

(i) the pair (X, S) is a cell complex such that each cell $\sigma \in S$ has a continuous characteristic function;

(ii) the space X has the weak topology with respect to the closed cells $\overline{S}$;

(iii) the cell complex (X, S) is closure finite.

If (X, S) is a cell complex such that X satisfies *(i) (ii) (iii)* with respect to S we say that X is a CW-complex with cells S.

From the definition, we obtain a lemma which links the topology of X with the topology of Euclidean spaces and a proposition which characterizes continuous map from a CW-complex to another topological space.

Lemma 4 *Let X be a CW-complex with cells S. Then*

(i) each cell $\overline{\sigma} \in S$ is a closed subset of X;

(ii) for each cell $\sigma \in S$, the restriction of the characteristic map ϕ_σ to $E^n - \dot{E}^n$ is an homeomorphism onto σ.

Proof. A characteristic map ϕ_σ has compact domain. The space X is Hausdorff, hence ϕ_σ has compact image and is a closed map. We get part (i). Since ϕ_σ is closed, its restriction $\phi_\sigma|(E_\sigma - \dot{E}_\sigma)$ is a closed continuous bijection onto σ, hence an homeomorphism. □

Proposition 20 *Let (X, S) be a CW-complex. If $f : X \to Y$ is a map from X to a topological space Y, then f is continuous if and only if $f|_{\overline{\sigma}}$ is continuous $\forall \sigma \in S$.*

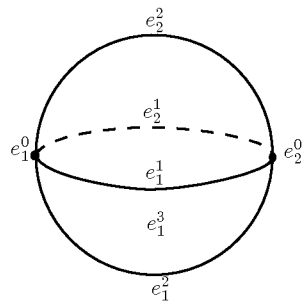

Figure 2.1: Regular cellular decomposition of the unit ball in $\mathbb{R}^3$

Proof. The condition is necessary (the restriction of a continuous map is a continuous map). Suppose now that $f : X \to Y$ is such that $f|_{\overline{\sigma}}$ is continuous for each cell σ and $C \subset Y$ is a closed set. Then $(f|_{\overline{\sigma}})^{-1}(C)$ is closed in $\overline{\sigma}$ for each σ, and therefore $f^{-1}(C)$ is closed in X. □

Remark. We must be careful with this topology. For example it is not true in general that an open cell is open in (X, S) (consider the regular cellular decomposition of the unit ball in $\mathbb{R}^3$ in 7 cells (2 of dimension 0, 2 of dimension 1, 2 of dimension 2 and one of dimension 3, cf figure 2.1).

2.2.3 Regular and Normal CW-complexes

Definition 59 *A CW-complex X is* regular *if each closed cell is homeomorphic to a a closed Euclidean n-cell. A CW-complex is* normal *if each closed cell is a sub-complex.*

We point out that if (X, S) is a regular CW-complex, then we can choose a cell structure on X in which each characteristic map is an homeomorphism without changing the cells (as subsets of X). Before proving that a regular CW complex is normal we need a definition and some preliminary results.

Definition 60 *If (X, S) is a cell complex, the* carrier *of $A \subset X$ is the intersection of all sub-complexes of (X, S) whose underlying set contains A. The carrier of A will be denoted by $(C(A), S|A)$, or just by $C(A)$ if the cellular decomposition is fixed.*

We remark that (X, S) can be equipped with a topology where the closed sets are the $C(A), A \in \mathcal{P}(X)$. As the union of any sub-complex is still a sub-complex, the union of any family of closed sets is still a closed set.

Lemma 5 *If (X, S) is a cell complex, and for $A \subset X$, $C(A)$ is the carrier of A, then*

1. $C(\emptyset) = \emptyset$;

2. $\forall A \subset X, A \subset C(A)$;

3. $\forall A \subset X, C(C(A)) = C(A)$;

4. $\forall A \subset X$ *and* $B \subset X, C(A \cup B) = C(A) \cup C(B)$;

5. $\forall A \subset X$ *and* $B \subset X$, *if* $A \subset B$, *then* $C(A) \subset C(B)$.

Proof. Strait forward from the definition. □

Lemma 6 *If* (X, S) *is a cell complex and* $A \subset X$ *then*

$$C(A) = \bigcup_{\substack{\sigma \in S \\ \sigma \cap A \neq \emptyset}} C(\overline{\sigma}) \ .$$

Proof. We use that the sets σ form a partition of X. We have that

$$A \subset \bigcup_{\substack{\sigma \in S \\ \sigma \cap A \neq \emptyset}} \overline{\sigma} \ .$$

By lemma 5 we get

$$C(A) \subset \bigcup_{\substack{\sigma \in S \\ \sigma \cap A \neq \emptyset}} C(\overline{\sigma}) \ .$$

On the other hand, if $\sigma \in S$ is such that $\sigma \cap A \neq \emptyset$, then σ meets any sub complex of (X, S) which contains A. By proposition 19, σ must be contained in such a complex, so that $\overline{\sigma} \subset C(A)$. Thus $C(\overline{\sigma}) \subset C(C(A)) = C(A)$, and we obtain

$$\bigcup_{\substack{\sigma \in S \\ \sigma \cap A \neq \emptyset}} C(\overline{\sigma}) \subset C(A) \ .$$

□

Lemma 7 *If* (X, S) *is a cell complex and* $\sigma \in S$, *then*

$$C(\overline{\sigma}) = C(\dot{\sigma}) \cup \sigma \ .$$

Proof. We have $C(\dot{\sigma}) \subset C(\overline{\sigma})$ by lemma 5, and since $\sigma \subset \overline{\sigma}$, we have $\sigma \cup C(\dot{\sigma}) \subset C(\overline{\sigma})$. Conversely, $\overline{\sigma} = \sigma \cup \dot{\sigma} \subset \sigma \cup C(\dot{\sigma})$, and $\sigma \cup C(\dot{\sigma})$ is a sub-complex of (X, S). Thus from the definition we have $C(\overline{\sigma}) \subset C(\dot{\sigma}) \cup \sigma$. □

We need more results before proving that a regular CW-complex is normal.

Lemma 8 *A CW-complex* X *is normal if and only if for each cell* σ *of* X, *one has* $C(\dot{\sigma}) \subset \overline{\sigma}$.

Proof. X is normal if and only if each closed cell $\overline{\sigma}$ is a sub-complex, which is equivalent to $\overline{\sigma} = C(\overline{\sigma}) = \sigma \cup C(\dot{\sigma})$ for each cell σ. □

Lemma 9 *A CW-complex X is normal if and only of for each cell σ of X, the boundary $\dot{\sigma}$ is a sub-complex of X.*

Proof: If X is normal, then $\overline{\sigma}^{(n)}$ and X^{n-1} are sub-complexes of X, so that $\dot{\sigma}^{(n)} = \overline{\sigma}^{(n)} \cap X^{n-1}$ is a sub-complex of X.
On the other hand, if for each cell σ of X, $\dot{\sigma}$ is a sub-complex of X, then $\overline{\sigma} = \sigma \cup \dot{\sigma} = \sigma \cup C(\dot{\sigma}) = C(\overline{\sigma})$, and $\overline{\sigma}$ is a sub-complex. Thus X is normal. □

Proposition 21 *If A is a sub-complex of a regular (resp. normal) CW-complex X, then A is a regular (resp. normal) CW-complex.*

The proof if immediate. □

We now quote without proving it the invariance of domain theorem, and one of its corollary (see e.g. [3, Section IV.19]).

Theorem 30 (Invariance of domain) *Let $f : X \to Y$ be a injective continuous map between the n-manifolds X and Y, then f is open.*

Corollary 5 (Invariance of dimension) *If an m-manifold X is homeomorphic to an n-manifold Y, then $m = n$.*

We will now prove the following.

Theorem 31 *If X is a regular CW-complex, then X is a normal CW-complex.*

Proof. By lemma 9, it is enough to prove that for any n-cell $\sigma^{(n)}$ of X, $\dot{\sigma}$ is a sub-complex. For that we have to show that, if $\tau^{(q)}$ is a cell of X and $\tau^{(q)} \cap \dot{\sigma}^{(n)} \neq \emptyset$, then we have $\overline{\tau}^{(q)} \subset \dot{\sigma}^{(n)}$. Since $\dot{\sigma}^{(n)} \subset X^{n-1}$, we know that $\tau^{(q)} \cap \dot{\sigma}^{(n)} \neq \emptyset$ implies $q \leq n-1$. We will prove the statement by descending induction on q.

Let τ be an $(n-1)$-cell of X such that $\tau \cap \dot{\sigma}^{(n)} \neq \emptyset$. Then $\tau \cap \dot{\sigma}^{(n)}$ is closed in τ. Since X is a regular CW-complex, $\dot{\sigma}^{(n)}$ is homeomorphic to an $(n-1)$-sphere $\mathbf{S}^{n-1}$. Also τ is homeomorphic to the Euclidean space $\mathbb{R}^{n-1}$. By invariance of domain the open subset $\tau \cap \dot{\sigma}^{(n)}$ of $\dot{\sigma}^{(n)}$ is homeomorphic to an open subset of τ. Thus $\tau \cap \dot{\sigma}^{(n)}$ is both open and closed in the connected space τ, and since $\tau \cap \dot{\sigma}^{(n)} \neq \emptyset$ we must have $\tau \subset \dot{\sigma}^{(n)}$. Finally, since $\dot{\sigma}^{(n)}$ is closed in X, we have $\overline{\tau} \subset \dot{\sigma}^{(n)}$.
Now let

$$C = \bigcup_{\substack{\tau \text{ is an } (n-1)\text{-cell of } X \\ \overline{\tau} \subset \dot{\sigma}^{(n)}}} \overline{\tau} \, .$$

We have shown that $\dot{\sigma}^{(n)} \subset C \cup X^{n-2}$. Suppose that $\dot{\sigma}^{(n)} \subset C \cup X^q$, where $q \leq (n-2)$, and suppose that $\tau^{(q)} \subset X^q - C$. Since $\tau^{(q)}$ is open in X^q, $\tau^{(q)}$ is open in $X^q - C$, which

is open in $X^q \cup C$, so that $\tau^{(q)}$ is open in $X^q \cup C$. By hypothesis, $\dot{\sigma}^{(n)} \subset C \cup X^q$, so that $\tau^{(q)} \cap \dot{\sigma}^{(n)}$ is open in $\dot{\sigma}^{(n)}$. But $q \leq (n-2)$, $\tau^{(q)}$ is homeomorphic to $\mathbb{R}^d$, and $\dot{\sigma}^{(n)}$ is homeomorphic to $\mathbf{S}^{n-1}$. By Corollary 5 we must have $\tau^{(q)} \cap \dot{\sigma}^{(n)} = \emptyset$. Thus $\dot{\sigma}^{(n)} \subset C \cup X^{q-1}$ and, by induction, $\dot{\sigma}^{(n)} \subset C \cup X^{-1} = C$ (with the convention $X^{-1} = \emptyset$) so that we have $\dot{\sigma}^{(n)} = C$. Also, if τ is a q-cell of X such that $\tau \cap \dot{\sigma}^{(n)} \neq \emptyset$, then $\overline{\tau} \subset C = \dot{\sigma}^{(n)}$, and $\dot{\sigma}^{(n)}$ is a sub-complex of X. □

Remark. The proof of theorem 31 show that if τ has dimension $p+1$ and M is a regular CW complex then the p-cells are dense in $\overline{\tau} - \tau$, i.e.

$$\overline{\bigcup_{\sigma^{(p)} < \tau} \sigma} = \overline{\tau} - \tau$$

Let M be a CW-complex and let $\mathbf{K}$ denote the set of (open) cells of M, with $\mathbf{K}_p$ the cells of dimension p. The notation $\sigma^{(p)}$ will indicate that σ is a p dimensional cell. We define a relationship between cells as follows. We write $\tau > \sigma$ (or $\sigma < \tau$) if $\tau \neq \sigma$ and $\sigma \subset \overline{\tau}$, where $\overline{\tau}$ is the closure of τ. In this case we say σ is a *face* of τ. We write $\sigma \leq \tau$ if either $\tau = \sigma$ or $\sigma < \tau$.
Suppose that $\sigma^{(p)}$ is a face of $\tau^{(p+1)}$. Let $\mathbf{B}$ be a closed Euclidean ball with dimension $p+1$, and let $h : \mathbf{B} \to M$ be the characteristic map for τ.

Definition 61 *The cell $\sigma^{(p)}$ is a* regular face *of $\tau^{(p+1)}$ if*

1. *the restriction $h : h^{-1}(\sigma) \to \sigma$ is an homeomorphism,*
2. *$\overline{h^{-1}(\sigma)}$ is a closed p-ball.*

We say that σ is an irregular face *of τ otherwise.*

Lemma 10 *A CW-complex is regular if and only if any face is a regular face.*

Proof. Any cell is a face for itself, hence if all faces are regular we obtain a regular CW-complex.

Suppose that we have a regular CW-complex. Let $\sigma^{(p)}$ be a face of $\tau^{(p+1)}$ and let h be a characteristic map for τ. Since h is an homeomorphism and $\sigma \subset \overline{\tau}$, we get $h : h^{-1}(\sigma) \to \sigma$ which is an homeomorphism for the induce topology. We also saw that $\overline{\sigma} \subset \overline{\tau}$ and h is an homeomorphism so

$$h^{-1}(\overline{\sigma}) \xrightarrow{\simeq} \overline{\sigma} \xrightarrow{\simeq} \mathbf{B}^p.$$

Finally σ is a regular face of τ. □

2.2.4 The Cellular Chain Complex

Let M be a CW-complex with oriented cells. Then we consider the cellular chain groups $C_n(M, \mathbb{Z})$ freely generated by the oriented n-cells. The relation between these groups and homology is as follows [7, chap.2].

a) $C_n(X) \simeq H_n(M^n, M^{n-1}; \mathbb{Z})$;
b) Let $\partial_n : C_n(M, \mathbb{Z}) \to C_{n-1}(M, \mathbb{Z})$ be the homomorphism defined by using the connecting operator in the homology exact sequence of the triad (M^n, M^{n-1}, M^{n-2}), then $H_*(M)$ is isomorphic to the homology of the cellular chain complex $(C_*(M), \partial)$.

The boundary operator operator can be written in the basis.

$$\partial\tau^{(n)} = \sum_{(n-1)\text{-cells } \sigma} < \tau^{(n)} : \sigma > \sigma \ .$$

The coefficients $< \tau^{(n)} : \sigma >$ are called the incidence numbers. In the case where the restriction of the characteristic map

$$\phi_| : \phi^{-1}(\sigma) \to \sigma \ ,$$

has a regular value y, then we have

$$< \tau^{(n)} : \sigma >= \sum_{\phi(x)=y} deg_x(\phi_|) \ .$$

In particular, if σ is a regular face, then

$$< \partial\tau, \sigma >= \pm 1 \ . \tag{2.1}$$

We will deduce the following property for CW-complexes:

Theorem 32 *Suppose $\tau^{(p+1)} > \sigma^{(p)} > \nu^{(p-1)}$ then one of the following properly is true.*

(i) σ is an irregular face of τ.

(ii) ν is an irregular face of σ.

(iii) There is a p-cell $\widetilde{\sigma} \neq \sigma$ such that $\tau > \widetilde{\sigma} > \nu$.

Proof. Suppose neither (i) nor (ii) is true. Choose an orientation for each cell of M. Since σ is a regular face of τ, (2.1) holds so that

$$\partial\tau = \pm\sigma + \sum_{\substack{\widetilde{\sigma}^{(p)} \neq \sigma \\ \widetilde{\sigma} < \tau}} c_{\widetilde{\sigma}} \widetilde{\sigma}$$

where $c_{\widetilde{\sigma}} \in \mathbb{Z}$. In the same way, since ν is a regular face of σ,

$$\partial\sigma = \pm\nu + \sum_{\widetilde{\nu} \neq \nu, \ \nu < \sigma} c_{\widetilde{\nu}} \widetilde{\nu}.$$

Therefore,

$$0 = \partial^2\tau = \pm\partial\sigma + \sum_{\tau>\widetilde{\sigma}\neq\sigma} c_\sigma \partial\widetilde{\sigma} = \mp\nu + \sum_{\tau>\widetilde{\sigma}\neq\sigma} c_{\widetilde{\sigma}}\partial\widetilde{\sigma} \pm \sum_{\widetilde{\nu}\neq\nu,\ \nu<\sigma} c_{\widetilde{\nu}}\widetilde{\nu}.$$

For this equation to hold, there must be some $\widetilde{\sigma}$, with $\tau > \widetilde{\sigma} \neq \sigma$ satisfying

$$\partial\widetilde{\sigma} = c\nu + (\text{sum de } (p-1)\text{-cells other than } \nu)$$

with $c \neq 0$ (we look at the the coefficient of $\widetilde{\sigma}$ corresponding to ν). So $\widetilde{\sigma} > \nu$ and then $\tau > \widetilde{\sigma} > \nu$ as desired. □

2.2.5 Collapse for CW-complexes

Definition 62 *Let M be a CW-complex, N a sub-complex and $\sigma^{(p)} < \tau^{(p+1)}$ be two cells of M which satisfy*

(i) σ is a regular face of τ;

(ii) $N = M - (\sigma \cup \tau)$.

We say that M collapse onto *N; σ is called a* free face *of τ.*
More generally, we say that M collapse onto a sub-complex N if N is obtained from M by a finite sequence of such operations. We write $M \searrow N$.

Wee have the following property.

Theorem 33 *Let M be a CW-complex. If $M \searrow N$ then N is a deformation retract of M.*

Proof. We consider σ and τ as in definition 62. We denote by h the characteristic map for τ. Since σ is a regular face of τ, we have $\overline{h^{-1}(\sigma)} \subset \dot{E}^{p+1}$ and $\overline{h^{-1}(\sigma)}$ is homeomorphic to E^p. We leave as an exercise to show that $\dot{E}^{p+1} - h^{-1}(\sigma)$ is a deformation retract of $E^{p+1} - h^{-1}(\sigma)$. The statement follows. □
Example. Here are two illustrations of collapsings (figures 2.2 and 2.3).

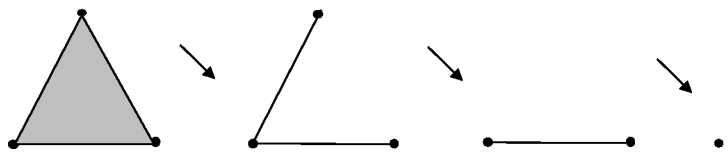

Figure 2.2: Collapsing a 2-simplex onto a vertex

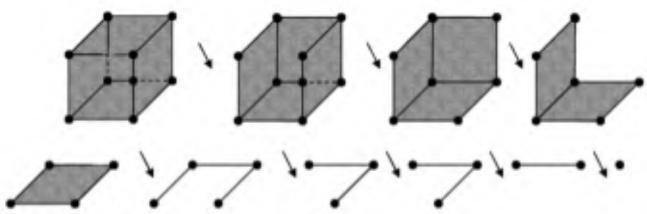

Figure 2.3: Collapsing the boundary of the cube but the upper face

Theorem 34 *Suppose M is a regular CW-complex, and for some p and $r \geq 1$ we have $\tau^{(p+r)} > \nu^{(p-1)}$. Then there are two $(p+r-1)$-cells $\sigma^{(p+r-1)}$ and $\widetilde{\sigma}^{(p+r-1)}$ such that $\widetilde{\sigma} \neq \sigma$ and*

$$\tau > \sigma > \nu, \qquad \tau > \widetilde{\sigma} > \nu.$$

Proof. The proof is by induction on r. Suppose $r = 1$, that is $\tau^{(p+1)} > \sigma^{(p-1)}$. Since M is regular, the p-cells in $\overline{\tau} - \tau$ are dense in $\overline{\tau} - \tau$ (remark 2.2.3), i.e.

$$\overline{\bigcup_{\sigma^{(p)} < \tau} \sigma} = \overline{\tau} - \tau.$$

Thus there exists a p-cell $\sigma^{(p)}$ with $\tau > \sigma$ such that $\sigma > \nu$. Theorem 32 guarantees the existence of $\widetilde{\sigma}^{(p)} \neq \sigma$ such that $\tau > \widetilde{\sigma} > \nu$.
For general r, we have again that the $(p+r-1)$-cells are dense in $\overline{\tau} - \tau$. Thus we can find a $(p+r-1)$-cell σ with $\tau > \sigma > \nu$. Continuing this way, we can find a $(p+r-2)$-cell $\widetilde{\nu}$ such that $\sigma > \widetilde{\nu} > \nu$. Applying theorem 32 to the triple $\tau > \sigma > \widetilde{\nu}$ we learn that there is a $(p+r-1)$-cell $\widetilde{\sigma} \neq \sigma$ such that $\tau > \widetilde{\sigma} > \widetilde{\nu}$. Then, the cells σ and $\widetilde{\sigma}$ satisfy the required properties. □

2.3 Discrete Morse Theory

In this section, we give a basic exposition of Robin Forman's discrete Morse theory [6]. We will mainly consider the case of regular CW-complexes. Note that simplicial complexes are regular CW-complexes. We will obtain the fundamental theorems of Morse theory 35, 36 and deduce Morse inequalities. We will also study the dynamics of discrete gradient vector fields and obtain a description of the Morse complex.

2.3.1 Discrete Morse function

In this section, we introduce the main definitions. Let M be a finite CW-complex. We denote by K the set of cells, and by K_p the set of p-dimensional cells.

Definition 63 *A* discrete Morse function *on M is a function $f : K \to \mathbb{R}$ satisfying for all $\sigma \in K_p$*

(i) If σ is an irregular face of $\tau^{(p+1)}$ then $f(\tau) > f(\sigma)$. Moreover,

$$\#\{\tau^{(p+1)} > \sigma | f(\tau) \leq f(\sigma)\} \leq 1.$$

(ii) If $\nu^{(p-1)}$ is an irregular face of σ then $f(\nu) < f(\sigma)$. Moreover,

$$\#\{\nu^{(p-1)} < \sigma | f(\nu) \geq f(\sigma)\} \leq 1.$$

Definition 64 *Let f a discrete Morse function on M. We say $\sigma \in K_p$ is a* critical point of index p *if*

(i) $\#\{\tau^{(p+1)} > \sigma | f(\tau) \leq f(\sigma)\} = 0.$

(ii) $\#\{\nu^{(p-1)} < \sigma | f(\nu) \geq f(\sigma)\} = 0.$

Example. Consider the circle decomposed in 3 0-cells and 3-1-cells, and the fonctions depicted in figure 2.4.

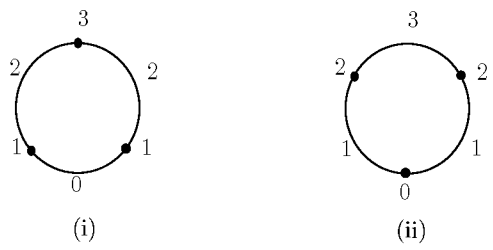

Figure 2.4: Two discrete functions on a CW-complex

On one hand, we see that the function in figure 2.4 (i) is not a discrete Morse function. More precisely the edge $f^{-1}(0)$ violates rule (ii) of definition 63, and the vertex $f^{-1}(3)$ violates rule (i) of definition 63. On the other hand,the function of figure 2.4 (ii) is a discrete Morse function.

Example. definitions 63 and 64 imply that if M is a regular CW-complex, then the minimum of f must occur on a vertex, which must be a critical point of index 0.
Indeed, if $p \geq 1$, then each p-cell has at least 2 $(p-1)$-faces (consequence of theorem 34).

It follows from definition 64 that a p-cell σ is not a critical cell if and only if either of the following conditions holds

(i) $\exists \tau^{(p+1)} > \sigma$ such that $f(\tau) \leq f(\sigma)$.

(ii) $\exists \nu^{(p-1)} < \sigma$ such that $f(\nu) \geq f(\sigma)$.

Lemma 11 *Statements (i) and (ii) cannot hold simultaneously.*

Proof. Condition (ii) requires $p \geq 1$ which we now assume.
Suppose (i) is true. Then σ must be a regular face of τ (since f respect partial order on irregular faces). From condition (i) of definition 63, if $\widetilde{\sigma} \neq \sigma$ is another p-face of τ, we must have

$$f(\widetilde{\sigma}) < f(\tau). \tag{2.2}$$

Therefore $f(\widetilde{\sigma}) < f(\sigma)$
Now suppose condition (ii) is true. Then ν must also be a regular face of σ. By theorem 32, there exists a p-cell $\widetilde{\sigma} \neq \sigma$ such that $\tau > \widetilde{\sigma} > \nu$.
From condition (ii) of definition 63, $f(\nu)$ cannot be $\geq$ both $f(\sigma)$ and $f(\widetilde{\sigma})$. Thus $f(\nu) < f(\widetilde{\sigma})$. Combining this with (2.2) we obtain

$$f(\sigma) \leq f(\nu) < f(\widetilde{\sigma}) < f(\tau) \leq f(\sigma)$$

which is a contradiction. □

2.3.2 Morse theorems for regular CW-complexes

In this section, we will prove Morse theorems in the case of finite regular CW-complexes. In the whole section, M will be a finite regular CW-complex and f a discrete Morse function on M.

Definition 65 *For $c \in \mathbb{R}$, we define*

$$M(c) = \bigcup_{\substack{\sigma \in K \\ f(\sigma) \leq c}} \bigcup_{\tau \leq \sigma} \tau.$$

That is $M(c)$ is the set of all cells on which f is $\leq c$, together with all their faces. In particular M is a sub-complex of M.
The lemma below gives a criterion in order to detect if some σ with $f(\sigma) > c$ lies in $M(c)$. A priori, one has to find τ with $\sigma < \tau$ and $f(\tau) \leq c$. The lemma says that considering only those τ with $dim\ \tau = dim\ \sigma + 1$ is enough.

Lemma 12 *Let σ be a p-cell of M and suppose $\tau > \sigma$. Then there exists a $(p+1)$-cell $\tilde{\tau}$ with $\sigma < \tilde{\tau} \leq \tau$ and $f(\tilde{\tau}) \leq f(\tau)$.*

Proof. We prove this by induction on the dimension of τ.
Since $\tau > \sigma$, we have $dim\ \tau > dim\ \sigma$. If $dim\ \tau = p+1$, we can consider $\tilde{\tau} = \tau$.
Suppose

$$dim\ \tau = p + r, \qquad r > 1$$

by theorem 34, there exists $(p+r-1)$-cells ν_1 and ν_2 satisfying $\tau > \nu_1 > \sigma$ and $\tau > \nu_2 > \sigma$. Condition (ii) of definition 63 gives either $f(\nu_1) < f(\tau)$ or $f(\nu_2) < f(\tau)$. By induction on one of this two cells for which the inequality is true we obtain the result. □

Theorem 35 *Let M be a finite regular CW-complex and f a discrete Morse function on M. If $a < b$ are real numbers such that $[a, b]$ contains no critical values of f, then $M(b) \searrow M(a)$.*

Proof. First of all, we will demonstrate that we can perturb f to obtain a $1-1$ function on K. If $\tau^{(p+1)} > \sigma^{(p)}$ satisfies $f(\tau) \leq f(\sigma)$ then we perturb f by replacing $f(\tau)$ by $f(\tau) - \varepsilon$, or $f(\sigma)$ by $f(\sigma) + \varepsilon$ for $\varepsilon \geq 0$ small enough, without changing which cells are critical. In the same way if $\sigma^{(p)}$ satisfies $f(\tau^{(p+1)}) \neq f(\sigma) \neq f(\nu^{(p-1)})$ for each $\tau^{(p+1)} > \sigma > \nu^{(p-1)}$, then we may perturb f by changing $f(\sigma)$ by $f(\sigma) \pm \varepsilon$, for $\varepsilon \leq 0$ small enough, without changing which cells are critical. Perturbing f this way, we don't change neither $M(a)$, nor $M(b)$. Thus

$$f : K \to \mathbb{R}$$

is $1-1$.
If $f^{-1}([a, b]) = \emptyset$ Then $M(a) = M(b)$ and there is nothing to prove. Otherwise, by partitioning $[a, b]$ into smaller intervals if necessary, we may assume there is a single non-critical cell σ with

$$f(\sigma) \in [a, b].$$

By lemma 11, exactly one of the following holds:

(i) $\exists \tau^{(p+1)} > \sigma$ with $f(\tau) \leq f(\sigma)$.

(ii) $\exists \nu^{(p-1)} < \sigma$ with $f(\nu) \geq f(\sigma)$.

In case (i), we must have $f(\tau) < a$ (for τ and σ are two non-critical cells in $[a, b]$). Thus $\tau \subseteq M(a)$. Since σ is a face of τ we have $\sigma \subseteq M(a)$ and consequently

$$M(a) = M(b).$$

Again, there is nothing to prove.

Suppose (ii) is true. Lemma 11 tells us that $\forall \tau^{(p+1)} > \sigma$, $f(\tau) > f(\sigma)$. In particular, $f(\tau) > b$ (otherwise σ and τ are in $[a, b]$ which contradicts the hypothesis). By lemma 12, we have $\forall \tau > \sigma$, $f(\tau) > b$. Thus

$$\sigma \cap M(a) = \emptyset.$$

Since (ii) is true, there exists $\nu^{(p-1)} < \sigma$ with $f(\nu) > f(\sigma)$, and in particular $f(\nu) > b$. If $\widetilde{\nu}^{(p-1)} \neq \nu$ is another $(p-1)$-face of σ, then $f(\widetilde{\nu}) < f(\sigma)$ (cf definition 63 (ii)). Therefore

$$f(\widetilde{\nu}) < a.$$

Thus $\widetilde{\nu}$ and all its faces are contained in $M(a)$.

Let $\widetilde{\sigma}^{(p)} \neq \sigma$ be another p-face of M with $\widetilde{\sigma} > \nu$. Condition (i) of definition 63 implies

$$f(\widetilde{\sigma}) > f(\nu) > b.$$

By lemma 12, if $\widetilde{\sigma}$ is any cell such that $\widetilde{\sigma} > \nu$, thus $f(\widetilde{\sigma}) > b$. So

$$\nu \cap M(a) = \emptyset.$$

It follows that $M(b)$ can be expressed as a disjoint union

$$M(b) = M(a) \cup \sigma \cup \nu,$$

where ν is a free face of σ. Finally $M(b) \searrow M(a)$. □

Theorem 36 *Let M be a finite regular CW-complex and f a discrete Morse function on M. Suppose $\sigma^{(p)}$ is a critical cell of index p with*

$$f(\sigma) \in [a, b]$$

and $f^{-1}([a, b])$ contains no other critical points. Then $M(b)$ is homotopy equivalent to

$$M(a) \bigcup_{\dot{e}^p} e^p,$$

where e^p denotes a p-dimensional cell with boundary $\dot{e}^p$.

Proof. As in proof of theorem 35 we may assume f is $1-1$. Thus we can find a' and b' with $a \leq a' < b' \leq b$ with

$$\sigma = f^{-1}([a', b']).$$

By theorem 35 $M(b) \searrow M(b')$ and $M(a') \searrow M(a)$, so it suffices to prove that $M(b')$ homotopy equivalent to

$$M(a') \bigcup_{\dot{e}^p} e^p.$$

Since σ is a critical cell, if $\tau^{(p+1)} > \sigma$ then $f(\tau) > f(\sigma)$ and so $f(\tau) > b'$. Lemma 12 tells us that if τ is a cell of M such that $\tau > \sigma$ then $f(\tau) > b'$. Therefore

$$\sigma \cap M(a') = \emptyset.$$

Since σ is a critical cell of M, then for any cell $\nu^{(p-1)} < \sigma$ we have $f(\nu) < f(\sigma)$ and so

$$f(\nu) < a'.$$

Consequently $\nu \subseteq M(a')$ and $\dot{\sigma} \subseteq M(a')$. Therefore

$$M(b') = M(a') \bigcup_{\dot{\sigma}} \sigma.$$

Since σ is homeomorphic to e^p, the proof is complete. □

Let $m_p(f)$ (or simply m_p if it will not cause confusion) be the number of critical cells of index p. The m_p's are called the *Morse number of* f. A consequence of theorems 35 and 36 is

Corollary 6 *Let M be a finite regular CW-complex and f a discrete Morse function on M. Then M is homotopy equivalent to a CW-complex with exactly $m_p(f)$ cells of dimension p.*

2.3.3 Morse Inequalities

References for this sections are [12] (p.28 to p.31) for the theorem and [5] (chap.I) for results on homology.
We first need some definitions and preliminary results.

Definition 66 *Let S be a function from certain pairs of spaces to the integers. S is* sub-additive *(respectively* additive*) if $\forall\ X \supset Y \supset Z$ we have $S(X,Z) \leq S(X,Y) + S(Y,Z)$ (respectively $=$).*

Here are two important examples:

Example 1. $b_\lambda(X,Y) = \lambda$th *Betti number* of (X,Y) = rank over a field $\mathbf{F}$ of $H_\lambda(X,Y)$, for any pair (X,Y) such that this rank is finite.
b_λ is sub-additive by examining the following portion of the exact sequence for the triple $X \supset Y \supset Z$

$$\ldots \to H_\lambda(Y,Z) \to H_\lambda(X,Z) \to H_\lambda(X,Y) \to \ldots$$

(for a proof see theorem 10.2 p.25 de [5]).

Example 2. If the integers $(b_\lambda(X,Y))$ are all zero but a finite number, then we define the *Euler characteristic* by

$$\chi(X,Y) = \sum_{\lambda=0}^{+\infty} (-1)^\lambda b_\lambda(X,Y).$$

The Euler characteristic is additive. As a matter of fact, if two at least of following Euler characteristic are defined, then the third is, and

$$\chi(X) = \chi(Y) + \chi(X, Y).$$

This comes from the exactness axiom (cf Axiom 4 p.11 [5]) and the fact that the exact sequence is finite.

Lemma 13 *Let S be sub-additive and $X_0 \subset X_1 \subset \ldots \subset X_n$. Then*

$$S(X_n, X_0) \leq \sum_{i=1}^{n} S(X_i, X_{i-1}).$$

If S is additive then the equality holds.

Proof. We prove this by induction on n. For $n = 1$, we have the equality and if $n = 2$, it's only the definition of sub-additivity.
If the result is true for $n-1$, then $S(X_{n-1}, X_0) \leq \sum_{i=1}^{n-1} S(X_i, X_{i-1})$. Therefore,

$$S(X_n, X_0) \leq S(X_{n-1}, X_0) + S(X_n, X_{n-1}) \leq \sum_{i=1}^{n} S(X_i, X_{i-1})$$

and the result is true for n.
We prove the result the same way for S additive. □

Let $S(X, \emptyset) = S(X)$. Taking $X_0 = \emptyset$ in lemma 13 we get

$$S(X_n) \leq \sum_{i=1}^{n} S(X_i, X_{i-1}) \tag{2.3}$$

with equality if S is additive.

Let M be a finite regular CW-complex and f a discrete Morse function on K with isolated critical points. Let $a_1 < \ldots < a_k$ be such that M^{a_i} contains exactly i critical cells, and $M^{a_k} = M$. By theorem 36 we have

$$\begin{aligned} H_*(M^{a_i}, M^{a_i-1}) &= H_*(M^{a_i-1} \cup e^{\lambda_i}, M^{a_i-1}) \\ &= H_*(e^{\lambda_i}, \dot{e}^{\lambda_i}) \\ &= \begin{cases} \text{coefficient group in dimension } \lambda_i \text{ i.e. } \mathbf{F} \\ 0 \text{ otherwise} \end{cases} \end{aligned}$$

where λ_i is the index of the critical cell (for a proof see theorem 16.4 p.45 [5] and excision axiom for a homology).

Applying (2.3) to $\emptyset = M^{a_0} \subset \ldots M^{a_k} = M$ with $S = b_\lambda$ we have

$$b_\lambda(M) \leq \sum_{i=1}^{k} b_\lambda(M^{a_i}, M^{a_{i-1}}) = m_\lambda;$$

where m_λ is the Morse number of index λ (as a matter of fact $dim_{\mathbf{F}} H_{\lambda_i}(e^{\lambda_i}, \dot{e}^{\lambda_i}) = 1$). Applying this formula to the case $S = \chi$ we obtain

$$\chi(M) = \sum_{i=1}^{k} \chi(M^{a_i}, M^{a_{i-1}}) = m_0 - m_1 + m_2 - \ldots \pm m_{dim\ M}$$

Thus we have proven:

Theorem 37 (Weak Morse Inequalities) *Let M be a finite regular CW-complex and f a discrete Morse function on M. If m_λ denotes the Morse number of index λ of f and b_λ the λth Betti number, then*

$$b_\lambda(M) \leq m_\lambda \tag{2.4}$$

$$\sum_{\lambda=0}^{+\infty} (-1)^\lambda b_\lambda(M) = \sum_{\lambda=0}^{+\infty} (-1)^\lambda m_\lambda \tag{2.5}$$

Slightly sharper inequalities can be proven by the following argument.

Lemma 14 *The function S_λ is sub-additive where*

$$S_\lambda(X,Y) = b_\lambda(X,Y) - b_{\lambda-1}(X,Y) + b_{\lambda-2}(X,Y) - \ldots \pm b_0(X,Y)$$

Proof. Given an exact sequence of vector space

$$\xrightarrow{h} A \xrightarrow{i} B \xrightarrow{j} C \xrightarrow{k} \ldots \longrightarrow D \longrightarrow 0$$

Then the rank of h plus the rank of i is equal to the rank of A. Therefore,

$$\begin{aligned} rank(h) &= rank(A) - rank(i) \\ &= rank(A) - rank(B) + rank(j) \\ &= rank(A) - rank(B) + rank(C) - rank(k) \\ &\ldots \\ &= rank(A) - rank(B) + rank(C) - \ldots \pm rank(D). \end{aligned}$$

Hence the last expression is ≥ 0. Now consider the homology exact sequence of a triple $X \supset Y \supset Z$. Then with

$$H_{\lambda+1}(X,Y) \xrightarrow{\partial} H_\lambda(Y,Z)$$

we obtain

$$rank(\partial) = b_\lambda(Y,Z) - b_\lambda(X,Z) + b_\lambda(X,Y) - b_{\lambda-1}(Y,Z) + \ldots \geq 0.$$

Collecting terms, $S_\lambda(Y,Z) - S_\lambda(X,Z) + S_\lambda(X,Y) \geq 0$, which complete the proof. □

Applying this sub-additive function to the spaces

$$\emptyset \subset M^{a_1} \subset \ldots \subset M^{a_k}$$

we obtain:

Theorem 38 (Strong Morse Inequalities) *Let M be a finite regular CW-complex and f a discrete Morse function on M. If m_λ denotes the Morse number of index λ of f and b_λ the λth Betti number, then*

$$\forall\ \lambda \geq 0, \qquad b_\lambda(M) - b_{\lambda-1}(M) + \ldots \pm b_0(M) \leq m_\lambda - m_{\lambda-1} + \ldots \pm m_0. \tag{2.6}$$

Remark. These inequalities are sharper than the previous ones.
In fact, adding (2.6) for λ and $\lambda - 1$ we obtain (2.4).
Comparing (2.6) for λ and $\lambda - 1$ where $\lambda > dim(M)$ we obtain (2.5).

2.3.4 Examples of Discrete Morse Functions

Every CW-complex M has a discrete Morse function. For example, define a Morse function f by setting, for each $\sigma \in K$,

$$f(\sigma) = dim(\sigma).$$

We see that every cell is critical. Corollary 6 in the case of regular CW-complexes tells us that a CW-complex with m_p faces of dimension p is homotopy equivalent to a CW complex with m_p cells of dimension p.

We now examine ways in which a Morse function on one CW-complex may induce a Morse function on another CW-complex

Lemma 15 *Let M be a CW-complex and $N \subseteq M$ be a sub-complex. Then any discrete Morse function on M restricts to a discrete Morse function on N. If $\sigma \subseteq N$ is a critical cell for f (the original function), then it is critical for the restriction.*

Proof. This follows directly from definitions 63 and 64. □

Here is a lemma which gives us a converse of the previous lemma.

Lemma 16 *Let M be a (finite) CW-complex and $N \subseteq M$ be a sub-complex. Then any discrete Morse function on N can be extended to a discrete Morse function on M i.e. if f is a discrete Morse function on N, then there is a discrete Morse function g on M such that*

$$\forall \sigma \subseteq N,\ g(\sigma) = f(\sigma).$$

Proof. Let $c = max_{\sigma \subseteq N} f(\sigma)$. Define g on M by setting,

$$g(\sigma) = \begin{cases} f(\sigma) & si\ \sigma \subseteq N \\ c + dim(\sigma) & si\ \sigma \nsubseteq N \end{cases}$$

One easily check that g is a discrete Morse function on M that extends f. □

The Morse function constructed in the above proof may be very insufficient. In particular, every cell of $M - N$ is a critical cell. There may exist extensions to M with many fewer critical cells. In the following lemma, we will see it is possible when M collapses onto N.

Lemma 17 *Let M be a (finite) CW-complex and $N \subseteq M$ be a sub-complex such that $M \searrow N$. Let f be a discrete Morse function on N and $c = max_{\sigma \subseteq N} f(\sigma)$. Then f can be extended to a Morse function on M with*

$$N = M(c)$$

and such there are no critical cells in $M - N$.

Proof. The proof is by induction on the number of elementary collapses and so it suffices to prove this when M collapses onto N by a single elementary collapse. Supposes τ is a cell of M with a free face $\sigma < \tau$ such that M is a disjoint union

$$M = N \cup \sigma \cup \tau.$$

Define a Morse function g on M by

$$\begin{array}{rcl} g(\nu) & = & f(\nu), \quad \nu \neq \sigma, \tau \\ g(\tau) & = & c + 1 \\ g(\sigma) & = & c + 2 \end{array}$$

Then g satisfies the required properties. □

Corollary 7 *Let Δ^n be a n-simplex with its standard triangulation, and $\dot{\Delta}^n$ its boundary. Then*

(i) Δ^n has a Morse function with exactly 1 critical point.

(ii) $\dot{\Delta}^n$ has a Morse function with exactly 2 critical points.

Proof. (i) follows from lemma 17 and the fact that Δ^n collapses onto one of its vertex.
(ii) follows from lemmas 16 and 17, and also the fact that for any $(n-1)$-cell σ of $\dot{\Delta}^n$, $\dot{\Delta}^n - \sigma$ collapses onto one of its vertex. □

2.3.5 The discrete gradient vector field and the associated flow

We will define in a natural way the notion of discrete gradient vector fields thanks to the discrete Morse function. To do this, we will draw one's inspiration from the definition of gradient vector field on smooth manifold.

Let M be a general CW-complex (not necessary regular) and f be a discrete Morse function on K the open cells of M.

We try to define an object $V_f : K \to K$ which is a discrete analog of $-\nabla f$ the gradient vector field, and also Φ_f the associated flow. For now, we assume f a discrete Morse function has been fixed, and we write V and Φ (we forget the suffix f).

To give a natural definition of the discrete vector field, we first consider the vertices of M. Let $v \in K_0$. If v is a critical cell, we wish its value to be zero as in smooth Morse theory. So we set

$$V(v) = 0.$$

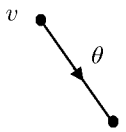

Figure 2.5: Direction of the gradient vector field

If v is not a critical cell, then there is a unique edge $e > v$ with $f(e) \leq f(v)$. The edge e specifies the unique direction in which f is not increasing (cf Fig. 2.5). So $-\nabla f(v)$ should be e. We must now precise this. We wish to think of $V(v)$ as a discrete tangent vector field pointing away from v. That is $V(v)$ is e with a chosen orientation.

Thus we introduce the chain complex of M. Fix an orientation for each cell $\sigma \in K$. Let $C_i(M, \mathbb{Z})$ be the free abelian group generated over $\mathbb{Z}$ by these oriented cells of dimension i of M. We identify $-\sigma$ with σ given with the opposite orientation. Let ∂ denote the usual boundary operator.

$$\partial : C_p(m, \mathbb{Z}) \to C_{p-1}(M, \mathbb{Z}).$$

Then

$$\partial\sigma = \sum_{\nu^{(p-1)} < \sigma} \epsilon(\sigma, \nu)\nu,$$

where the ϵ's are integers called the incidence numbers.
It is convenient to introduce an inner product $<,>$ on C_* by declaring the cells of M to be an orthonormal basis. Thus we obtain

$$\partial\sigma = \sum_{\nu^{(p-1)} < \sigma} < \partial\sigma, \nu > \nu.$$

Thanks to this, we complete our definition of the discrete gradient vector field on the vertices. If $v \in K_0$ is not a critical cell, and $e \in K_1$ verifies $e > v$, $f(e) \leq f(v)$, we set

$V(v) = \pm e$, where the sign is determined so that

$$< \partial(V(v)), v >= -1.$$

now we define the (discrete time) flow Φ also in a natural way. If v is a critical cell, then $V(v) = 0$ and so v is fixed under the gradient flow, i.e.

$$\Phi(v) = v.$$

If v is not critical, and $V(v) = \pm e$, then v should flow to the "other end" of e. That is,

$$\Phi(v) = v + \partial(V(v)).$$

Note that this formula holds for all vertices, whether critical or not. Moreover, if the cell is not critical, then it has actually moved since $< \Phi(v), v >= 0$.

We extend V and Φ linearly to maps on chains

$$V : C_0(M, \mathbb{Z}) \longrightarrow C_1(M, \mathbb{Z})$$

$$\Phi : C_0(M, \mathbb{Z}) \longrightarrow C_0(M, \mathbb{Z}).$$

We now extend V to higher dimension cells.

Definition 67 *Let σ be a p-cell de M (with a fixed orientation). If there is $\tau^{(p+1)} > \sigma$ with $f(\tau) \leq f(\sigma)$, we set*

$$V(\sigma) = - < \partial\tau, \sigma > \tau.$$

(note that σ must be a regular face of τ (for f respects the partial order on irregular faces of τ) and so $< \partial\tau, \sigma >= \pm 1$).
If there is no such τ, then we set

$$V(\sigma) = 0.$$

For each p, we extend V linearly to a map

$$V : C_p(M, \mathbb{Z}) \longrightarrow C_{p+1}(M, \mathbb{Z}).$$

In figure 2.6 we have $V(e_1) = 0$. Nevertheless e_1 must not be fixed under the flow Φ. The boundary of e_1 (i.e. its two vertices) "move downwards". This illustrates that there are a tangent and a transversal component of the gradient $-\nabla f(v)$. $V(\sigma)$ is the transversal component and $V(\partial\sigma)$ is the tangent one (here we catch the movement of the vertices).
Remark. Note that $V(\sigma) = 0$ without σ fixed under the flow. We have represented in figure 2.7 a discrete Morse function on the torus (opposite edges of the square are identified). The arrows indicate the cells with non zero gradient, together with the value of this gradient up to sign. The signs are fixed according to the choices of orientations of cells.
Remark. As a consequence of theorem 39 below, the gradient vector field, as defined above, will give a matching of non critical cells, corresponding to the non zero entries of the matrix for V. This matching is the *combinatorial gradient vector field.* More general (partial) matching of the incidence graph of a simplicial complex is called a *combinatorial vector field*; such a graph encodes interesting dynamics.

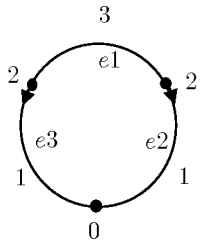

Figure 2.6: Cell for which the gradient is null without being fixed under the flow

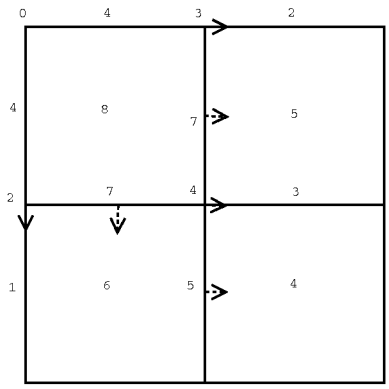

Figure 2.7: A discrete Morse function on the torus

Definition 68 *For any oriented face σ we define the (discrete time) gradient flow Φ by*

$$\Phi(\sigma) = \sigma + \partial V(\sigma) + V(\partial\sigma)$$

or, more succinctly,

$$\Phi = 1 + \partial V + V\partial.$$

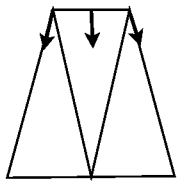

Figure 2.8: Gradient vector field of the upper edge

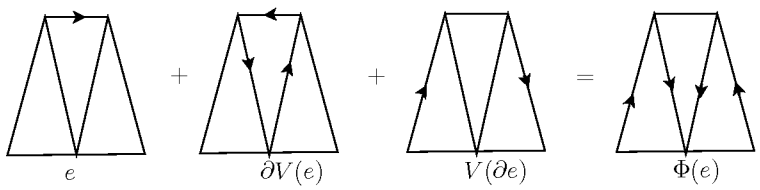

Figure 2.9: Expression of the flow $\Phi(e)$

Example Figures 2.8 and 2.9 illustrate the gradient vector field and the flow for a cell e oriented from the left to the right.

The main properties of V and Φ are contained in the following theorems.

Theorem 39 *(1)* $V \circ V = 0$.

(2) If σ is an oriented p-cell of M, then

$$\#\{\nu^{(p-1)} | V(\nu) = \pm\sigma\} \leq 1.$$

(3) If σ is an oriented p-cell of M, then

$$\sigma \text{ is critical} \Leftrightarrow [\sigma \notin Im(V) \text{ and } V(\sigma) = 0].$$

Proof. (1) If $V(\nu^{(p-1)}) = \pm\sigma^{(p)}$ then $\nu < \sigma$ and $f(\nu) \geq f(\sigma)$. By lemma 11, there is no $\tau > \sigma$ with $f(\tau) \leq f(\sigma)$. Thus $V(\sigma) = 0$.
(2) If $V(\nu^{(p-1)}) = \pm\sigma^{(p)}$ then $\nu < \sigma$ and $f(\nu) \geq f(\sigma)$. By condition (ii) of definition 63, ν is unique.
(3) From definition 64, σ is critical if and only if

(i) $\nexists\ \nu^{(p-1)} < \sigma,\ f(\nu) \geq f(\sigma)$ and

(ii) $\nexists\ \tau^{(p+1)} > \sigma,\ f(\tau) \leq f(\sigma)$.

These conditions are equivalent to

(i) $\nexists\ \nu^{(p-1)},\ V(\nu) = \pm\sigma$ and

(ii) $\nexists\ \tau^{(p+1)},\ V(\sigma) = \pm\tau$.

i.e.

(i) $\sigma \notin Im(V)$ and

(ii) $V(\sigma) = 0$.

□

Theorem 40 *(1)* $\Phi\partial = \partial\Phi$.
Let $\sigma_1, \ldots, \sigma_r$ denote the p-cells of M with a chosen orientation. Write

$$\Phi(\sigma_i) = \sum_j a_{ij}\sigma_j.$$

(2) $\forall\ i,\ a_{ii} = 0$ or 1, and $a_{ii} = 1$ if and only if σ_i is a critical cell.

(3) If $i \neq j$, then $a_{ij} \in \mathbb{Z}$. If $i \neq j$ and $a_{ij} \neq 0$ then $f(\sigma_j) < f(\sigma_i)$.

Proof. (1) Since $\Phi = 1 + \partial V + V\partial$, we have

$$\Phi\partial = (1 + \partial V + V\partial)\partial = \partial + \partial V\partial + V\partial^2 = \partial + \partial V\partial$$

$$\partial\Phi = \partial(1 + V\partial + \partial V) = \partial + \partial V\partial + \partial^2 V = \partial + \partial V\partial.$$

We prove (2) and (3) simultaneously. First, since ∂ and V both map integer chains to integer chains, each $a_{ij} \in \mathbb{Z}$.
By theorem 39, each cell σ satisfies exactly one of the following properties:

(i) σ is a critical cell

(ii) $\pm\sigma \in Im(V)$

(iii) $V(\sigma) \neq 0$.

We examine each case independently to prove the theorem.
(i) If σ is critical then $V(\sigma) = 0$, so

$$\Phi(\sigma) = \sigma + V(\partial\sigma) = \sigma + \sum_{\nu^{(p-1)} < \sigma} < \partial\sigma, \nu > V(\nu).$$

Since σ is critical, $\forall\ \nu^{(p-1)} < \sigma$, $f(\nu) < f(\sigma)$. For each such ν, either $V(\nu) = 0$ or $V(\nu) = \widetilde{\sigma}^{(p)}$ with

$$f(\widetilde{\sigma}) \leq f(\nu) < f(\sigma).$$

Thus $\Phi(\sigma) = \sigma + \sum a_{\widetilde{\sigma}}\widetilde{\sigma}$ where $a_{\widetilde{\sigma}} \neq 0 \Rightarrow f(\widetilde{\sigma}) < f(\sigma)$.
(ii) Suppose $\pm\sigma \in Im(V) \subseteq Ker(V)$ (and so $V(\sigma) = 0$). Then

$$\Phi(\sigma) = \sigma + V(\partial\sigma) = \sigma + \sum_{\nu^{(p-1)} < \sigma} < \partial\sigma, \nu > V(\nu).$$

By theorem 39 (2), there is exactly one $(p-1)$-cell $\widetilde{\nu} < \sigma$ with $V(\widetilde{\nu}) = \pm\sigma$ (for $\sigma \in Im(V)$) and $< \partial\sigma, \widetilde{\nu} > V(\widetilde{\nu}) = -\sigma$ (cf definition 67). Moreover, if $\widetilde{\nu} \neq \nu < \sigma$ then either $V(\nu) = 0$ or $V(\nu) = \widetilde{\sigma}$ with $f(\widetilde{\sigma}) \leq f(\nu) < f(\sigma)$. Thus

$$\Phi(\sigma) = \sum_{\widetilde{\sigma}^{(p)}} a_{\widetilde{\sigma}}\widetilde{\sigma},$$

oÃź $a_{\widetilde{\sigma}} \neq 0 \Rightarrow f(\widetilde{\sigma}) < f(\sigma)$.
(iii) Suppose $V(\sigma) = - < \partial\tau, \sigma > \tau \neq 0$. Then

$$\Phi(\sigma) = \sigma + \partial(V(\sigma)) + V(\partial\sigma).$$

Since $V(\sigma) \neq 0$, $\pm\sigma \notin Im(V)$. Thus $\forall\ \nu^{(p-1)} < \sigma$, either $V(\nu) = 0$ or $V(\nu) = \pm\widetilde{\sigma}$ with $f(\widetilde{\sigma}) \leq f(\nu) < f(\sigma)$. Moreover

$$\partial(V(\sigma)) = - < \partial\tau, \sigma > \partial\tau = - < \partial\tau, \sigma >^2 \sigma + \sum b_{\widetilde{\sigma}}\widetilde{\sigma} = -\sigma + \sum b_{\widetilde{\sigma}}\widetilde{\sigma}$$

oÃź $b_{\widetilde{\sigma}} \neq 0 \Rightarrow f(\widetilde{\sigma}) < f(\tau) \leq f(\sigma)$.
This complete the proof. □

One can say (by theorem 40) that Φ decreases f, and $\sigma \subseteq \Phi(\sigma)$ if and only if σ is critical. For example consider Fig. 2.6 then

$$\Phi(e_1) \supseteq e_1$$

$$\Phi(e_2) \subseteq e_2$$

$$\Phi(e_3) \subseteq e_3$$

More precisely,

$$\Phi(e_1) = e_1 + e_2 + e_3$$

$$\Phi(e_2) = \Phi(e_3) = 0.$$

2.3.6 The Morse Complex and Invariant Chains

Let M be a finite CW-complex. Let $C_p^{\Phi}(M,\mathbb{Z})$ denote the Φ-invariant p-chains of M, that is

$$C_p^{\Phi}(M,\mathbb{Z}) = \{c \in C_p(M,\mathbb{Z}) | \Phi(c) = c\}.$$

From theorem 40, the boundary operator ∂ maps C_p^{Φ} to C_{p-1}^{Φ}. Thus we have a differential complex

$$\mathcal{C}^{\Phi} : 0 \longrightarrow C_n^{\Phi}(M,\mathbb{Z}) \xrightarrow{\partial} C_{n-1}^{\Phi}(M,\mathbb{Z}) \xrightarrow{\partial} \ldots \xrightarrow{\partial} C_0^{\Phi}(M,\mathbb{Z}) \longrightarrow 0. \tag{2.7}$$

The complex $\mathcal{C}^{\Phi}$ is called the *Morse complex.*
The goal of this section is to prove that the homology of $\mathcal{C}^{\Phi}$ is precisely the cell homology of M. The first step is to investigate the stabilization map $\Phi^{\infty} : C_* \longrightarrow C_*$ given by

$$\Phi^{\infty} = \lim_{N\to\infty} \Phi^N$$

More precisely, we will prove that there is $N \in \mathbb{N}$ large enough such that $\Phi^{\infty} = \Phi^N$.

Lemma 18 *Let $c \in C_p^{\Phi}(M,\mathbb{Z})$ and write $c = \sum_{\sigma\in K_p} a_\sigma \sigma$. Let*

$$\sigma^* = \textit{any maximizer of } \{f(\sigma) | a_\sigma \neq 0\}.$$

Then σ^* is a critical cell of f.

Proof. Since c is Φ-invariant, $c = \Phi(c) = \sum_{\sigma\in K_p} a_\sigma \Phi(\sigma)$. Therefore,

$$a_{\sigma^*} =< c, \sigma^* >= \sum_{\sigma\in K_p} a_\sigma < \Phi(\sigma), \sigma^* > .$$

From theorem 40 (3), if $\sigma \neq \sigma^*$ et $f(\sigma) \leq f(\sigma^*)$ then

$$< \Phi(\sigma), \sigma^* >= 0.$$

Thus $0 \neq a_{\sigma^*} = a_{\sigma^*} < \Phi(\sigma^*), \sigma^* >$. So $< \Phi(\sigma^*), \sigma^* >\neq 0$. Theorem 40 (2) implies that σ^* is a critical cell. □

Theorem 41 *$\exists N \in \mathbb{N}$ large enough $\Phi^N = \Phi^{N+1} = \ldots = \Phi^{\infty}$.*

Proof. The proof is by induction on $r = \#\{\widetilde{\sigma} \in K | f(\widetilde{\sigma}) < f(\sigma)\}$.
Suppose $r = 0$, then by theorem 40, either $\Phi(\sigma) = \sigma$ or $\Phi(\sigma) = 0$. In either case $\Phi(\sigma) = \Phi^{\infty}(\sigma)$.
Suppose the property holds for $r - 1$. Then for r we have two cases:
(i) σ is not critical. Then by theorem 40

$$\Phi(\sigma) = \sum_{f(\widetilde{\sigma})<f(\sigma)} a_{\widetilde{\sigma}} \widetilde{\sigma}.$$

By induction, $\exists N_{\widetilde{\sigma}} \in \mathbb{N}$ such that $\Phi^{N_{\widetilde{\sigma}}}(\widetilde{\sigma})$ is Φ-invariant whenever $f(\widetilde{\sigma}) < f(\sigma)$. Therefore $\Phi^{Max\{N_{\widetilde{\sigma}}\}}$ is Φ-invariant ($Max\{N_{\widetilde{\sigma}}\}$ exists for M is finite).
(ii) σ is critical. Let $c = V(\partial\sigma)$. Then

$$\Phi^m(\sigma) = \sigma + c + \Phi(c) + \ldots + \Phi^{m-1}(c).$$

It follows that $\Phi^m(\sigma)$ is Φ-invariant if and only if $\exists N \in \mathbb{N}$, $\Phi^N(c) = 0$. A seen in proof of theorem 40, c is the sum of p-cells $\widetilde{\sigma}$ such that $f(\widetilde{\sigma}) < f(\sigma)$. By induction, $\exists \widetilde{N} \in \mathbb{N}$, $\Phi^{\widetilde{N}}(c)$ is Φ-invariant.
We now observe that $c \in Im(V)$ and $Im(V)$ is Φ-invariant, since

$$\Phi V = (1 + \partial V + V\partial)V = V(1 + \partial V)$$

(cf theorem 39 (1)). Thus $\Phi^{\widetilde{N}} \in Im(V)$. By theorem 39 (3), the image of V is orthogonal to the critical faces and therefore $\Phi^{\widetilde{N}}(c)$ is a p-chain Φ-invariant which is orthogonal to the critical faces. By lemma 18, $\Phi^{\widetilde{N}}(c) = 0$. $\square$ **Example.** The figure 2.10 illustrates that N is finite for large enough (here N=2 for the cell e).

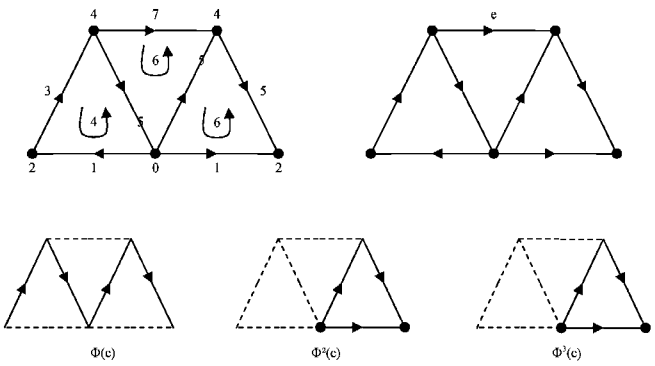

Figure 2.10: For the cell e, $N = 2$

By theorem 41, there is an $N \in \mathbb{N}$ large enough so that every chain c,

$$\Phi^N(c) = \Phi^{N+1}(c) = \Phi^{N+2}(c) = \ldots.$$

Let $\Phi^\infty(c)$ denote this Φ-invariant chain. Then for each $p \in \mathbb{N}$, we have maps

$$\Phi^\infty : C_p(M, \mathbb{Z}) \longrightarrow C_p^\Phi(M, \mathbb{Z})$$

$$i : C_p^{\Phi}(M,\mathbb{Z}) \hookrightarrow C_p(M,\mathbb{Z})$$

where i is the natural inclusion. Note that $\Phi^\infty \circ i$ is the identity on $C_p^{\Phi}(M,\mathbb{Z})$. We have now the following theorem:

Theorem 42 *Let C_*^{Φ} denote the Morse complex (2.7). Then $\forall p \in \mathbb{N}$*

$$H_p(C_*^{\Phi}) \cong H_p(M,\mathbb{Z}).$$

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & C_n(M,\mathbb{Z}) & \xrightarrow{\partial} & C_{n-1}(M,\mathbb{Z}) & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & C_0(M,\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow_{\Phi^\infty}\uparrow^i & & \downarrow_{\Phi^\infty}\uparrow^i & & & & \downarrow_{\Phi^\infty}\uparrow^i & & \\
0 & \longrightarrow & C_n^{\Phi}(M,\mathbb{Z}) & \xrightarrow{\partial} & C_{n-1}^{\Phi}(M,\mathbb{Z}) & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & C_0^{\Phi}(M,\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

Let

$$\Phi_*^\infty : H_*(M,\mathbb{Z}) \longrightarrow H_*(C_*^{\Phi})$$

$$i_* : H_*(C_*^{\Phi}) \longrightarrow H_*(M,\mathbb{Z})$$

denote the induced maps on homology. Our goal is to show that i_* and Φ_*^∞ are isomorphisms. In fact, we will that they are inverses of each other. Since $\Phi^\infty \circ i = 1$ we have

$$1 = (\Phi^\infty \circ i)_* = \Phi_*^\infty \circ i_*$$

To see that $i_* \circ \Phi_*^\infty = 1$, it is sufficient to find an operator

$$L : C_*(M,\mathbb{Z}) \longrightarrow C_{*-1}(M,\mathbb{Z})$$

such that

$$1 - i \circ \Phi^\infty = \partial L + L\partial.$$

As a matter of fact, closed forms would be mapped to exact forms by $1 - i \circ \Phi^\infty$, and this map would be the zero map on homology. Since i is the identity map on chains and $\Phi^\infty = \Phi^N$ for some N large enough,

$$\begin{aligned}
1 - i \circ \Phi^\infty &= 1 - \Phi^N = (1-\Phi)(1+\Phi+\Phi^2+\ldots+\Phi^{N-1}) \\
&= (-\partial V - V\partial)(1+\Phi+\Phi^2+\ldots+\Phi^{N-1}) \\
&= \partial[-V(1+\Phi+\Phi^2+\ldots+\Phi^{N-1})] + [-V(1+\Phi+\Phi^2+\ldots+\Phi^{N-1})]\partial
\end{aligned}$$

(we used $\Phi\partial = \partial\Phi$).
$L = [-V(1+\Phi+\Phi^2+\ldots+\Phi^{N-1})]$ is as desired. □

2.3.7 The Morse Complex and Critical Points

Let M be a finite CW-complex. The goal of this section is to prove that the Morse complex can be expressed thanks to critical cells. We first prove that the space of Φ-invariant chains is canonically isomorphic to the span of the critical cells. Then, we will find an explicit expression of the boundary operator of the Morse complex thanks to the critical cells and the notion of gradient path.

For each $p \in \mathbb{N}$, let $\mathcal{M}_p$ denote the span of the critical p-cells, i.e.

$$\mathcal{M}_p = \{ \sum_{\sigma \in K_p} a_\sigma \sigma | a_\sigma \in \mathbb{Z} \text{ and } a_\sigma \neq 0 \Longrightarrow \sigma \text{ is critical} \}.$$

By restricting the map Φ^∞ defined in the previous section, we get a map

$$\Phi^\infty : \mathcal{M}_p \longrightarrow C_p^\Phi(M, \mathbb{Z}). \tag{2.8}$$

Fix an orientation for each p-cell σ and identify $-\sigma$ with σ given the opposite orientation.

Lemma 19 *Let σ be a critical p-cell. If $\widetilde{\sigma} \neq \sigma$ is critical, then*

$$< \Phi^\infty(\sigma), \widetilde{\sigma} >= 0.$$

Proof. As seen in the proof of lemma 18

$$\Phi^\infty(\sigma) = \sigma + c,$$

where $c \in Im(V) \subset \mathcal{M}_p^\perp$. □

Theorem 43 *The map 2.8 is an isomorphism.*

Proof. (onto) Suppose $c \in C_p^\Phi(M, \mathbb{Z})$, and let

$$\widetilde{c} = \sum_{\sigma \text{ critical}} < c, \sigma > \sigma \in \mathcal{M}_p.$$

We shall see that $\Phi^\infty(\widetilde{c}) = c$. From lemma 19, for any critical cell σ

$$< \Phi^\infty(\widetilde{c}), \sigma >=< c, \sigma > .$$

So $\Phi^\infty(\widetilde{c}) - c$ is a Φ-invariant chain such that for any critical cell σ

$$< \Phi^\infty(\widetilde{c}) - c, \sigma >= 0.$$

By lemma 18 we have $\Phi^\infty(\widetilde{c}) - c = 0$.
(1-1) Suppose $c \in \mathcal{M}_p$ satisfies $\Phi^\infty(c) = 0$. Then, for any critical cell σ,

$$< \Phi^\infty(c), \sigma >= 0.$$

We use again the equality $\Phi^\infty(c) = c + c_V$, where $c_V \in Im(V) \subseteq \mathcal{M}_p^\perp$. Therefore, for any critical cell σ,

$$< c, \sigma >= 0,$$

which implies $c = 0$. □

Theorem 43 implies that the Morse complex is isomorphic to

$$\mathcal{M} : 0 \longrightarrow \mathcal{M}_n \xrightarrow{\widetilde{\partial}} \mathcal{M}_{n-1} \xrightarrow{\widetilde{\partial}} \dots \xrightarrow{\widetilde{\partial}} \mathcal{M}_0 \to 0 \tag{2.9}$$

Consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{M}_p & \xrightarrow{\widetilde{\partial}} & \mathcal{M}_{p-1} \\ \Phi^\infty \downarrow & & \Phi^\infty \downarrow \\ C_p^\Phi & \xrightarrow{\partial} & C_{p-1}^\Phi \end{array}$$

We obtain, for $c \in \mathbf{M}_p$, $(\Phi^\infty)^{-1}(\partial\Phi^\infty(c)) = \widetilde{\partial}c$. Thanks to the beginning of the proof of theorem 43, we have

$$\widetilde{\partial}c = \sum_{\widetilde{\sigma}\in K_{p-1} \text{ and critical}} < \partial\Phi^\infty(c), \widetilde{\sigma} > \widetilde{\sigma}.$$

So, if $c \in \mathcal{M}_p$, σ is a critical $(p-1)$-face

$$< \widetilde{\partial}c, \sigma >=< \partial\Phi^\infty c, \sigma >=< \Phi^\infty \partial c, \sigma > . \tag{2.10}$$

Since $H_*(\mathcal{M}) \cong H_*(M, \mathbb{Z})$, we learn from the Universal Coefficient Theorem that for any field $\mathbf{F}$,

$$H_*(\mathcal{M} \otimes \mathbf{F}) \cong H_*(\mathcal{M}) \otimes \mathbf{F} \cong H_*(M, \mathbb{Z}) \otimes \mathbf{F} \cong H_*(M, \mathbf{F}).$$

Thus $\mathcal{M} \otimes \mathbf{F}$ is a differential complex of vector spaces over $\mathbf{F}$ with the same homology as M. Moreover, $dim_{\mathbf{F}}\mathcal{M}_p \otimes \mathbf{F} = m_p(f)$, it follows that

Corollary 8 *If M is a finite CW-complex, f a discrete a discrete Morse Function on M and $\mathbf{F}$ is any coefficient field, then the Strong Morse Inequalities and hence the Weak Morse Inequalities hold.*

Now we look for a suitable expression for $\widetilde{\partial}$. We will show that for $\tau^{(p+1)}$ and $\sigma^{(p)}$ two critical cells, $< \widetilde{\partial}\tau, \sigma >$ can be expressed in term of *gradient paths* from $\partial\tau$ to σ.

Definition 69 *A gradient path of dimension p is a sequence γ of p-cells of M*

$$\gamma = \sigma_0, \sigma_1, \dots, \sigma_r$$

such that for every $i = 0, \dots, r-1$,

(i) if $V(\sigma_i) = 0$ then $\sigma_{i+1} = \sigma_i$.

(ii) if $V(\sigma_i) \neq 0$ then $\sigma_{i+1} < V(\sigma_i)$ and $\sigma_{i+1} \neq \sigma_i$.

We say γ is a gradient path *from σ_0 to σ_r. The* length *of γ, denoted $|\gamma|$ is equal to r.*

Thus we have the two following properties

Lemma 20 *(i) If $\gamma = \sigma_0, \sigma_1, \ldots, \sigma_r$ is a gradient path then for each $i = 0, \ldots, r-1$ either $\sigma_i = \sigma_{i+1}$ or $f(\sigma_i) > f(\sigma_{i+1})$.*

(ii) If $\gamma_1 = \sigma_0, \sigma_1, \ldots, \sigma_r$ and $\gamma_2 = \sigma_{r+1}, \sigma_1, \ldots, \sigma_{r+s}$ are two sequences of p-cells, then

$$\sigma_0, \sigma_1, \ldots, \sigma_{r+1}, \ldots, \sigma_{r+s}$$

is a gradient path if and only if γ_1 and γ_2 are gradient paths.

We now introduce the notion of multiplicity of a gradient path to "measure" the way orientation is carried along a gradient path.

Suppose that $\sigma \neq \widetilde{\sigma}$ are two p-cells of M and τ is a $(p+1)$-cell with $\sigma < \tau$ and $\widetilde{\sigma} < \tau$ and both are regular faces of τ. Then an orientation on σ induces an orientation on $\widetilde{\sigma}$ in the following way. An orientation on σ induces an orientation on τ so that $< \partial\tau, \sigma >= -1$. Given the orientation on τ, we choose the orientation on $\widetilde{\sigma}$ so that $< \partial\tau, \widetilde{\sigma} >= 1$. Equivalently, fixing an orientation on σ and τ, an orientation is induced on $\widetilde{\sigma}$ so that

$$< \partial\tau, \sigma >< \partial\tau, \widetilde{\sigma} >= -1.$$

One can say that we induce an orientation on $\widetilde{\sigma}$ by "sliding" σ across τ to $\widetilde{\sigma}$. Figure 2.11 presents this process, with a fixed orientation of σ.

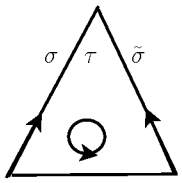

Figure 2.11: Induced orientation

Moreover, if $\sigma = \widetilde{\sigma}$, then the orientation induced on $\widetilde{\sigma}$ must be the same orientation on σ. Thus, if $\gamma = \sigma_0, \sigma_1, \ldots, \sigma_r$ is a gradient path, σ_0 induces an orientation by degrees on σ_i an in particular on σ_r. Since we have fixed an orientation for each cell of M, we set $m(\gamma)$ by $m(\gamma) = 1$ if the induced orientation by σ_0 on σ_r is the orientation of this one, and $m(\gamma) = -1$ otherwise. Equivalently, we set

$$m(\gamma) = \prod_{\substack{i=0 \\ V(\sigma_i) \neq 0}}^{r-1} < \partial V(\sigma_i), \sigma_{i+1} > . \tag{2.11}$$

We can use this formula to define the multiplicity of any gradient path.

Definition 70 *Let $\gamma = \sigma_0, \sigma_1, \ldots, \sigma_r$ be a gradient path of dimension p. We define the* multiplicity *of γ, $m(\gamma)$ by the formula (2.11).*

We note that this notion is compatible with the composition of gradient paths. More precisely, if $\gamma_0 = \sigma_0, \sigma_1, \ldots, \sigma_r$ and $\gamma_1 = \sigma_{r+1}, \ldots, \sigma_{r+s}$ are two gradient paths of same dimension, then

$$m(\gamma_1)m(\gamma_0) = m(\gamma_1 \circ \gamma_0) \tag{2.12}$$

where $\gamma_1 \circ \gamma_0 = \sigma_0, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_{r+s}$.
For p-cells σ and $\widetilde{\sigma}$, let $\Gamma_r(\sigma, \widetilde{\sigma})$ the set of all gradient paths from σ to $\widetilde{\sigma}$ of length r.

We now show that if $\tau^{(p+1)}$ and $\sigma^{(p)}$ are two critical cells, then

$$< \widetilde{\partial}\tau, \sigma >= \sum_{\widetilde{\sigma}^{(p)} < \tau} < \partial\tau, \widetilde{\sigma} > \sum_{\gamma \in \Gamma_N(\widetilde{\sigma}, \sigma)} m(\gamma)$$

for any N large enough.

Definition 71 *Define a* reduced flow $\widetilde{\Phi} : C_p(M, \mathbb{Z}) \to C_p(M, \mathbb{Z})$ *by*

$$\widetilde{\Phi} = 1 + \partial V$$

Though $\widetilde{\Phi}$ differs from Φ, we will prove that we can replace Φ by $\widetilde{\Phi}$ in equation (2.10). The gain is that $\widetilde{\Phi}$ is easier than Φ to compute.

Lemma 21 *For any critical cells $\tau^{(p+1)}$ and $\sigma^{(p)}$*

$$< \widetilde{\partial}\tau, \sigma >=< \widetilde{\Phi}^{\infty}\partial\tau, \sigma > .$$

Proof. It is sufficient to prove that for every $r \geq 0$

$$< \widetilde{\Phi}^r\partial\tau, \sigma >=< \Phi^r\partial\tau, \sigma > .$$

This follows from the observation that for every chain c and every $r \geq 0$

$$\Phi^r(c) - \widetilde{\Phi}^r(c) \in Im(V) \subseteq \mathcal{M}_*^{\perp}.$$

We prove this by induction on r.
If $r = 0$ then there is nothing to prove.
Suppose the property is true for $r - 1$. For r,

$$\Phi^r(c) = \Phi(\Phi^{r-1}(c)) = \Phi(\widetilde{\Phi}^{r-1}(c) + V(\widetilde{c}))$$

for some chain $\widetilde{c}$ (by the inductive hypothesis).

$$\begin{aligned} \Phi^r(c) &= (\widetilde{\Phi} + V\partial)(\widetilde{\Phi}^{r-1}(c) + V(\widetilde{c})) \\ &= \widetilde{\Phi}^r(c) + \widetilde{\Phi}(V(\widetilde{c})) + V\partial\widetilde{\Phi}^{r-1}(c) + V\partial V(\widetilde{c}) \\ &= \widetilde{\Phi}^r(c) + V(\widetilde{c} + \partial\widetilde{\Phi}^{r-1}(c) + \partial V(\widetilde{c})) \end{aligned}$$

where the last equality follows from $\widetilde{\Phi}V = V + \partial V^2 = V$. □

Lemma 22 $\forall(\sigma_1^{(p)}, \sigma_2^{(p)}) \in (K_p)^2$,

$$< \widetilde{\Phi}\sigma_1, \sigma_2 >= \sum_{\gamma \in \Gamma_1(\sigma_1,\sigma_2)} m(\gamma). \tag{2.13}$$

Proof. We distinguish elementary cases to prove this.
First, suppose $V(\sigma_1) = 0$. Then

$$< \widetilde{\Phi}\sigma_1, \sigma_2 >=< \sigma_1, \sigma_2 >= \begin{cases} 1, & si\ \sigma_1 = \sigma_2 \\ 0, & si\ \sigma_1 \neq \sigma_2 \end{cases}$$

On the other hand, the only gradient path of length 1 beginning at σ is the trivial one $\gamma = \sigma_1, \sigma_1$ if $\sigma_2 = \sigma_1$ and so $m(\gamma) = 1$. If $\sigma_1 \neq \sigma_2$, $\Gamma_1(\sigma_1, \sigma_2) = \emptyset$ so

$$\sum_{\gamma \in \Gamma_1(\sigma_1,\sigma_2)} m(\gamma) = 0.$$

We have the equality between the two expressions.
Now, suppose $V(\sigma_1) \neq 0$. If $\sigma_1 = \sigma_2$, we calculate the left hand side of (2.13) to find

$$< \widetilde{\Phi}\sigma_1, \sigma_1 >=< \sigma_1, \sigma_1 > + < \partial V(\sigma_1), \sigma_1 >= 1 - 1 = 0.$$

We calculate the right hand side of (2.13). Since $V(\sigma_1) \neq 0$, there is no gradient path of length 1 from σ_1 to σ_1 so $\Gamma_1(\sigma_1, \sigma_1) = \emptyset$ and

$$\sum_{\gamma \in \Gamma_1(\sigma_1,\sigma_1)} m(\gamma) = 0.$$

Now, suppose $\sigma_1 \neq \sigma_2$ then

$$< \widetilde{\Phi}\sigma_1, \sigma_2 >=< \sigma_1, \sigma_2 > + < \partial V(\sigma_1), \sigma_2 >=< \partial V(\sigma_1), \sigma_2 > .$$

If σ_2 is not a face of $V(\sigma_1)$, then $< \widetilde{\Phi}\sigma_1, \sigma_2 >= 0$. In this case there are no gradient paths of length 1 from σ_1 to σ_2 so that

$$\sum_{\gamma \in \Gamma_1(\sigma_1,\sigma_2)} m(\gamma) = 0.$$

If σ_2 is a face of $V(\sigma_1)$, then there exactly one gradient path of length 1 from σ_1 to σ_2: $\gamma = \sigma_1, \sigma_2$. So

$$m(\gamma) =< \partial V(\sigma_1), \sigma_2 >$$

as desired. □

Theorem 44 $\forall(\tau^{(p+1)}, \sigma^{(p)}) \in (K_{p+1} \times K_p)$ *which are critical cells*

$$< \widetilde{\partial}\tau, \sigma >= \sum_{\widetilde{\sigma}^{(p)} < \tau} < \partial\tau, \widetilde{\sigma} > \sum_{\gamma \in \Gamma_N(\widetilde{\sigma},\sigma)} m(\gamma)$$

for N large enough.

Proof. By lemma 21, for N large enough

$$< \widetilde{\partial}\tau, \sigma >=< \widetilde{\Phi}^N \partial\tau, \sigma > .$$

Since $\partial\tau = \sum_{\widetilde{\sigma}^{(p)}<\tau} < \partial\tau, \widetilde{\sigma} > \widetilde{\sigma}$, We find

$$< \widetilde{\partial}\tau, \sigma >= \sum_{\widetilde{\sigma}^{(p)}<\tau} < \partial\tau, \widetilde{\sigma} >< \widetilde{\Phi}^N \widetilde{\sigma}, \sigma >$$

for N large enough. We prove by induction on $r \geq 1$

$$< \widetilde{\Phi}^r \widetilde{\sigma}, \sigma >= \sum_{\gamma \in \Gamma_r(\widetilde{\sigma},\sigma)} m(\gamma).$$

The case $r = 1$ is lemma 22.
Suppose the property is true for $r - 1$.

$$\begin{aligned}
< \widetilde{\Phi}^r \widetilde{\sigma}, \sigma > &= < \widetilde{\Phi}(\widetilde{\Phi}^{r-1}\widetilde{\sigma}), \sigma > \\
&= \sum_{\sigma'^{(p)}} < \widetilde{\Phi}^{r-1}\widetilde{\sigma}, \sigma' >< \widetilde{\Phi}\sigma', \sigma > && (\widetilde{\Phi}^{r-1}\widetilde{\sigma} = \sum_{\sigma'^{(p)}} < \widetilde{\Phi}^{r-1}\widetilde{\sigma}, \sigma' > \sigma') \\
&= \sum_{\sigma'} \sum_{\gamma \in \Gamma_{r-1}(\widetilde{\sigma},\sigma')} m(\gamma) < \widetilde{\Phi}\sigma', \sigma > && \text{(by induction hypothesis)} \\
&= \sum_{\sigma'} \sum_{\gamma \in \Gamma_{r-1}(\widetilde{\sigma},\sigma')} m(\gamma) \sum_{\gamma' \in \Gamma_1(\sigma',\sigma)} m(\gamma') && \text{(by lemma 21)} \\
&= \sum_{\gamma \in \Gamma_r(\widetilde{\sigma},\sigma)} m(\gamma) && \text{(by lemma 20 (ii) and (2.12))}
\end{aligned}$$

So the property is true for r and the proof is complete. □

2.4 Applications to PL Manifolds and Perspective

A triangulation of a PL manifold gives a regular CW-complex structure. It will be convenient to use for PL manifolds, and more generally for polyhedra [27], CW-complexes structures which are more convenient that triangulations.

Definition 72 *A CW-complex structure on X is polyhedral, if X can be embedded in some Euclidean space in such a way that every p-cell is convex, and lies in a single p-dimensional affine space.*

We can define subdivision of polyhedral CW-complexes as we did for triangulations, and refer to it as polyhedral subdivision.

Definition 73 *A bisection of a polyhedral CW structure is a polyhedral subdivision which subdivides one n-cell e^n in two parts. The cell e^n is replaced by two n-cells e_1^n, e_2^n, and one $n-1$-cell in between.*

The theorem below show that we may restrict to bisections [27, 1.10.5].

Theorem 45 *If $\tilde{X}$ is a finite polyhedral subdivision of a polyhedral CW-complex X, then $\tilde{X}$ can be obtained from X by a finite sequence of bisections.*

Theorem 46 *Let M be a compact connected PL n-manifold equipped with a polyhedral CW structure.*
a) There exists a discrete Morse function with exactly one critical 0-cell, which can be chosen arbitrarily.
b) If $\partial M = \emptyset$ then there exists a discrete Morse function with exactly one critical 0-cell and exactly one critical n-cell which can be chosen arbitrarily.

Theorem 47 (Sphere Theorem) *Let M be a closed PL n-manifold, $n > 0$. Suppose that M has a discrete Morse function with two critical cells, then M is PL homeomorphic to a sphere.*

Proof. The manifold M is connected, and has a critical 0-cell v and a critical n-cell σ. From theorem 35 we get that $M \setminus \sigma$ collapse on v. Applying 4 we get that $M \setminus \sigma$ is a PL n-cell, hence M is a PL n-sphere. □

Remark. Poincaré conjecture can be formulated as follows.

Poincaré Conjecture. Let M be a polyhedral CW-complex which is a closed connected 3-manifold with trivial fundamental group. Then M can be transformed, by a finite sequence of polyhedral bisections, into a CW-complex which admits a discrete Morse function with exactly 2 critical cells.

The above sphere theorem gives a flavor of the kind of result that can be obtained via discrete Morse theory. The famous h-cobordim and s-cobordism theorems ([12] for a proof in the Diff category, [15] for a proof in the PL category) can also be established in the PL category via discrete Morse theory. This fact is announced in Forman's paper. It should be interesting to write down the details.

The usual proof uses *handle calculus*. By using the regular neighborhood theorem, one can refine theorem 36 and show that in the case of a PL manifolds, passing across a critical k-cell amounts to gluing a PL k-handle $D^k \times D^{n-k}$ along $S^{k-1} \times D^{n-k}$. The handle calculus should be interpreted as a set of elementary modifications for the discrete Morse function. Note that existence of a discrete Morse function establishes existence of a handle decomposition for a compact PL manifold.

Question 1 *Describe a set of elementary modifications sufficient to connect any two (polyhedral) discrete Morse functions on a compact PL manifold by a finite sequence of moves. Is there an algorithm for finding such a sequence (at least for some specific manifolds) ?*

In the classical theory, considering *generic path* between two Morse functions is a central question in Cerf theory [4]. This is the starting point towards deep results in low dimensional topology.

Question 2 *Develop a discrete counterpart of Cerf theory.*

In dimension 3, a simple application of Morse theory is existence of a so called Heegaard splitting. This state that any oriented 3-manifolds is the union of two handle-bodies. This can be seen by considering a Morse function with one critical point of index zero and one critical point of index three. This can be done in such a way that the Morse function is self indexed, i.e. values of critical points are equal to their index. Heegaard splitting is obtained by cutting along level $\frac{3}{2}$.

It can be shown that any oriented closed 3-manifold M bounds a 4-manifold W (e.g. via Thom's theory). By considering a Morse function on W, one can obtain a so called surgery description of M, which can be encoded by a link in the 3 sphere. It was shown by Kirby [10] that two such presentation are related by a finite sequence of elementary moves. The proof uses Cerf theory.

Question 3 *Write a proof of Kirby theorem based on discrete Morse theory.*

Question 4 *Find a discrete description of different kinds of structures on manifolds, such that framings, spin or spin^c structures.*

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Differential Geometry and Topology, Discrete and Computational Geometry
M. Boucetta and J.-M. Morvan (Eds.)
IOS Press, 2005
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Chapter 3

Dynamics of Groups of Homeomorphisms

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These notes provide a survey of certain results concerning dynamical and spectral properties of some countable subgroups of homeomorphisms of a metric space. Most of them were obtained in recent years. For many results we only give the references or at most a glimpse of the method used in the proof.

Let E be a metric space and Homeo(E) be its group of homeomorphisms. We study in this chapter certain results concerning dynamical and spectral properties of some countable subgroups G of the group Homeo(E).

We shall point out in a first paragraph some general topological notions which will be used in the following paragraphs. We shall present through the second paragraph some classical dynamical properties of the group G when E is the line $\mathbb{R}$; in order to state several properties of the existence of minimal sets and of exceptional orbits, we precise the nature of orbits under additional assumptions on the group G or on the class of its elements.

The class of an orbit O is the union of all orbits O' which have the same closure as O. We denote by $X = E/\widetilde{G}$ the space of classes of orbits (called quasi-orbits space). In the third paragraph, we shall study some of the relations between these groups G on one hand, and approximately finite-dimensional C*-algebra's and unitary commutative rings on the other hand. We shall look at the case when E is the line $\mathbb{R}$. The purpose of the fourth paragraph will be to give some dynamical properties when these groups are equicontinuous.

Throughout this chapter we will give many examples which illustrate the studied situation and show that the hypothesis are necessary.

I would expect that this chapter only assumes knowledge of general topology.

3.1 General Notions

Recall that a topological space is called a T_0-space if for any two points $x \neq y$ there is an open subset containing one of them but not the other. A closed subset C is irreducible if it is not the union of two proper closed subsets ; this means that the intersection of two nonempty subsets is nonempty. An element x of C is called a generic point if the closure of the singleton $\{x\}$ is equal to C. A topological space is said to be quasi-compact if it satisfies the property of Borel-Lebesgue, but it is not necessarily a Hausdorff space.

The main reference concerning this paragraph is [26].

3.1.1 Quasi-homeomorphisms

According to A. Grothendieck and J. Dieudonné [24] a continuous map $f : X \to Y$ between two topological spaces is called a *quasi-homeomorphism* if the map which assigns to each open subset $V \subset Y$ the open subset $U = f^{-1}(V) \subset X$ is a bijective map. Equivalently, the map which assigns to each closed subset $G \subset Y$ the closed subset $F = f^{-1}(V) \subset X$ is a bijective map. In the same way, the initial topology on X coincides with the inverse topology on X under f and the space $f(X)$ is strongly dense (i.e. $f(X)$ meets any locally dense subset of Y).

An onto quasi-homeomorphism satisfies the following statements [6] :

1) The map f is open closed and for every locally closed subset $A \subset X$ we have $A = f^{-1}(f(A))$.

2) For any x, y we have the following implication :

$$f(x) = f(y) \Longrightarrow \overline{\{x\}} = \overline{\{y\}}.$$

Moreover, we deduce that if X is a T_0-space then f is a homeomorphism.

3) The notions of quasi-compact open subset, irreducible closed subset are stable by direct image and by inverse image under f.

We obtain from these properties the following statements :

4) Since f is a continuous open closed map, for each $A \subset X$, $B \subset Y$ we have

$$f(\overline{A}) = \overline{f(A)} \quad \text{and} \quad f^{-1}(\overline{B}) = \overline{f^{-1}(B)}.$$

5) For every open subsets $U \subset X$, $V \subset Y$ and for every $A \subset X$, $B \subset Y$ we have

$$f(U \cap A) = f(U) \cap f(A).$$

Thus we obtain

$$f(U \cap \overline{A}) = f(U) \cap \overline{f(A)} \quad \text{and} \quad f^{-1}(V \cap \overline{B}) = f^{-1}(V) \cap \overline{f^{-1}(B)}.$$

Immediately, we deduce

6) - If A is a locally closed (resp. locally dense) subset of X, then $f(A)$ is a locally closed (resp. locally dense) subset of Y.

- If B is a locally closed (resp. locally dense) subset of Y, then $f(B)$ is a locally closed (resp. locally dense) subset of X.

7) Using the previous property, we obtain this equivalence :

$$X \text{ is a Baire space} \Leftrightarrow Y \text{ is a Baire space}$$

3.1.2 Dynamical Notions of an Open Equivalence Relation

Let $\mathcal{R}$ be an open equivalence relation on a topological space X. Throughout this paragraph, we will call *trajectory at a point* x the equivalence class at x and we write it $\mathcal{R}(x)$.

Remind that if $A \subset X$, *the saturation* $Sat(A)$ of A is the union of all trajectories meeting A. The subset A is called *saturated* (or *invariant*) if we have $Sat(A) = A$. Since the relation is open, the interior, the closure and the boundary of each saturated subset are also saturated.

In the following, we shall introduce some dynamical notions which will be used in the following paragraphs.

1) *The class* $Cl(T)$ of a trajectory T is the union of all trajectories having the same closure as T.

2) A *minimal set* (M.S) is a minimal element of the family of nonempty saturated closed subsets (ordered by inclusion). Equivalently, it is a nonempty saturated closed subset $S \subset X$ such that the closure of every trajectory $T \subset S$ is equal to S.

3) A *minimal set in a nonempty saturated open subset* U is a minimal set for the restriction of the relation $\mathcal{R}$ to U. Then, a minimal set in U is a nonempty saturated subset $S \subset U$ such that for any trajectory $T \subset S$ we have $\overline{T} \cap U = S$.

4) A *local minimal set* (L.M.S) is a minimal set in some nonempty saturated open subset. We can verify that a trajectory T is contained in a local minimal set if and only if $\overline{T} - Cl(T)$ is closed, in this case the class $Cl(T)$ of T is the (L.M.S) containing T. If it is not the case, we have $\overline{\overline{T} - Cl(T)} = \overline{T}$.

5) A trajectory T is called *locally closed at a point* $x \in T$ if there exists a non empty open subset U containing x such that $\overline{T} \cap U = T \cap U$. We say that a trajectory T is *locally closed* if it is locally closed at each point $x \in T$. We have the following equivalences :

The trajectory T is locally closed $\Longleftrightarrow$ the subset $\overline{T} - T$ is closed $\Longleftrightarrow$ there exists an open subset U such that $\overline{T} \cap U = T$.

6) A trajectory T is called *proper* if it is locally closed such that the subset $int(\overline{T})$ is empty.

7) A trajectory T is called *locally dense at a point* $x \in T$ if there exists a non empty open subset U containing x such that $\overline{T} \cap U = U$. We say that a trajectory T is *locally dense* if it is locally dense at each point $x \in T$. We have the following equivalences :

The trajectory T is locally dense $\Longleftrightarrow$ the subset $int(\overline{T})$ is nonempty $\Longleftrightarrow$ there exists an open subset U such that $T \subset U \subset \overline{T}$.

8) A trajectory T is called *a minimal trajectory* if any trajectory $T' \subset \overline{T}$ has the same closure as T. In other words, the closure of T is a minimal set.

9) A trajectory T is called *a maximal trajectory* if any trajectory T' with $T \subset \overline{T'}$ has the same closure as T.

Remark 1 *1. Two trajectories T and T' which have the same closure have the same nature (locally closed, proper, locally dense, maximal, minimal, contained in a local minimal set,...).*

2. Consider a minimal set S in a nonempty saturated open connected subset U and a trajectory $T \subset S$. The following statements hold :

- *If the trajectory T is locally closed, then T is closed in U and $T = S$.*
- *If the trajectory T is locally dense, then $S = U = Cl(T)$.*

3. (Quasi-trajectories space). The trajectory classes determine an equivalence relation $\widetilde{\mathcal{R}}$ on X defined by $\widetilde{\mathcal{R}}(x) = Cl(\mathcal{R}(x))$. For any open subset $U \subset X$ we have

$$Sat_{\mathcal{R}}(U) = Sat_{\widetilde{\mathcal{R}}}(U).$$

Thus the new equivalence relation $\widetilde{\mathcal{R}}$ is also open.

The space of trajectories $X/\mathcal{R}$ is not in general a T_0-space but the quotient space $X/\widetilde{\mathcal{R}}$ (the space of trajectory classes) is always a T_0-space. It is the universal T_0-space associated to the space $X/\mathcal{R}$. We call it the quasi-trajectories space.

The map $\varphi : X/\mathcal{R} \to X/\widetilde{\mathcal{R}}$ which assigns to each trajectory T its class $Cl(T)$ is an onto quasi-homeomorphism. We denote by $q : X \to X/\mathcal{R}$ and $p : X \to X/\widetilde{\mathcal{R}}$ the canonical projections.

The saturated topology T_s on X is formed by the saturated open subsets of X. A saturated open subset $U \subset X$ is called *compact by saturation* if it is quasi-compact for the saturated topology T_s. This means that every covering (U_i) of U by saturated open subsets U_i contains a finite sub-covering.

Lemma 23 *Let $\mathcal{R}$ by an open equivalence relation on a topological space X. We have the following properties :*

1. An open subset $V \subset X/\mathcal{R}$ is quasi-compact if and only if the saturated open subset $U = q^{-1}(V)$ is compact by saturation.

2. If the space X is locally compact and second countable, then for any saturated open subset U the following two statements are equivalent :

- *U is compact by saturation.*
- *There exists a compact subset $K \subset U$ with $U = Sat(K)$.*

Proof. We can verify directly the property a). In order to show the property b) we use the fact that there exists an increasing sequence (K_n) of compact subsets in U with $U = \bigcup_{n\in\mathbb{N}} int(K_n)$. $\triangle$

Remark 2 (the identity relation) *Let $\mathcal{R}$ be the identity relation on a topological space X. We say that a point $x \in X$ satisfies a property $\mathcal{P}$ if the trajectory $\mathcal{R}(x) = \{x\}$ satisfies this property. For example, we define the class $Cl(x)$ of a point x, a locally closed point, a locally dense point, a minimal set in X, a local minimal set in X,... Sometimes we loosely write x to indicate the singleton $\{x\}$.*

The following remark links the notions of onto quasi-homeomorphisms and of open equivalence relations.

Remark 3 *Let $f : X \to Y$ be an onto quasi-homeomorphism. Then there exists an open equivalence relation $\mathcal{R}$ on X and a homeomorphism $\widetilde{f} : X/\mathcal{R} \to Y$ such that $\widetilde{f} \circ \pi = f$, where $\pi : X \to X/\mathcal{R}$ is the canonical projection. Moreover, π is an onto quasi-homeomorphism.*

We define the relation $\mathcal{R}$ on X by : $x\mathcal{R}y$ if $f(x) = f(y)$ and the map $\widetilde{f} : X/\mathcal{R} \to Y$ by : $\widetilde{f}(\mathcal{R}(x)) = f(x)$.

3.2 Dynamics of Groups of Increasing Homeomorphisms

Let $\mathrm{Diffeo}^r_+(\mathbb{R})$ be the group of increasing C^r-diffeomorphisms on the line $\mathbb{R}$, $(r \geq 0)$. We remark that a C^0-diffeomorphim means a homeomorphism. Consider a countable subgroup G of the group $\mathrm{Homeo}_+(\mathbb{R})$. It will always be assumed that every orbit is unbounded in both sides, this is equivalent to the fact that the group G does not have any fixed point.

We denote by G_1 the subset of elements $g \in G$ fixing an element $x \in \mathbb{R}$ and by $\mathrm{Fix}(G_1)$ the subset of $x \in \mathbb{R}$ such that every $g \in G_1$ satisfies $g(x) = x$.

An orbit $O = G(x)$ is called

- *exceptional* if it is nowhere dense and nonproper.
- *semi-proper* if there exists $\epsilon \rangle 0$ such that at least one of the two intersections $O \cap]x - \epsilon, x[$ or $O \cap]x, x + \epsilon[$ is empty. Such an orbit is either proper or exceptional.
- *resilient* if there exists $g \in G_1$ and $y \in O$ with $g(x) = x$ and such that the sequence $(g^n(y))$ converges to x when $n \to +\infty$ or $n \to -\infty$. In this case O is a nonproper orbit.

These three previous definitions are independent of the choice of the point x in O.

We present through this paragraph some classical dynamical properties of the group G, in order to state several properties concerning the existence of minimal sets and of exceptional orbits. We will precise also the nature of orbits under additional assumptions on the group G or on the class of its elements. These properties are well-known and relatively old but they have an interest for studying the following. These questions were registered in the topic studied by Poincaré(1881), Denjoy(1932), Reeb(1961), Sackesteder(1965), Koppel(1970), Hector(1972), Imanishi(1974), Plante(1975), Conlon-Cantwell(1978, 1881),...

This paragraph is inspired by [41] and of [42] p.122-128.

3.2.1 Groups of Homeomorphisms and Minimal Sets

The following properties are related with existence and nature of minimal sets.

1. Every minimal set m of G satisfies one of the following properties :
   - m is a closed orbit.
   - m coincides with the whole space $\mathbb{R}$; in this case, every orbit is everywhere dense.
   - m is an uncountable union of exceptional orbits ; in such a case, m cuts every nonempty open interval on a Cantor set and we say that it is of *exceptional type*.
2. If G has a closed orbit, then we have the following properties :
   - The closure of any orbit contains at most two closed orbits and at least one closed orbit.
   - The subset $\mathrm{Fix}(G_1)$ is equal to the union of all closed orbits. Thus the union of all closed orbits is a closed subset.
   - G_1 is a normal subgroup and the quotient group G/G_1 is cyclic.
3. If $\mathrm{Fix}(G_1)$ is nonempty, then G_1 is a normal subgroup and the quotient group G/G_1 is abelian torsion free. In fact, $\mathrm{Fix}(G_1)$ contains every minimal set without resilient orbit.

   We remark that the support of any G-invariant measure which is finite on compact sets is a closed saturated subset and it contains a minimal set without resilient orbit.
4. If G has an exceptional minimal set m, then m is contained in the closure of any orbits and so it would be the only minimal set.
5. Each following condition implies the existence of a minimal set :
   - There exists a compact $K \subset \mathbb{R}$ met by all orbits : $\mathrm{Sat}(K) = \mathbb{R}$.

- G contains a subgroup H having an orbit which is unbounded at least from one side (above or below) such that $\mathrm{Fix}(H)$ is a nonempty subset.
- G contains a finitely generated subgroup H having an orbit which is unbounded at least from one side. This is the case if G is finitely generated or if it contains an element without fixed point.
- G is an abelian subgroup of $\mathrm{Diffeo}^2_+(\mathbb{R})$ (finitely generated or not).
- G is a subgroup of $\mathrm{Diffeo}^\omega_+(\mathbb{R})$ (finitely generated or not, abelian or not).

6. There exists a countable abelian subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ without any minimal set. There exists a countable subgroup of $\mathrm{Diffeo}^\infty_+([0,1])$ without any minimal set in $]0,1[$.

7. If G is a finitely generated subgroup of $\mathrm{Diffeo}^2_+([0,1])$ without fixed point in $]0,1[$, then from Cantwell-Conlon we have the following statements :

   - The class of every semi-proper nonproper orbit contains a resilient orbit.
   - The group G contains a minimal set in each nonempty saturated open subset ; in particular in $]0,1[$.
   - Every totally ordered family of orbits (ordered by the inclusion of their closures) is well-ordered of order type less than or equal to the first ordinal limit ω.

8. The closure of an orbit O is equal to a finite union of classes of orbits, if we have one of the following properties :

   - The orbit O is semi-proper and the group G is contained in $\mathrm{Diffeo}^2_+([0,1])$.
   - The orbit O has a polynomial growth (in the sense of Plante).
   - G is a finitely generated abelian subgroup of $\mathrm{Homeo}_+(\mathbb{R})$.
   - G is an abelian subgroup of $\mathrm{Diffeo}^2_+(\mathbb{R})$ (finitely generated or not).
   - G is a subgroup of $\mathrm{Diffeo}^\omega_+(\mathbb{R})$ (finitely generated or not, abelian or not).

3.2.2 Abelian Groups and Groups of Analytic Diffeomorphisms

In this subsection, we shall suppose that G is either a countable abelian subgroup of $\mathrm{Homeo}_+(\mathbb{R})$, or is an arbitrary countable subgroup contained in $\mathrm{Diffeo}^\omega_+(\mathbb{R})$.

1. First, we will recall some well-known results :

   - Let f, g be two elements of $\mathrm{Diffeo}^2_+([0,1[)$ satisfying $fog = gof$ and such that f is without fixed point in $]0,1[$. If g has a fixed point in $]0,1[$, then g is equal to the identity on $[0,1[$ (Koppel). According to this result, one can check the following :
   - If G is a non cyclic abelian subgroup of $\mathrm{Diffeo}^2_+([0,1])$ without fixed point in $]0,1[$, then every orbit is everywhere dense (Reeb).

- If G is a non abelian subgroup of $\mathrm{Diffeo}_+^\omega([0,1])$ without fixed point in $]0,1[$, then every orbit is everywhere dense (Hector).

2. From the three previous results, we can show the following : Let G be a subgroup of $\mathrm{Diffeo}_+^\omega(\mathbb{R})$ or an abelian subgroup of $\mathrm{Diffeo}_+^2(\mathbb{R})$. The following properties hold :

   - G has a minimal set.
   - The closure of an orbit O is equal to a finite union of classes of orbits.
   - The class of every exceptional orbit (if it exists) is a minimal set. Thus, there exists at most one class of exceptional orbits.
   - If G has a closed orbit, then G does not have any exceptional orbit.
   - The length of every chain of orbits is less than or equal to 3.

   We note that a chain of orbits is a finite sequence $(O_1, ..., O_n)$ of orbits satisfying

   $$\overline{O_1} \subset ... \subset \overline{O_n}$$

   and $\overline{O_i} \neq \overline{O_j}$, for every $1 \leq i \neq j \leq n$.

3. We have also the following properties : If G is a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ which acts freely on $\mathbb{R}$ (i.e. the identity map is the only homeomorphism having a fixed point), then one of the following properties holds :

   - Every orbit is closed ; in this case G is a cyclic subgroup.
   - Every orbit is everywhere dense.
   - G has a minimal set m such that every orbit $O \subset \mathbb{R} - m$ meets any connected component of $\mathbb{R} - m$ in one point at most.

   The first and second properties are equivalent to the fact that the group G is topologically conjugate to a group of translations.

4. There exists a subgroup G of $\mathrm{Homeo}_+(\mathbb{R})$ acting freely on $\mathbb{R}$, having an exceptional minimal set and satisfying one of the two additional following hypotheses :

   - G is a subgroup of $\mathrm{Diffeo}_+^1(\mathbb{R})$ generated by two elements (Denjoy).
   - G is a countable subgroup of $\mathrm{Diffeo}_+^\infty(\mathbb{R})$ (Imanishi).

5. If G is a subgroup of $\mathrm{Diffeo}_+^\omega(\mathbb{R})$, then one of the following properties holds :

   - G acts freely on $\mathbb{R}$.
   - G has a closed orbit. In this case there exists at most finitely many closed orbits and either all other orbits are proper or all other orbits are locally dense.

- G has a minimal set m containing a resilient orbit. In such a case, either every orbit is everywhere dense, or m is an exceptional minimal and any orbit $O \subset \mathbb{R} - m$ is proper.

6. Let G be a subgroup of $\text{Diffeo}^{\omega}_{+}(\mathbb{R})$. If G has a minimal set without resilient orbit and not reduced to a closed orbit, then G is a resoluble group (under theses hypotheses G_1 must be an abelian subgroup such that the quotient group G/G_1 is abelian).

3.2.3 Groups of Homeomorphisms on the Unit Circle S^1

We denote by $\text{Diffeo}^{r}_{+}(S^1)$, $r \geq 0$, the group of all C^r-diffeomorphisms of the unit circle S^1 conserving its orientation. The passage from homeomorphisms of S^1 to homeomorphisms of the line $\mathbb{R}$ is obtained in the following way. Let $p : \mathbb{R} \to S^1$ be the universal cover defined by $p(t) = e^{2\pi i t}$. The lifting of any homeomorphism $f : S^1 \to S^1$ is a homeomorphism $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ with $\widetilde{f}(t+1) = \widetilde{f}(t) + 1$, for all $t \in \mathbb{R}$. Thus, we can associate, by lifting, to every subgroup $G \subset \text{Diffeo}^{r}_{+}(S^1)$ a subgroup $\widetilde{G} \subset \text{Diffeo}^{r}_{+}(\mathbb{R})$, $r \geq 0$, such that a finite, proper, locally dense exceptional orbit of G corresponds to a closed, proper, locally dense exceptional orbit of $\widetilde{G}$. Conversely, by putting $\overline{\mathbb{R}} = [-\infty, +\infty]$ and by identifying S^1 with $\overline{\mathbb{R}}/-\infty \approx +\infty$, we can associate to every subgroup $\widetilde{G} \subset \text{Homeo}_{+}(\mathbb{R})$ a subgroup $G \subset \text{Homeo}_{+}(S^1)$ fixing the point which corresponds to ∞.

The above remark with the preceding sub-paragraph allows us to formulate what follows:

1. First, we point out the case when the group G is generated by one element f in $\text{Diffeo}^{r}_{+}(S^1)$, $r \geq 0$. If f is without periodic points and if we have $r \geq 2$, then f is topologically conjugate to a rotation. Thus either all orbits are everywhere dense, or all orbits are finite (Poincaré-Denjoy). However, Denjoy constructed a C^1-diffeomorphism $f : S^1 \to S^1$ without periodic points and having an exceptional minimal set.

2. A finitely generated abelian group $G \subset \text{Diffeo}^{2}_{+}(S^1)$ does not have any exceptional orbits (Reeb). But, there exists a group $G \subset \text{Diffeo}^{\infty}_{+}(S^1)$ generated by two elements without any periodic point and having an exceptional minimal set (Sackesteder). A similar example can be chosen in $\text{Diffeo}^{\omega}_{+}(S^1)$ (Hector).

3. We deduce that a group $G \subset \text{Diffeo}^{r}_{+}(S^1)$, $r \geq 0$, satisfies the properties of the paragraph 2.2), if we replace $\mathbb{R}$ by S^1.

3.3 Groups of Homeomorphisms and Spectral Topology

A topological space is called a spectral space (definition 4.9 of [29]) if it is a T_0-space and if each irreducible closed subset has a generic point. From 4.17 of [29] such a space is homeomorphic to the prime spectrum of the lattice of its open subsets.

M. Hochster proved in [28] that a topological space X is homeomorphic to the prime spectrum of a unitary commutative ring equiped with the Zariski topology if and only if it satisfies the following properties :

1. X is a quasi-compact space.
2. X is a T_0 -space.
3. Each irreducible closed subset has a generic point.
4. X has a basis of quasi-compact open subsets.
5. The intersection of two quasi-compact open subsets is quasi-compact.

These five properties are called *properties of spectrality.*

In [4] it was shown that an ordered set $(Y, \leq)$ is order-isomorphic to the prime spectrum of a unitary commutative ring equipped with the inclusion if and only if there exists a topology on the set Y compatible with the order and satisfying the properties 1), 2), 3), 4) and 5) above. According to [31] such a set satisfies the following conditions :

(K_1) Each totally ordered family of elements in $(Y, \leq)$ has a supremum and an infimum.

(K_2) For every elements $a \lneqq b$ in Y there exists two consecutive elements a_1, b_1 with $a \leq a_1 \lneqq b_1 \leq b$.

O. Bratteli and G. A. Elliot showed in [9] that a topological space X is homeomorphic to the primitive spectrum of an approximately finite-dimensional separable C*-algebra (briefly, an A.F C*-algebra) equipped with the Jacobson topology if and only if it has a countable basis and it satisfies the previous properties 2), 3) and 4), but not necessarily the properties 1) and 5).

An approximately finite-dimensional separable C*-algebra is an inductive limit of a sequence of finite-dimensional C*-algebras. In this paragraph, we study some of the relationships between countable groups of homeomorphisms on a topological space E on one side, and approximately finite-dimensional C*-algebra's and unitary commutative rings on the other side. We study the case where E is the line R. For more details concerning this paragraph one can see [26].

3.3.1 General Case

Let E be a topological space and Homeo(E) be its group of homeomorphisms. Let G be a countable subgroup of Homeo(E). For a point $x \in E$ the set $G(x) = \{g(x) : g \in G\}$ is called *the orbit of* x, and the subgroup $G_x = \{g \in G : g(x) = x\}$ is called *the isotropy or stability group at* x. Because two orbits are either disjoint or identical, the action of G on E determines an equivalence relation on E. This relation is necessarily open.

Consider $Z = E/G$ the orbits space and $X = E/\widetilde{G}$ the quasi-orbits space (an element of $E/\widetilde{G}$ is an orbit class). The canonical projections $q : E \to E/G$ and $p : E \to E/\widetilde{G}$ are open. The map $\varphi : E/G \to E/\widetilde{G}$, which associates its class to any orbit, is an onto quasi-homeomorphism.

The fact that the quasi-orbits space X is a T_0-space allows us to define an ordering on X by

$$a = p(x) \leq p(y) = b \quad \text{if} \quad \overline{G(x)} \subset \overline{G(y)}.$$

It is easy to see that if O and O' are two orbits, then $O \subset \overline{O'}$ if and only if every nonempty saturated open subset U containing O contains O'. Hence the quotient topology on X is compatible with the inverse ordering : that is, every element $a \in X$ satisfies

$$\overline{\{a\}} = \{b \in X : b \leq a\}$$

A point $a \in X$ is a *separated point* if for every $b \notin \overline{\{a\}}$ there exists an open neighborhood U_a (resp. U_b) of a (resp. b) such that the intersection $U_a \cap U_b$ is empty.

An orbit $G(x)$ is a *separated orbit* if $q(x)$ is a separated point of E/G ; this is equivalent to the fact that for every orbit $G(y)$ not contained in $\overline{G(x)}$ there exist two saturated open subsets U and V such that $G(x) \subset U$, $G(y) \subset V$ and $U \cap V = \emptyset$.

We denote by $Sep(X)$ the union of all separated points of X and by X_0 its interior. The space $X - X_0$ is called *the singular part* of X.

We have the following properties :

- Each locally dense orbit is a separated orbit.
- Each separated orbit is a maximal orbit.
- The space Sep(X) is a Hausdorff space.

Remark 4 *1. If there exists a compact subset $K \subset E$ meeting every orbit, then it follows that the space E is compact by saturation, so the quasi-orbits space $X = E/\widetilde{G}$ is quasi-compact. Also by using Zorn's lemma, we verify that the closure of each orbit contains a minimal set.*

2. If E is a locally compact topological space, then the quasi-orbits space $X = E/\widetilde{G}$ and the orbits space $Z = E/G$ are Baire spaces.

Proof. First of all, we check directly that X is a Baire space. After, we apply the property 7) of the paragraph 3.1 in order to show that Z is also a Baire space. $\triangle$

In the following we suppose that E is a locally compact second countable topological space. First of all, we establish two lemmas, which we will use below :

Lemma 24 *Let E be a locally compact second countable topological space and let G be a countable subgroup of Homeo(E). Then every irreducible closed subset of $X = E/\widetilde{G}$ has a generic point.*

Proof. We know, from ([7], exercise 7, page 171), that if E is a compact metrisable topological space such that the quasi-orbits space X is irreducible, then X has a generic point. Under the hypothesis of this lemma and by using the one-compactification $\hat{E} = E \cup \{\omega\}$ of E we can return to the situation above. We may find a direct proof in ([26], lemma 2.2). $\triangle$

Lemma 25 *Let E be a locally compact second countable topological space and let G be a countable subgroup of Homeo(E). If $\{a_i\}_{i\in I}$ is a totally ordered family of the quasi-orbits space X, then there exists $a \in X$ satisfying $\overline{\{a\}} = \overline{\{a_i : i \in I\}}$ and $a = \sup\{a_i : i \in I\}$.*

The space X is equipped with the ordering defined above.

Proof. First, we prove that the set $A = \overline{\{a_i : i \in I\}}$ is an irreducible closed subset. Thus, by the previous lemma, this subset A has a generic point a which satisfies this lemma. $\triangle$

Proposition 22 *Let E be a locally compact second countable topological space and let G be a countable subgroup of Homeo(E). Then every orbit is contained in the closure of a maximal orbit.*

Proof. Let O_0 be an arbitrary orbit and put $a_0 = p(O_0)$. We denote by $\mathcal{C}a_0$ a maximal totally ordered family formed by elements $b \in X$ such that $a_0 \leq b$. From lemma 25, there exists $a_m \in X$ satisfying $\overline{\{a_m\}} = \overline{\{a : a \in \mathcal{C}a_0\}}$ and $a_m = \sup\{a : a \in \mathcal{C}a_0\}$. An orbit O_m contained in $p^{-1}(\{a_m\})$ is a maximal orbit containing O_0 in its closure. $\triangle$

Applying (4.17 of [29]) and lemma 24, we obtain the following proposition :

Proposition 23 *Let E be a locally compact second countable topological space and let G be a countable subgroup of Homeo(E). Then $X = E/\widetilde{G}$ is homeomorphic to the prime spectrum of the lattice of its open subsets.*

Theorem 48 *Let E be a locally compact second countable topological space and let G be a countable subgroup of Homeo(E). If every decreasing sequence of saturated closed subsets of E is finite, then we have the following properties :*

1. *The quasi-orbits space X is homeomorphic to the primitive spectrum of an A.F C*-algebra and to the prime spectrum of a unitary commutative ring.*
2. *The space E is the closure of a finite union of orbits $O_1, ..., O_n$: $E = \overline{O_1 \cup ... \cup O_n}$.*
3. *The closure of each orbit contains a minimal set.*

Proof. Under the hypothesis of this theorem, every open subset of X is quasi-compact and so the properties 1) 2), 4) and 5) of spectrality are satisfied. Therefore the lemma 24 allows us to obtain the property 1).

The space X is a noetherian space. Let $X_1, ..., X_n$ be finite irreducible components of X such that $X = X_1 \cup ... \cup X_n$ (by [7], these components exist and are finite). From lemma 24 every X_i has a generic point α_i. If, for every $1 \leq i \leq n$, we take an orbit O_i contained in $p^{-1}(\{\alpha_i\})$, we verify that we have $E = \overline{O_1 \cup ... \cup O_n}$; so the property 2) holds.

Since X is a quasi-compact space, the property 3) follows from Zorn's lemma. $\triangle$

Examples

1. Let us consider the set $Y = \mathbb{N} \cup \{\omega\}$ equipped with the topology of basis $\{U_n\}_{n\in\mathbb{N}}$ such that for each integer n we have $U_n = \{n, n+1, ...\} \cup \{\omega\}$. This space corresponds to the singular part of the quasi-orbits space of a countable group $G \subset \text{Diffeo}^{\infty}_{+}(\mathbb{R})$ (example 4-14 of [21]). This space satisfies the hypothesis of the above theorem and so its conclusions.

2. Let us consider the set $Y = \{\frac{1}{n}, n \geq 1\} \cup \{0\}$ equipped with the topology of basis $\{U_n\}_{n\in\mathbb{N}}$ such that for each integer $n \geq 1$ we have $U_n = \{\frac{1}{n}, \frac{1}{n-1}, ..., 1\}$, $U_0 = Y$. This space corresponds to the singular part of the quasi-orbits space of a countable group $G \subset \text{Diffeo}^{\infty}_{+}(\mathbb{R})$ (example of remark 4-17-iii of [21]). Since the sequence $\{U_n\}_{n\in\mathbb{N}}$ is an increasing sequence of open subsets of Y which is not finite, this space Y does not satisfies the hypothesis of the previous theorem, but it satisfies its conclusions.

3. The next example does not correspond to any quasi-orbits space. Let $Y = A \cup B$ be a subset of $\mathbb{R}^2$ defined by $A = \{a_0 = (0,0)\}$ and $B = \{b_n = (n,1) : n \in \mathbb{Z}\}$. We consider a topology $\mathcal{T}$ on Y such that every nonempty closed subset F distinct from Y is given by $F = A \cup B_1$ where B_1 is a finite subset of B. This topology satisfies

$$\overline{\{a_0\}} = \{a_0\} \text{ and } \overline{\{b_n\}} = \{a_0, b_n\}, \ n \in \mathbb{Z}.$$

   This space $(Y, \mathcal{T})$ is a noetherian space such that every closed subset $F \neq Y$ has a generic point, but the whole space Y is irreducible without any generic point. We conclude, by lemma 24, that it cannot be homeomorphic to any quasi-orbits space.

4. The next example gives a group generated by one diffeomorphism on the unit two-sphere such that the singular part $X - X_0$ of its quasi-orbits space X does not have any basis of quasi-compact open subsets:
   Let $S^2 = \{(z,u) \in \mathbb{C} \times \mathbb{R} : |z|^2 + u^2 = 1\}$ be the unit two-sphere. We consider the diffeomorphism f on S^2 defined by $f(z,u) = (ze^{2\pi iu}, u)$. We can see that each orbit of this diffeomorphism is either finite (if $u \in \mathbb{Q}$) or everywhere dense in a parallel circle (if $u \in [-1,1] - \mathbb{Q}$). The interior X_0 of the union of all separated points of X is empty.

We show that the sphere S^2 cannot have any basis of compact by saturation open subsets, thus the space X does not have any basis of quasi-compact open subsets.$\triangle$

3.3.2 Groups of Homeomorphisms on the Line $E = \mathbb{R}$ and Spectral Topology

Let $\mathcal{F}$ be a transversally oriented codimension-one foliation of class C^r, $r \geq 0$, on a closed manifold M. The foliation $\mathcal{F}$ is in particular an open equivalence relation on M such that each trajectory corresponds to a leaf. Thus, we may define the class of a leaf and so the quasi-leaves space X, as in paragraph 3.2. Let X_0 be the union of all open subsets of X homeomorphic to $\mathbb{R}$ or S^1.

In [5] the authors proved that the singular part $X - X_0$ of X satisfies the following properties of spectrality :

1. $X - X_0$ is a quasi-compact space.
2. $X - X_0$ is a T_0 space.
3. Each irreducible closed subset of $X - X_0$ has a generic point.
4. $X - X_0$ has a basis of quasi-compact open subsets.
5. And the property :
   If $\mathcal{F}$ has a well defined height, then the intersection of two quasi-compact open subsets of $X - X_0$ is also quasi-compact.

Recall that a foliation has a well defined height (in the sense of [45]) if and only if every totally ordered family of leaves (ordered by inclusion of their closures) is well-ordered ; that is, it has a minimal element.

The following remark links countable subgroups of $\text{Homeo}_+(\mathbb{R})$ and codimension-one foliations on closed manifolds.

Remark 5 *We can associate to each countable subgroup G of $\text{Homeo}_+(\mathbb{R})$ a subgroup H of $\text{Homeo}_+(\mathbb{R})$ generated by two elements T and h, such that the orbits of G and those of H have the same nature. By taking $S^1 = \overline{\mathbb{R}} / -\infty \simeq +\infty$ where $\overline{\mathbb{R}} = [-\infty, +\infty]$ and by suspension of the new group H we obtain a transversally oriented codimension-one foliation defined on the closed manifold $M = V_2 \times S^1$ (where V_2 is a closed surface of genus 2) and having the same dynamics as the group H and so as the group G.*

Proof. By taking $\overline{\mathbb{R}} = [-\infty, +\infty]$, we can suppose that G is contained in $\text{Homeo}_+([0,1])$. Since G is countable, we write $G = \{g_n : n \in \mathbb{N}\}$. Now, we put

$$T(x) = x + 1, \text{ for every } x \in \mathbb{R},$$
$$h(x) = x \text{ if } x \leq 0 \text{ and } h(x) = g_n(x - n) + n \text{ if } x \in [n, n+1],\ n \in \mathbb{N}.$$

The orbit $H(0)$ is equal to $\mathbb{Z}$ and the other orbits of H are determined by :
$H(x) \cap]0,1[= G(x-n)$, for $x \in]n, n+1[, n \in \mathbb{N}$.
We conclude that the orbits $G(x)$ and $H(x-n)$ have the same nature. $\triangle$

In the following, we consider a countable subgroup $G \subset \text{Homeo}_+(\mathbb{R})$ such that each orbit is unbounded in both sides ; equivalently, G does not have any fixed point.

Let $Z = \mathbb{R}/G$ be the space of orbits and $X = \mathbb{R}/\widetilde{G}$ be the space of quasi-orbits. The maps $p : \mathbb{R} \to X$ and $q : \mathbb{R} \to Z$ are the canonical projections. We denote by Z_0, X_0 the union of all open subsets of Z, X homeomorphic to $\mathbb{R}$ or to S^1. We have $X_0 = Z_0$.

Lemma 26 *With the above notations we have the following properties :*

1. *If G has a minimal set, then $X - X_0$ is quasi-compact.*
2. *$X - X_0$ is a T_0-space.*
3. *Each closed irreducible subset of $X - X_0$ has a generic point.*

Proof. The property 2) is trivial and the property 3) follows from lemma 24. By remark 3.1.1), the property 1) is obtained from the fact that the existence of a minimal set is equivalent to the existence of a compact set $K \subset \mathbb{R}$ meted by every orbit (for showing this equivalence, we can use the properties 1), 2), and 4) of the paragraph 3.2.1. $\triangle$

We denote by $I^{\eta}_{x,\epsilon}$ the open interval $]x, x+\epsilon[$ if $\eta = +$ and $]x-\epsilon, x[$ if $\eta = -$.

We say that a proper orbit $O = G(x)$ is *attractant from the side* $\eta = \pm$ if there exists $\epsilon \rangle 0$ such that for every $y \in I^{\eta}_{x,\epsilon}$ we have $G_x(y) \neq \{y\}$; equivalently, for every $y \in I^{\eta}_{x,\epsilon}$ we have $x \in \overline{G(y) \cap I^{\eta}_{x,\epsilon}}$.

This proper orbit $O = G(x)$ is said to be *stable from the side* $\eta = \pm$ if there exists $\epsilon \rangle 0$ such that for every $y \in I^{\eta}_{x,\epsilon}$ we have $G_x(y) = \{y\}$. In this case we have the inclusion $p(\text{Sat}(I^{\eta}_{x,\epsilon})) \subset X_0$.

It is clear that the notions "attractant" and "stable" are independent of the choice of the point x in the orbit O.

We say that O^{η} $(\eta = \pm)$ is *attractant* or *stable* instead of attractant or stable from the side η.

If U is a different nonempty saturated open subset from $\mathbb{R}$ and if $]a, b[$ is a component connected of U, then a and b are finite reals, and we have :

$$G_a = G_b \text{ and } G(y) \cap]a, b[= G_a(y), \text{ for every } y \in]a, b[.$$

Lemma 27 *Let G be a countable subgroup of Homeo$_+(\mathbb{R})$ without any fixed point. The singular part $X - X_0$ of the quasi-orbits space X has a countable basis of quasi-compact open subsets.*

Proof It suffices to show that for every nonempty open subset V of $X - X_0$ and for every $\alpha \in V$ there exists a nonempty quasi-compact open subset W of $X - X_0$ satisfying the condition $(*) : \alpha \in W \subset V$.

Let V' be an open subset of X verifying $V = V' - X_0$. We put $U' = p^{-1}(V')$ and we take a point x in a given orbit O contained in $p^{-1}(\{\alpha\})$. If the orbit $O = G(x)$ is not proper, then there exists $g \in G$ with $g(x) \neq x$ (we can suppose that $g(x) > x$) such that the closed interval $[x, g(x)]$ is contained in U. . We deduce that the quasi-compact open subset $W = p(\text{Sat } ([x, g(x)])) - X_0$ satisfies the above condition (*) ; indeed, any closed subset contained in a quasi-compact subset is quasi-compact.
If the orbit O is proper, then we make a discussion according to the fact that O is attractant in both sides, attractant in one side and not attractant in the other side, or not attractant in both sides. We use also the remark that if an open subset U is saturated by stable proper orbits, then we have $p(U) \subset X_0$. (A proper orbit is called *stable* if it is stable in both sides). $\triangle$

Let U be a nonempty saturated open subset different from $\mathbb{R}$. We define $\delta^\eta U$ $(\eta = \pm)$ by the set of $x \in \mathbb{R} - U$ such that there exists $\epsilon > 0$ satisfying the inclusion $I^\eta_{x,\epsilon} \subset U$. We put $\delta U = \delta^+ U \cup \delta^- U$.

Obviously, the subsets $\delta^+ U$, $\delta^- U$ and so δU are saturated with $\overline{\delta U} = \overline{U} - U$.

Let $\mathcal{I} = \{I_j =]a_j, b_j[\}_{j \in J}$ be the family of connected components of U. We have $\delta^+ U = \{a_j : j \in J\}$ and $\delta^- U = \{b_j : j \in J\}$.

If U is compact by saturation (i.e. is quasi-compact for the saturated topology T_s), then there exists a finitely many bounded open intervals $I_1, ..., I_n$ with $U = Sat(I_1) \cup ... \cup Sat(I_n)$. Indeed, we have $U = \bigcup_{j \in J} Sat(I_j)$.

Finally, we remark that we have the two following equivalences :

1. δU is a finite union of orbits.

2. There exists finitely many bounded open intervals $I_1, ..., I_n$ with $U = \text{Sat}(I_1) \cup ... \cup \text{Sat}(I_n)$.

Now, we are able to state the following lemma characterizing the compact by saturation open subsets.

Lemma 28 *Let U be a nonempty saturated open subset different from $\mathbb{R}$. Then U is compact by saturation if and only if we have the following statements:*

1. *There exists finitely many bounded open intervals $I_1 =]a_1, b_1[, ..., I_n =]a_n, b_n[$ with $U = Sat(I_1) \cup ... \cup Sat(I_n)$.*

2. *For each $1 \leq i \leq n$, the orbits $O^+_{a_i}$ and $O^-_{b_i}$ are attractant.*

3. *For every orbit $O \subset U$, the intersection $\overline{O} \cap U$ contains a minimal set in U ; equivalently, for each $1 \leq i \leq n$, G_{a_i} has a minimal set in the interval I_i.*

Proof. From the previous discussion, a compact open subset U by saturation satisfies the property 1). Now, if the property 2) or the property 3) is not satisfied, then we may construct an infinite increasing sequence $(U_n)_{n\in\mathbb{N}}$ of nonempty saturated open subsets such that $U = \bigcup_{n\in\mathbb{N}} U_n$, so U would not be compact by saturation.

Conversely, let us suppose that U =Sat(I) where $I =]a, b[$ is a bounded open interval. The fact that O_a^+ and O_b^+ are attractant together with the fact that the subgroup G_a has a minimal set in the interval I implies the existence of a compact subset $K \subset U$ such that Sat(K)= U, therefore U will be compact by saturation. △

The lemma 28 combined with the fact that the closure of any orbit contains at most two minimal sets in a given nonempty saturated open subset allows us to state the following lemma.

Lemma 29 *If the group G has a minimal set, then the intersection of two quasi-compact open subsets of $X - X_0$ is quasi-compact if and only if every totally ordered family of orbits (ordered by inclusion of their closures) has an infimum.*

Proof. By lemmas 26 and 27 the space $X - X_0$ satisfies the properties 1), 2), 3) and 4) of spectrality. Moreover, if it satisfies also the property 5), then it satisfies the condition (K_1) of Kaplansky as in the beginning of this paragraph, so every totally ordered family of orbits has an infimum.

For the converse, we use the lemma 28. Since $\delta^\eta W \subset \delta^\eta U \cup \delta^\eta V$ ($\eta = \pm$) it suffices to show the property 3) of this lemma. We consider an orbit $O \subset W$ and a maximal totally ordered family $\mathcal{C}$ of orbits contained in $\overline{O} \cap W$. By hypothesis, there exists on one hand an orbit O_m which is an infimum of $\mathcal{C}$ and on the other hand a minimal set m_1 (resp. m_2) in U (resp. V) contained in the closure of each orbit in $\mathcal{C}$. If O_1, O_2 are two orbits with $O_1 \subset m_1$ and $O_2 \subset m_2$, then we have $m_1 = \overline{O_1} \cap U$ and $m_2 = \overline{O_2} \cap V$. We conclude that $O_1 \subset \overline{O_m}$ and $O_2 \subset \overline{O_m}$, and so O_m is contained in $W = U \cap V$ which implies that $\overline{O_m} \cap W$ is a minimal set in $\overline{O} \cap W$; indeed, the family $\mathcal{C}$ is maximal. Thus W satisfies the property 3) of lemma 28. △

From lemmas 26, 27, 29 and from the characterization of primitive and prime spectrum spaces, given in the beginning of this paragraph, we can state the following theorem :

Theorem 49 *Let G be a countable subgroup of $Homeo_+(\mathbb{R})$ and X be its quasi-orbits space. Let X_0 be the union of all open subsets of X homeomorphic to $\mathbb{R}$ or to S^1. We have the following statements :*

1. *The singular part $X - X_0$ of the space X is always homeomorphic to the primitive spectrum of an A.F C^*-algebra equipped with the Jacobson topology.*

2. *The singular part $X - X_0$ of the space X is homeomorphic to the prime spectrum of a unitary commutative ring equipped with the Zariski topology if and only if every totally ordered family of orbits (ordered by inclusion of their closures) has a greatest lower bound.*

The lemmas 26 and 27 are stated in terms of foliation in [5], *idem* for the lemma 26 in [26]. Thus, we have also the following theorem :

Theorem 50 *Let $\mathcal{F}$ be transversally oriented codimension-one foliation of class C^r $(r \geq 0)$ on a closed manifold and X be its quasi-leaves space. Let X_0 be the union of all open subsets of X homeomorphic to $\mathbb{R}$ or to S^1. We have the following statements :*

1. *The singular part $X - X_0$ of the space X is always homeomorphic to the primitive spectrum of an A.F C*-algebra equipped with the Jacobson topology.*
2. *The singular part $X - X_0$ of the space X is homeomorphic to the prime spectrum of a unitary commutative ring equipped with the Zariski topology if and only if every totally ordered family of leaves (ordered by inclusion of their closures) has a greatest lower bound.*

Applying ([9], theorem page 79), we can prove the following :

Proposition 24 *Under the hypotheses of theorem 49 (or theorem 50, if every orbit (or every leaf) is contained in a local minimal set, then the space $X - X_0$ is homeomorphic to the primitive spectrum of a postliminary A.F C*-algebra.*

Since each nonempty quasi-compact topological T_0-space contains a closed point, the fact that the singular part $X - X_0$ in the theorem 49 (or in the theorem 50 has a basis of quasi-compact open subsets implies the following :

Remark 6 *Under the hypotheses of theorem 49 (or theorem 50, the union Y of all locally closed points of X is strongly dense. In particular, Y is everywhere dense. A subset is said to be strongly dense if it meets every nonempty locally closed subset.*

Examples

1. The example 4 of 3.3.1 shows that the two conclusions of theorem 49 fail if we replace the line $\mathbb{R}$ by a surface. By suspension of the diffeomorphism in this example, we see also that the two conclusions of theorem 50 fail if we replace "codimension-one foliation" by "codimension-two foliation".
2. The examples 1 and 2 of 3.3.1 satisfy the two conclusions of theorem 49.
3. Consider the set $Y = \{\frac{1}{n} : n \geq 1\} \cup \{\alpha, \beta\}$ equipped with the topology of basis the family $\{U_n : n \in N\} \cup \{V_\alpha, V_\beta\}$ where $V_\alpha = X - \{\beta\}$, $V_\beta = X - \{\alpha\}$ and $U_n = \{\frac{1}{n}, \frac{1}{n-1}, ..., 1\}$ for each integer $n \geq 1$. This space corresponds to the singular part of the quasi-orbits space of the group of (example 2.9 in [B.E.S2]), it satisfies the property a) but not the property b) of theorem 3.2.6). Indeed, the intersection $V_\alpha \cap V_\beta = Y - \{\alpha, \beta\}$ of the two quasi-compact open subsets V_α and V_β is not quasi-compact.

4. In the next step, we suppose that $a, c, b_n, n \in \mathbb{N}$ are elements of the plane $\mathbb{R}^2$ such that $a = (0,0)$, $c = (0,2)$ and $b_n = (n,1)$.
   Consider the set $Y = \{a, c\} \cup \{b_n : n \in \mathbb{N}\}$ equiped with the topology of basis the family $\{X, \{c\}\} \cup \{U_n, n \in \mathbb{N}\}$ where $U_n = \{b_n, c\}$, $n \in \mathbb{N}$. We have $\overline{\{a\}} = \{a\}$, $\overline{\{c\}} = X$ and $\overline{\{b_n\}} = \{a, b_n\}$, for $n \in \mathbb{N}$. This space satisfies the five properties of spectrality but it cannot be homeomorphic to any space $X - X_0$ neither as in theorem 3.2.6) nor as in theorem 3.2.7). Indeed, the closure of $\{c\}$ contains infinitely minimal sets (all $b_n, n \in \mathbb{N}$) in the open subset $U = Y - \{a\}$. But, under the hypothesis of theses theorems, the closure of any leaf (or of any orbit) contains at most finitely many minimal sets in a given nonempty saturated open subset [43].

5. Consider the set $Y = \{a, c\} \cup \{b_1, b_2, b_3\}$ equipped with the topology such that $\overline{\{a\}} = \{a\}$, $\overline{\{c\}} = X$ and $\overline{\{b_n\}} = \{a, b_n\}$ for $n = 1, 2, 3$. This space satisfies the five properties of spectrality but it cannot be homeomorphic to any space $X - X_0$ as in theorem 3.2.6). Indeed, the closure of {c} contains three distinct minimal sets (b_1, b_2, b_3) in the open set $Y - \{a\}$. But the closure of any orbit contains at most two minimal sets in a given non empty saturated open subset U [43].

   Obviously, each T_0-space with finitely many elements satisfies the five properties of spectrality. Then one can put the following question :

   Under what conditions is a finite T_0-space homeomorphic to a space $X - X_0$ corresponding to a foliation as in theorem 50 ?

6. A countable subgroup G of $\mathrm{Homeo}_+(\mathbb{R})$ satisfies the two properties of theorem 3.2.6) if one of the following holds :
   - G is polycyclic ; in particular, if G is abelian of finite type.
   - G is an abelian subgroup of $\mathrm{Diffeo}^2_+(\mathbb{R})$ (of finite type or not).
   - G is a subgroup of $\mathrm{Diffeo}^\omega_+(\mathbb{R})$ (of finite type or not, abelian or not).
   - G is a finite type subgroup of $\mathrm{Diffeo}^2_+([0,1])$.
   - The action of G on $\mathbb{R}$ is free.

7. Using the examples told in the property 3.2.1-(6), the remark 5 and the method used in example 2.9) of [5], we can build some countable groups G in $\mathrm{Homeo}_+(\mathbb{R})$ satisfying the property a) without satisfying the property b) of theorem 49 under one of the following conditions :
   - G is abelian (necessarily of non finite type).
   - G is a subgroup of $\mathrm{Diffeo}^\infty_+(\mathbb{R})$ generated by two elements (necessarily non abelian).
   - G is a subgroup of $\mathrm{Diffeo}^\infty_+([0,1])$ (necessarily of non finite type).

3.4 Equicontinuous Groups of Homeomorphisms

Let (E, d) be a metric space and Homeo(E) be its group of homeomorphisms. We suppose that Homeo(E) is equipped with the compact-open topology. We denote by G a countable equicontinuous subgroup of Homeo(E) :

for every $x \in E$, and for every $\epsilon > 0$ there exists $\eta > 0$ such that for every $g \in G$ and $y \in E$ we have $(d(x, y) \leq \eta \Longrightarrow d(g(x), g(y)) \leq \epsilon)$.

We remark that finite groups of homeomorphisms and groups of isometries are equicontinuous groups. As we said previously, we denote by E/G the orbits space and by $E/\widetilde{G}$ the quasi-orbits space.

3.4.1 General Notions

A homeomorphism f of a metric space (E, d) is called :

- *periodic* if there is an integer k such that $h^k(x) = x$ for all $x \in E$.
- *pointwise periodic* if for every $x \in E$, there exists an integer k (depending on x) such that $h^k(x) = x$.
- *almost periodic* if for every $\epsilon > 0$ there exists an integer $N \rangle 0$ such that if $k \in \mathbb{Z}$, then there exists $i \in \{k, k+1, ..., k+N\}$ such that for every $x \in E$ we have $d(f^i(x), x) \leq \epsilon$.
- *regular* if the group $\{f^n : n \in Z\}$ generated by $\{f\}$ is equicontinuous.

Remark 7 *We have the following implications :*

$$\textit{periodic} \Longrightarrow \textit{almost periodic} \Longrightarrow \textit{regular}$$

A nontrivial translation $T_a : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T_a(x) = x + a$, $a \neq 0$, is regular but it is not almost periodic. A rotation $r_\alpha : S^1 \to S^1$ defined by $r_\alpha(z) = ze^{2\pi i \alpha}$, where $\alpha \in \mathbb{R} - \mathbb{Q}$, is almost periodic but it is not periodic.

There are pointwise periodic nonperiodic homeomorphisms defined on a compact arcwise connected metric space. These homeomorphisms can be regular (example 2 of 3.4.2, or nonregular (example 3 of 3.4.2).

The property of almost periodicity is metric dependent. For example an irrational rotation of the plane is not almost periodic with the usual metric, but it is almost periodic when $\mathbb{R}^2$ is equipped with a metric inherited from the compact 2-sphere $\mathbb{R}^2 \cup \{\infty\}$ (one can see [34]).

If the space E is compact we have the following equivalences :

Theorem 51 ***(Gottschalk)*** *Let f be a homeomorphism of a compact metric space (E, d). The following statements are equivalent :*

- *f is almost periodic.*

- *f is regular.*
- *The closure $\overline{\{f^n : n \in \mathbb{Z}\}}$ of the group generated by $\{f\}$ is a compact subgroup of Homeo(E).*
- *There exists a distance on E compatible with its topology in which f is an isometry.*

The following theorem characterizes the regular homeomorphisms on a closed surface.

Theorem 52 *[3] Let f be an orientation-preserving regular homeomorphism of a closed orientable surface S. Then :*

- *If $S = S^2$, f is conjugate to a linear fractional transformation.*
- *If $S = T^2$, f is periodic or f is conjugate to the map $(x, y) \to (x + \alpha, y + \beta)$.*
- *If $\mathcal{X}(S) < 0$, f is periodic.*

Remark 8 *A regular homeomorphism f of the plane such that all its orbits are relatively compact is conjugate to rotation or reflection. In this case, we can extend f to a regular homeomorphism of the sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$ fixing ∞.*

Now, we will give some examples showing the great difference between "studying diffeomorphisms of the line $\mathbb{R}$" and "studying diffeomorphisms of a manifold of dimension $q \geq 2$".

- The matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ determines the well-known diffeomorphism of Anosov $\overline{A}$ on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. This diffeomorphism $\overline{A}$ has an everywhere dense orbit O_0 and the union of all finite orbits is also dense everywhere. Thus the union of all minimal sets is not a closed subset and the closure of O_0 contains many infinite minimal sets.
- The example 4 of 3.3.1 gives a diffeomorphim of the unit two-sphere such that all its orbits are minimal. Moreover, the union of all finite orbits and the union of all nonproper orbits are two everywhere dense subsets. In this example, the singular part $X - X_0$ of the quasi-orbits space X does not have any basis of quasi-compact open subsets.
- The example 4 of 3.4.2 gives an orientation-preserving, fixed point free and nonregular diffeomorphism of the plane $\mathbb{R}^2$ such that all its orbits are closed nonperiodic.

We remark that every increasing fixed point free diffeomorphism of the line $\mathbb{R}$ is topologically conjugate to a translation.

From theorem 53, it follows that these three diffeomorphisms of the previous examples are all nonregular ; indeed, the quasi-orbits space of each of them is not Hausdorff.

These examples check that many dynamical properties which are true in the line $\mathbb{R}$ are false in a surface ; for example, the following statements :

1. The union of all minimal sets is a closed subset.
2. The closure of any orbit contains at most a finitely many minimal sets in a given nonempty saturated open subset.
3. If any orbit is a minimal orbit, then either any orbit is closed or any orbit is everywhere dense.
4. The singular part $X - X_0$ of X is always homeomorphic to the primitive spectrum of an A.F C*-algebra equipped with the Jacobson topology (theorem 49).
5. The upper structure $US(O)$ of any orbit O is an open subset ([45]).

The upper structure $US(O)$ is the union of all orbits O' satisfying $O \subset \overline{O'}$ and $\overline{O} \neq \overline{O'}$. Concerning the properties 1), 2) and 3) one can see [43].

An orientation-preserving and fixed point free homeomorphism f on the plane $\mathbb{R}^2$ (called Brouwer homeomorphism) satisfies the following properties :

- f does not have any periodic points.
- Every point $x \in \mathbb{R}^2$ is wandering (i.e. there exists an open neighborhood U of x such that for each integer $n \in \mathbb{Z} - \{0\}$ we have $U \cap f^n(U) = \emptyset$).
- For every point $x \in \mathbb{R}^2$ there exists a proper embedding $h : \mathbb{R} \to \mathbb{R}^2$ such that $x \in h(\mathbb{R})$ and such that for each integer $n \in \mathbb{Z} - \{0\}$ we have $h(\mathbb{R}) \cap f^n(h(\mathbb{R})) = \emptyset$.

The Ferrand-Obata theorem says that the only non essential groups of conformal diffeomorphisms (i.e. non equal to groups of isometries) are groups of $\mathbb{R}^n$ or of S^n equiped with their standard metric. Those groups are well-known, the first is the group of similitudes and the second is the Möbius group.

Thurston studied diffeomorphisms of a compact connected surface. He showed that if we consider a compact connected oriented surface S (possibly with boundary) with Euler-characteristic $\mathcal{X}(S) < 0$, then any homeomorphism $f : S \to S$ can be isotopic to a unique homeomorphism $g : S \to S$ such that exactly one of the following holds :

- g is periodic.
- g is pseudo-Anosov.
- g is reducible : g leaves a finite family of closed curves $\{\gamma_1, ..., \gamma_n\}$ such that no pair of curves are homotopic and no curve is homotopic to a point or to a boundary component of S.

Newman, Montgomery-Zippin, Dress studied the pointwise homeomorphisms of a connected manifold. They showed, for example, that :

- Every periodic homeomorphism which coincides with the identity on a nonempty open subset is the identity everywhere (Newman).
- Every pointwise periodic homeomorphism is periodic (Montgomory-Zippin).

By using these two results, and with the aim to study the stability of foliations with all leaves compact, Epstein showed the following theorem :

If a group G of homeomorphisms on a connected manifold has all its orbits with finitely many elements, then the group G is finite.

The examples 1,2,and 3 of 3.4.2 show that the hypothesis of "manifold" in the results of Newman, Montgomory-Zippin and Epstein cited above is necessary.

We know also that any periodic, orientation-preserving transformation of a closed surface is conjugate to a conformal transformation (Brouwer, Epstein). We have the same result concerning periodic homeomorphism of the plane ([12]). In the last case, we can extend f to a homeomorphism of the sphere S^2 by identifying the plane $\mathbb{R}^2$ with the complement of the North pole using the stereographic projection.

Bing [2] has given an example of a two-periodic homeomorphism h of S^3 onto itself which cannot be linear in any coordinate and cannot be equivalent to an orthogonal transformation.

3.4.2 Dynamical Properties of Equicontinuous Groups

The purpose of this paragraph is to study some dynamical properties of countable equicontinuous groups of homeomorphisms of a metric space. This paragraph is inspired by [16].

The following theorem gives some dynamical properties of countable equicontinuous groups.

Theorem 53 *Let G be a countable equicontinuous group of homeomorphisms of a metric space (E, d). The following properties hold :*

1. *The closure of each orbit is a minimal set .*
2. *If each orbit is closed, then the orbit space E/G is Hausdorff.*
3. *The quasi-orbits space $E/\widetilde{G}$ is always Hausdorff.*
4. *If E is a compact space, then he closure $\overline{G}$ of G is a compact subgroup of homeomorphisms such that for all $x \in E$ we have $\overline{G}(x) = \overline{G(x)}$.*

Proof. By a direct calculation, we establish the property 1).

It is clear that the orbits space is not Hausdorff if and only if there exists two distinct orbits $G(x) \neq G(y)$ and two sequences (u_n) and (v_n) converging respectively to x and to y such that for each integer $n \in \mathbb{N}$, $v_n \in G(u_n)$.

Suppose that any orbit is closed and the orbits space is not Hausdorff. Since G is equicontinuous we can verify the fact : $x \in \overline{G(y)} = G(y)$ which contradicts the fact that

the orbits $G(x)$ and $G(y)$ are distinct, thus we obtain the property 2). In the same way, we can show the property 3). The property 4) follows from Ascoli's theorem. $\triangle$

Corollary 9 *Let G be a countable equicontinuous group of homeomorphisms on a connected metric space (E, d). Then :*

1. *If G has a locally dense orbit O, then every orbit is everywhere dense.*

2. *Each proper orbit O is closed.*

Proof. If O is a locally dense orbit, then its closure $\overline{O}$ is a minimal set with non empty interior, thus $\overline{O}$ is equal to E ; indeed, E is connected. We deduce that every orbit is everywhere dense.

Now, if O is proper, then $\overline{O} - O$ is a saturated closed subset strictly contained in the minimal set $\overline{O}$ and so $\overline{O} - O$ is empty which implies that O is a closed orbit. $\triangle$

If G is a countable group of homeomorphisms of a compact space E, then every orbit is relatively compact and so every closed orbit has only a finite number of elements. We can also state the analogous theorem.

Theorem 54 *Let G be a countable equicontinuous group of homeomorphisms on a n-Euclidean space E. If G has at least one relatively compact orbit, then every orbit is relatively compact In particular, if G has a finite orbit then every closed orbit is finite.*

In order to show this theorem, we check that the set $A = \{x \in E : \overline{G(x)} \text{ is compact}\}$ is open and closed. The fact that A is nonempty implies the equality $A = E$.

Remark 9 *The hypothesis of "equicontinuous" in theorem 4.2.1), corollary 4.2.2) and theorem 4.2.3) is necessary ; for this, one may see examples* 3, 4, 5 *which follow..*

One can ask the following question :

- Let $f : E \to E$ be a regular homeomorphism of the Euclidean space $E = R^n$ such that f coincides with the identity map on a nonempty open subset $U \subset E$. Is f equal to the identity map on the whole space E ?

Since each proper orbit is closed (corollary 9) and each closed orbit is finite (theorem 54). Thus, by the results of Newman and Montgomory-Zippin in the paragraph 3.4.1, we may answer positively to this question if all orbits of f are proper. This question is also true if we have $n = 1$ or $n = 2$. Since f is *a regular homeomorphism and every orbit is relatively compact* (theorem 54, the metric defined on E by :

$$d^*(x, y) = \sup_{p \in \mathbb{N}} d(f^p(x), f^p(y))$$

is an equivalent metric and f is an isometry of (E, d^*).

There are nonequicontinuous groups such that all their orbits are finite (see 3.4.2 examples 1 and 3. But, we have the following theorem :

Theorem 55 *[25] Let E be a metric space and G be a finitely generated subgroup of Homeo(E). Suppose that there exists an integer $p \in \mathbb{N}^*$ such that the cardinality of each orbit is less than or equal to p. Then G is finite.*

Proof.. First, we prove that for each orbit O with $\text{card}(O) \geq 2$ there exists a non trivial homomorphism $\varphi_O : G \to S_p$ (S_p is the symmetric group on p elements) such that the kernel of φ_O acts trivially on O. Using the facts that G is finitely generated and that S_p is a finite group, we deduce that there are only finite many possible homomorphisms $\varphi_1, ..., \varphi_n$ from G to S_p. Let K_i be the kernel of φ_i $(1 \leq i \leq n)$ and let $K = \bigcap_{i=1}^{n} K_i$. Since G/K is isomorphic to a subgroup of $G/K_1 \times ... \times G/K_n$ the group G/K is finite. Also, K acts trivially on all the orbits. This implies that $K = \{1\}$ and so G is finite. $\triangle$

An orbit $O = G(x)$ is said *regular* if there exists an open neighborhood U of x and an integer k depending on x such that for all $y \in U$ the cardinal of the orbit $G(y)$ is less than or equal to k.

Corollary 10 *[25] Let E be a metric space and G be a finitely generated subgroup of Homeo(E). If each orbit is regular, then the group G is equicontinuous.*

Proof. The open subset U of regular orbits in the previous definition can be chosen to be regular saturated. Thus from the previous theorem the group $G_U = \{g/_U : g \in G\}$ is finite, and so it is equicontinuous. This implies that G is equicontinuous. $\triangle$

Corollary 11 *[25] Let E be a metric space and G be a finitely generated subgroup of Homeo(E). If each orbit is regular and intersects a compact subset $K \subset E$, then G is finite.*

Examples

1. In this example the group G is not equicontinuous but all its orbits are of cardinality less than or equal to 2.

   In the plane $\mathbb{R}^2$ we consider the points $A(0,1)$, $B(0,-1)$ and $M_n(1 - \dfrac{1}{n+2}, 0)$ where n is a natural integer. We denote by C_n the arc of the circle which ends in the points A, B and passes through M_n and by $\overline{C_n}$ the symmetric image of C_n relatively to the y-axis. Let

$E = S^1 \bigcup \left(\bigcup_{n\geq 0} (C_n \cup \overline{C_n})\right)$. This space E is a compact, arcwise connected metric space but it is not locally arcwise connected. We define the map $f_n : E \to E$ by :

$$f_n(x,y) = \begin{cases} (-x,y) & if \quad (x,y) \in C_n \cup \overline{C_n} \\ (x,y) & if \quad not \end{cases}$$

For all integers n, f_n is a homeomorphism with $f_n \circ f_n = id$. Consider the group G generated by the family $\{f_n : n \in \mathbb{N}\}$. The cardinal of each orbit of this group is less than or equal to 2. Precisely, for each $(x,y) \in E - S^1$ the orbit $O_{(x,y)}$ at (x,y) is equal to the set $\{(-x,y),(x,y)\}$ and for each $(x,y) \in S^1$ we have $O_{(x,y)} = \{(x,y)\}$. The quotient space E/G is not Hausdorff : for each $(a,b) \in S^1$ the orbits $O_{(a,b)}$ and $O_{(-a,b)}$ are nonseparated. Thus, G is not equicontinuous.

In the following examples (z,u) indicates the coordinate of a point (x,y,u) in $\mathbb{R}^3 = \mathbb{C}\times\mathbb{R}$.

2. In this example we shall build a regular, pointwise periodic and nonperiodic homeomorphism f of a compact, arcwise connected and locally arcwise connected metric subspace E of $\mathbb{R}^3$.

   In $\mathbb{C}\times\mathbb{R}$ we consider the points $A(0,1)$ and $M_n(0, 1 - \dfrac{1}{n+1})$ where n is a natural integer, and the spheres S_n the diameter of which are the segments $[AM_n]$. The space E is the union of all spheres S_n and the homeomorphism f is defined by :

$$f(z,u) = (ze^{\frac{2i\pi}{n+1}}, u) \text{ for every } (z,u) \in S_n \text{ and every } n \geq 0.$$

   For each integer n the restriction of f to S_n is a rotation of axis $(u'u)$ and of angle $\dfrac{2\pi}{n+1}$. Thus, for every $z \in S_n$ the orbit O_z at z is contained in S_n and its cardinal is equal to n. So, this space E and this homeomorphism f satisfy the desired assumption of this example.

3. In this example we shall build a pointwise periodic nonregular homeomorphism f of a compact, arcwise connected and not locally arcwise connected metric subspace E of $\mathbb{R}^3$.

   In $\mathbb{C}\times\mathbb{R}$ we consider the points $A(0,1)$ and $M_n(0, \dfrac{1}{n+1})$ where n is a natural integer. Let S_n be the sphere of diameter the segment $[AM_n]$ and S_∞ be the sphere of diameter $[OA]$. The space E is the union of S_∞ and of all spheres S_n, and the homeomorphism f is defined by :

$$f(z,u) = \begin{cases} (ze^{\frac{2i\pi}{n+1}}, u) & \text{if} \quad (z,u) \in S_n \\ (z,u) & \text{if} \quad (z,u) \in S_\infty \end{cases}$$

As in the previous example, f is pointwise periodic but it is not periodic.We verify that S_∞ is the union of all nonseparated orbits ; so f is not regular. Thus this space E and this homeomorphism f answer to our question.

As in examples 2 and 3 :

- can one construct a pointwise periodic homeomorphism $f : E \to E$ which is not regular such that E is a compact, arcwise connected and locally arcwise connected metric subspace of R^2 ?

4. In this example we give a nonregular diffeomorophism $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that all orbits are closed nonperiodic.

   We consider the vector field on the plane $\mathbb{R}^2$ defined by $V(x,y) = ((1-x)(1+x), x)$. Each orbit of V is closed nonperiodic non singular and the quotient space $\mathbb{R}^2/V$ is not Hausdorff ; the vertical lines $\{x = -1\}$ and $\{x = 1\}$ are two nonseparated orbits. Let $(\varphi_t)_{t\in\mathbb{R}}$ be the flow associated to this vector field V.

   The diffeomorphism $f = \varphi_1 : R^2 \to R^2$ has all its orbits closed nonperiodic and the quotient space $\mathbb{R}^2/f$ is not Hausdorff ; so f is not regular.

5. By the same method as in the previous example, via a well choosen vector field on $\mathbb{R}^2$, one can give a diffeomorphism $f : R^2 \to R^2$ having a fixed point, a closed orbit which is not relatively compact and a proper orbit which is not closed.

   By means of rotations one can construct several examples of nontrivial periodic diffeomorphisms f of tori T^n and of spheres S^n. Here, we will give a nontrivial periodic diffeomorphism on the hyperbolic torus T^3_A.

6. The matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ determines the well-known diffeomorphism of Anosov $\overline{A}$ of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The hyperbolic torus T^3_A is the suspension of $\overline{A}$, it is the quotient space of $\mathbb{R}^3$ by the equivalence relation $\mathcal{R}$ defined by $(x,y,z)\mathcal{R}(x',y',z')$ if there exists a natural integer n satisfying

$$(x',y',z') = (A^n(x,y), z-n).$$

   Let B be a real square matrix of order 2 with coefficients in $\mathbb{Z}$ such that $B^2 = A$; for example, one can take $B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

   The map $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(x,y,z) = (B(x,y), z - \frac{1}{2})$ is compatible with the above equivalence relation ; that is, we have the following implication :

$$(x,y,z)\mathcal{R}(x',y',z') \Longrightarrow f(x,y,z)\mathcal{R}f(x',y',z').$$

Thus, f determines a mapping $\overline{f} : T_A^3 \to T_A^3$ defined by :

$$\overline{f}(\mathcal{R}(x,y,z)) = \mathcal{R}(f(x,y,z))$$

where $\mathcal{R}(a,b,c)$ is the trajectory of the point $(a,b,c,) \in \mathbb{R}^3$.

This map $\overline{f}$ is a nontrivial diffeomorphism satisfying $\overline{f} \circ \overline{f} = id$.

For any natural integer $q \in \mathbb{N}^*$, by the same manner, one can build a nontrivial periodic diffeomorphism $g : T_A^3 \to T_A^3$ of order q.

3.5 Groups of Homeomorphisms of a Metric Space

This paragraph is written by Hattab Hawete and it is inspired by [25].

A well known theorem [17] states that, if E is a connected manifold, then a group of homeomorphisms of E with all orbits finite is finite. In [16] the authors gave significative examples of infinite groups of homeomorphisms of connected, locally connected compact metric spaces whose orbits are all finite. The goal of this paragraph is to give some conditions on a group of homeomorphisms of a metric space with all orbits finite to be finite.

First, we study the group G in case $inf\{d(x,gx)\} > \varepsilon_0$ where ε_0 is a real positive number and the infimum is taken over all $x \in E$ and $g \in G \setminus G_x$. Secondly we show that, if E is connected and if there exists $p \in \mathbb{N}^*$ such that each orbit has cardinality p, then G itself has cardinality p. Moreover, we prove that if G is finitely generated and if there exists $p \in \mathbb{N}^*$ such that the cardinality of each orbit is less than or equal to p, then G is finite.

3.5.1 G satisfies a property $\wp$

We say that the group G satisfies the property $\wp$ if we have :
$\exists \epsilon_0 > 0$ such that $\forall$ x $\in E$, $\forall g \in G \setminus G_x$ we have $d(x, g(x)) > \epsilon_0$

Theorem 56 *Suppose that E is connected and that G has a finite orbit and satisfies property $\wp$. Then G is finite.*

Proof.

1. In this part we prove by two methods that if E is connected and G satisfies the property $\wp$, then the action of G on E is free.

(a) Let $x \in E$ and $g_0 \in G$ such that $g_0(x) = x$. Suppose that there exists $y \in E$ such that $g_0(y) \neq y$. Consider the function $f : z \in E \mapsto d(z, g_0(z)) \in \mathbb{R}_+$. This function is clearly continuous; since E is connected, $f(E)$ is an interval of $\mathbb{R}_+$ containing $[0, \varepsilon_0]$. Then there exists $z \in E$ such that $f(z)\frac{\varepsilon_0}{2}$; we conclude that $g_0(z) \neq z$ and $d(z, g_0(z) < \varepsilon_0$; but this contradicts the property $(\wp)$; this implies that $g_0 = id$ *i.e* the action of G on E is free.

(b) Let $g \in G$; we denote by $\mathrm{Fix}(g) = \{x \in E : gx = x\}$. We can easily see that $\mathrm{Fix}(g)$ is closed. It is also open. Indeed, let $x \in \mathrm{Fix}(g)$; because g is a continuous map, there exists $0 < \alpha < \frac{\epsilon_0}{3}$ such that $g(B(x, \alpha)) \subset B(x, \frac{\epsilon_0}{3})$. Then for all $y \in B(x, \alpha)$ we have $d(y, g(y)) < \epsilon_0$; this implies that $g(y) = y$ for all $y \in B(x, \alpha)$. Therefore $B(x, \alpha) \subset \mathrm{Fix}(g)$, so $\mathrm{Fix}(g)$ is open. Since E is connected, for all $g \in G$ $\mathrm{Fix}(g) = \emptyset$ or $\mathrm{Fix}(g) = E$. This implies that the action of G on E is free.

2. Suppose that G is infinite. If $G(x)$ is a finite orbit, then the isotropy group G_x is nontrivial; but this contradicts the fact that the action of G on E is free. Then G is necessarily finite. △

An orbit $G(x)$ is said to be *wandering* if there exists a neighborhood U_x of x such that, for all $g \in G \setminus \{id\}$, $U_x \cap g(U_x) = \emptyset$.

Corollary 12 *Suppose that E is connected and that G satisfies property $\wp$. Then each orbit is wandering.*

Proof. By (1) given in the proof of the above theorem, the action of G on E is free. Then for all $g \in G \setminus \{id\}$ we have $d(x, g(x)) > \epsilon_0$, so $g(B(x, \frac{\epsilon_0}{3})) \cap B(x, \frac{\epsilon_0}{3}) \neq \emptyset$; this proves that $G(x)$ is a wandering orbit. △

Theorem 57 *Suppose that E is compact and that G satisfies the property $\wp$ and that all the orbits are finite. Then the group G is finite in one of the following conditions:*

1. *G is equicontinuous;*
2. *E is locally connected.*

Proof.

1. (a) In this part we prove that if G is equicontinuous and satisfies property $\wp$, then for all $x \in E$ there exists a neighborhood U_x of x such that:

$$y \in U_x \Rightarrow G_x = G_y$$

We suppose to the contrary that there exists $x \in E$ and y close to x, but for some $g_0 \in G$, $g_0(x) = x$ and $g_0(y) \neq y$. Since G is equicontinuous, then there exists $0 < \alpha < \frac{\epsilon_0}{3}$ such that if $d(x, y) < \alpha$ then $d(g(x), g(y)) < \frac{\epsilon_0}{3}$ for any g $\in G$. If we choose $y \in B(x, \alpha)$, then $d(y, g_0(y)) \leq d(y, x) + d(x, g_0(y)) < \epsilon_0$ which contradicts the fact that $d(y, g_0(y)) > \epsilon_0$.

(b) By (a) for all $x \in E$ there exists an invariant neighborhood W_x of x (we can take $Sat(U_x)$, where U_x is the open neighborhood of x given in part (i)) such that, if $y \in W_x$ then $G_x = G_y$. Because E is a compact space, there exists a finite sequence $x_1,...,x_p$ such that $E \bigcup_{i=1}^{p} W_{x_i}$. We denote by $G_i = \{g \in G : g \neq id \text{ on } W_{x_i}\}$. By (i), all the points $x \in W_{x_i}$ have the same isotropy group. Because the orbits are finite, G_i is necessarily finite for all $1 \leq i \leq p$ which implies that G is finite.

2. Because E is compact and locally connected, E is a finite union of connected components $E_1,...,E_p$. We denote by $G_i\{g \in G : g \neq id \text{ on } E_i\}$. Suppose that there exists $x \in E_i$ and $g_0 \in G_i$ such that $g_0(x) = x$. Then there exists $y \in E_i$ such that $g_0(y) \neq y$. Consider the function $f : z \in E_i \mapsto d(z, g_0(z) \in \mathbb{R}_+$. This function is clearly continuous. Since E_i is connected, $f(E_i)$ is an interval of $\mathbb{R}_+$ containing $[0, \varepsilon_0]$. By the same argument given in (i) (proof of theorem 5.1.1) we got a contradiction. Then for all $x \in E_i$ and for all $g \in G_i$ we have $g(x) \neq x$. Since the orbits are finite, G_i is necessarily finite for all $1 \leq i \leq p$; this implies that G is finite. $\triangle$

Let $\overline{G}$ be the closure of G in $Homeo(E)$ equipped with the C^0 topology.

Corollary 13 *Suppose that E is compact and that G is equicontinuous and does not satisfies the property $(\wp)$. Then there exists $g \in \overline{G}$ having a fixed point.*

Proof. If G does not satisfy the property $(\wp)$ then for all $n \in \mathbb{N}^*$ there exits $x_n \in E$ and $g_n \in G \setminus G_{x_n}$ such that $d(x_n, g_n(x_n)) < \frac{1}{n}$. Since E is compact, we can suppose that the sequence (x_n) converges to x (if not, we can take a subsequence). By [16] $\overline{G}$ is compact and then there exists a subsequence (g_{n_p}) which converges to g. Then for a rather sufficiently large the sequence $(g_{n_p}(x_{n_p}))$ converges to $g(x)$. Since

$$d(g(x), x) \leq d(g(x), g_{n_p}(x_{n_p})) + d(g_{n_p}(x_{n_p}), x_{n_p}) < \frac{2}{n_p}$$

then $g(x) = x$. This ends the proof of the corollary. $\triangle$

If G is a finite group of homeomorphisms of a Hausdorff topological space E, then the set U of points with trivial isotropy group is open.

In [15] (Theorem 1) the author proves that, if a finite group acts effectively on a connected manifold, then the set of points with trivial isotropy group is open and is everywhere dense.

As it is shown in the example below the property above is not satisfied in the case when E is not a manifold but only a Hausdorff topological space.

Example In $\mathbb{R}^3$ we consider the points $O(0,0,0)$, $A(0,0,1)$ and $I(0,0,\frac{1}{2})$. We denote by S_2 the sphere of center I and diameter $[OA]$ and S_3 is the sphere of diameter $[IA]$. Let E $= S_2 \cup S_3$. We define the homeomorphism $f : E \to E$ such that the restriction of f on S_3 is the rotation with angle π around the z-axis and the identity elsewhere. Let U be the set

of points such that the isotropy subgroup is $S_3 \setminus \{I, A\}$ is open but not everywhere dense. $\triangle$

Theorem 58 *If G is a group of homeomorphisms of a connected metric space E such that each orbit has cardinality p, then $|G| = p$.*

Proof. Suppose that $|G| > p$. Let $x \in E$; then the isotropy group G_x of x is nontrivial. Let $g_1 \in G_x$ be different from the unity. Then the closure $\overline{U}$ of the (nonempty) set $U = \{t \in E : g_1(t) \neq t\}$ is a proper subset of E. Because E is connected, the intersection of $\overline{U}$ with the complement U^c of U is nonempty; let $z \in U^c \cap \overline{U}$; we have $g_1(z) = z$. Let $G(z)\{g_0(z) = g_1(z), ..., g_{p-1}(z)\}$ (where g_0 is the identity of G) be the orbit of z. Let $\delta = \inf_{i \neq j} d(g_i(z), g_j(z))$ and $\epsilon \in]0, \delta[$. By the equicontinuity of the (finite) family $\{g_1, ..., g_p\}$ there exists $\alpha \in]0, \epsilon[$ such that:

$$d(z, y) < \alpha \Rightarrow \sup_{i=1,...,p} d(g_i(z), g_i(y)) < \frac{\epsilon}{3}.$$

Let $t \in U$ such that $d(z, t) < \alpha$; then $\sup_i d(g_i(z), g_i(t)) < \frac{\epsilon}{3}$ which shows that all the points $g_i(t)$, $i = 1, ..., p$ are distinct one from each other. But also $g_1(t) \neq t$, $g_2(t) \neq t$,...,$g_p(t) \neq t$. Then the orbit of t has $p + 1$ elements; this contradicts the hypothesis. So $|G| = p$. $\triangle$

3.5.2 Finitely Generated Group

A Group G is said to be *periodic* if for any $g \in G$ there exists $n \in \mathbb{N}^*$ with $g^n = 1$. (Note that the integer n may depend on the element g).

One of the oldest problems in group theory was first posed by William Burnside in a paper published in 1902. Let G be a finitely generated periodic group. Is G necessarily a finite group?

The answer is negative (as shown by E.S. Golod [22] in 1964).

As a related question which seems as if it might have an easier answer, consider a periodic group G with the additional property that there exists a single integer p such that for all g in G, $g^p = 1$. A group with this property is said to be periodic with bounded exponent p, or just a group with exponent p. Then the Burnside problem is stated as, is a finitely generated group with exponent p finite?

This problem also has a negative answer, as shown by an example due to S.I Adian and P.S.Novikov in 1968 [1].

In this section we study a related question. Consider a finitely generated group of homeomorphisms of a metric space E such that the orbit of any point of E is finite with the additional property that there is a single integer p such that for all $x \in E$, $|G(x)| \leq p$. A group with this property is a finitely generated group with bounded exponent.

Theorem 59 *Let G be a finitely generated group of homeomorphisms of a metric space E. Suppose that there exists $p \in \mathbb{N}^*$ such that the cardinality of each orbit is less than or equal to p. Then G is finite.*

Proof.

1. The action of G on O implies that there exists a homomorphism φ from G to $S(O)$ ($S(O)$ is the set of self-bijections of O). But there exists an isomorphism f from $S(O)$ to $S_{|O|}$ and also there exists an injective homomorphism i from $S_{|O|}$ to S_p (S_p is the symmetric group on p elements) such that for any $\sigma \in S_{|O|}$ we have $i(\sigma)(j) = \begin{cases} \sigma(j) & if \quad 1 \leq j \leq |O| \\ j & if \quad |O| < j \leq p. \end{cases}$

   Hence, there is an homomorphism φ_O from G to S_p defined by $\varphi_O = i \circ f \circ \varphi$.

   If O is the orbit of a point $x \in E$ and if $|O| > 1$ then there exists $g \in G$ such that $g(x) \neq x$. Because $\varphi(g)$ is different from the identity of $S(O)$ and i and f are injective, $\varphi_O(g)$ is different from the identity of S_p. This implies that φ_O is nontrivial.

2. It is easy to show that if G is a finitely generated group having n generators and H be a finite group of order q, then there exists at most p^n homomorphisms from G to H.

3. For each orbit O ($|O| > 1$), there exists a nontrivial homomorphism $\varphi_O : G \to S_p$. Let g in the kernel of φ_O. By i) $\varphi_O(g) = i \circ f \circ \varphi_O(g) = 1_{S_p}$. Since i and f are injective, $\varphi(g) = 1_{S(O)}$. Then for all $x \in O$ we have $\varphi(g)(x) = x$; this implies that $g(x) = x$ for all $x \in O$ which shows that the kernel of φ_O acts trivially on O. Using part ii), there can only be finite many possible homomorphisms φ_1,..., φ_n from G to S_p. Let K_i be the kernel of φ_i ($1 \leq i \leq n$) and let $K = \bigcap_{i=1}^n K_i$. Since G/K_i is isomorphic to $\varphi_i(G) \subset S_p$ for $1 \leq i \leq n$, G/K_i is finite. Let π be the homomorphism of G in $G/K_1 \times ... \times G/K_n$ defined by $\pi(g) = (gK_1, ..., gK_n)$. The kernel of π is K, then G/K is isomorphic to a subgroup of $G/K_1 \times ... \times G/K_n$, so G/K is finite. For each orbit O ($|O| > 1$) there exists i_0 such that $\varphi_O = \varphi_{i_0}$, then K_{i_0} acts trivially on O which shows that K acts trivially on O. Then K acts trivially on all the orbits; this implies that $K = \{1\}$. Thus G is finite. $\triangle$

Remark 10 1. *We have the same result as the above theorem if G is a finitely generated group acting faithfully by permutations on a set S such that all orbits have size at most p.*

2. *A periodic group with bounded exponent can have infinite orbits. Indeed, Professor Robert Israel (University of British Columbia, Canada) confirms that any finitely generated group (and in particular a counterexample to Burnside's conjecture i.e an infinite periodic group with bounded exponent) can be faithfully represented as a group of homeomorphisms of a connected 2-dimensional manifold; so this group of homeomorphisms has necessarily an infinite orbit (otherwise, by [17] it will be finite).*

3. *The conditions "finitely generated and "each orbit with cardinality less than or equal to a given integer p" are necessary for the group G to be finite as it is shown in the following examples:*

   In the example 1 of 3.4.2 the group G is infinitely generated but each orbit is of cardinality less than or equal to 2.

In the example 3 of 3.4.2 the group G is finitely generated but the cardinality of orbits is not bounded.

Definition 74 *A point x of E is regular if there exists a neighborhood U_x of x and an integer k_x such that for all $y \in U_x$, $|G(y)| \leq k_x$.*

An orbit O of G is regular if it contains a regular point.

Theorem 60 *Suppose that G is a finitely generated group and that each orbit of G is regular. Then the group G is equicontinuous. In particular the quotient space E/G is Hausdorff.*

Proof. Let $a \in E$. Since the orbit $G(a)$ is regular, there exists an invariant neighborhood U of a and an integer k such that for all $x \in U$ we have $|G(x)| \leq k$. The group $G_U = \{g_{/U} : g \in G\}$ is a finitely generated group such that the cardinality of each orbit is less than or equal to k; this implies that G_U is finite. We suppose that $G_U\{f_0, ..., f_n\}$, where $f_0 = id$ and $f_i = g_{i/U}$ such that $g_i \in G$.

Let $\epsilon > 0$; by the equicontinuity of the finite family $\{f_0, ..., f_n\}$ there exists $\alpha > 0$ such that $B(a, \alpha) \subset U$ and

$$d(a, x) < \alpha \Rightarrow \sup_{i=0,...,n} d(f_i(a), f_i(x)) < \epsilon.$$

Let $g \in G$, there exists $f_i \in G_U$ such that $g_{/U} = f_i$. Then

$$d(a, x) < \alpha \Rightarrow d(g(a), g(x)) = d(f_i(a), f_i(x)) < \epsilon.$$

This implies that G is equicontinuous at a. Thus G is equicontinuous.

Because the group G is equicontinuous, the quotient space E/G is Hausdorff [16]. $\triangle$

Remark 11 *There exists a group G with irregular orbits and such that the quotient space E/G is Hausdorff (see example 1.4,[16]).*

Corollary 14 *Suppose that G is finitely generated group and that each orbit is regular and intersects a compact $K \subset E$, then the group G is finite.*

Proof. Each orbit is regular, for all $x \in E$ there exists an invariant neighborhood U_x of x and an integer k_x such that for all $y \in U_x$, $|G(y)| \leq k_x$. If every orbit of G cuts $K \subset E$ then $E = Sat(K)$. Because K is compact, there exists a finite sequence x_1,...,x_p such that $K \subset \bigcup_{i=1}^{p} U_{x_i}$. This implies that $E \bigcup_{i=1}^{p} U_{x_i}$. Let $k = \sup_{1 \leq i \leq p} k_{x_i}$. Then for all $x \in E$ we have $|G(x)| \leq k$; from Theorem 5.2.1 we conclude that G is finite. $\triangle$

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Differential Geometry and Topology, Discrete and Computational Geometry
M. Boucetta and J.-M. Morvan (Eds.)
IOS Press, 2005
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Chapter 4

Examples of Transverse Structures of Foliations

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These notes give an elementary introduction, trough some simple and significant examples, to the rich and beautiful subject of transverse structures of foliations.

4.1 Generalities

One of the most important object in mathematics is the vector space $\mathbb{R}^d$ with its usual Euclidean product $\langle x, y\rangle = \sum_{i=1}^d x_i y_i$. The fact that it admits a global system of coordinates makes easy the formulation of problems in analysis. But many problems may be formulated with some constraints like for instance:

Solve the problem (P) *in* $\mathbb{R}^d$ *with the constraints* $x_1^2 + \cdots + x_d^2 = 1$.

The problem (P) is in fact formulated in a (closed) subset M of $\mathbb{R}^d$ (the $(d-1)$-sphere $\mathbb{S}^{d-1}$) and not in the hull space. Locally topological space M given by similar conditions looks like an Euclidean one: for every point $x \in M$, there exist an open neighborhood U and a homeomorphism $\varphi : \mathbb{R}^k \longrightarrow U$; we say that (U, φ) is a *local chart* or a *local system* of coordinates: $(x_1, \cdots, x_k) = \varphi^{-1}(x)$ are the coordinates of the point x in the chart (U, φ). If (V, ψ) is another chart around x for which $(x'_1, \cdots, x'_k)$ are the coordinates, the map $\psi^{-1} \circ \varphi : \varphi^{-1}(U \cap V) \longrightarrow \psi^{-1}(U \cap V)$ is a homeomorphism such that $\psi^{-1} \circ \varphi(x_1, \cdots, x_k) = (x'_1, \cdots, x'_k)$. We say that $\psi^{-1} \circ \varphi$ is the *coordinate change* from the chart (U, φ) to the chart (V, ψ). If such property is satisfied for every point $x \in M$, we say that M is a *topological manifold.* In that case, there exist an open cover $\{U_i\}$ of M and a family of homeomorphisms $\varphi_i : \mathbb{R}^n \longrightarrow U_i$; $\{(U_i, \varphi_i)\}$ is called an *atlas* defining M. Let M be

a topological manifold of dimension k defined by an atlas (U_i, φ_i); we say that M is a *differentiable manifold* if, for every pair (i,j) such that $U_i \cap U_j \neq \emptyset$, the homeomorphism $\varphi_j^{-1} \circ \varphi_i$ is of class C^∞. We say that a differentiable manifold M is *connected, compact...* if the underlying topological space M is connected, compact... The notion of a "differentiable manifold" is central in differential geometry. Many geometrical structures can be defined on it; in this text we shall study a special type: *foliated structure.*

Let M be the Euclidean space $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ with canonical coordinates denoted $(x,y) = (x_1, \ldots, x_m, y_1, \ldots, y_n)$ and consider the family of affine subspaces F_y of M where $y \in \mathbb{R}^n$, defined by the differential system: $dy_1 = \ldots = dy_n = 0$. Then M, considered as a disjoint union of these spaces, is a non connected manifold of dimension m. Its topology is the product of the usual topology on $\mathbb{R}^m$ and the discrete one on $\mathbb{R}^n$. We say that M, with this structure, is a *foliated manifold* of *dimension* m and *codimension* n. It constitutes the *local model* of a *foliation* of codimension n on a manifold of dimension $m+n$. Let us denote this structure by $\mathfrak{F}$ on $\mathbb{R}^{m+n}$. Let $\mathcal{O}$ be an open set of $\mathbb{R}^{m+n}$; a $\mathfrak{F}$-*plaque* of $\mathcal{O}$ is any intersection of $\mathcal{O}$ with a horizontal space F_y. A diffeomorphism $\phi : \mathcal{O} \longrightarrow \mathcal{O}'$ is said to be $\mathfrak{F}$-*foliated* if it is of the form $\phi(x,y) = (\phi_1(x,y), \phi_2(y))$ that is, it sends any $\mathfrak{F}$-plaque of $\mathcal{O}$ into a $\mathfrak{F}$-plaque of $\mathcal{O}'$. This is summerazed in the following definition.

Definition 75 *Let M be a manifold of dimension $m+n$. A codimension n <u>foliation</u> $\mathcal{F}$ on M is given by an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ and for each i, a diffeomorphism $\varphi_i : \mathbb{R}^{m+n} \longrightarrow U_i$ such that, on each non empty intersection $U_i \cap U_j$, the coordinate change $\varphi_j^{-1} \circ \varphi_i : (x,y) \in \varphi_i^{-1}(U_i \cap U_j) \longrightarrow (x',y') \in \varphi_j^{-1}(U_i \cap U_j)$ is a $\mathfrak{F}$-foliated diffeomorphism, that is it has the form $x' = \varphi_{ij}(x,y)$ and $y' = \gamma_{ij}(y)$.*

The manifold M is decomposed into connected submanifolds of dimension m. Each of them is called a *leaf* of $\mathcal{F}$. A subset U of M is *saturated* for $\mathcal{F}$ if it is a union of leaves: if $x \in U$ then the leaf passing through x is contained in U.

Coordinate patches (U_i, φ_i) satisfying conditions of definition 1.1 are said to be *distinguished* for the foliation $\mathcal{F}$.

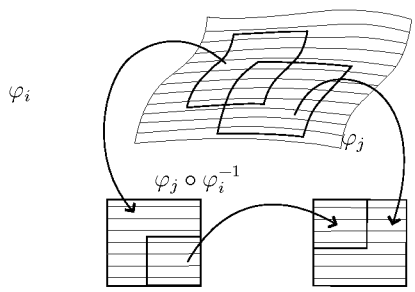

Fig.1

Let $\mathcal{F}$ be a codimension n foliation on M defined by a maximal atlas $\{(U_i, \varphi_i)\}_{i\in I}$ like in definition 1. Let $\pi : \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the second projection. Then the map $f_i : U_i \xrightarrow{\pi\circ\varphi_i^{-1}} \mathbb{R}^n$ is a submersion. On $U_i \cap U_j \neq \emptyset$ we have $f_j = \gamma_{ij} \circ f_i$. The fibres of the submersion f_i are the $\mathcal{F}$-plaques of U_i. The submersions f_i and the local diffeomorphisms γ_{ij} of $\mathbb{R}^n$ give a complete characterization of $\mathcal{F}$.

Definition 76 *A codimension n <u>foliation</u> on M is given by an open cover $(U_i)_{i\in I}$, submersions $f_i : U_i \longrightarrow T$ over a n dimensional transverse manifold T and, for $U_i \cap U_j \neq \emptyset$, a diffeomorphism $\gamma_{ij} : f_i(U_i \cap U_j) \subset T \longrightarrow f_j(U_i \cap U_j) \subset T$ satisfying $f_j(x) = \gamma_{ij} \circ f_i(x)$ for $x \in U_i \cap U_j$. We say that $\{U_i, f_i, T, \gamma_{ij}\}$ is a <u>foliated cocycle</u> defining $\mathcal{F}$.*

The foliation $\mathcal{F}$ is said to be *transversely orientable* if T can be given an orientation preserved by all the local diffeomorphisms γ_{ij}.

4.1.1 Induced Foliations

Let N and M be two manifolds; suppose that we are given a codimension n foliation $\mathcal{F}$ on M. We say that a map $f : N \longrightarrow M$ is *transverse* to $\mathcal{F}$ if, for each point $x \in N$, the tangent space T_yM of M at $y = f(x)$ is generated by $T_y\mathcal{F}$ and $(d_xf)(T_xN)$ (where d_xf is the tangent linear map of f at x) *i.e.*:

$$T_yM = T_y\mathcal{F} + (d_xf)(T_xN).$$

Equivalently, if we suppose that M is of dimension $m+n$, f is transverse to $\mathcal{F}$ if, for each $x \in N$, there exists a local system of coordinates $(x_1, \cdots, x_m, y_1, \cdots, y_n) : \mathbb{R}^{m+n} \longrightarrow V$ around y such that the map $g_U : (y_1^{-1}\circ f, \cdots, y_n^{-1}\circ f) : U = f^{-1}(V) \longrightarrow \mathbb{R}^n$ is a submersion. The collection of the local submersions (U, g_U) defines a codimension n foliation denoted $f^*(\mathcal{F})$ on N called the *pull-back foliation* of $\mathcal{F}$ by f.

If f is a submersion and $\mathcal{F}$ is the foliation by points, f is transverse to $\mathcal{F}$; in that case the leaves of $f^*(\mathcal{F})$ are exactly the fibres of f.

If $N = \widetilde{M}$ is the universal covering of M and f is the covering projection $f : \widetilde{M} \longrightarrow M$, then $f^*(\mathcal{F})$, denoted $\widetilde{\mathcal{F}}$, has the same dimension as $\mathcal{F}$; the two foliations $\widetilde{\mathcal{F}}$ and $\mathcal{F}$ have the same local properties.

4.1.2 Morphisms of Foliations

Let M and M' be two manifolds endowed respectively with two foliations $\mathcal{F}$ and $\mathcal{F}'$. A map $f : M \longrightarrow M'$ will be called *foliated* or a *morphism* between $\mathcal{F}$ and $\mathcal{F}'$ if, for every leaf L of $\mathcal{F}$, $f(L)$ is contained in a leaf of $\mathcal{F}'$; we say that f is an *isomorphism* if, in addition, f is a diffeomorphism; in this case the restriction of f to any leaf $L \in \mathcal{F}$ is a diffeomorphism on the leaf $L' = f(L) \in \mathcal{F}'$.

Suppose now that f is a diffeomorphism of M. Then for every leaf $L \in \mathcal{F}$, $f(L)$ is a leaf of a codimension n foliation $\mathcal{F}'$ on M; we say that $\mathcal{F}'$ is the *image* of $\mathcal{F}$ by the diffeomorphism f and we write $\mathcal{F} = f^*(\mathcal{F}')$. Two foliations $\mathcal{F}$ and $\mathcal{F}'$ on M are said to be C^r*-conjugated* (*topologically* if $r = 0$, *differentiably* if $r = \infty$ and *analytically* in the case $r = \omega$) if there exists a C^r-homeomorphism $f : M \longrightarrow M$ such that $f^*(\mathcal{F}') = \mathcal{F}$.

The set $\mathrm{Diff}^r(M, \mathcal{F})$ of C^r-diffeomorphisms of M which preseve the foliation $\mathcal{F}$ is a subgroup of the group $\mathrm{Diff}^r(M)$ of all the C^r-diffeomorphisms of M.

4.1.3 Frobenius Theorem

Let M be a manifold of dimension $m+n$. Denote by TM the tangent bundle of M and let E be a subbundle of rank m. Let U be an open set of M such that on U, TM is equivalent to the product $U \times \mathbb{R}^{m+n}$. At each point $x \in U$, the fibre E_x can be considered as the kernel of n differential 1-forms $\omega_1, \ldots, \omega_n$ linearly independent:

$$E_x = \bigcap_{j=1}^{n} \ker \omega_j(x). \tag{S}$$

The subbundle E is called an *m-plane field* on M. We say that E is *involutive* if, for every vector fields X and Y tangent to E (*i.e.* sections of E), the bracket $[X, Y]$ is also tangent to E. We say that E is *completely integrable* if, through each point $x \in M$, there exists a submanifold P_x of dimension m which admits $E_{|P_x}$ (the restriction of E to P_x) as tangent bundle. The maximal connected submanifolds satisfying this property are called the *integral submanifolds* of the differential system (S). They define a partition of M *i.e.* a codimension n foliation. We have the following theorem: *Let E be a subbundle of rank m given locally by a differential system like in* (S). *Then the following assertions are equivalent:*

– *E is involutive,*

– *E is completely integrable,*

– *there exist differential* 1*-forms (defined locally)* (β_{ij}) *, $i, j = 1, \ldots, n$ such that* $d\omega_i = \sum_{j=1}^{n} \beta_{ij} \wedge \omega_j \quad i = 1, \ldots, n$. Let ω be a non ingular 1-form on M. Then ω defines a codimension 1 foliation if and only if there exists a 1-form β such that $d\omega = \beta \wedge \omega$; this is equivalent to $\omega \wedge d\omega = 0$. In particular this is the case if ω is closed.

On the other hand the non singular 1-form on $\mathbb{R}^3$ given by $\omega = dx - zdy$ satisfies the relation $\omega \wedge d\omega = dx \wedge dy \wedge dz$ and cannot define a foliation. The plane field $E \subset T\mathbb{R}^3$, kernel of the 1-form ω, has the remarkable following property: given two points a and b in $\mathbb{R}^3$, there exists a differentiable curve $\gamma : [0, 1] \longrightarrow \mathbb{R}^3$ such that $\gamma(0) = a$ and $\gamma(1) = b$ and γ is tangent to E at every point. We say that ω defines a *contact structure*. Contact structures are the opposite of foliated structures.

4.1.4 Holonomy of a Leaf

This is a very important notion in foliation theory. In many situations it determines completely the structure of the foliation. In this subsection, we will introduce this concept. We will give later some examples.

Let $\mathcal{F}$ be a codimension n foliation on M, let L be a leaf of $\mathcal{F}$ and $x \in L$. Let T be a small transversal to $\mathcal{F}$ passing through x. Let $\sigma : [0,1] \longrightarrow L$ be a continuous path such that $\sigma(0) = \sigma(1) = x$. Then there exist a finite open cover U_i, $i = 0, 1, \ldots, k$ of M with $U_0 = U_k$ and a subdivision $0 = t_0 < t_1 < \ldots < t_k = 1$ of $[0,1]$ such that:

– $\sigma([t_{i-1}, t_i]) \subset U_i$,

– if $U_i \cap U_j \neq \emptyset$ then $U_i \cup U_j$ is contained in a distinguished chart of $\mathcal{F}$.

We say that $\{U_i\}$ is a *subordinated chain* to σ. For $i = 0, 1, \ldots, k$ let T_i be a small transversal to $\mathcal{F}$ passing through $\sigma_i(t)$ with $T_0 = T_k = T$. For every point $z \in T_i$, sufficiently close to $\sigma(t_i)$, the plaque of $\mathcal{F}$ passing through z intersects T_{i+1} in a unique point $f_i(z)$. The domain of f_i contains a transversal T_i' passing through $\sigma(t_i)$ and homeomorphic to an open ball of $\mathbb{R}^n$. Then, it is clear that the map: $f_\sigma = f_{k-1} \circ f_{k-2} \circ \ldots \circ f_0$ is well defined on an open neighbourhood of x; it is called the *holonomy map* associated to σ. We can prove (see [14] for instance) that the germ of f_σ:

does not depend on the chain U_i, $i = 1, \ldots, k$ and in the choice of σ in its homotopy class in the group $\pi_1(L, x)$ of the homotopy classes of loops based at x,

– satisfies $f_\sigma(x) = x$.

So we get a homomorphism $h : [\sigma] \in \pi_1(L, x) \longrightarrow f_\sigma \in G(T, x)$ where $G(T, x)$ is the group of germs of diffeomorphisms of T fixing the point x. This representation h is called the *holonomy* of the leaf L at x; the image of $\pi_1(L)$ (L is path connected) is called the *holonomy group* of the leaf L. The foliation $\mathcal{F}$ is said to be *without holonomy* if every leaf L of $\mathcal{F}$ has a trivial holonomy group.

4.2 Transverse Structures

Let us fix some notations. Let $\mathcal{F}$ be a codimension n foliation on M. We denote by $T\mathcal{F}$ the tangent bundle to $\mathcal{F}$ and $\nu\mathcal{F}$ the quotient $TM/T\mathcal{F}$ which is the *normal bundle* to $\mathcal{F}$; $\mathfrak{X}(\mathcal{F})$ will denote the space of sections of $T\mathcal{F}$ (elements of $\mathfrak{X}(\mathcal{F})$ are vector fields $X \in \mathfrak{X}(M)$ tangent to $\mathcal{F}$). A differential form $\alpha \in \Omega^r(M)$ is said to be *basic* if it satisfies $i_X\alpha = 0$ and $L_X\alpha = 0$ for every $X \in \mathfrak{X}(\mathcal{F})$. (Here i_X and L_X denote respectively the inner product and the Lie derivative with respect to the vector field X.) For a function $f : M \longrightarrow \mathbb{R}$, these conditions are equivalent to $X \cdot f = 0$ for every $X \in \mathfrak{X}(\mathcal{F})$ *i.e.* f is constant on the leaves of $\mathcal{F}$; we denote by $\Omega^r(M/\mathcal{F})$ the space of basic forms of degree r on the foliated manifold $(M, \mathcal{F})$; this is a module over the algebra $\mathfrak{B}$ of basic functions. A vector field $Y \in \mathfrak{X}(M)$ is said to be *foliated*, if for every $X \in \mathfrak{X}(\mathcal{F})$, the bracket $[X, Y] \in \mathfrak{X}(\mathcal{F})$. We can easily see that the set $\mathfrak{X}(M, \mathcal{F})$ of foliated vector fields is a Lie algebra and a $\mathfrak{B}$-module; by definition $\mathfrak{X}(\mathcal{F})$ is an ideal of $\mathfrak{X}(M, \mathcal{F})$ and the quotient $\mathfrak{X}(M/\mathcal{F}) = \mathfrak{X}(M, \mathcal{F})/\mathfrak{X}(\mathcal{F})$ is called the Lie

algebra of *transverse* (or *basic*) vector fields on the foliated manifold $(M, \mathcal{F})$. Also, it has a module structure over the algebra $\mathfrak{B}$.

Let M be a manifold of dimension $m+n$ endowed with a codimension n foliation $\mathcal{F}$ defined by a foliated cocycle $\{U_i, f_i, T, \gamma_{ij}\}$ like in definition 2.

Definition 77 *A* <u>*transverse structure*</u> *to $\mathcal{F}$ is a geometric structure on T invariant by the local diffeomorphisms γ_{ij}.*

This is a very important notion in foliation theory. To make it clear, let us give the main examples of such structures.

4.2.1 Lie Foliations

We say that $\mathcal{F}$ is a *Lie G-foliation*, if T is a Lie group G and γ_{ij} are restrictions of left translations on G. Such foliation can also be defined by a 1-form ω on M with values in the Lie algebra $\mathcal{G}$ such that:

i) $\omega_x : T_xM \to \mathcal{G}$ is surjective for every $x \in M$,

ii) $d\omega + \frac{1}{2}[\omega, \omega] = 0$. If $\mathcal{G}$ is Abelian, ω is given by n linearly independant closed scalar 1-forms $\omega_1, \ldots, \omega_n$. In the general case, the structure of a Lie foliation on a compact manifold, is given by the following theorem due to E. Fédida [11]: *Let $\mathcal{F}$ be a Lie G-foliation on a compact manifold M. Let $\widetilde{M}$ be the universal covering of M and $\widetilde{\mathcal{F}}$ the lift of $\mathcal{F}$ to $\widetilde{M}$. Then there exist a homomorphism $h : \pi_1(M) \longrightarrow G$ and a locally trivial fibration $D : \widetilde{M} \longrightarrow G$ whose fibres are the leaves of $\widetilde{\mathcal{F}}$ and such that, for every $\gamma \in \pi_1(M)$, the following diagram is commutative:*

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\gamma} & \widetilde{M} \\ D \downarrow & & \downarrow D \\ G & \xrightarrow{h(\gamma)} & G(*) \end{array}$$

where the first line denotes the deck transformation of $\gamma \in \pi_1(M)$ on $\widetilde{M}$. The subgroup $\Gamma = h(\pi_1(M)) \subset G$ is called the *holonomy group* of $\mathcal{F}$ although the holonomy of each leaf is trivial. The fibration $D : \widetilde{M} \longrightarrow G$ is called the *developing map* of $\mathcal{F}$. This theorem gives also a way to construct a Lie foliation. Let us see explicitly a particular example. Let M be the 2-torus $\mathbb{T}^2$; its universal covering $\widetilde{M}$ is $\mathbb{R}^2$ and its fundamental group is $\Gamma = \mathbb{Z}^2$. Denote by h the morphism from Γ to the Lie group $G = \mathbb{R}$ given by $h(m,n) = n + \alpha m$ where α is a real positive number. For convenience we will consider that the action of an element $(m,n) = \gamma \in \Gamma$ on $\mathbb{R}^2$ is given by:

$$(x+y) \in \mathbb{R}^2 \overset{\gamma}{\longmapsto} (x-m, y+n) \in \mathbb{R}^2.$$

Let $D : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the submersion defined by $D(x, y) = y - \alpha x$. It is not difficult to see that, for any $\gamma \in \Gamma$, the diagram:

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\gamma} & \mathbb{R}^2 \\ D \downarrow & & \downarrow D \\ \mathbb{R} & \xrightarrow{h(\gamma)} & \mathbb{R} \end{array}$$

is commutative *i.e.* the fibration

$$D : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

is equivariant under the action of Γ on $\mathbb{R}^2$ and then induces a Lie foliation on $\mathbb{T}^2$ transversely modeled on the Lie group $\mathbb{R}$.

4.2.2 Transversely Parallelizable Foliations

We say that $\mathcal{F}$ is *transversely parallelizable* if there exist on M foliated vector fields $Y_1, \ldots, Y_n$, transverse to $\mathcal{F}$ and everywhere linearly independent. This means that the manifold T admits a parallelism $(Y_1, \ldots, Y_n)$ invariant by all the local diffeomorphisms γ_{ij} or, equivalently, that the $\mathfrak{B}$-module $\mathfrak{X}(M/\mathcal{F})$ is free of rank n. The structure of a transversely parallelizable foliation on a compact manifold is given by the following theorem due to L. Conlon [4] for $n = 2$ and in general to P. Molino [19]. *Let $\mathcal{F}$ be a transversely parallelizable foliation of codimension n on a compact manifold M. Then:*

(1) *the closures of the leaves are submanifolds which are fibres of a locally trivial fibration $\pi : M \longrightarrow W$ where W is a compact manifold,*

(2) *there exists a simply connected Lie group G_0 such that the restriction $\mathcal{F}_0$ of $\mathcal{F}$ to any leaf closure F is a Lie G_0-foliation,*

(3) *the cocycle of the fibration $\pi : M \longrightarrow W$ has values in the group of diffeomorphisms of F preserving $\mathcal{F}_0$.*

The fibration $\pi : M \longrightarrow W$ and the manifold W are called respectively the *basic fibration* and the *basic manifold* associated to $\mathcal{F}$. This theorem says that if, in particular, the leaves of $\mathcal{F}$ are closed, then the foliation is just a fibration over W. This is still true even if the leaves are not closed: the manifold M is a fibration over the leaf space $M/\mathcal{F}$ which is, in this case, a *Q-manifold* in the sense of [1]. The preceding theorem is still valid for *transversely complete foliations* on non compact manifolds (cf. [19]).

Any Lie foliation is transversely parallelizable. This is a consequence of the fact that a Lie group is parallelizable and that the parallelism can be chosen invariant by left translations.

4.2.3 Riemannian foliations

The foliation $\mathcal{F}$ is said to be *Riemannian* if there exists on T a Riemannian metric such that the local diffeomorphisms γ_{ij} are isometries. Using the submersions $f_i : U_i \longrightarrow T$ one can construct on M a Riemannian metric which can be written in local coordinates:

$$ds^2 = \sum_{i,j=1}^{m} \theta_i \otimes \theta_j + \sum_{k,\ell=1}^{n} g_{k\ell}(y) dy_k \otimes dy_\ell.$$

(This metric is said to be *bundle like*.) Equivalently, $\mathcal{F}$ is Riemannian, if any geodesic orthogonal to the leaves at a point is orthogonal to the leaves everywhere. See the paper [20] by B. Reinhart who introduced firstly the notion of Riemannian foliation. Let $\mathcal{F}$ be Riemannian. Then there exists a Levi-Civita connection, transverse to the leaves which, by unicity argument, coincides on any distinguished open set, with the pull-back of the Levi-Civita connection on the Riemannian manifold T. This connection is said to be *projectable*. Let:

$$\mathrm{O}(n) \longrightarrow M^{\#} \xrightarrow{\tau} M$$

be the principal bundle of orthonormal frames transverse to $\mathcal{F}$. The following theorem is due to P. Molino [19]:

Suppose M is compact. Then, the foliation $\mathcal{F}$ can be lifted to a foliation $\mathcal{F}^{\#}$ on $M^{\#}$ of the same dimension and such that:

(1) *$\mathcal{F}^{\#}$ is transversely parallelizable,*

(2) *$\mathcal{F}^{\#}$ is invariant under the action of $O(n)$ on $M^{\#}$ and projects, by τ, on $\mathcal{F}$.*

The basic manifold $W^{\#}$ and the basic fibration $F^{\#} \longrightarrow M^{\#} \xrightarrow{\pi^{\#}} W^{\#}$ are called respectively the *basic manifold* and the *basic fibration* of $\mathcal{F}$. We have the following properties:

– the restriction of τ to a leaf of $\mathcal{F}^{\#}$ is a covering over a leaf of $\mathcal{F}$. So all leaves of $\mathcal{F}$ have the same universal covering (*cf.* [20]),

the closure of any leaf of $\mathcal{F}$ is a submanifold of M and the leaf closures define a *singular foliation* (the leaves have different dimensions) on M. (For more details about this notion see [19]) Another interesting result for Riemannian foliations is the Global Reeb Stability Theorem which is valid even if the codimension is greater than 1. *Let $\mathcal{F}$ be a Riemannian foliation on a compact manifold M. If there exists a compact leaf with finite fundamental group, then all leaves are compact with finite fundamental group.*

The property $\mathcal{F}$ is Riemannian means that the leaf space $Q = M/\mathcal{F}$ is a Riemannian manifold even if it this space does not support any differentiable structure !

4.2.4 $\mathcal{G}/\mathcal{H}$-foliations

This is a class of foliations which possess interesting transverse properties (see [6]). Let $\mathcal{G}$ be a Lie algebra of dimension d and $\mathcal{H}$ a Lie subalgebra of $\mathcal{G}$. We fix a basis $e_1, \ldots, e_d$ of $\mathcal{G}$ such that $e_{n+1}, \ldots, e_d$ span $\mathcal{H}$ and denote by $\theta^1, \ldots, \theta^d$ the corresponding dual basis. One has $[e_i, e_j] = \sum_k K_{ij}^k e_k$, where the *structure constants* K_{ij}^k fulfill the relations

$$\text{(C1)} \qquad K_{ij}^k = -K_{ji}^k$$

$$\text{(C2)} \qquad \sum_i (K_{ij}^k K_{rs}^i + K_{ir}^k K_{sj}^i + K_{is}^k K_{jr}^i) = 0 \text{ (Jacobi identity)}$$

$$\text{(C3)} \qquad K_{ij}^k = 0 \text{ if } k \leq n \text{ and } n+1 \leq i, j$$

The set of constants K_{ij}^k satisfying (C1) and (C2) determines the Lie algebra structure of $\mathcal{G}$ while (C3) states that $\mathcal{H}$ is a Lie subalgebra of $\mathcal{G}$. We denote by G the simply connected Lie group with Lie algebra $\mathcal{G}$ and by H the connected Lie subgroup of G corresponding to the Lie subalgebra $\mathcal{H}$.

We shall denote by θ the $\mathcal{G}$-valued 1-form on G which is the identity over the left invariant vector fields on G, i.e. $\theta = \sum_k \theta^k \otimes e_k$. Let $\omega = \sum_k \omega^k \otimes e_k$ be a $\mathcal{G}$-valued 1-form on a manifold M. An element $g \in G$ transforms ω into the $\mathcal{G}$-valued form $\mathrm{Ad}_g\omega$ where $\mathrm{Ad}_g\omega(X) = \mathrm{Ad}_g \cdot (\omega(X))$ for any vector field X on M. Once the basis $e_1, \ldots, e_d$ of $\mathcal{G}$ has been fixed, we shall identify ω with the n-tuple of scalar 1-forms $(\omega^1, \ldots, \omega^d)$. In particular $\theta = (\theta^1, \ldots, \theta^d)$. *Let a $\mathcal{G}$-valued 1-form $\omega = (\omega^1, \ldots, \omega^d)$ on a connected manifold M be given. Assume that ω fulfills the Maurer-Cartan equation* $d\omega + \frac{1}{2}[\omega, \omega] = 0$, i.e.

$$\text{(C4)} \qquad d\omega^k = -\frac{1}{2} \sum_{i,j=1}^{d} K_{ij}^k \omega^i \wedge \omega^j$$

and that $\omega^1, \ldots, \omega^n$ are linearly independent. Then the differential system $\omega^1 = \cdots = \omega^n = 0$ is integrable and defines a codimension n foliation $\mathcal{F}$. We shall say that $\mathcal{F}$ is a $\mathcal{G}/\mathcal{H}$-foliation defined by the $\mathcal{G}$-valued form ω.

Main example. Let $M = G$. Then $\theta = (\theta^1, \ldots, \theta^d)$ defines a $\mathcal{G}/\mathcal{H}$-foliation $\mathcal{F}_{G,H}$ whose leaves are the left cosets of H.

Remark. The notion of $\mathcal{G}/\mathcal{H}$-foliation includes several classes of geometric structures: (a) If $n = \dim M$ and H is closed then a $\mathcal{G}/\mathcal{H}$-foliation $\mathcal{F}$ defines a structure of locally homogeneous space on M; that is, the manifold M is locally modeled on the homogeneous space G/H with coordinate changes given by left translations by elements of G and $\mathcal{F}$ is the foliation by points. The homogeneous space G/H is endowed with a $\mathcal{G}/\mathcal{H}$-foliation when the projection $G \to G/H$ admits a global section. (b) When $\mathcal{H} = 0$, $\mathcal{G}/\mathcal{H}$-foliations are just Lie foliations modeled over G. For instance a non-singular closed 1-form ω on M defines a Lie foliation modeled over $\mathbb{R}$. (c) If H is closed then a $\mathcal{G}/\mathcal{H}$-foliation is a transversely homogeneous foliation modeled over the homogeneous space G/H. Every transversely homogeneous foliation is given locally by a collection of 1-forms $\omega^1, \ldots, \omega^d$ fulfilling (C4) (cf. [2]). If these forms are global then they define a $\mathcal{G}/\mathcal{H}$-foliation. This is the case if $H^1(M, H) = 0$ (cf. [2]). (d) In general, when H is not necessarily closed, a $\mathcal{G}/\mathcal{H}$-foliation is a locally transversely homogeneous foliation as it is defined in [19].

Let $\mathcal{F}$ be a $\mathcal{G}/\mathcal{H}$-foliation on M defined by ω. A map $\varphi : N \to M$ transverse to $\mathcal{F}$ induces a $\mathcal{G}/\mathcal{H}$-foliation $\varphi^*\mathcal{F}$ on N which is defined by $\varphi^*\omega$. We say that $\varphi^*\mathcal{F}$ is the *pull-back*

of $\mathcal{F}$ by φ. In particular, the universal covering space $\widetilde{M}$ of M is endowed with the $\mathcal{G}/\mathcal{H}$-foliation $\widetilde{\mathcal{F}}$ defined by $\pi^*\omega$ where $\pi : \widetilde{M} \to M$ is the canonical projection. The following proposition states that the $\mathcal{G}/\mathcal{H}$-foliation $\widetilde{\mathcal{F}}$ on $\widetilde{M}$ is a pull-back of the $\mathcal{G}/\mathcal{H}$-foliation $\mathcal{F}_{G,H}$ on G which was considered as the main example.

Proposition [2]. *Let $\mathcal{F}$ be a $\mathcal{G}/\mathcal{H}$-foliation on M defined by ω and let $\widetilde{\mathcal{F}} = \pi^*\mathcal{F}$ be its pull-back to the universal covering space $\widetilde{M}$ of M. There are a map $\mathcal{D} : \widetilde{M} \to G$ and a group representation $\rho : \pi_1(M) \to G$ such that (i) $\mathcal{D}$ is $\pi_1(M)$-equivariant, i.e. $\mathcal{D}(\gamma \cdot \tilde{x}) = \rho(\gamma) \cdot \mathcal{D}(\tilde{x})$ for any $\gamma \in \pi_1(M)$, and (ii) $\widetilde{\omega} := \pi^*\omega = \mathcal{D}^*\theta$, i.e. $\widetilde{\mathcal{F}} = \mathcal{D}^*\mathcal{F}_{G,H}$. The map $\mathcal{D}$ is called the developing map of $\mathcal{F}$ and it is uniquely determined up to left translations by elements of G.*

4.2.5 Transversely Holomorphic Foliations

The foliation $\mathcal{F}$ is said to be *transversely holomorphic* if T is a complex manifold and the γ_{ij} are local biholomorphisms. Particular case is a *holomorphic foliation*: the manifolds M and T are complex, all the f_i are holomorphic and all γ_{ij} are local biholomorphisms.

If T is Kählerian and γ_{ij} are biholomorphisms preserving the Kähler form on T we say that $\mathcal{F}$ is *transversely Kählerian*. For example, any codimension 2 Riemannian foliation which is transversely orientable is transversely Kählerian.

Let us give concrete examples of such foliations. Let M be the unit sphere in the Hermitian space $\mathbb{C}^{n+1}$:

$$M = \mathbb{S}^{2n+1} = \left\{ (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : \sum_{k=1}^{n+1} |z_k|^2 = 1 \right\}.$$

Let Z be the holomorphic vector field on $\mathbb{C}^{n+1}$ given by the formula: $Z = \sum_{k=1}^{n+1} a_k z_k \frac{\partial}{\partial z_k}$ where $a_k = \alpha_k + i\beta_k \in \mathbb{C}$. There exists a good choice of the numbers a_k such that the orbits of Z intersect transversely the sphere M; then Z induces on M a real vector field X which defines a foliation $\mathcal{F}$. It is not difficult to see that $\mathcal{F}$ is transversely holomorphic. It is transversely Kählerian if we choose in addition $\alpha_k = 0$ for any $k = 1, \ldots, n+1$.

4.3 More Examples

4.3.1 Simple Foliations

Two trivial foliations can be defined on a manifold M: the first one is obtained by considering that all the leaves are the points; the second one has only one leaf, namely, M itself.

Every submersion $M \xrightarrow{\pi} B$ with connected fibres defines a foliation. The leaves being the fibres $\pi^{-1}(b)$, $b \in B$. In particular, every product $F \times B$ is a foliation with leaves $F \times \{b\}$, $b \in B$. These foliations are transversely orientable if, and only if, the manifold B is orientable.

4.3.2 Linear Foliation on the Torus $\mathbb{T}^2$

This example already differently described in the subsection 2.2. Let $\widetilde{M} = \mathbb{R}^2$ and consider the linear differential equation $dy - \alpha dx = 0$ where α is a real number. This equation has $y = \alpha x + c$, $c \in \mathbb{R}$ as general solution. When c varies, we obtain a family of parallel lines which defines a foliation $\widetilde{\mathcal{F}}$ in $\widetilde{M}$. The natural action of $\mathbb{Z}^2$ on $\widetilde{M}$ preserves the foliation $\widetilde{\mathcal{F}}$ (*i.e.* the image of any leaf of $\widetilde{\mathcal{F}}$ by an integer translation is a leaf of $\widetilde{\mathcal{F}}$). Then $\widetilde{\mathcal{F}}$ induces a foliation $\mathcal{F}$ on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The leaves are all diffeomorphic to the circle $\mathbb{S}^1$ if α is rational and to the real line if α is not rational (*cf.* Fig. 2). In fact, if α is not rational, every leaf of $\mathcal{F}$ is dense; this shows that even if locally a foliation is simple, globally it can be complicated.

Fig. 2

4.3.3 One Dimensional Foliations

Let M be a closed (i.e. without boundary) manifold of dimension n. Let X be a non singular vector field on M that is, for every $x \in M$, the vector T_xM is non zero. Then its integral curves are leaves of a 1-dimensional foliation. This is also the case for a line bundle on M (not necessarily a vector field). In fact there is a natural one-to-one correspondence between the set of C^∞-line bundles and the set of 1-dimensional C^∞-foliations. The fact that M admits a one dimensional foliation depends on its topology. For each $r = 0, 1, \ldots, n$, let $H^r(M, \mathbb{R})$ denote the real r-th *cohomology space* of M which is finite dimensional. Then the number:

$$\chi(M) = \sum_{r=0}^{n} (-1)^r \dim H^r(M, \mathbb{R})$$

is a *topological invariant* called the *Euler-Poincaré number* of M. We have the following theorem: *The manifold M admits a one dimensional foliation if, and only if, $\chi(M) = 0$.*

4.3.4 Reeb Foliation on the 3-sphere $\mathbb{S}^3$

Let M be the 3 dimensional sphere $\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$. Denote by $\mathbb{D}$ the open unit disc in $\mathbb{C}$ and $\overline{\mathbb{D}}$ its closure which is the closed unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$. The two subsets:

$$M_+ = \left\{(z_1, z_2) \in \mathbb{S}^3 : |z_1|^2 \leq \frac{1}{2}\right\} \text{ and } M_- = \left\{(z_1, z_2) \in \mathbb{S}^3 : |z_2|^2 \leq \frac{1}{2}\right\}$$

are diffeomorphic to $\overline{\mathbb{D}} \times \mathbb{S}^1$. They have $\mathbb{T}^2$ as common boundary:

$$\mathbb{T}^2 = \partial M_+ = \partial M_- = \left\{(z_1, z_2) \in \mathbb{S}^3 : |z_1|^2 = |z_2|^2 = \frac{1}{2}\right\}$$

and their union is equal to $\mathbb{S}^3$. Then $\mathbb{S}^3$ can be obtained by gluing M_+ and M_- along their boundaries by the diffeomorphism $(z_1, z_2) \in \partial M_+ \longrightarrow (z_2, z_1) \in \partial M_-$, i.e. we identify (z_1, z_2) with (z_2, z_1) in the disjoint union $M_+ \coprod M_-$. Let $f : \mathbb{D} \longrightarrow \mathbb{R}$ be the function defined by:

$$f(z) = \exp\left(\frac{1}{1 - |z|^2}\right).$$

Let t denote the second coordinate in $\mathbb{D} \times \mathbb{R}$. The family of surfaces $(S_t)_{t \in \mathbb{R}}$ obtained by translating the graph S of f along the t-axis defines a foliation on $\mathbb{D} \times \mathbb{R}$. If we add the cylinder $\mathbb{S}^1 \times \mathbb{R}$, where $\mathbb{S}^1$ is viewed as the boundary of $\overline{\mathbb{D}}$, we obtain a codimension one foliation $\widetilde{\mathcal{F}}$ on $\overline{\mathbb{D}} \times \mathbb{R}$. By construction, $\widetilde{\mathcal{F}}$ is invariant by the transformation $(z, t) \in \overline{\mathbb{D}} \times \mathbb{R} \longmapsto (z, t+1) \in \overline{\mathbb{D}} \times \mathbb{R}$; so it induces a foliation $\mathcal{F}_0$ on the quotient:

$$\overline{\mathbb{D}} \times \mathbb{R}/(z, t) \sim (z, t+1) \simeq \overline{\mathbb{D}} \times \mathbb{S}^1.$$

It has the boundary $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ as a closed leaf. The others are diffeomorphic to $\mathbb{R}^2$ (see Fig. 3).

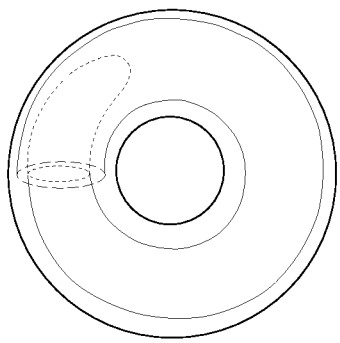

Fig. 3

Because M_+ and M_- are diffeomorphic to $\overline{\mathbb{D}} \times \mathbb{S}^1$, $\mathcal{F}_0$ defines on M_+ and M_- respectively two foliations $\mathcal{F}_+$ and $\mathcal{F}_-$ which give a codimension one foliation $\mathcal{F}$ on $\mathbb{S}^3$ called the *Reeb foliation.* All the leaves are diffeomorphic to the plane $\mathbb{R}^2$ except one which is the torus, the common boundary L of the two components M_+ and M_-. Its holonomy is given by the representation:

$$h : \pi_1(L) = \mathbb{Z}^2 \longrightarrow G(\mathbb{R}, 0)$$

($G(\mathbb{R}, 0)$ is the group of germs at 0 of diffeomorphisms of $\mathbb{R}$ fixing the point 0) which sends the generator $(1, 0)$ (resp. the generator $(0, 1)$) to the germ of the diffeomorphism h_1 (resp. h_2) equal to the identity on $[0, +\infty[$ (resp. on $]-\infty, 0]$) different from the identity on $]-\infty, 0]$ and infinitely tangent to it at 0 (resp. on $[0, +\infty[$).

4.3.5 Lie Group Actions

Let M be a manifold of dimension $m + n$ and G a connected Lie group of dimension m. An *action* of G on M is a map $G \times M \xrightarrow{\Phi} M$ such that:

– $\Phi(e, x) = x$ for every $x \in M$ (where e is the unit element of G),

– $\Phi(g', \Phi(g, x)) = \Phi(g'g, x)$ for every $x \in M$ and every $g, g' \in G$. Suppose that, for every point $x \in M$, the dimension of the *isotropy subgroup*:

$$G_x = \{g \in G : \Phi(g, x) = x\}$$

is independent of x. Then the action Φ defines a foliation $\mathcal{F}$ of dimension $= m - \dim G_x$; its leaves are the orbits $\{\Phi(g, x) : g \in G\}$. In particular this is the case if Φ is *locally free* i.e. if, for every $x \in M$, the isotropy subgroup G_x is discrete. An explicit example is given when M is the quotient H/Γ of a Lie group H by a discrete subgroup Γ and G is a connected Lie subgroup of H; the action of G on M being induced by the left action of G on H. We say that $\mathcal{F}$ is a *homogeneous foliation.* Let us give an example. Let $A \in \mathrm{SL}(m + n - 1, \mathbb{Z})$, where $m + n \geq 3$, be a matrix diagonalizable on the field $\mathbb{R}$ and having all its eigenvalues $\mu_1, \ldots, \mu_{m-1}, \lambda_1, \ldots, \lambda_n$ positive. Let $u_1, \cdots, u_{m-1}, v_1, \cdots, v_n$ be the corresponding eigenvectors in $\mathbb{R}^{m+n-1}$. As we can think of A as a diffeomorphism of the $(m + n - 1)$-torus $\mathbb{T}^{m+n-1}$, the vectors $u_1, \ldots, u_{m-1}$, $v_1, \ldots, v_n$ can be considered as linear vector fields on $\mathbb{T}^{m+n-1}$ such that:

$$A_* u_j = \mu_j u_j, \quad A_* v_k = \lambda_k v_k \quad \text{for} \quad j = 1, \ldots, m-1 \quad \text{and} \quad k = 1, \ldots, n.$$

Let $(x_1, \cdots, x_{m-1}, y_1, \cdots, y_n, t)$ be the coordinates of a vector in $\mathbb{R}^{m+n} = \mathbb{R}^{m+n-1} \times \mathbb{R}$. Then the vector fields $u_1, \cdots, u_{m-1}, v_1, \cdots, v_n, u_m = \frac{\partial}{\partial t}$ generate the Lie algebra (over the ring of C^∞-functions) $\mathfrak{X}(\mathbb{R}^{m+n})$. The vector fields:

$$X_i = \mu_i^t u_i, \quad Y_j = \lambda_j^t v_j \quad \text{and} \quad X_m = \frac{\partial}{\partial t} \quad (\text{for } i = 1, \cdots, m-1,\ j = 1, \cdots, n)$$

satisfy the bracket relations:

$$[X_i, X_\ell] = [X_i, Y_j] = [Y_j, Y_k] = 0$$

and

$$[X_m, X_i] = \ln(\mu_i)X_i, \ \ [X_m, Y_j] = \ln(\lambda_j)Y_j$$

(for $i, \ell = 1, \cdots, m-1$ and $j, k = 1, \cdots, n$,) and then generate over the field $\mathbb{R}$ a finite dimensional Lie algebra $\mathcal{H}$. It is the semi-direct product of the abelian algebra $\mathcal{H}_0$ generated by $X_1, \cdots, X_{m-1}, Y_1, \cdots, Y_n$ and the one dimensional Lie algebra generated by X_m; $\mathcal{H}$ is solvable. The Lie subalgebra $\mathcal{G}$ defined by $X_1, \cdots, X_m$ is also solvable and it is an ideal of $\mathcal{H}$. The simply connected Lie groups H and G corresponding respectively to $\mathcal{H}$ and $\mathcal{G}$ can be constructed as follows. As the eigenvalues of the matrix A are real positive, the group $\mathbb{R}$ acts on $\mathbb{R}^{m+n-1}$: $(t, z) \in \mathbb{R} \times \mathbb{R}^{m+n-1} \longmapsto A^t z \in \mathbb{R}^{m+n-1}$ (where $z = (x_1, \cdots, x_{m-1}, y_1, \cdots y_n)$) leaving invariant the eigenspace E corresponding to $\mu_1, \cdots, \mu_{m-1}$; this action defines the groups H and G respectively as the semi-direct products $\mathbb{R} \ltimes \mathbb{R}^{m+n-1}$ and $\mathbb{R} \ltimes E$. Because the coefficients of A are in $\mathbb{Z}$, the preceding action restricted to $\mathbb{Z}$ preserves the subgroup $\mathbb{Z}^{m+n-1}$; this gives a subgroup $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^{m+n-1}$ which is a cocompact lattice of H. The quotient $\mathbb{T}_A^{m+n} = H/\Gamma$ is a compact manifold of dimension $m+n$. As we have already pointed, any subgroup of H induces a locally free action on H/Γ which defines a foliation. In our example we have two subgroups: G and the normal abelian subgroup K whose Lie algebra is the ideal generated by $Y_1, \cdots, Y_n$. Their actions on $\mathbb{T}_A^{m+n}$ give respectively two foliations $\mathcal{F}$ and $\mathcal{V}$; $\mathcal{V}$ is a Lie foliation transversely modeled on the Lie group G.

4.4 Suspension of Diffeomorphism Groups

One of the main class of foliations is obtained by the suspension procedure of groups of diffeomorphisms. This section will be devoted to the definition of this procedure and to give some examples of groups of diffeomorphisms which give interesting foliations.

4.4.1 General Construction

Let B and F be two manifolds, respectively of dimensions m and n. Suppose that the fundamental group $\pi_1(B)$ of B is finitely generated. Let $\rho : \pi_1(B) \longrightarrow \mathrm{Diff}(F)$ be a representation, where $\mathrm{Diff}(F)$ is the diffeomorphism group of F. Denote by $\widetilde{B}$ the universal covering of B and $\widetilde{\mathcal{F}}$ the horizontal foliation on $\widetilde{M} = \widetilde{B} \times F$, i.e., the foliation whose leaves are the subsets $\widetilde{B} \times \{y\}$, $y \in F$. This foliation is invariant by all the transformations $T_\gamma : \widetilde{M} \longrightarrow \widetilde{M}$ defined by $T_\gamma(\widetilde{x}, y) = (\gamma \cdot \widetilde{x}, \rho(\gamma)(y))$ where $\gamma \cdot \widetilde{x}$ is the natural action of $\gamma \in \pi_1(B)$ on $\widetilde{B}$; then $\widetilde{\mathcal{F}}$ induces a codimension n foliation $\mathcal{F}_\rho$ on the quotient manifold:

$$M = \widetilde{M}/(\widetilde{x}, y) \sim (\gamma \cdot \widetilde{x}, \rho(\gamma)(y)).$$

We say that $\mathcal{F}_\rho$ is the *suspension* of the diffeomorphism group $\Gamma = \rho(\pi_1(B))$. The leaves of $\mathcal{F}_\rho$ are transverse to the fibres of the natural fibration induced by the projection on the first factor $\widetilde{B} \times F \longrightarrow \widetilde{B}$.

Conversely, suppose that $F \longrightarrow M \xrightarrow{\pi} B$ is a fibration with compact fibre F and that $\mathcal{F}$ is a codimension n foliation (n = dimension of F) transverse to the fibres of π. Then there exists a representation $\rho : \pi_1(B) \longrightarrow \mathrm{Diff}(F)$ such that $\mathcal{F} = \mathcal{F}_\rho$. The geometric transverse structures of the foliation $\mathcal{F}$ are exactly the geometric structures on the manifold F invariant by the action of Γ. So to give examples of foliations obtained by suspension, it is sufficient to construct examples of diffeomorphism groups. This is what we shall do now.

4.4.2

Let G be any compact Lie group of dimension n and Γ a finitely presented subgroup of G (in particular Γ is numerable). Then there exists a compact manifold B such that $\pi_1(B) = \Gamma$. Of course, the subgroup Γ can be viewed as a group of diffeomorphisms of G (Γ acts on G by left multiplication). The suspension of Γ gives a codimension n foliation; it is in fact a Lie foliation transversely modeled on G. Concrete examples are given for instance by the following representation. - The compact Abelian Lie group $\mathbb{T}^n$ is the maximal torus of the compact Lie group $\mathrm{SO}(2n, \mathbb{R})$. Any element g can be represented by a matrix:

$$\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{pmatrix}$$

where:

$$A_j = \begin{pmatrix} \cos 2\pi\theta_j & \sin 2\pi\theta_j \\ -\sin 2\pi\theta_j & \cos 2\pi\theta_j \end{pmatrix}$$

and all the other entries are equal to 0. Let $\Gamma = \mathbb{Z}^m$; this group has a system of generators $(\gamma_1, \cdots, \gamma_m)$ where $\gamma_1 = (1, 0, \cdots, 0)$, $\gamma_2 = (0, 1, \cdots, 0)$, $\cdots$, $\gamma_m = (0, \cdots, 0, 1)$. Let $\rho : \Gamma \longrightarrow \mathbb{T}^n$ be the representation which sends the generator γ_j (for $j = 1, \cdots, n$) to the matrix:

$$\begin{pmatrix} \ddots & & \\ & \begin{pmatrix} \cos 2\pi\alpha_j & \sin 2\pi\alpha_j \\ -\sin 2\pi\alpha_j & \cos 2\pi\alpha_j \end{pmatrix} & \\ & & \ddots \end{pmatrix}$$

where $\alpha_1, \cdots, \alpha_n \in]0, 1[$. The remaining entries in the diagonal are 1 and the other are equal to 0. The foliation obtained by suspending this representation is the linear codimension n foliation on the torus $\mathbb{T}^{m+n}$.

4.4.3

Now let n an integer ≥ 2 and A a matrix of order n with coefficients in $\mathbb{Z}$ an determinant equal to 1 i.e. A is an element of $\mathrm{SL}(n,\mathbb{Z})$. Suppose that A admits n real positive eigenvalues $\lambda_1,\cdots,\lambda_n$ such that, for each $\lambda\in\{\lambda_1,\cdots,\lambda_n\}$, the components $(v^1,\cdots,v^n)$ in $\mathbb{R}^n$ of an eigenvector v associated to λ are linearly independent over $\mathbb{Q}$ i.e. , for $\mathbf{m}\in\mathbb{Z}^n$, every relation $\langle\mathbf{m},v\rangle=0$ implies $\mathbf{m}=0$ (where $\langle\ ,\ \rangle$ is the Euclidean product in $\mathbb{R}^n$). Such matrices exist ; take for instance(cf. [7]):

$$A=\begin{pmatrix}1&1&1&\dots&1\\1&2&0&\cdots&0\\1&0&3&\dots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\1&0&0&\dots&d_n\end{pmatrix}$$

with $d_1=1$ and $d_{i+1}=1+d_1d_2\cdots d_i$ for $i=1,\cdots,n-1$. This fact is easy to verify for $n\leq 3$. Let G be the solvable Lie group semi-direct $\mathbb{R}\ltimes\mathbb{R}^n$ where $\mathbb{R}$ acts on $\mathbb{R}^n$ by the matrix A:

$$(t,u)\in\mathbb{R}\times\mathbb{R}^n\longmapsto A^tu\in\mathbb{R}^n.$$

Because the coefficients of A are integers, this action preserves the lattice $\mathbb{Z}^n$, so we can construct the semi-direct product $\mathbb{Z}\ltimes\mathbb{Z}^n$ exactly in the same way; then we obtain a cocompact discrete subgroup Γ of G. The quotient manifold $B=G/\Gamma$ is a flat fibre bundle with fibres the n-torus $\mathbb{T}^n$ over the circle $\mathbb{S}^1$.

Now let $\lambda\in\{\lambda_1,\cdots,\lambda_n\}$ and v an associated eigenvector. Since $\lambda\langle\mathbf{m},v\rangle=\langle\mathbf{m}',v\rangle$ where $A'(\mathbf{m})=\mathbf{m}'\in\mathbb{Z}^n$ (A' is the transpose matrix of A), Γ can be embedded in $\mathrm{SL}(n,\mathbb{C})$ as follows: choose integers $a_1,\cdots,a_{n-1}$, set $a=a_1+\cdots+a_{n-1}$ and associate to $(\mathbf{m},\ell)\in\mathbb{Z}\ltimes\mathbb{Z}^n$ the matrix $n\times n$:

$$\lambda^{-\frac{al}{n}}\begin{pmatrix}\lambda^{a_1l}&\cdots&0&<\mathbf{m},v>\\\vdots&\ddots&\vdots&\vdots\\\vdots&&\lambda^{a_{n-1}l}&0\\0&\cdots&0&1\end{pmatrix},$$

(only the terms in the diagonal and the term in the first line and the nth column are nonzero). So we obtain a representation

$$\rho:\pi_1(B)=\Gamma\longrightarrow\mathrm{Aut}(P^{n-1}(\mathbb{C})).$$

The action of Γ on $P^{n-1}(\mathbb{C})$ extends to the point ∞ the affine action:

$$(z_1,\cdots,z_{n-1})\in\mathbb{C}^{n-1}\longmapsto(\lambda^{a_1\ell}z_1+\langle\mathbf{m},\ell\rangle,\lambda^{a_2\ell}z_2,\cdots,\lambda^{a_{n-1}\ell}z_{n-1})\in\mathbb{C}^{n-1}$$

for every $(\mathbf{m},\ell)\in\Gamma$. The suspension of this representation gives a transversely holomorphic foliation $\mathcal{F}$ of codimension $n-1$ on the compact differentiable manifold M, quotient of

$\widetilde{M} = P^{n-1}(\mathbb{C}) \times G$ by the equivalence relation which identifies (z, x) to $(\rho(\gamma)(z), \gamma x)$ with $\gamma \in \Gamma$ (Γ acts on G by left translation). The leaves of $\mathcal{F}$ are homogeneous spaces of G by discrete subgroups. Note that $\mathcal{F}$ is not transversely Kählerian because the image of the representation ρ does not preserve the Kaehlerian metric on $P^{n-1}(\mathbb{C})$.

4.4.4

Let $\mathrm{SL}(n, \mathbb{R})$ be the group of real matrices of order n and determinant 1. This is a real form of the group $\mathrm{SL}(n, \mathbb{C})$ (complex matrices of order n and determinant 1); This group acts by projective tranformations on $P^{n-1}(\mathbb{C})$ (complex projectice space of dimension $n-1$). Then every subgroup of $\mathrm{SL}(n, \mathbb{C})$ acts by the same transformations on $P^{n-1}(\mathbb{C})$. The construction of the following group Γ and the study of its properties can be found in [18]. In the upper half plane $\mathbb{H} = \{z = x+iy : y > 0\}$ with the Poincaré metric $\frac{dx^2+dy^2}{y^2}$ we consider a geodesic triangle $T(p, q, r)$ with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$ et $\frac{\pi}{r}$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. We denote by σ_1, σ_2 et σ_3 the reflections associated respectively to the sides of this triangle; they generate an isometry group Σ^*; elements which preserve the orientation form a subgroup Σ of Σ^* of index 2 called the *triangle group* and denoted $T(p, q, r)$. It is a subgroup of $\mathrm{SL}(2, \mathbb{R})$ and its pull-back Γ by the projection $\widetilde{\mathrm{SL}}(2, \mathbb{R}) \longrightarrow \mathrm{SL}(2, \mathbb{R})$ ($\widetilde{\mathrm{SL}}(2, \mathbb{R})$ is the universal covering of $\mathrm{SL}(2, \mathbb{R})$) is a central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma \longrightarrow \Sigma \longrightarrow 1.$$

The group Γ has the presentation

$$\Gamma = \langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_1^p = \gamma_2^q = \gamma_3^r = \gamma_1\gamma_2\gamma_3 \rangle.$$

The quotient $B = \widetilde{\mathrm{SL}}(2, \mathbb{R})/\Gamma$ is a compact manifold of dimension 3. If the integers p, q et r are mutually prime the cohomology (with coefficients in $\mathbb{Z}$) of B is exactly the cohomology of the sphere $\mathbb{S}^3$. Since Γ is a subgroup of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, it acts on $P^1(\mathbb{C})$. So we obtain a (non injective) representation $\rho : \pi_1(B) = \Gamma \longrightarrow \mathrm{Aut}(P^1(\mathbb{C}))$. The suspension of such representation gives a transversely holomorphic foliation $\mathcal{F}$ of codimension 1 on the differentiable manifold M of dimension 5, which is the quotient of $\widetilde{M} = P^1(\mathbb{C}) \times \widetilde{\mathrm{SL}}(2, \mathbb{R})$ by the equivalence relation which identifies (z, x) with $(\rho(\gamma)(z), \gamma x)$ (Γ acts on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ by left translation). The leaves of $\mathcal{F}$ are homogeneous spaces of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ by discrete subgroups.

4.4.5

The 1-dimensional real projective space $P^1(\mathbb{R})$ is obtained by adding the point ∞ to the real line $\mathbb{R}$; it is also isomorphic to the circle $\mathbb{S}^1$. The group $\mathrm{SL}(2, \mathbb{R})$ of 2-order real matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$ acts analytically on $\mathbb{S}^1$ by:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, x \right) \in \mathrm{SL}(2, \mathbb{R}) \times \mathbb{S}^1 \longmapsto \frac{ax+b}{cx+d} \in \mathbb{S}^1.$$

For any integer m such that $m \geq 2$, the two elements $\gamma_1 = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$ generate a free non Abelian subgroup Γ (*cf.* [16]) of the group $\mathrm{Diff}(\mathbb{S}^1)$ of diffeomorphisms of the circle $\mathbb{S}^1$.

Let B_1 and B_2 be two copies of $\mathbb{S}^2 \times \mathbb{S}^1$; each one of them has its fundamental group isomorphic to $\mathbb{Z}$. By Van Kampen theorem, the connected sum $B = B_1 \# B_2$ (which is a 3-dimensional manifold) has the non Abelian free group on two generators α_1 and α_2 as fundamental group. Let $\rho : \pi_1(B) \longrightarrow \Gamma$ be the representation defined by $\rho(\alpha_1) = \gamma_1$ and $\rho(\alpha_2) = \gamma_2$. As usual, the suspension of this representation gives rise to a codimension 1 foliation on the 4-manifold M which is a flat bundle $\mathbb{S}^1 \longrightarrow M \longrightarrow B$. This foliation is transversely homogeneous (in fact transversely *projective*).

4.5 The Basic Cohomology

Let M be a manifold equipped with a codimension n foliation $\mathcal{F}$. Recall that a differential form α is said to be *basic* if it satisfies $i_X\alpha = L_X\alpha = 0$ for every vector field X tangent to $\mathcal{F}$. If α is basic so is $d\alpha$; then basic forms constitute a differential complex $(\Omega^r(M/\mathcal{F}), d)$ called the *basic complex* of the foliation $\mathcal{F}$; its homology $H^*(M/\mathcal{F})$ is the *basic cohomology* of $\mathcal{F}$. It can be viewed as the de Rham cohomology of the leaf space $M/\mathcal{F}$ even it does not admit any differentiable structure. If $\mathcal{F}$ is a fibration $M \longrightarrow B$, $H^*(M/\mathcal{F})$ is exactly the de Rham cohomology of B. So the following question is natural: *in which sense the basic cohomology looks like the de Rham cohomology of a manifold* ? For instance *is $H^*(M/\mathcal{F})$ finite dimensional when the manifold M is compact* ? This is immediate for $H^0(M/\mathcal{F})$ and $H^1(M/\mathcal{F})$ which injects in $H^1(M)$. But in general this is not the case for $H^p(M/\mathcal{F})$ with $p \geq 2$ as it was shown by G.W. Schwarz [21] and E. Ghys [12] who gave an example of a real analytic foliation whose $H^2(M\mathcal{F})$ is infinite dimensional. But the answer is positive when the foliation is Riemannian.

4.5.1 The Case of a Lie Foliation

Suppose that $\mathcal{F}$ is a Lie G-foliation. Let $h : \pi_1(M) \longrightarrow G$ be the holonomy representation and $D : \widetilde{M} \longrightarrow G$ the developing map associated to $\mathcal{F}$. Denote by $\widetilde{\mathcal{F}}$ the pull-back of $\mathcal{F}$ to the universal covering $\widetilde{M}$. For every $\gamma \in \pi_1(M)$ the following diagramm is commutative:

$$(*) \qquad \begin{array}{ccc} \widetilde{M} & \xrightarrow{\gamma} & \widetilde{M} \\ D \downarrow & & \downarrow D. \\ G & \xrightarrow{h(\gamma)} & G \end{array}$$

The basic forms for $\mathcal{F}$ on M are the basic forms for $\widetilde{\mathcal{F}}$ on $\widetilde{M}$ invariant by the action of $\pi_1(M)$. By the diagram $(*)$ the space $\Omega^*(M/\mathcal{F})$ is canonically isomorphic to the space of differential forms on G invariant by $\Gamma = h(\pi_1(M))$ and then (by continuity) to the space $\Omega^*_K(G)$ of differential forms on G invariant by the closure K of Γ in G. The two differential

complexes $(\Omega^*(M/\mathcal{F}), d)$ and $(\Omega^*_K(G), d)$ are canonically isomorphic; so their homologies $H^*(M/\mathcal{F})$ and $H^*_K(G)$ are the same. In particular if the leaves of $\mathcal{F}$ are dense, $K = G$ and $H^*(M/\mathcal{F})$ is the cohomology $H^*(\mathcal{G})$ of the Lie algebra $\mathcal{G}$ of G. Let us give an explicite example where all the leaves are dense i.e. $K = G$. Let $X = \frac{\partial}{\partial x_0} + a_1\frac{\partial}{\partial x_1} + \ldots + a_n\frac{\partial}{\partial x_n}$ be a linear vector field on the torus $\mathbb{T}^{n+1}$. The orbits of X define a one dimensional Lie $\mathbb{R}^n$-foliation $\mathcal{F}$. If the real numbers $1, a_1, \ldots, a_n$ are linearly independent over $\mathbb{Q}$, the leaves of $\mathcal{F}$ are all dense. Then $H^*(\mathbb{T}^{n+1}/\mathcal{F})$ is the exterior algebra $\Lambda^*\mathbb{R}^n$ of $\mathbb{R}^n$.

The result is the same if $\mathcal{F}$ is a linear foliation on $\mathbb{T}^{m+n}$ with all leaves dense and defined by m vector field $X_1, \cdots, X_n$ which are linearly independent.

4.5.2 The case of a transversely parallelizable foliation

Suppose $\mathcal{F}$ is transversely parallelizable and let $F \hookrightarrow M \xrightarrow{\pi} W$ be its basic fibration. We know that there exists a simply connected Lie group G_0 such that the induced foliation $\mathcal{F}_0$ on each fibre F is a Lie G_0-foliation and the cocycle of the fibration π is with values in the group $\mathrm{Diff}(F, \mathcal{F}_0)$. Let $\mathcal{G}_0$ be the Lie algebra of G_0. Then there exists a spectral sequence (E_r) (cf. [10]) with term:

$$E_2^{pq} = H^p(W, \mathcal{H}^q)$$

converging to $H^*(M/\mathcal{F})$. Here $\mathcal{H}^q$ is the locally constant presheaf on W with fibre $H^q(\mathcal{G}_0)$. As W is compact and $H^q(\mathcal{G}_0)$ is finite dimensional, the basic cohomology $H^*(M/\mathcal{F})$ is also finite dimensional. The vector space $H^n(M/\mathcal{F}) = E_2^{st}$ (where s =dim(W) and t =dim(G_0)) is trivial or isomorphic to $\mathbb{R}$; in the last case we say that $\mathcal{F}$ is *homologically orientable.* This property has an interesting geometric interpretation: *there exists a bundle-metric on M for which the leaves are minimal submanifolds if and only if $\mathcal{F}$ is homologically orintable.* (cf. [17]).

Let us compute an explicit example. Let GA be the Lie group of affine transformations $x \in \mathbb{R} \longmapsto ax + b \in \mathbb{R}$ where $a \in]0, +\infty[$ and $b \in \mathbb{R}$. It can be embedded in the group G =SL$(2, \mathbb{R})$ as follows:

$$(x \longrightarrow ax + b, a > 0) \in GA \longmapsto \frac{1}{\sqrt{a}}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

There exists (cf. [9]) a compact manifold M equipped with a Lie G-foliation $\mathcal{F}$ with GA as the closure of its holonomy group. Then, the basic cohomology of $\mathcal{F}$ is the cohomology of differential forms on G invriant by GA. The quotient $\mathrm{SL}(2, \mathbb{R})/GA$ is diffeomorphic to the circle $\mathbb{S}^1$. Then we have a spectral sequence:

$$E_2^{pq} = H^p(\mathbb{S}^1, \mathcal{H}^q(\mathcal{GA}))$$

converging to $H^*(M/\mathcal{F})$. Because the fibre bundle $GA \hookrightarrow \mathrm{SL}(2, \mathbb{R}) \longrightarrow \mathbb{S}^1$ is principal and GA connected, the action of $\pi_1(\mathbb{S}^1)$ on $H^q(\mathcal{GA})$ is trivial. So:

$$E_2^{pq} = H^p(\mathbb{S}^1) \otimes H^q(\mathcal{GA}).$$

As $\dim(\mathbb{S}^1) = 1$, this spectral sequence converges at the E_2 term, that is:

$$H^r(M/\mathcal{F}) = \bigoplus_{p+q=r} H^p(\mathbb{S}^1) \otimes H^q(\mathcal{GA}).$$

But $H^q(\mathcal{GA}) = \mathbb{R}$ for $q = 0, 1$ and 0 otherwise. So we get:

$$H^*(M/\mathcal{F}) = \mathbb{R} \text{ if } * = 0, 2, \mathbb{R} \oplus \mathbb{R} \text{ if } * = 1, \text{ and } 0 \text{ otherwise.}$$

4.5.3 The Case of a Riemannian Foliation

We will use the same idea as in the case of a transversely parallelizable foliation even if the leaf closures does not have the same dimension. For details see [8]. The topological space X, quotient of M by the leaf closures is not a manifold in general and the projection $\pi : M \longrightarrow X$ is not a fibration. However, locally the foliation $\mathcal{F}$ has a good behaviour (cf. [14]). It was proved in [8] that X admits a *good cover* i.e. an open cover $\mathcal{U} = \{U_i\}$ (which can be chosen finite $\{U_1, \cdots, U_k\}$ because X is compact) such that each intersection $U_{i_1} \cap \cdots \cap U_{i_p}$ is contractible. Then there exists a spectral sequence:

$$E_2^{pq} = H^p(\mathcal{U}, \mathcal{H}_b^q)$$

converging to $H^*(M/\mathcal{F})$. Here $\mathcal{H}_b^q$ is the preasheaf on X which associates to any open set U the vector space $H^q(\pi^{-1}(U)/\mathcal{F})$. If L is a leaf closure with minimal dimension contained in $\pi^{-1}(U)$, the injection $L \hookrightarrow \pi^{-1}(U)$ induces an isomorphism $H^q(\pi^{-1}(U)/\mathcal{F}) \simeq H^q(L/\mathcal{F})$. Since the foliation in L is with dense leaves, its basic cohomology is finite dimensional (the space of basic forms itself is finite dimensional) and so is $H^*(M/\mathcal{F})$ (because the cover $\mathcal{U}$ is finite). It was also proved in [8] that the basic cohomology is a *topological invariant* in the category of complete Riemannian foliations. (Any Riemannian foliation on a compact manifold is complete.)

To illustrate concretely these methods, let us give an example. Let $\mathcal{F}$ be the transversely Kählerian flow defined on :

$$\mathbb{S}^3 = \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\right\}$$

by the restriction of the holomorphic vector field:

$$Z = i\beta_1 \frac{\partial}{\partial z_1} + i\beta_2 \frac{\partial}{\partial z_2}$$

(cf. subsection 2.6) where β_1 and β_2 are real numbers such that $\frac{\beta_1}{\beta_2} \notin \mathbb{R}^-$. If $\beta_1 = \beta_2 = 1$, $\mathcal{F}$ is just the Hopf fibration:

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \longrightarrow P^1(\mathbb{C}).$$

We will compute the basic cohomology of $\mathcal{F}$. If all the leaves of $\mathcal{F}$ are compact, the quotient $X = M/\mathcal{F}$ is an orbifold homeomorphic to the 2-sphere $\mathbb{S}^2$; then $H^*(M/\mathcal{F}) = H^*(\mathbb{S}^2)$.

Suppose that the leaves are not closed. Then the leaf closures are diffeomorphic to the torus $\mathbb{T}^2$ except two of them L_1 and L_2 corresponding to the points $z_1 = 0$ and $z_2 = 0$. The space X is homeomorphic to $[0,1]$. Take the open sets $U_1 = [0, 2/3[$ and $]1/3, 1]$; U_1, U_2 and the intersection $U_{12} = U_1 \cap U_2 =]1/3, 2/3[$ are contractible so $\mathcal{U} = \{U_1, U_2\}$ is a good cover of $[0,1]$. Let $V_1 = \pi^{-1}(U_1)$, $V_2 = \pi^{-1}(U_2)$ and $V_{12} = \pi^{-1}(U_1 \cap U_2)$. It is easy to see that:

$$H^*(V_1/\mathcal{F}) = H^*(V_2/\mathcal{F}) = H^*(\text{ point }) = \mathbb{R} \text{ for } * = 0, 0 \text{ otherwise}$$

and

$$H^*(V_{12}/\mathcal{F}) = H^*(\mathbb{S}^1) = \mathbb{R} \text{ for } * = \{0,1\}, \text{ and } 0 \text{ otherwise.}$$

Since the cover $\mathcal{U}$ contains only two open sets, the spectral sequence is reduced to the exact sequences:

$$0 \longrightarrow H^0(\mathbb{S}^3) \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \longrightarrow H^1(\mathbb{S}^3/\mathcal{F}) \longrightarrow 0$$
$$0 \longrightarrow \mathbb{R} \longrightarrow H^2(\mathbb{S}^3/\mathcal{F}) \longrightarrow 0.$$

Since $H^1(\mathbb{S}^3/\mathcal{F})$ injects in $H^1(\mathbb{S}^3) = 0$, $H^1(\mathbb{S}^2/\mathcal{F}) = 0$ and this exact sequence gives:

$$H^*(\mathbb{S}^3/\mathcal{F}) = \mathbb{R}, \text{ for } * = 0, 2, \text{ and } 0 \text{ otherwise,}$$

which is exactly the de Rham cohomology of $P^1(\mathbb{C})$; in particular $H^*(\mathbb{S}^3/\mathcal{F})$ does not depend on β_1 and β_2. The following example is a generalization of this situation.

4.5.4 Flow of Isometries

Suppose that $\dim(M) = n+1$ and that $\mathcal{F}$ is defined by a locally free isometric action of the Lie group $\mathbb{R}$ (the action preserves a Riemannian metric g on M). This is always the case if M is simply connected (cf. [13]: a Riemannian flow on a simply connected manifold is an isometry flow). The image of $\mathbb{R}$ by the representation $\mathbb{R} \longrightarrow \mathrm{Isom}(M)$ which defines the action is a connected abelian subgroup of the compact group $\mathrm{Isom}(M)$ of isometries of Riemannian manifold (M, g); its closure is a compact abelian subgroup K (K is a torus). Then the de Rham cohomology of M is isomorphic to the cohomology of the complex $(\Omega_K(M), d)$ of differential forms on M invariant by the action of K. Let X be the fundamental vector field defining $\mathcal{F}$. It is easy to see that:

$$\Omega^*(M/\mathcal{F}) = \{\omega \in \Omega^*_K(M) : i_X\omega = 0\}.$$

For $r = 1, \cdots, n+1$ let $\phi : \Omega^r_K(M) \longrightarrow \Omega^{r-1}_K(M)$ be defined by $\phi(\omega) = (-1)^r i_X\omega$. The image of ϕ is $\Omega^{r-1}(M/\mathcal{F})$ and the sequence:

$$0 \longrightarrow \Omega^r(M/\mathcal{F}) \hookrightarrow \Omega^r_K(M) \xrightarrow{\phi} \Omega^{r-1}(M/\mathcal{F}) \longrightarrow 0$$

is exact. Then it gives a long exact cohomology sequence:

$$0 \longrightarrow H^0(M/\mathcal{F}) \longrightarrow H^0(M) \longrightarrow H^{-1}(M/\mathcal{F}) = 0 \longrightarrow$$

$$H^1(M/\mathcal{F}) \longrightarrow H^1(M) \longrightarrow H^0(M/\mathcal{F}) = \mathbb{R} \longrightarrow \cdots$$
$$\cdots \longrightarrow H^r(M/\mathcal{F}) \longrightarrow H^r(M)H^{r-1}(M/\mathcal{F}) \longrightarrow \cdots$$

If M is a homology sphere (i.e. it has the homology of a sphere), the integer n is necessarily even and the long exact sequence gives:

$$H^*(M/\mathcal{F}) = \mathbb{R} \text{ if } r \text{ is even and } 0 \leq r \leq n, 0 \text{ otherwise.}$$

The basic cohomology of $\mathcal{F}$ is exactly the de Rham cohomology of the n-dimensional complex projective space $P^n(\mathbb{C})$. An explicite example is the transversely Kählerian flow on $\mathbb{S}^{2n+1}$ induced by the holomorphic vector field in $\mathbb{C}^{N+1}$:

$$Z = \sum_{k=1}^{n+1} i\beta_k z_k \frac{\partial}{\partial z_k}$$

for a good choice of the real numbers $\beta_1, \cdots, \beta_{n+1}$. For $\beta_1 = \cdots = \beta_{n+1} = 1$, this flow is just the Hopf fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \longrightarrow P^n(\mathbb{C})$.

The study of the basic cohomology by Hodge theory was done by many authors. See for instance [5] where it appears as a particular case of the general theory of transversely elliptic operators on Riemannian foliations.

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Differential Geometry and Topology, Discrete and Computational Geometry
M. Boucetta and J.-M. Morvan (Eds.)
IOS Press, 2005
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Chapter 5

Integrability of Lie Algebra Action

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In this lecture we are interested in the way used to build a Lie group action from certain Lie algebra actions. More precisely, we will give a complete proof of the following result of Palais(1957): "*Every weakly-complete Lie algebra action can be seen as the infinitesimal description of a global Lie group action*".

Our objective here is the understanding of the passage from the infinitesimal phenomenon to the global one, what seems a big scientific and educational interest for us. Therefore, we tried to make this text self-contained and this while avoiding to drag in generalities that are not useful to us.

The main prerequisites are the following: Frobenius theorem (theory of foliations); elements of the theory of Lie groups and Lie algebras, and mainly the fundamental theorem of Sophus Lie (the Lie Functor defines a bijective correspondence between simply connected Lie groups and Lie algebra). I have made no attempt to discuss basic concepts: manifolds, bundles, covering, Lie groups, Lie algebras. Nevertheless, I review a few facts about vector fields, Lie group actions, Lie algebra actions, and the manifold of linear frames.

5.1 Introduction

The Lie theory of transformation groups, to which this lecture is devoted, is a fundamental part of mathematics, and it is now seen to touch many fields of mathematics and science, including differential geometry, topology, ordinary and partial differential equations, complex analysis, group theory and physics. The local aspect of the theory was conceived and developed by Sophus Lie himself ([21] 1893), whose aim was to apply algebraic methods to the theory of differential equations. In his work, S. Lie connects Lie algebras of vector fields on a differentiable manifold with local Lie group actions on the manifold. More precisely: "*there is a bijective correspondence between Lie algebra actions and* **local** *Lie group*

actions". However, the development of the global aspects of the theory is due to R. Palais ([28] 1957); the main result of this study can be formulated as follows:
" *there is a bijective correspondence between weakly-complete Lie algebra actions and* **global** *Lie group actions* ".
We focus our course on this theorem (see section 3 for the precise statement).
In a second step (section 6), we will make the link with a theorem due to Kobayashi according to which *the group of diffeomorphisms preserving a certain geometric structure of a manifold is often a Lie transformation group* . This generalizes the classic result of Meyers and Steenrod [1939], which states that the isometries of a Riemannian manifold form a Lie group $Iso_g(M)$. The group $Diff_\nabla(M)$ of affine transformations (i.e. which preserves a linear connection ∇ on a manifold M) is also a Lie group (these are investigations of Nomizu, Hano and Morimoto). Let us note that H. Cartan was the first to give that kind of result: (*the group Aut*(D) *of analytic automorphisms of a bounded domain* $D \subset \mathbb{C}^n$ *carries the structure of a Lie group*) (see [25]). The methods introduced by Cartan were used by Bochner-Montgomery [24] to prove that *if* G *is a locally compact topological transformation group of* M*, then* G *is a Lie transformation group of* M.

Throughout this text we use the words "differentiable", "smooth" , and "C^∞" as synonymous. For us the word "mapping" means smooth mapping. All structures are assumed to be differentiable.

5.2 Preliminary

5.2.1 Vector Fields and Flows

This section contains a brief introduction to vector fields and the flows they determine. They appear as the first objects in the theory of dynamical systems and the simplest example of a local Lie group action.

Let M be a manifold, and $TM \to M$ be its tangent bundle. Recall that a vector field on M is a section of the tangent bundle TM of M. Thus a vector field X on M is a mapping $X : M \to TM$ such that $X(x) \in T_xM$ for all $x \in M$.
In various fundamental courses in differential geometry, one shows that a vector field on a manifold M can be considered as a derivation of the algebra of smooth functions $C^\infty(M)$, that is a linear operator $X : C^\infty(M) \to C^\infty(M)$ satisfying the Leibniz rule $X(fg) = X(f)g + fX(g)$. Conversely, every derivation X on $C^\infty(M)$ induces a vector field on M.
- The set of all vector fields on M is denoted by $\chi(M)$; this is a $C^\infty(M)$-module.
- If $\varphi : M \to M$ is a diffeomorphism and $X \in \chi(M)$, the push-forward of X by φ is defined by $\varphi.X := T\varphi \circ X \circ \varphi^{-1} \in \chi(M)$.

Example.
(i) Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function without critical points. For any real number c in the image of f, we can associate the smooth n-dimensional manifold $\Sigma_c := \{x \in$

$\mathbb{R}^{n+1}$ / $f(x) = c\}$. The tangent bundle of Σ_c is:

$$T\Sigma_c := \{(x, v) \in \Sigma_c \times \mathbb{R}^{n+1} \ / \ < v, \nabla f(x) >= 0\},$$

where $\nabla f(x) := (\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_{n+1}}(x))$. Thus a vector field X on Σ_c is a mapping $X : \Sigma_c \to \mathbb{R}^{n+1}$ such that $< X(x), \nabla f(x) >= 0$. For example a vector field on the sphere S^2 is given by $X : (x_1, x_2, x_3) \mapsto (-x_2, x_1, 0)$ (denoted also by $X := -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}$).
(ii) The group $G := O(n)$ of orthogonal real matrices, is a manifold of dimension $\frac{n(n-1)}{2}$. Let us describe vector fields on G: Put $\mathcal{G} := so(n)$ the algebra of skew-symmetric matrices (this is the Lie algebra of the Lie group G). For any mapping $F : G \to \mathcal{G}$, we can associate the vector field X^F on G given by the formula

$$X_g^F = \frac{d}{dt}_{|t=0} g.exp(tF(g)).$$

Conversely, each vector field on G is obtained by this way.□
Remark. It is easy to see that the set $Diff_X(M)$ of diffeomorphisms preserving the vector field X (that is the set of $\varphi \in Diff(M)$ such that $\varphi.X = X$) is not a Lie group in general; for example we can consider $M := \mathbb{R}^2$ and $X := \frac{\partial}{\partial y}$ (the details are let to the reader). □

To any vector fields $X \in \chi(M)$ is associated a differential equation:

$$\frac{d\gamma}{dt}(t) = X_{\gamma(t)} \qquad (\varepsilon)$$

The maximal solutions $\gamma : t \in \mathbb{R} \mapsto \gamma(t) \in M$, are called the *integral curves* or the *trajectories* of X. For any $x \in M$, the integral curve of X at x is defined on an open interval $I_x \subset \mathbb{R}$, and will be denoted by $\varphi^X(\cdot, x) : t \mapsto \varphi^X(t, x)$; that is $\varphi^X(\cdot, x)$ is the unique maximal solution of the equation (ε) with initial condition $\varphi^X(0, x) = x$.
The *flow* of X is the mapping $\varphi^X : (t, x) \mapsto \varphi^X(t, x)$ defined on an open subset $D_X \subset \mathbb{R} \times M$ with values in M.
Example. Let M be the unit sphere $S^2 = \{x \in \mathbb{R}^3 / x_1^2 + x_2^2 + x_3^2 = 1\}$, and the vector fields $X \in \chi(S^2)$ given by $X := (-\gamma x_2 + \beta x_3)\frac{\partial}{\partial x_1} + (\gamma x_1 - \alpha x_3)\frac{\partial}{\partial x_2} + (-\beta x_1 + \alpha x_2)\frac{\partial}{\partial x_3}$. The flow of X is given by the mapping $\varphi^X(t, x) = exp(tA).x$, where $A := \begin{pmatrix} 0 & -\gamma & +\beta \\ +\gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{pmatrix}$. □

For t a fixed real number, we denote by φ_t^X the local transformation of M given by $\varphi_t^X(x) := \varphi^X(t, x)$. We show that if (t, x), $(s, \varphi^X(t, x))$ and $(s + t, x)$ are all in D_X, then $\varphi^X(s, \varphi^X(t, x)) = \varphi^X(s + t, x)$.
The transformations $(\varphi_t^X)_t$ are local diffeomorphisms of M, the inverse of φ_t^X is φ_{-t}^X, and we have $\varphi_s^X \circ \varphi_t^X = \varphi_{s+t}^X$ (for points of M where the two members of this equality are defined). The family $(\varphi_t^X)_t$ is called the *pseudo-group* of transformations (or the *local* 1*-parameter subgroup*) associated to the field X.
A vector field X is said to be **complete** if its flow is defined on $\mathbb{R} \times M$; in other words, the integral curves of X are all defined on $\mathbb{R}$. This is equivalent to the data of a representation

$\mathbb{R} \to Diff(M)$ of $\mathbb{R}$ in the diffeomorphism group of M, such that the induced mapping $\mathbb{R} \times M \to M$ is differentiable.
On a compact manifold, each vector field is complete.
Examples-Exercises :
• On $M := \mathbb{R}$, the vector fields $X := x^2 \frac{d}{dx}$ is not complete: its flows is the mapping given by $\varphi^X(t,x) = \frac{x}{1-tx}$ defined on $D_X = \{(t,x) \in \mathbb{R} \times \mathbb{R} \ / \ tx < 1\}$.
• Let $M := \mathbb{R} \times \mathbb{R}$, the rotation field $Z := -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ is complete: its trajectories are the circles $\gamma_{(x,y)} : t \mapsto (xcost - ysint, xsint + ycost)$. □
Remark. The bracket or the sum of two complete vector fields can be non complete, for example:
Let $M := \mathbb{R} \times \mathbb{R}$, and consider the two fields: $X := y\frac{\partial}{\partial x}$ and $Y := (\frac{x^2}{2})\frac{\partial}{\partial y}$. These are complete, their respective flows are the following: $\varphi_t^X(t,x) = (x + ty, y)$ and $\varphi_t^Y(t,x) = (x, y + x^2\frac{t}{2})$. Show that the two vectors fields $X + Y$ and $[X,Y]$ are not complete. □

5.2.2 Actions of Lie Algebras on Manifold

Let $\mathcal{G}$ be a finite dimensional Lie algebra and let M be a smooth manifold. We say that $\mathcal{G}$ acts on M or that M is a $\mathcal{G}$-manifold if there is a Lie algebra homomorphism

$$\tau : \mathcal{G} \to \chi(M) \ ,$$

from $\mathcal{G}$ into the Lie algebra of all vector fields on M.
Note that such an action is uniquely determined for any ordered linearly independent set $\{X^1, \cdots, X^p\}$ of vectors fields on M satisfying the commutation relations

$$[X^i, X^j] = \sum_{1 \leq k \leq p} C_{ij}^k X^k$$

where C_{ij}^k are the structure contants of the Lie algebra $\mathcal{G}$ with respect to a basis.

If a Lie algebra $\mathcal{G}$ acts on a manifold M, then it defines an integrable distribution on M, which does not need to be of constant rank; it is called a singular foliation (see [32][31]).
An action of a Lie algebra $\mathcal{G}$ on a manifold M is called:
- *effective* if $\tau : \mathcal{G} \to \chi(M)$ is injective. So for each $h \in \mathcal{G}$ there is some $x \in M$ such that $\tau(h)_x \neq 0$;
- *complete* if there exists a basis $\beta = \{e_1, \ldots, e_r\}$ of the real vector space $\mathcal{G}$ such that the vectors fields $\tau(e_1), \ldots, \tau(e_r)$ are complete;
- *weakly-complete* if there exists a subset $S \subset \mathcal{G}$ which generates $\mathcal{G}$ (i.e. each subalgebra of the algebra $\mathcal{G}$ containing S coincides with $\mathcal{G}$), and such that for each $h \in S$ the vectors fields $\tau(h)$ is complete.
We will show a lemma according to which there is an *equivalence between "weakly-complete action" and "complete action"*.

5.2.3 Lie Group Actions on Manifolds

In this paragraph we review some basic facts about actions of Lie groups.

Definition 78 *Let M be a smooth manifold. An action (global action) of a Lie group G on M is a smooth mapping $\rho : G \times M \to M$ such that (i) for all $x \in M$, $\rho(e, x) = x$ (ii) for every $a, b \in G$, $\rho(a, \rho(b, x)) = \rho(ab, x)$ for all $x \in M$.*

Examples.
- A complete flow on M is an action of $\mathbb{R}$ on M.
- To any matrix group G (that is a closed subgroup of the group $GL(F)$ of linear automorphisms of a finite vector space F), corresponds a natural action of G on F. It is important to note that there are Lie groups which can not be realized as matrix groups. One example of such a group is $G := \widetilde{SL(2, \mathbb{R})}$ the universal covering group of $SL(2, \mathbb{R})$.
- The group $SL(2, \mathbb{R})$ acts on the upper half-plane $\mathbb{H} := \{z \in \mathbb{C}/Imz > 0\}$ by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = \frac{az + b}{cz + d}$$

- If G is a subgroup of a Lie group H, and if K is a closed subgroup of H, then the canonical map $\rho : G \times H/K \to H/K$ defined by $\rho(g, \overline{h}) = \overline{gh}$ is an action of G on the homogeneous space H/K.

Given an action ρ, to every $g \in G$, corresponds a diffeomorphism $\rho_g : x \mapsto \rho(g, x)$ of the manifold M and the map $\rho : g \to \rho_g$ is a homomorphism of the group G into the group $Diff(M)$ of diffeomorphism of M. Conversely, any homomorphism $\rho : G \to Diff(M)$ such that the mapping $(g, x) \mapsto \rho(g, x) := \rho(g)(x)$ from $G \times M$ to M is of class C^∞, defines a smooth action of G on M. The action ρ is said to be *effective* if the kernel of the homomorphism ρ is trivial (that is if $\rho(g)x = x$ for all x, then $g = e$).
Let ρ be an action of G on M. For $x \in M$, the G-orbit Gx through x and the isotropy subgroup G_x of x are defined respectively by: $Gx = \{\rho(g)(x) \ / \ g \in G\}$, $G_x = \{g \in G \ / \ \rho(g)(x) = x\}$. The action ρ is said to be *transitive* if there is only one orbit: $M = Gx$. For instance, the natural action of the orthogonal group $O(n + 1)$ on the sphere S^n is transitive.
The G-orbit map $\rho_x : G \to M$ is the map $g \mapsto \rho(g)(x)$. It defines a map $G/G_x \to M$ that is clearly injective, with image Gx. Since G/G_x has a smooth quotient structure, this means that we can regard each orbit as a smooth manifold by carrying over the differentiable structure from G/G_x. It is important to note that the orbits are immersed sub-manifold in M, but, in general, they will not be imbedded in M (i.e. the manifold topology that Gx inherits from G/G_x will not be the topology induced from M).
The action G defines an equvalence relation on M. We let M/G be the set of equivalence classes, that is, M/G is the set of G-orbits. Let $\pi : x \mapsto Gx$ denote the canonical projection $G \to M/G$. On M/G we put the quotient topology, that is, $U \subset M/G$ is open if and only if the G-invariant subset $\pi^{-1}(U)$ is open in M. The following simple properties of quotients by Lie group actions, can be shown as exercises:
- For any smooth (or continuous) action of a Lie group G on a manifold M, the quotient map $\pi : M \to M/G$ is open.
- The quotient topology on M/G is Hausdorff if and only if the orbit relation $\{(x, y) \in$

$M \times M \ / \ y \in Gx\}$ is a closed subset of $M \times M$. The action $(t,(x_1,x_2)) \mapsto (x_1 + tx_2, x_2)$ of the Abelian group $(\mathbb{R},+)$ on $M := \mathbb{R}^2$ gives an example of a non-Hausdorff quotient topology. A *proper action* leads to a Hausdorff quotient space M/G (recall that an action ρ is said to be proper if the mapping $G \times M \to M \times M$: $(g,x) \mapsto (\rho(g)(x),x)$ is proper). In particular, for each action of a **compact** Lie group G on a manifold M, the orbit space M/G is Hausdorff.

Definition 79 *Let M be a smooth manifold, G a Lie group. By a local action of G on M or a local G-transformation group on M we mean a smooth map ρ of a subset D of $G \times M$ into M such that*
(i) D is an open neighborhood of $\{1\} \times M$,
(ii) $\rho(e,x) = x$ for all $x \in M$,
(iii) if E is the set of all $(a,b,x) \in G \times G \times M$ such that (a,x), (ab,x), $(a,\rho(b,x))$ are all in D and $\rho(a,\rho(b,x)) = \rho(ab,x)$, then E is a neighborhood of $\{1\} \times \{1\} \times M$.

▶ Let $\mathcal{G}$ be the Lie algebra of a Lie group G.
To every action (local or global) of the Lie group G on a manifold M is associated a Lie algebra action of $\mathcal{G}$ on M :
for any $h \in \mathcal{G}$, we introduce the vector field X^h on M whose integral curves are of the form $t \mapsto \rho(exp(-th),x)$. This vector field on M, given by

$$X^h_x = \frac{d}{dt}_{|t=0} \rho(exp(-th),x)$$

is called the *infinitesimal generator* (or the fundamental vector field) of the action corresponding to h. We define the *differential* of the action $\rho : G \to Diff(M)$ as the mapping

$$\rho' : \mathcal{G} \to \chi(M)$$

given by the formula $\rho'(h) = X^h$. The mapping ρ' is a homomorphism of Lie algebras. Now if λ is an action of $\mathcal{G}$ on M, and if there exists a smooth Lie group action ρ of G on M such that $\rho' = \lambda$, we will say that ρ is a *primitive* of λ. It is obvious that Lie algebra actions do not always admit a primitive; a necessary condition is that this action is complete.
The main theorem in this talk (Palais's theorem) claims that any weakly-complete Lie algebra action admits a primitive, and so it can be viewed as the differential of an action of a Lie group.
Examples.
• The affine group GA of $\mathbb{R}$ is the group of transformations: $x \in \mathbb{R} \mapsto ax + b \in \mathbb{R}$. It is a 2-dimensional Lie group; its Lie algebra is identified with the space $\mathbb{R}^2$ endowed with the bracket $[e_1,e_2] = e_2$. Hence, every action of GA on a manifold M gives rise to a two complete vector fields $\{X^1,X^2\}$ where $[X^1,X^2] = X^2$.
• The Lie algebra of the orthogonal group $G = SO(3)$, that is the algebra $so(3)$ of skew-symmetric matrices, is identified to $\mathbb{R}^3$ endowed with the vector product via the following isomorphisms

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & -x^1 \\ -x^2 & x^1 & 0 \end{pmatrix}.$$

Now, consider the natural linear action of $SO(3)$ on $M = \mathbb{R}^3$: $\rho(g,x) = g.x$. Its associated infinitesimal action $\mathcal{G} \to \chi(\mathbb{R}^3)$ is given by $\rho'(h) = h \wedge ..$ □

Exercise. The Heisenberg group H is the matrix group:

$$\begin{pmatrix} 1 & s & t \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix}.$$

For the natural linear action $\rho : H \to Diff(\mathbb{R}^3)$, determine the infinitesimal associated action ρ' as well as the orbits of this action.

5.3 Statement of the Theorem ★

With the notations and the definitions above, we have the following theorem of Palais.

Theorem 61 *Let $\mathcal{G}$ be a finite dimensional Lie algebra, and $\tau : \mathcal{G} \to \chi(M)$ be a weakly-complete action of $\mathcal{G}$ on the manifold M.*
We denote by:

- *G the simply connected Lie group with $\mathcal{G}$ as Lie algebra.*
- *$T(\mathcal{G})$ the subgroup of the group $Diff(M)$ generated by all diffeomorphisms of the form (φ_t^X), where $t \in \mathbb{R}$ and X is an arbitrary complete field from $\tau(\mathcal{G})$.*

Then:

1. *There exists an action $\rho : G \to Diff(M)$ such that $\rho' = \tau$ and $\rho(G) = T(\mathcal{G})$.*
2. *The group $T(\mathcal{G})$ admits a smoothness with respect to which it is a connected Lie group with Lie algebra $\tau(\mathcal{G})$.*
3. *If the action τ is effective, the homomorphism $G \to T(\mathcal{G})$ induced from ρ is a smooth covering map.*

Example. Let $so(n)$ be the subset of $n \times n$ real matrices A such that $A^T + A = 0$; it is the Lie algebra of the orthogonal group $O(n)$. To every weakly complete action of this Lie algebra, is associated an action of the Lie group $Spin_n$.
We have the following identifications: $Spin_3 \cong SU_2 \cong S^3$, $Spin_4 \cong SU_2 \times SU_2$. ([LM]).□

5.4 Proof of the Theorem

We can suppose without loss of generalities that τ is injective $\tau : \mathcal{G} \hookrightarrow \chi(M)$, and the $\mathcal{G}$-action will be assumed to be complete. It is in a second stage that we will show that a weakly-complete action is automatically complete.
The Lie algebra $\mathcal{G}$ acts on $G \times M$ via the map:

$$\widehat{\tau} : \mathcal{G} \to \chi(G \times M)$$

given by $\widehat{\tau}(h) = (L^h, X^h)$, where L^h is the left invariant vector field on G associated to $h \in \mathcal{G}$. The flow of the vector field $\widehat{X}^h := \widehat{\tau}(h)$ is given by

$$\varphi_t^{\widehat{X}^h}(a, x) = (a.exp(th), \varphi_t^{X^h}(x)) \ .$$

For all $(a, x) \in G \times M$, the linear map

$$\widehat{\tau}_{(a,x)} : \mathcal{G} \to T(a, x)(G \times M)$$

given by $\widehat{\tau}_{(a,x)}(h) = (L_a^h, X_x^h)$ is injective !
We then consider the homomorphism of vector bundles (which we denote also by $\widehat{\tau}$ with an abuse of notation)

$$\mathcal{G} \times (G \times M) \xrightarrow{\widehat{\tau}} T(G \times M),$$

defined by: $\widehat{\tau} : (h, (a, x)) \mapsto (L_a^h, X_x^h)$. This is a homomorphism of constant rank, and we obtain a regular foliation $\mathcal{F}$ on $G \times M$:

$$\mathcal{F}_{(a,x)} = \widehat{\tau}_{(a,x)}(\mathcal{G}) = \{(L_a^h, X_x^h) \ / \ h \in \mathcal{G}\}.$$

We call it the *graph foliation associated to the $\mathcal{G}$-action.*
Let us analyze this foliation:
According to Frobenius theorem $\rightsquigarrow$ Through any point $(a, x) \in G \times M$ passes a unique maximal integral submanifold of the distribution $\mathcal{F}$, this is called the leave passing through (a, x) and will be denoted by $F_{(a,x)}$ (it is an immersed submanifold).
On the other hand, the Lie group G acts on $G \times M$ by the action given by $g.(a, x) = (ga, x)$. As for all action, and by differentiation, we obtain an action of G on the tangent bundle $T(G \times M)$:

$$G \to Aut(T(G \times M)) \ .$$

This is a representation of G in the group of automorphisms of the tangent bundle, and it is given by

$$g.(Z_b, X_x) = (((l_g)_*(Z_b), X_x).$$

This action preserves the foliation $\mathcal{F}$: for all $g \in G$

$$g.(\widehat{\tau}(h))_{(b,x)} = \widehat{\tau}(h)_{(gb,x)},$$

and hence

$$g.\mathcal{F}_{(b,x)} = \mathcal{F}_{(gb,x)}.$$

The distribution $\mathcal{F}$ is then stable by the action of G. An immediate consequence is the following :

$$g.F_{(a,x)} = F_{(ga,x)} \ .$$

Hence the group G acts on the leaf spaces:

$$(\mathbf{G} \times \mathbf{M})/\mathcal{F} \ .$$

▶ Every leaf is of the form $a.F_{(e,x)}$ where $a \in G$ and $x \in M$. In particular every orbit of the G-action on $(G \times M)/\mathcal{F}$ contains a leaf of the form $F_{(e,x)}$.
▶ For all $a \in G$ and $x \in M$, the mapping $F_{(e,x)} \to F_{(a,x)}$ given by $(b, y) \mapsto (ab, y)$ is a diffeomorphism (the argument is the fact that the leaves of a foliation are conservative submanifolds).
▶ For all $x \in M$, the mapping

$$\mathbf{p_{G,x}} : \mathbf{F_{(e,x)}} \to \mathbf{G}$$

given by $p_{G,x}(b, y) = b$, is a local diffeomorphism. (It is smooth because it is the restriction of the first projection $G \times M \to G$ to the leaf $F_{(e,x)}$. And the differential of this mapping is given by $(dp_{G,x})_{(b,y)}(L^h_b, X^h_x) = L^h_b$ for all $h \in \mathcal{G}$.)
Hence for all $x \in M$ there exists an open neighborhood U_x of e in G and a smooth mapping $s_x : U_x \to F_{(e,x)}$ such that $p_{G,x} \circ s_x = id_{U_x}$. [i.e. $s_x : U_x \to G \times M$, $s_x(g) \in F_{(e,x)}$ et $s_x(g) = (g, -)$ for all $g \in U_x$.]

Lemma 30 *(The neighborhood U_x can be chosen one and the same for all points $x \in M$.) i.e. There exists an open neighborhood U of e in G such that for all $x \in M$, there exists a mapping $s_x : U \to G \times M$ such that $s_x(g) \in F_{(e,x)}$ and $s_x(g) = (g, \star)$ for all $g \in U$.*

For this, we will need the following lemma.

Lemma 31 *For any complete field $X^{e_1}, \ldots, X^{e_k}$, and for any point $x \in M$, the formula*

$$\gamma_x(t_1, \ldots, t_k) = (exp(t_1 e_1) \ldots exp(t_k e_k) \ , \ \varphi^{X^{e_k}}_{t_k} \circ \cdots \circ \varphi^{X^{e_1}}_{t_1}(x))$$

defines a smooth mapping from $\gamma_x : \mathbb{R}^k \to F_{(e,x)}$.

Proof of lemma 2. (*By induction on k*)
Because the submanifold $F_{(e,x)}$ (as leaf of foliation) is conservative, we just have to prove that $\gamma_x(t_1, \ldots, t_k) \in F_{(e,x)}$ for all $(t_1, \ldots, t_k) \in \mathbb{R}^k$.
. We first let $k = 1$. In this case γ_x is a curve

$$\gamma_x(t) = (\alpha(t), \varphi(t))$$

where $\alpha(t) = exp(th)$, $\varphi(t) = \varphi^{X^h}_t(x)$, and $h \in \mathcal{G}$. X^h is a complete field. Moreover,

$$\dot{\gamma}_x(t) = (\dot{\alpha}(t), \dot{\varphi}(t)) = (L^h_{\alpha(t)}, X^h_{\varphi(t)}) = \widehat{\tau}(h)_{(\alpha(t),\varphi(t))}$$

for any $t \in \mathbb{R}$. Therefore

$$\dot{\gamma}_x(t) = \widehat{\tau}(h)_{\gamma_x(t)} = \widehat{X}^h_{\gamma_x(t)}.$$

This means that γ_x is an integral curve of the vector field $\widehat{X}^h$ (and in particular it is an integral curve of the distribution $\mathcal{F}$), and because $\gamma_x(0) = (e, x)$, this curve is contained entirely in the leaf $F_{(e,x)}$: $\gamma_x(\mathbb{R}) \subset F_{(e,x)}$.

. Suppose now that the lemma is already proved until the order $k-1$, and let

$$(exp(t_1e_1)\dots exp(t_{k-1}e_{k-1})\ ,\ \varphi^{X^{e_{k-1}}}_{t_{k-1}} \circ \dots \circ \varphi^{X^{e_1}}_{t_1}(x)) = (a, y).$$

. Because of the induction assumption, we have $(a, y) \in F_{(e,x)}$. Then

$$a.F_{(e,y)} = F_{(e,x)}\ .$$

We have: $\gamma_x(t_1, \dots, t_k) = (a.exp(t_ke_k), \varphi^{X^{e_k}}_{t_k}(y))$.
On the other hand, applying the case $k = 1$ (already proved)to the field X^{e_k}, we obtain

$$(exp(t_ke_k), \varphi^{X^{e_k}}_{t_k}(y)) \in F_{(e,y)}.$$

Therefore, $\gamma_x(t_1, \dots, t_k) \in F_{(e,x)}$. ♦

Proof of lemma 1. Let $\{e_1, \dots, e_r\}$ be a basis of the space $\mathcal{G}$ so that the vector fields $X^{e_1}, \dots, X^{e_r}$ are all complete.
Let us consider the mapping

$$\beta : \mathbb{R}^r \to G$$

given by $\beta(t_1, \dots, t_r) = exp(t_1e_1)\dots exp(t_re_r)$. This is a local diffeomorphism at 0, then there exists an open neighborhood V of 0 in $\mathbb{R}^r$ and an open neighborhood U of e in G such that $\beta_{|_V} : V \to U$ is a diffeomorphism. By lemma 2., the mapping

$$s_x := \gamma_x \circ (\beta_{|_V})^{-1} :\ U \to G \times M$$

allows to finish the proof. ♦

Lemma 32 *For any $x \in M$, the mapping*

$$p_{G,x} : F_{(e,x)} \to G$$

is surjective.

Proof of lemma 3. Let U as in lemma 1. Then

$$U \subset p_{G,x}(F_{(e,x)})\ .$$

The Lie group G being connected, the neighborhood U of e generates the group G:

$$\bigcup_m U^m = G\ .$$

We show by induction that for all $m \in \mathbb{N}$,

$$U^m \subset p_{G,x}(F_{(e,x)}) \ .$$

. For $m = 1$, it is what we have just seen.
. We suppose that: $\forall z \in M$, $U^m \subset p_{G,z}(F_{(e,z)})$.
. Let $x \in M$. We have:

$$\forall a \in U^m \ , \exists y \in M \ , \ \ (a, y) \in F_(e, x).$$

On the other hand, the fact that $(a, y) \in F_{(e,x)}$ means that $a.F_{(e,y)} = F_{(e,x)}$. Hence, the following diagram commutes:

$$\begin{array}{ccc} F_{(e,y)} & \xrightarrow{p_{G,y}} & G \\ a \downarrow & & \downarrow l_a \\ F_{(e,x)} & \xrightarrow{p_{G,x}} & G \end{array}$$

and since $U \subset p_{G,y}(F_{(e,y)})$, we will have $a.U = l_a(U) \subset p_{G,x}(F_{(e,x)})$; this holds for any $a \in U^m$. Therefore, $U^{m+1} \subset p_{G,x}(F_{(e,x)})$. ♦

As a consequence of lemma 3:

$$\forall a \in G, \forall x \in M, \exists z \in M, (a^{-1}, z) \in F_{(e,x)}.$$

Then $(e, z) \in a.F_{(e,x)}$, and therefore $F_{(e,z)} = a.F_{(e,x)} = F_{(a,x)}$.

Conclusion I.: Every leaf of the foliation $\mathcal{F}$ is of the form $F_{(e,z)}$.
In other words the mapping

$$\mathbf{M} \longrightarrow (\mathbf{G} \times \mathbf{M})/\mathcal{F}$$

given by $x \mapsto F_{(e,x)}$, is surjective.

Lemma 33 *For any $x \in M$, the mapping $p_{G,x}$ is a* **covering**.

Proof of lemma 4. We consider again the open set U in lemma 1. Put $V_x := s_x(U)$. This is an open set of the leaf $F_{(e,x)}$, which is contained in $p_{G,x}^{-1}(U) = F_{(e,x)} \cap (U \times M)$, and on which the mapping $p_{G,x}$ is a diffeomorphism onto U ($p_{G,x} : V_x \to U$).
Let $(a, y) \in [p_{G,x}^{-1}(U) - V_x]$. This means that $a \in U$ but $(a, y) \neq s_x(a)$.
$F_{(e,x)}$ being a Hausdorff space, there exists θ_1 an open neighborhood of (a, y) in $F_{(e,x)}$ and $\theta_2 \subset V_x$ an open neighborhood of $s_x(a)$ such that $\theta_1 \cap \theta_2 = \emptyset$. The open set $p_{G,x}(\theta_1) \cap p_{G,x}(\theta_2)$ is a neighborhood of a. Then we put

$$\theta_1' := \theta_1 \cap p_{G,x}^{-1}(p_{G,x}(\theta_1) \cap p_{G,x}(\theta_2))$$

and

$$\theta_2' := \theta_2 \cap p_{G,x}^{-1}(p_{G,x}(\theta_1) \cap p_{G,x}(\theta_2)).$$

While replacing θ_1 by θ_1' (and θ_2 by θ_2'), we obtain again two disjoint open sets of $p_{G,x}^{-1}(U)$ in such a way that

$$p_{G,x}(\theta_1') = p_{G,x}(\theta_2') \ .$$

By the injectivity of $p_{G,x|_{V_x}}$, we have:

$$\theta_1' \cap V_x = \emptyset \ .$$

[Indeed, let $p \in \theta_1'$, then there exists $q \in \theta_2'$ such that $p_{G,x}(p) = p_{G,x}(q)$; therefore $p \notin V_x$].
In summary, we have just established that for all $(a, y) \in [p_{G,x}^{-1}(U) - V_x]$, there exists θ_1' an open neighborhood of (a, y) in $p_{G,x}^{-1}(U)$ such that

$$\theta_1' \subset [p_{G,x}^{-1}(U) - V_x] \ .$$

This means that the set V_x is not only open but also closed in $p_{G,x}^{-1}(U)$. Since the set V_x (being homeomorphic to the neighborhood U) is connected, this proves that the set V_x **is the connected component of the set** $p_{G,x}^{-1}(U)$ **containing the point** (e, x).
Affirmation: *In every connected component of the set $p_{G,x}^{-1}(U)$, there exists a point of the form (e, z).*
Indeed, let C be a connected component of the space

$$X := p_{G,x}^{-1}(U) = (U \times M) \cap F_{(e,x)} \ .$$

Let us take $(a, y) \in C$. The mapping

$$\beta : \mathbb{R}^r \to G$$

given by $\beta(t_1, \ldots, t_r) = e^{t_1 e_1} \ldots e^{t_r e_r}$, being a local diffeomorphism from a neighborhood V of 0 onto U, there exists $(s_1, \ldots, s_r) \in \mathbb{R}^r$ such that $e^{s_1 e_1} \ldots e^{s_r e_r} = a$. Put

$$z = \varphi_{-s_1}^{X^{e_1}} \circ \cdots \circ \varphi_{-s_r}^{X^{e_r}}(y).$$

We consider the mapping

$$\gamma : V \to (U \times M) \cap F_{(e,z)}$$

given by

$$\gamma(t_1, \ldots, t_r) = (e^{t_1 e_1} \ldots e^{t_r e_r} \ , \ \varphi_{t_r}^{X^{e_r}} \circ \cdots \circ \varphi_{t_1}^{X^{e_1}}(z)).$$

Since $\gamma(t_1, \ldots, t_r) = (a, y)$, we have $(a, y) \in F_{(e,z)}$, then $F_{(a,y)} = F_{(e,z)}$. On the other hand $F_{(a,y)} = F_{(e,x)}$ [because $(a, y) \in C \subset F_{(e,x)}$); and then $F_{(e,z)} = F_{(e,x)}$.
Hence $\gamma(V)$ is a connected set in $(U \times M) \cap F_{(e,x)} = X$, containing the point (a, y), we have $\gamma(V) \subset C$. In particular $\gamma(0) \in C$, and $(e, z) \in C$.
In summary, we have established that for all connected component C in $p_{G,x}^{-1}(U)$, there exists a point $z \in M$ such that $C = V_z$ and $F_{(e,z)} = F_{(e,x)}$. Conversely, let $z \in M$ such that $F_{(e,z)} = F_{(e,x)}$; the set V_z is then the connected component of $F_{(e,x)} \cap (U \times M)$ containing the point (e, z). Hence

$$p_{G,x}^{-1}(U) = \coprod_{z_x / F_{(e,z)} = F_{(e,x)}} V_{z_x}.$$

[One can see easily that if z, z' are such that $F_{(e,z)} = F_{(e,z')} = F_{(e,x)}$ but $z \neq z'$, then $V_z \neq V_{z'}$].

Now let us consider the neighborhood of another point $a \in G$. The open set $a.U$ is a neighborhood of a in G. We have

$$p_{G,x}^{-1}(a.U) = \coprod_{q/F_{(a,q)}=F_{(e,x)}} a.V_q.$$

Indeed, $(b, y) \in p_{G,x}^{-1}(a.U)$ if and only if ($a^{-1}b \in U$ and $(b, y) \in F_{(e,x)}$); this means that

$$a^{-1}(b, y) \in a^{-1}.F_{(e,x)} \cap (U \times M).$$

Since $a^{-1}.F_{(e,x)} = F_{(e,z)}$, the point $a^{-1}(b, y)$ is contained in a connected component C of $F_{(e,z)}$, then in a set V_{q_z}. Hence

$$p_{G,x}^{-1}(a.U) = \coprod_{q/F_{(e,q_z)}=F_{(e,z)}} a.V_{q_z}.$$

The fact that $F_{(e,q_z)} = F_{(e,z)} = a^{-1}.F_{(e,x)}$ is equivalent to the fact that $F_{(a,q_z)} = F_{(e,x)}$. This allows us to finish the proof of lemma 4 ♦

Some consequences of lemma 4:

▶ Since the Lie group G is simply connected by assumption and therefore has only a trivial covering, for all $x \in M$ the mapping

$$p_{G,x} : F_{(e,x)} \to G$$

is a diffeomorphism.

▶ For any $x, x' \in M$, we have the implication

$$(F_{(e,x)} = F_{(e,x')}) \quad \Rightarrow \quad (\ x = x'\ )$$

[The reason is that (e, x') and (e, x) are two points in $F_{(e,x)}$, and have the same image by the mapping $p_{G,x}$.]

Conclusion II: The mapping

$$\mathbf{\Phi} \ : \ \mathbf{M} \longrightarrow (\mathbf{G} \times \mathbf{M})/\mathcal{F}$$

given by $\Phi(x) = F_{(e,x)}$, is **bijective**.

This bijection transports the G-action from $(G \times M)/\mathcal{F}$ to the manifold M. More explicitly this action is given by

$$\rho(\mathbf{g}, \mathbf{x}) = \mathbf{g}.\mathbf{x} \ := \ \mathbf{\Phi}^{-1}(\mathbf{g}.\mathbf{F}_{(\mathbf{e},\mathbf{x})}) = \mathbf{\Phi}^{-1}(\mathbf{F}_{(\mathbf{g},\mathbf{x})}),$$

that is $g.x$ is defined by

$$\mathbf{F}_{(\mathbf{e},\mathbf{g}.\mathbf{x})} = \mathbf{F}_{(\mathbf{g},\mathbf{x})} \ .$$

►The G-action ρ just defined satisfies the following assertions:
i) It is a smooth action.
ii) It is a primitive of the Lie algebra action τ.

To prove i) we just have to show it on U. We shall need to come back on the expressions of the mapping $s_x : U \to F_{(e,x)}$, and to explicit the "$\star$" of lemma 1.
Let us remind that the mapping

$$\beta : \mathcal{G} \to G$$

given by $\beta(t_1e_1 + \cdots + t_re_r) = exp(t_1e_1)\ldots exp(t_re_r)$, is a diffeomorphism from a neighborhood V of 0 onto U.
Let

$$\phi : \mathcal{G} \to T(\mathcal{G})$$

be the mapping defined by

$$\phi(t_1e_1 + \cdots + t_re_r) = \varphi_{t_r}^{X^{e_r}} \circ \cdots \circ \varphi_{t_1}^{X^{e_1}} .$$

Lemmas 1 and 2 can then be formulated as follows:

Lemma 34 *For any $x \in M$, the mapping $s_x : U \to F_{(e,x)}$ is given by*

$$s_x(g) = (g, [(\phi \circ \beta^{-1})(g)](x)).$$

Let us use this lemma: Let $g \in U$ and put $y := [(\phi \circ \beta^{-1})(g)]^{-1}(x)$. By this lemma $s_y(g) = (g, [(\phi \circ \beta^{-1})(g)](y))$.
Then $s_y(g) = (g, x)$, and therefore $(g, x) \in F_{(e,y)}$; that is $F_{(g,x)} = F_{(e,y)}$. Hence $g.x = y$.
In summary, the expression of the action ρ on $U \times M$ is given by

$$\rho(g, x) = [(\phi \circ \beta^{-1})(g)]^{-1}(x).$$

The composition map of $\beta \times id_M : V \times M \to U \times M$ and $\rho : U \times M \to M$, is given by

$$(t_1e_1 + \cdots + t_re_r, x) \mapsto \varphi_{-t_1}^{X^{e_1}} \circ \cdots \circ \varphi_{-t_r}^{X^{e_r}}(x).$$

In particular the mapping ρ is a smooth.
Moreover, for any $j = 1, \ldots, r$ the fundamental vector field associated to the vector e_j and defined by

$$-\rho'(e_j)_x = -\frac{d}{dt}_{|t=0}\rho(expte_j, x)$$

coincides with X^{e_j}. Then by linearity, this implies that ρ is a primitive of τ [For any $h \in \mathcal{G}$, we have $X^h = -\rho'(h)$]. Therefore, we have established ii). In particular all $X^h \in \tau(\mathcal{G})$ are complete [the flow of X^h is given by $\varphi_t^{X^h}(x) = \rho(exp(-th), x)$].

► Consider now the homomorphism

$$\widehat{\rho} : G \longrightarrow T(\mathcal{G}) \subset Diff(M),$$

associated to the action ρ, that is $\widehat{\rho}(g) = \rho(g, \cdot)$.
The vector fields X^h are all complete. Then the subgroup $T(\mathcal{G}) \subset Diff(M)$ is the one generated by the flow $\varphi_t^{X^h}$; and since $\widehat{\rho}(exp(-th)) = \varphi_t^{X^h}$, the homomorphism $\widehat{\rho}$ is a surjective map. The kernel $\Gamma := ker\widehat{\rho}$ (i.e. the intersection of isotropy groups) being a closed subgroup in G, it is then a Lie group. Its Lie algebra is the set of elements $h \in \mathcal{G}$ such that $\forall t \in \mathbb{R}$ and $\forall x \in M$, $\rho(exp(-th), x) = x$, that is $\rho'(h) = 0$ i.e. $\tau(h) = 0$; then $h = 0$ (since the action τ is effective). Hence Γ is a distinguished discrete subgroup in G. Now, we have a bijection

$$\widetilde{\rho} : \mathbf{G}/\mathbf{\Gamma} \overset{\cong}{\to} \mathbf{T}(\mathcal{G})$$

By this identification we bring the smoothness from G/Γ to $T(\mathcal{G})$ in a such a way that the bijection $\widetilde{\rho}$ becomes a diffeomorphism. The homomorphism $\widehat{\rho}$ is then a covering map. Moreover, it is clear that the canonical effective action $T(\mathcal{G}) \times M \to M$ is smooth. This concludes the proof of the theorem. □

5.5 Weak-Completeness implies Completeness

Lemma 35 *Let $\tau : \mathcal{G} \hookrightarrow \chi(M)$ be an effective action. Suppose that there exists a subset $S \subset \tau(\mathcal{G})$ such that:*
- *$\forall Y \in S$, the field Y is complete.*
- *S generates the Lie algebra $\tau(\mathcal{G})$: $\mathcal{A}lg(S) = \tau(\mathcal{G})$.*

Then there exists a basis $\{X^{e_1}, \ldots, X^{e_r}\}$ of complete fields.

Proof.
We shall show that the set

$$C := \{X \in \tau(\mathcal{G}) \ / \ X \text{ is complete}\}$$

generates linearly $\tau(\mathcal{G})$.
By assumption $\mathcal{A}lg(C) = \tau(\mathcal{G})$ (this is because $S \subset C$ and $\mathcal{A}lg(S) = \tau(\mathcal{G})$).
The set C of complete vector fields satisfies the following two properties:
(i)

$$\forall t \in \mathbb{R},\ \forall X \in C,\ tX \in C\ .$$

(ii)

$$\forall X \in C,\ \forall Y \in C,\ e^{ad_X}(Y) \in C\ .$$

We are going to show the assertions (ii), which seem not to be obvious.

Let $h, k \in \mathcal{G}$ and $X^h = \tau(h)$, $X^k = \tau(k)$. Put $a := exp(h) \in G$, and consider the curve $\beta : \mathbb{R} \to G$ given by

$$\beta(t) = aexp(tk)a^{-1}.$$

We have the following equalities

$$\beta(t) = exp(tAd_a(k)) = exp(tAd_{exph}(k)) = exp(te^{ad_h}(k))\ .$$

For $x \in M$, we denote by $\varphi : \mathbb{R} \to M$ the curve

$$\varphi(t) := \varphi_{-1}^{X^h} \circ \varphi_t^{X^k} \circ \varphi_1^{X^h}(x).$$

By Lemma 2, the curve $\gamma_x : \mathbb{R} \to G \times M$, given by

$$\gamma_x(t) := (exp(h)exp(tk)exp(-h)\ ,\ \varphi_{-1}^{X^h} \circ \varphi_t^{X^k} \circ \varphi_1^{X^h}(x))$$

has its images in the leaf $F_{(e,x)}$; then

$$\dot{\gamma}_x(0) \in T_{(e,x)} = \widehat{\tau}_{(e,x)}(\mathcal{G})$$

Now $\dot{\gamma}_x(0) = (\dot{\beta}(0), \dot{\varphi}(0))$ and $\dot{\beta}(0) = e^{ad_h}(k)$. We have

$$\dot{\varphi}(0) = [\tau(e^{ad_h}(k))]_x.$$

Since $\tau : \mathcal{G} \to \tau(\mathcal{G})$ is a Lie algebra isomorphism, we have

$$\tau(e^{ad_h}(k)) = e^{ad_{\tau(h)}}\tau(k)\ ;$$

and therefore $\dot{\varphi}(0) = [e^{ad_{\tau(h)}}\tau(k)]_x$.
Conclusion: the flow of the vector field $e^{ad_{\tau(h)}}\tau(k) \in \chi(M)$, is then given by

$$\varphi(t,x) = \varphi_{-1}^{X^h} \circ \varphi_t^{X^k} \circ \varphi_1^{X^h}(x)\ .$$

Then, it is complete. Hence (ii)is proved.
The last step: we are going to show that the set C must satisfy the two properties (i) and (ii) above, which imply that $vect(C) = \tau(\mathcal{G})$.
Put $\mathcal{V} := vect(C)$. We shall show that the set $\mathcal{V}$ is a Lie subalgebra of $\tau(\mathcal{G})$.
Since

$$[X,Y] = (adX)Y = lim_{t\to 0}\frac{e^{tadX}Y - Y}{t},$$

we have $[X,Y] \in C$ for any $X, Y \in C$, then $[C,C] \subset \mathcal{V}$, and therefore $[\mathcal{V},\mathcal{V}] \subset \mathcal{V}$ (because of linearity). Therefore, $\mathcal{V}$ is a subalgebra of the Lie algebra $\tau(\mathcal{G})$ that contains C, and then $\mathcal{V} = \tau(\mathcal{G})$. ♦

5.6 Lie Transformation Groups

Let M be a smooth manifold, and let $H \subset Diff(M)$ be a subgroup. The theorem below which is assigned to Kobayashi allows to give sufficient conditions so that such a subgroup must be a Lie group. This will be a consequence of Palais theorem as we are going to see.

Theorem 62 *Let $H \subset Diff(M)$ be a subgroup of the diffeomorphism group of a manifold M. Let $\mathcal{S} \subset \chi(M)$ be the subset:*

$$\mathcal{S} := \{X complet \ / \ \varphi_t^X \in H \ , \ \forall t \in \mathbb{R}\}.$$

Suppose that $\quad dim(\mathcal{A}lg(\mathcal{S})) < \infty$.
Then, H is a Lie group with Lie algebra $\mathcal{S}$. [In particular: $\mathcal{S} = \mathcal{A}lg(\mathcal{S})$].

Proof. By assumption, the natural action of the Lie algebra $\mathcal{G} := \mathcal{A}lg(\mathcal{S})$ on M is weakly-complete and therefore complete (as consequence of a Palais theorem), besides the group

$$T(\mathcal{G}) :=< \{\varphi_t^X \ / \ X \ \in \mathcal{G}\} >$$

is a Lie group with Lie algebra $\mathcal{G}$.
Claim : $T(\mathcal{G}) \subset H$.
Indeed, for any $X \in \mathcal{G}$, $x \in M$ and $t \in \mathbb{R}$ we have

$$exp(tX).x = \varphi_t^X(x),$$

where exp denotes the exponential map of the Lie group $T(\mathcal{G})$. In particular, $exp(Y) \in H$ for any $Y \in \mathcal{S}$. Since $exp(\mathcal{S})$ generates the group $T(\mathcal{G})$ (cf. the lemma below), we have $T(\mathcal{G}) \subset H$.

Lemma 36 *Let G be a connected Lie group with Lie algebra $\mathcal{G}$. Let $\mathcal{S}$ be a subset of $\mathcal{G}$ such that*
- $\mathcal{A}lg(\mathcal{S}) = \mathcal{G}$.
- $t.\mathcal{S} \subset \mathcal{S}$ *for all* $t \in \mathbb{R}$.

Then $exp(\mathcal{S})$ generates G.

Proof. Let K be the subgroup of G generated by $exp(\mathcal{S})$. Consider the set

$$U := \{X \in \mathcal{G} \ / \ exp(tX) \in K, \ \forall t \in \mathbb{R}\}.$$

We have the inclusions $\mathcal{S} \subset U \subset V := vect(U) \subset \mathcal{G}$.
Claim: $\forall a \in K$, $\forall X \in U$, $ad_a(X) \in U$. In fact,

$$exp(tad_a(X)) = exp(ad_a(tX)) = aexp(tX)a^{-1} \ .$$

Then (by linearity) $ad_a(V) \subset V$ for any $a \in K$; and therefore $ad_{exp(tX)}(Y) \in V$ for any $X \in U$ and $Y \in V$. This implies that $[U, V] \subset V$, and using linearity, $[V, V] \subset V$. Then V is a Lie subalgebra of $\mathcal{G}$, which contains $\mathcal{S}$; this implies $V = \mathcal{A}lg(\mathcal{S}) = \mathcal{G}$.
Therefore, we have established that $\mathcal{G} = vect(U)$ and there exists a basis $\beta := \{X_1, \ldots, X_r\}$ of $\mathcal{G}$ such that $X_1, \ldots, X_r \in U$, that is $exp(tX_j) \in K$ for all $t \in \mathbb{R}$. The image of the mapping $(t_1, \ldots, t_r) \mapsto e^{t_1 X_1} \ldots e^{t_r X_r}$ is then contained in K; since this mapping is a local diffeomorphism from a neighborhood of zero in $\mathbb{R}^r$ onto a neighborhood θ of e in G, we have $\theta \subset K$; therefore $K = G$ since θ generates G.

Summary: we have $T(\mathcal{G}) \subset H$. In another way, for any $X \in \mathcal{G}$ and $\forall t \in \mathbb{R}$, we have $\varphi_t^X \in H$. Hence $\mathcal{G} = \mathcal{S}$.
On the other hand, for any $X \in \mathcal{G}$ and $g \in H$, we can consider the vector field $\widehat{X}$ whose flow (which is global) is given by

$$\varphi_t^{\widehat{X}} := g^{-1} \circ \varphi_t^X \circ g.$$

The fact that these diffeomorphisms are all in H, allows us to deduce that $\widehat{X} \in \mathcal{S} \subset \mathcal{G}$, and therefore

$$\varphi_t^{\widehat{X}} \in T(\mathcal{G}).$$

Now, since the diffeomorphisms of the form φ_t^X generate $T(\mathcal{G})$, we have

$$\forall g \in H \ , \ g^{-1} \circ T(\mathcal{G}) \circ g \subset T(\mathcal{G}).$$

That is $T(\mathcal{G})$ is a normal subgroup in H.

We are now in the following situation: . $K := T(\mathcal{G})$ is a connected Lie group, . K is a normal subgroup of H, and . for any $g \in H$ the automorphism $\lambda_g : x \mapsto g^{-1}.x.g$ of K is smooth (this is due to the fact that it sends a one-parameter subgroup "$exp(tX)$" onto a one-parameter subgroup "$exp(t\widehat{X})$"). Under such conditions, there exists a smoothness on H with respect to which the group H is a Lie group whose identity component is K (this is left as an exercise to the reader). This finishes the proof. □

5.7 The Classical Transformation Groups in Geometry

As an application of Theorem 6.1. above, we can prove that automorphism groups of many geometric structures are Lie groups. For more information on the subject, see [19].

Definition 80 *Let M be a manifold, and ∇ a linear connection on the tangent bundle TM of M.*
• *A diffeomorphism $\varphi \in Diff(M)$ will be called an affine transformation if and only if it preserves the connection ∇, that is*

$$\varphi(\nabla_Y Z) = \nabla_{\varphi Y} \varphi Z.$$

The set of all affine transformations is a group which will be denoted by $Diff_\nabla(M)$ or by $Aff(M, \nabla)$.
• *A vector field $X \in \chi(M)$ is called an infinitesimal affine transformation if it generates a local 1-parameter group of local affine transformations; this is equivalent to say that we have*

$$[X, \nabla_Y Z] = \nabla_Y [X, Z] + \nabla_{[X,Y]} Z \quad \forall X, Y, Z \in \chi(M),$$

that is

$$L_X \circ \nabla_Y - \nabla_Y \circ L_X = \nabla_{[X,Y]} \ \ \forall X, Y \in \chi(M),$$

in other words

$$\nabla_{ad(X)Y} = [adX, \nabla_Y] \ \ \forall X, Y \in \chi(M) \ .$$

The set of all infinitesimal affine transformations will be denoted by $\mathcal{G}_\nabla(M)$.

It is easy to see (because of the Jacobi identity) that $\mathcal{G}_\nabla(M)$ is a Lie subalgebra of $\chi(M)$.

Theorem 63 *Let M be a connected n dimensional manifold endowed with a connection ∇.*
Then $dim(\mathcal{G}_\nabla(M)) \leq n^2 + n$.

The proof of this theorem will be discussed in the following section. As a consequence, and by applying II, we obtain:

Theorem 64 *The group $Diff_\nabla(M)$ is a Lie group of dimension at most $n^2 + n$.*

Besides, we can show that if the connection is *complete* (c.f. subsection 8.3), then the Lie algebra of the Lie group $Diff_\nabla(M)$ is identified with $\mathcal{G}_\nabla(M)$ [*this follows from the fact that under this assumption, each affine vector field is complete*].
Example-exercise. Let $M := \mathbb{R}^n$ endowed with the canonical linear connection ∇, then

$$Diff_\nabla(M) \simeq GL(n, \mathbb{R}) \ltimes \mathbb{R}^n = Aff(n, \mathbb{R})$$

is the affine transformation group, acting transitively on $\mathbb{R}^n$ by

$$(A, b).x := Ax + b \quad \forall (A, b) \in GL(n, \mathbb{R}) \ltimes \mathbb{R}^n \ , \ x \in \mathbb{R}^n.$$

[Note that because the isotropy subgroup at the origin $0 \in \mathbb{R}^n$ is isomorphic to $GL(n, \mathbb{R})$, the manifold $\mathbb{R}^n$ is represented as the homogenous manifold $Aff(n, \mathbb{R})/GL(n, \mathbb{R})$].

Definition 81 *Let M be a manifold endowed with a Riemannian metric $<>$.*
• *An isometry of M is a transformation of M which leaves the metric $<>$ invariant, that is*

$$< \varphi_* X_x, \varphi_* Y_x > = < X_x, Y_x >,$$

$\forall x \in M, \forall X_x, Y_x \in T_x M$.
• *The set of all isometries is a group which will be denoted by $Diff_{<,>}(M)$ or $Iso(M, <, >)$.*
• *A vector field $X \in \chi(M)$ is called an infinitesimal isometry or a Killing vector field if it generates a local 1-parameter group of local isometries; this is equivalent to say that we have*

$$X. < Y, Z > = < [X, Y], Z > + < Y, [X, Z] > \ .$$

The set of all Killing field denoted by $\mathcal{G}_{<,>}(M)$ (or $Kill_{<,>}(M)$) is a Lie subalgebra of $\chi(M)$. By an analogue proof of the previous theorem we can show the following result.

Theorem 65 *Let M be a connected n-dimensional manifold endowed with a Riemannian metric $<, >$.*
Then $dim(\mathcal{G}_{<,>}(M)) \leq \frac{1}{2}n(n+1)$.

Example-exercise. Let $M := \mathbb{R}^n$ endowed with the canonical metric $<>$. Then

$$Diff_{<>}(M) \simeq O(n) \ltimes \mathbb{R}^n = Iso(n, \mathbb{R})$$

is the Euclidean motion group, acting transitively on $\mathbb{R}^n$ by

$$(A, b).x := Ax + b \quad \forall (A, b) \in O(n, \mathbb{R}) \ltimes \mathbb{R}^n \ , \ x \in \mathbb{R}^n.$$

[Thus the manifold $\mathbb{R}^n$ is represented as the homogenous manifold $Iso(n, \mathbb{R})/O(n)$].
Remark. If ∇ denotes the Levi-Civita connection associated to the metric $<>$, then every transformation φ preserving the metric $<>$ preserves automatically the connection ∇ (show it as an exercise); therefore we have the inclusion

$$\mathcal{G}_{<>}(M) \subset \mathcal{G}_{\nabla}(M).$$

Now, if M is a **compact** Riemannian manifold, then we have the equalities: $\mathcal{G}_{<>}(M) = \mathcal{G}_{\nabla}(M)$, that is, each affine field is Killing, (this is a result of K. Yano [35]). More recently, for other results in the pseudo-Riemannian case, one can see the papers of A. Zeghib [37][38]. □

Corollary 15 *1) The group $Iso(M, <, >)$ is a Lie group of dimension at most $\frac{1}{2}n(n+1)$.*
2) If the metric is complete, then the Lie algebra of $Iso(M, <, >)$ is identified with $\mathcal{G}_{<,>}(M)$.

The main fact in the second point of this theorem is that *on a complete Riemannian manifold, every Killing field is complete.*

5.8 Notes on the Manifold of Linear Frames

In this section, we review some of the standard properties about the manifold of linear frames; these are useful to complete the proofs of previous result in this course. For more details see [18].
Let M be an n-dimensional manifold. Let us remind that a frame is a point x on the manifold together with a linear isomorphism $z_x : \mathbb{R}^n \to T_xM$; the frame bundle is the manifold of all frames of M, and it will be denoted by P_M. The group $G := GL(n, \mathbb{R})$ acts on the right on P_M in a natural way: $z_x.g := z_x \circ g$. The bundle of linear frames $P_M \xrightarrow{\pi} M$ is a G-principal bundle. It is important to note that the first interesting property of this bundle is the following: "*the pull-back of the tangent bundle $TM \to M$ via the map π is the trivial bundle $P_M \times \mathbb{R}^n \to P_M$*"; this follows from the existence of the natural mapping $P_M \times \mathbb{R}^n \to T(M)$ given by $(z, v) \mapsto z(v)$, which defines an homomorphism between the two bundles $P_M \times \mathbb{R}^n \to P_M$ and $T(M) \to M$.

5.8.1 The canonical differential $\mathbb{R}^n$-valued 1-form

Let M be a n-dimensional manifold, and put $P := P_M$. There exists on the manifold P a canonical $\mathbb{R}^n$-valued differential 1-form:

$$\theta \in \Omega^1(P; \mathbb{R}^n),$$

defined by

$$\theta(Y_z) := z^{-1}(\pi_*(Y_z)),$$

where $Y_z \in T_z(P)$ et $\mathbb{R}^n \xrightarrow{z} T_xM$.
This is a semi-basic 1-form (i.e. $\theta(Y) = 0$ for any vertical field Y) and it is G-equivariant (i.e. $\theta(Z.g) = g^{-1}.\theta(Z)$ for any $g \in G$ and $Z \in T_z(P)$).
We have a short exact sequence of G-vector bundles:

$$0 \to V \to TP \xrightarrow{\theta} P \times \mathbb{R}^n \to 0 \qquad (\epsilon)$$

where $V := ker(d\pi) \to P$ is the vertical bundle which is a G-vector subbundle of TP. The action of G on TP is obtained by differentiation of the right G-action on P, and G acts on $P \times \mathbb{R}^n$ by $g.(z, v) = (z.g^{-1}, g.v)$.
• Any diffeomorphism $f \in Diff(M)$ induces in a natural way an automorphism $\widetilde{f}$ of the main bundle $P_M \to M$, which preserves the 1-form θ (i.e. $\widetilde{f} \in Diff(P)$, $\pi \circ \widetilde{f} = f \circ \pi$ and $\widetilde{f}^\star(\theta) = \theta$). It is easy to establish that we have

$$Diff(M) \cong Aut_\theta(P_M).$$

• Every vector field $X \in \chi(M)$ induces in a natural manner (i.e. without connection !) a lifting $\widetilde{X} \in \chi(P_M)$ (such that $\pi_*(\widetilde{X}_z) = X_{\pi(z)}$ for any $z \in P$). The field $\widetilde{X}$ is G-invariant and preserves the 1-form θ (that is $L_{\widetilde{X}}\theta = 0$).
The space of vector fields $\widetilde{X} \in \chi(P_M)$ which are G-invariant and which preserve the form θ, constitutes a Lie algebra: $\chi^G_\theta(P_M)$. One can show: $\chi(M) \cong \chi^G_\theta(P_M)$.

5.8.2 Finiteness of the dimension of $\mathcal{G}_\nabla(M)$.

Let ∇ be a connection on M. It can be interpreted as a G-equivariant splitting of the above short exact sequence (ϵ). The reason is that the connection 1-form $\omega^\nabla \in \Omega^1(P; \mathcal{G}l(n, \mathbb{R}))$ which is usually associated to ∇ can be viewed as a G-equivariant projector of the tangent bundle TP onto the vertical bundle V). This leads to the existence of a bundle homomorphism:

$$TP \xleftarrow{B} P \times \mathbb{R}^n$$

such that $\theta \circ B = Identity$ and $B(z.g, g^{-1}v) = B(z, v).g$.
For each $v \in \mathbb{R}^n$, we associate the horizontal vector field $B^v \in \chi(P)$ which is given by $B^v_z = B(z, v)$, in such way that we have

$$\pi_*(B^v_z) := z(v) \ \ \forall z \in P\ .$$

We call $(B^v)_{v\in\mathbb{R}^n}$ the *standard horizontal vector fields.*

Lemma 37 *Let (M, ∇) be a connected manifold with a connection, and let $X \in \chi(M)$.*
(i) The two following assertions are equivalent:

• $X \in \mathcal{G}_\nabla(M)$.
• $[\widetilde{X}, B^v] = 0$, *for every* $v \in \mathbb{R}^n$.
(ii) Let $z \in P$ *be an arbitrary point. The evaluation mapping:*

$$\mathbf{X} \in \mathcal{G}_\nabla(\mathbf{M}) \longrightarrow \widetilde{\mathbf{X}}_\mathbf{z} \in \mathbf{T_z P}$$

is injective.

Proof.
(i) First, it is easy to see that we have $X \in \mathcal{G}_\nabla(M)$ if and only if $L_{\widetilde{X}}\omega^\nabla$, where ω^∇ is the associated connection 1-form. Then, for every $X \in \mathcal{G}_\nabla(M)$ we have

$$0 = (L_{\widetilde{X}}\omega^\nabla)(B^v) = \widetilde{X}(\omega^\nabla(B^v)) - \omega^\nabla([\widetilde{X}, B^v]) = -\omega^\nabla([\widetilde{X}, B^v]).$$

On the other hand, since the field $\widetilde{X}$ preserves θ, and since we have $0 = \widetilde{X}(\theta(B^v))$, we deduce that $\theta([\widetilde{X}, B^v]) = 0$. Conversely, if $[\widetilde{X}, B^v] = 0$, then the flow $\varphi_t^{\widetilde{X}}$ preserves the horizontal distribution $H := ker(\omega^\nabla)$; it follows that $\varphi_t^{\widetilde{X}}$ preserves the connection. Hence $X \in \mathcal{G}_\nabla(M)$.
(ii) If $\widetilde{X}_z = 0$, then $\widetilde{X}_{zg} = 0$ for any $g \in G$ (it ensues of the G-invariance of $\widetilde{X}$). Consider now the set F of points $x \in M$ such that $X_x = 0$; this is a non empty closed subset of M. Since M is connected, it is sufficient to establish that if X vanishes at some point $x \in M$ then it vanishes locally around x. Since $\widetilde{X}$ commutes with the vector fields B^v, the flows $\varphi_t^{B^v}$ preserves the field $\widetilde{X}$; therefore $\widetilde{X}_{\varphi_t^{B^v}(z)} = 0$ for any $v \in \mathbb{R}^n$ (z is a point in the fiber $\pi^{-1}(x)$).
Now, it is easy to see that points of the $\pi(\varphi_{t_1}^{B^{e_1}} \circ \cdots \circ \varphi_{t_n}^{B^{e_n}}(z))$ cover a neighborhood of x (where $(t_1, \ldots, t_n)$ are varying in a small neighborhood of 0 in $\mathbb{R}^n$). This allows to finish the proof of (ii). □

5.8.3 Completeness of Affine Fields

The aim of this section is to show that if ∇ *is a complete connection on* M*, then each affine vector field is complete.*

Definition 82 *A connection* ∇ *on* M *is said to be complete if every standard horizontal field on* P_M *is complete.*

We will assume that ∇ is a complete connection on M. Hence, $\forall v \in \mathbb{R}^n$ and $\forall z \in P_M$, the quantities $\varphi_t^{B^v}(z)$ have a sense $\forall t \in \mathbb{R}$.
• Let $x \in M$ and $z \in \pi^{-1}(x)$ (which will be seen as an arbitrary frame in x). To each $X_x \in T_xM$ corresponds the vector $v := z^{-1}(X_x) \in \mathbb{R}^n$; therefore, we can consider the curve

$$\gamma_x : \mathbb{R} \to M \ ,$$

given by $\gamma_x(t) = \pi(\varphi_t^{B^v}(z))$; this is a well defined mapping (it doesn't depend on the choice of the point z on the fiber $\pi^{-1}(x)$) and it is a smooth.
• The curve γ_x is called the *geodesic* with initial conditions (x, X_x); we have

$$\gamma_x(0) = x \quad , \quad \dot{\gamma}_x(0) = X_x.$$

• For each $x \in M$, we define the exponential map:

$$exp_x : T_xM \to M$$

by

$$exp_x(X_x) := \gamma_x(1) = \pi(\varphi_1^{B^{z^{-1}X_x}}(z)).$$

It is of class C^∞, and it is easy to see that the differential in 0_x of exp_x is equal to the identity $((dexp_x)_{0_x}(X_x) = X_x)$. By the implicit function theorem, there is a neighborhood of 0_x in T_xM which is mapped diffeomorphically onto a neighborhood U_x of x in M. The couple (U_x, exp_x^{-1}) is then a natural *coordinate system*, which will be called the normal coordinate system determined by the frame z.
We need the following lemma.

Lemma 38 *Let X be a vector field on a manifold M, and $(\varphi_t^X)_t$ its flow.*
If there exists a number $\epsilon > 0$ such that $\varphi_t^X(x)$ is defined for any $x \in M$ and any $t \in]-\epsilon, \epsilon[$, then the field X is complete ($\varphi_t^X(x)$ is defined for all $t \in \mathbb{R}$, $x \in M$).

Proof. For each $t \in \mathbb{R}$, there exists a unique integer $m \in \mathbb{Z}$ such that $t = m\frac{\epsilon}{2} + s$ and $s \in [0, \frac{\epsilon}{2}[$. We set

$$c_x(t) := \begin{cases} \overbrace{\varphi_{\frac{\epsilon}{2}} \circ \cdots \circ \varphi_{\frac{\epsilon}{2}}}^{m} \circ \varphi_s(x) & if \;\; m > 0 \\ \overbrace{\varphi_{\frac{-\epsilon}{2}} \circ \cdots \circ \varphi_{\frac{-\epsilon}{2}}}^{-m} \circ \varphi_s(x) & if \;\; m < 0 \\ \varphi_s(x) & if \;\; m = 0 \end{cases}$$

We let the reader to verify that for any $x \in M$, the curve $c_x : t \mapsto c_x(t)$ is differentiable on $\mathbb{R}$ and it is a trajectory of the vector field X. This finishes the proof of the lemma.□
Exercise. If X is a vector field with compact support, then it is a complete field.

Theorem 66 *Let ∇ be a complete connection on a manifold M.*
Then each affine vector field is complete.

Proof. Without loss of generality we can assume that the manifold is connected. Let $X \in \mathcal{G}_\nabla(M)$, and $\widetilde{X} \in \chi(P_M)$ be its natural lifting. Let $x_0 \in M$ and $z_0 \in \pi^{-1}(x_0)$ be an arbitrary point. There exists a neighborhood V of z_0 and a positive number ϵ such that the local 1-parameter group of local transformation $\varphi_t^{\widetilde{X}} : V \to P_M$, is defined for any $t \in]-\epsilon, \epsilon[$.
Because of the lemma above, it is sufficient to show that for any $x \in M$, the curve

$t \mapsto \varphi_t^{\widetilde{X}}(z)$ is defined on $]-\epsilon,\epsilon[$. This is equivalent to show that trajectories are all defined on $]-\epsilon,\epsilon[$.
Let $x \in M$ and $z \in \pi^{-1}(x)$. The two points $x = \pi(z)$ and $x_0 = \pi(z_0)$ can be joined by a succession of geodesics (this is due to the existence of normal coordinates, and the fact that M is connected). This means that there exist $t_1, \ldots, t_k \in \mathbb{R}$ and $v_1, \ldots, v_k \in \mathbb{R}^n$ such that $x = \pi(\varphi_{t_1}^{B^{v_1}} \circ \cdots \circ \varphi_{t_k}^{B^{v_k}}(z_0))$. Then consider the curve $\beta :]-\epsilon,\epsilon[\to M$ given by $\beta(t) = \pi(\varphi_{t_1}^{B^{v_1}} \circ \cdots \circ \varphi_{t_k}^{B^{v_k}}(\varphi_t^{\widetilde{X}} z_0))$. Such a curve is an integral curve of the field X (this is due to fact that the flow $\varphi_t^{B^v}$ preserves the field $\widetilde{X}$, which implies $\dot{\beta}(t) = \pi_*(\widetilde{X}_{\varphi_{t_1}^{B^{v_1}} \circ \cdots \circ \varphi_{t_k}^{B^{v_k}}(\varphi_t^{\widetilde{X}} z_0)}) = X_{\beta(t)}$).
The proof is then complete by applying the above lemma.□

Corollary 16 *If (M, g) is a pseudo-Riemannian manifold, then each Killing vector field is complete.*

5.9 Notes and Comments

This section contains some supplementary results and historical facts, allowing a more global vision on the main topic of this course.
N1• Throughout this course, we have seen that the group of transformations preserving a certain structure of a manifold is often a Lie group. This is due to the theorem of S. Kobayashi; it provides a criterion for a subgroup $G \subset Diff(M)$ to carry a structure of Lie group. On this topic, it is important to remind the following result of Montgomery-Zippin-Bochner [24](p.208-p.212) :
Theorem: *If G is a locally compact topological transformation group ($G \subset Diff(M)$), then G, with the compact-open topology, is a Lie transformation group.*
This approach is used in [18] to show that the groups $Iso(M, g)$ and $Aff(M, \nabla)$ are a Lie group. In [11](1983), the author applies the two theorems to study the group $G(P)$:=the group of diffeomorphisms commuting with an elliptic operator $P : C^\infty(M) \to C^\infty(M)$, where M be is a compact manifold. The main result of this article is the following: "*The group $G(P)$ with the compact-open-topology is a compact Lie group.* "
N2• It is clear that a necessary condition for a topological group to be a Lie group is that it must be a locally Euclidean space, that is a topological manifold. The question to know if this condition is sufficient constitutes the fifteenth problem of Hilbert (1900). The answer is affirmative and this was proved by Gleason, Montgomery, Zippin, Iwazawa and Yamabe (1955): *If a topological group is locally Euclidean, then it is a Lie group.*
The case where G is compact was proved by von Neumann (1933). The fact that every Lie group carries an analytical structure is due to Pontrjagin (1939).
More recently, (1995 [16]) S. Illman gives a contribution to the fifteenth Hilbert problem in the context of proper actions. .
N3• Here we remind some elementary facts:
- If $\phi : G \to H$ is continuous homomorphism between two Lie groups, then ϕ is of class C^∞.
- If two Lie groups are isomorphic as topological groups, then they are isomorphic as Lie

groups.
- If a topological group carries a structure of Lie group, then this structure is unique.
- Every finite-dimensional Lie algebra is isomorphic to a Lie subalgebra of $\mathbb{M}(n, \mathbb{R})$ for some n. This result is known as **Ado's theorem** (1936). It is important to note that the analogous result for Lie groups is false: There are Lie groups that are not isomorphic to any Lie subgroup of $GL(n, \mathbb{R})$; one example of such a group is the universal cover of $SL(2, \mathbb{R})$.

N4• For a recent proof of *Lie's third theorem*, we refer to [9]. The methods given in this book, by Duistermaat and Kolk, were used by M. Crainic and R. L. Fernandes to prove a Lie's third theorem for **Lie algebroids** [8](2003).

N5• If M is a *compact* manifold with a Riemannian metric g, then the Lie group $Iso(M, g)$ is compact. This compactness is not a direct consequence of the compactness of M as one could believe. This follows from the famous Ascoli's theorem (about this theorem one can see [40]), and the fact that it is a closed subset of $Homeo(M)$ with the distance $d(f, h) := sup_{x \in M} d_M(f(x), h(x))$ (where d_M denotes the distance induced from the metric g, by taking the minimal length of curves joining two points on M).
The affine group of a compact Riemannian manifold as well as the group $Iso(M, g)$ where g is a pseudo-Riemannian metric, doesn't have any reason to be compact (see [37][38][39]).

N6• For every Riemannian manifold (M, g), the action of the group $G := Iso(M, g)$ on M is *proper* (this means that the mapping $(a, x) \in G \times M \mapsto (a.x, x) \in M \times M$ is a proper map i.e. the inverse image of a compact subset is compact); in particular all the isotropy subgroups G_x are compact.
The Riemannian manifolds (M, g) such that the group $Iso(M, g)$ acts transitively on M are necessarily homogenous spaces of the form G/K where $K \subset G$ is a compact subgroup of G).
It is important to know that for every connected manifold M, the group $Diff(M)$ acts transitively on M (that is any two points $x, y \in M$ are taken into one another by a diffeomorphism $\psi \in Diff(M)$).

N7• Throughout this course, we didn't develop the qualitative aspect of the $\mathcal{G}$-actions. Note that supporting action with some restrictions could have obstructions (topological or other). We know for instance that the sphere S^2 doesn't support any regular action of $\mathbb{R}$ (because of the Euler-class); and it is impossible to define a locally free action of the commutative group $\mathbb{R}^2$ on the sphere S^3 (this is a theorem of Lima [20][6]).

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Differential Geometry and Topology, Discrete and Computational Geometry
M. Boucetta and J.-M. Morvan (Eds.)
IOS Press, 2005
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Chapter 6

Background on Riemannian Geometry

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6.1 A Survey of the Main Results

Let $<,>$ denote the usual scalar product on $\mathbb{R}^n$ given by

$$< x, y >= \sum_{i=1}^{n} x_i y_i$$

for any $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. According to Whithney classical theorem, it is known that any manifold M can be embedded in some $\mathbb{R}^n$. Suppose that $i : M \hookrightarrow \mathbb{R}^n$ is an embedding of a given manifold M in $\mathbb{R}^n$. If we put, for any $p \in M$ and for any tangent vectors $u, v \in T_pM$,

$$g(u, v) :=< i_* u, i_* v >, \tag{6.1}$$

we get a tensor field g of type $(0, 2)$ which endows each tangent space T_pM with a scalar product. This geometric situation has been studied at the beginning of Riemannian geometry and has been generalized later to abstract manifolds. We will give here a background on Riemannian geometry in a general setting and we will specialize it to the particular case of embedded manifolds in Euclidean spaces. Using the construction above, one can define easily the notion of Riemannian metric on an abstract manifold: a *Riemannian metric* on a smooth manifold M is a 2-tensor field g that is *symmetric* (i.e. $g(X, Y) = g(Y, X)$ and *positive definite* (i.e., $g(X, X) > 0$ if $X \neq 0$). Thus, a Riemannian metric determines an inner product on each tangent space T_pM. A manifold endowed with a given Riemannian metric is called a *Riemannian manifold* .
As we said above, any smooth manifold carries a Riemannian metric. The reader can also prove this fact by using a partition of unity. There are many Riemannian metrics on a given manifold.

If (M_1, g_1) and (M_2, g_2) are Riemannian manifolds, a diffomorphism ϕ from M_1 to M_2 is called an *isometry* if $\phi^* g_2 = g_1$. We say that (M_1, g_1) and (M_2, g_2) are *isometric* if there exists an isometry between them. It is easy to verify that "being isometric" is an equivalence relation on the class of Riemannian manifolds. *Riemannian geometry* is concerned with properties that are preserved by isometries.
An isometry $\phi : (M, g) \longrightarrow (M, g)$ is called an *isometry of* M. It is easy to verify that the set of isometries of M is a group, called the *isometry group of* (M, g); it is denoted $Isom(M)$. (It can be shown that the isometry group is always a finite dimensional Lie group acting smoothly on M; see, for example, [4]).

Examples.

1. By the far, the most important Riemannian manifold and the obvious one is $\mathbb{R}^n$ with its *Euclidean metric* $\bar{g}$, which is just the usual inner product on each tangent space $T_p\mathbb{R}^n$ under the natural identification $T_p\mathbb{R}^n = \mathbb{R}^n$.

2. The sphere $S^n \subset \mathbb{R}^{n+1}$ inherits a Riemannian metric from $\mathbb{R}^{n+1}$ as in (1). This is the standard metric of S^n.

3. **(POINCARÉ HALF-SPACE MODEL)** U^n is the upper half-space of $\mathbb{R}^n$ defined in coordinates $(x_1, \ldots, x_n)$ by $\{x_n > 0\}$, with the metric $h = \frac{1}{x_n^2}\bar{g}$ ($\bar{g}$ is the Euclidean metric of $\mathbb{R}^n$).

Remark. The three examples above constitute the model spaces of Riemannian geometry. One of the first things one notices about those model spaces is that they are highly symmetric. The interested reader can find a detailed description of the properties of these model spaces in many classical books on Riemannian geometry.
Just as in Euclidean geometry, if p is a point in a Riemannian manifold (M, g), we define the *length* or *norm* of any tangent vector $v \in T_pM$ to be

$$|v| := \sqrt{g(v, v)}.$$

We can also define the length of curves. A *curve* in a manifold M always means here a smooth, parameterized curve; that is, a smooth map $\gamma : I \longrightarrow M$, where $I \subset \mathbb{R}$ is some interval. A *curve segment* is a curve whose domain is a closed, bounded interval $[a, b] \subset \mathbb{R}$. Let $\gamma : [a, b] \longrightarrow M$ be a curve segment on M, then *the length of* γ is the real number $l(\gamma)$ defined by

$$l(\gamma) := \int_a^b |\dot{\gamma}(t)| dt. \tag{6.2}$$

This length is independent of the parametrization. If $\gamma' = \gamma \circ \phi$ is a reparametrization of γ where $\phi : [c, d] \longrightarrow [a, b]$ is a smooth map with a smooth inverse, is easy to verify that

$$l(\gamma) = l(\gamma').$$

For technical reasons, we will consider a large class of curves. An *admissible curve* is a continuous map $\gamma : [a, b] \longrightarrow M$ such that there exists a finite subdivision $a = a_0 < a_1 < \ldots < a_k = b$ for which $\gamma \mid [a_{i-1}, a_i]$ is an immersion of the segment $[a_{i-1}, a_i]$ into M, $i = 1, \ldots, k$. We will allow the trivial constant curve $\gamma : \{a\} \longrightarrow M$, $\gamma(a) = p$, to be considered as an admissible curve.
Let $\gamma : [a, b] \longrightarrow M$ be an admissible curve, and $a = a_0 < a_1 < \ldots < a_k = b$ a subdivision as above. The length of γ is defined as the sum of length of the smooth subsegments $\gamma \mid [a_{i-1}, a_i]$.
For any pair of points $p, q \in M$, we denote by $\mathcal{A}_{p,q}$ the space of admissible curves from p to q and define the *Riemannian distance* $d(p, q)$ by

$$d(p, q) = \inf_{\gamma \in \mathcal{A}_{p,q}} l(\gamma). \tag{6.3}$$

It is easy to check that d is well defined and verifies

$$d(p, q) = d(q, p),\ d(p, q) \geq 0 \text{ and } d(p, q) \leq d(p, r) + d(r, q),$$

for all $p, q, r \in M$.

To prove that d is a metric of M, it remains to show that if p and q are distinct points of M then $d(p, q) > 0$. It is a surprising fact that, at this level, we can not prove this simple property. For a proof of this simple fact see [5].
In Euclidean geometry, straight lines play an important role. A straight line in Euclidean space can be defined as a curve which minimizes the length.
In a Riemannian manifold, a curve $\gamma \in \mathcal{A}_{p,q}$ is said to be *minimizing* if $l(\gamma) \leq l(\bar{\gamma})$ for any other curve $\bar{\gamma} \in \mathcal{A}_{p,q}$. So it is natural to look for the minimizing curves in a Riemannian manifold.
One can also define a straight line as a curve in Euclidean space whose acceleration is identically zero. Thus, it is also natural to ask if one can define a notion of acceleration of curves in a Riemannian manifold and, if one can, to look for curves whose acceleration vanishes. The notion of acceleration of curves in a Riemannian manifold can be defined and the curves with vanishing acceleration will be called *geodesics* . To define the acceleration of curves one needs to introduce the notion of connection . It is essentially a natural method of differentiation of vector fields on the manifolds. The notion of connection is the most important tool in Riemannian geometry. Basically, on any Riemannian manifold there exists a unique connection called *Riemannian connection* or *Levi-Civita connection* which is particulary suited to computations in Riemannian geometry.
When all the notions above will be defined, we will show that any minimizing curve is a geodesic (maybe after a change of parametrization) and any geodesic is locally minimizing. The completeness of the Riemannian distance given by (6.3) will be studied and we will give without proof the classical Hopf-Rinow theorem which states the equivalence between the completeness of the Riemannian distance and the completeness of geodesics (this notion will be defined later).
The Riemannian connection allows to define the curvature tensor. The basic properties of

the Riemannian curvature will be given and we will focus on the special case of hypersurfaces in $\mathbb{R}^{n+1}$. The second fundamental form of hypersurfaces will be defined. It allows to define the principal curvatures. The Gauss formula relating the Riemannian connection on the hypersurface to that of $\mathbb{R}^{n+1}$, and the Gauss equation relating their curvatures will be given. Finally, we prove Theorema Egregium of Gauss which shows in particular that the Gaussian curvature of a surface in $\mathbb{R}^3$ can be computed from the intrinsic curvature tensor. It is very difficult to be original in this area according to the huge number of very good books devoted to this field of mathematics. Indeed, our purpose is to give a survey of the results cited above, so many proofs will be omitted.

6.2 Riemannian Connections

Riemannian connections constitute a subclass of the class of connections. So, it is appropriate to begin by deriving some classical properties of general connections on manifolds.

6.2.1 Connections

Definition 83 *Let M be a manifold. A connection on M is a map*

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$$

$(X, Y) \mapsto \nabla_X Y$, *satisfying the following properties:*

1. $\nabla_X Y$ *is linear over* $C^\infty(M)$ *in* X*:*

$$\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} + g\nabla_{X_2} Y, \qquad f, g \in C^\infty(M);$$

2. $\nabla_X Y$ *is linear over* $\mathbb{R}$ *in* Y*:*

$$\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2, \qquad a, b \in \mathbb{R};$$

3. ∇ *satisfies the following product rule:*

$$\nabla_X(fY) = f\nabla_X Y + X(f)Y \qquad f \in C^\infty(M).$$

$\nabla_X Y$ *is called the covariant derivative of* Y *in the direction of* X.

Example. By far the most important connection and the obvious one is the canonical connection on $\mathbb{R}^n$ denoted by $\overline{\nabla}$ and defined, for any vector fields X and Y on $\mathbb{R}^n$, by the formula

$$\overline{\nabla}_X Y(p) = T_p Y(X_p) \tag{6.4}$$

where $T_p Y$ is the differential of Y considered as a map from $\mathbb{R}^n$ to $\mathbb{R}^n$. The connection $\overline{\nabla}$ will be called the *Euclidean connection* of $\mathbb{R}^n$.

Although a connection is defined by its action on global vector fields, it follows from the definition that it is actually a local operator, as the next lemma shows.

Lemma 39 *Suppose that M is endowed with a connection ∇ and let U be an open subset of M. Let $X, Y \in \mathcal{X}(M)$.*

i) If X or Y vanishes identically on U, then so does $\nabla_X Y$.

ii) If X vanishes at a point p in M, then so does $\nabla_X Y$.

Proof. *i)* Suppose Y vanishes on U and let $p \in U$. Let $\phi \in C^\infty(M)$ such that $\phi(p) = 1$ and $\phi = 0$ outside U. We have $\phi Y \equiv 0$ and then $\nabla_X(\phi Y) \equiv 0$. By the product rule, we have

$$0 \equiv \phi \nabla_X Y + X(\phi) Y,$$

which implies that $\nabla_X Y$ vanishes at p. The statement about X follows similarly.

ii) Suppose $X_p = 0$. Let $\{x^1, \ldots, x^n\}$ be a system of coordinates defined on an open neighborhood U of p. On the set U we have $X = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ where $f_i \in C^\infty(U)$ and $f_i(p) = 0$, $i = 1, \ldots, n$. Using *i)*, we find

$$(\nabla_X Y)(p) = \sum_{i=1}^n f_i(p)(\nabla_{\frac{\partial}{\partial x_i}} Y)(p) = 0.$$

□

It follows immediately from this Lemma that a connection ∇ on M induces a connection ∇_U on an arbitrary open subset U of M. In particular, we suppose that U is a coordinate neighborhood where a system of coordinates $\{x^1, \ldots, x^n\}$ is defined. We define n^3 functions Γ_{ij}^k on U, called the *Christoffel symbols* of ∇ with respect $\{x^1, \ldots, x^n\}$, by

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x_k}. \tag{6.5}$$

Remark. It is easy to show that the Christoffel symbols of the Euclidean connection with respect to the canonical system of coordinates in $\mathbb{R}^n$ vanish identically.

Example. We can construct a connection on any manifold M in the following way. Let $M \subset \mathbb{R}^n$ be an embedding of M in some Euclidean space $\mathbb{R}^n$. For any $p \in M$, we have

$$T_p\mathbb{R}^n = T_pM \oplus (T_pM)^\perp,$$

where $(T_pM)^\perp$ is the vector subspace orthogonal to T_pM with respect to the usual scalar product. Let $\pi : T_p\mathbb{R}^n \longrightarrow T_pM$ denote the orthogonal projection. For any vector fields X and Y on M, choose vector fields $\widetilde{X}$ and $\widetilde{Y}$ on $\mathbb{R}^n$ which extend X and Y and put, for any $p \in M$,

$$(\nabla_X Y)_p := \pi\left((\overline{\nabla}_{\widetilde{X}} \widetilde{Y})(p)\right). \tag{6.6}$$

It is easy to check that ∇ is well defined, and is a connection on M. The connection ∇ will be called the *tangential connection* .

Remark. Using a partition of unity, we can also prove that every manifold admits (many) connections.

Let M be a manifold with a connection ∇. We put, for any $X, Y \in \mathcal{X}(M)$,

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]. \tag{6.7}$$

Note that $T(X,Y) = -T(Y,X)$ and it is easy to verify that $T(fX, gY) = fgT(X,Y)$ for all $f, g \in C^\infty(M)$, $X, Y \in \mathcal{X}(M)$. So T is a tensor field of type (1,2) called the *torsion tensor*. The connection ∇ will be called *symmetric* if T vanishes identically. A connection ∇ is symmetric if and only if the Christoffel symbols (Γ_{ij}^k) of ∇ with respect to any system of coordinates $\{x^1, \ldots, x^n\}$ verify

$$\Gamma_{ij}^k = \Gamma_{ji}^k, \quad \text{for all} \quad i, j, k. \tag{6.8}$$

Remark. It is an easy exercise to check that both the Euclidean connection $\overline{\nabla}$ defined by the formula (6.4) and the tangential connection ∇ defined by the formula (6.6) are symmetric.

Now, we will show that a connection is an appropriate tool in order to define the acceleration of curves (a notion which, of course, will depend on the connection).
Let $\gamma : I \longrightarrow M$ be a curve in M. Differentiation with respect to the parameter will often be denoted by a dot, in particular, $\dot{\gamma}(t)$ stands for the velocity of γ at time t.

A *vector field along a curve* $\gamma : I \longrightarrow M$ is a smooth map $V : I \longrightarrow TM$ such that $V(t) \in T_{\gamma(t)}M$ for every $t \in I$. Let $\mathcal{X}(\gamma)$ denote the space of vector fields along γ. A vector field V along γ is said to be *extendible* if there exists a vector field $\widetilde{V}$ on a neighborhood of the image of γ such that $V(t) = \widetilde{V}(\gamma(t))$ for any $t \in I$.

Lemma 40 *Let ∇ be a connection on M. For each curve $\gamma : I \longrightarrow M$, ∇ determines a unique operator*

$$D_{\dot{\gamma}} : \mathcal{X}(\gamma) \longrightarrow \mathcal{X}(\gamma),$$

satisfying the following properties:

1. *Linearity over $\mathbb{R}$:*

$$D_{\dot{\gamma}}(aV + bW) = aD_{\dot{\gamma}}V + bD_{\dot{\gamma}}W \qquad for \quad a, b \in \mathbb{R}.$$

2. *Product rule:*

$$D_{\dot{\gamma}} fV = \dot{f}V + fD_{\dot{\gamma}}V \qquad for \quad f \in C^\infty(I).$$

3. *If V is extendible, then for any extension $\widetilde{V}$ of V,*

$$D_{\dot{\gamma}}V(t) = \nabla_{\dot{\gamma}(t)}\widetilde{V}.$$

For any $V \in \mathcal{X}(\gamma)$, $D_{\dot{\gamma}}V$ is called the covariant derivative of V along γ.

Proof. Let $\{x^1, \ldots, x^n\}$ be a system of coordinates on an open neighborhood U of $\gamma(t_0)$. Then near t_0 we can write

$$V(t) = \sum_{i=1}^{n} V^i(t) \frac{\partial}{\partial x_i}.$$

If (1), (2), (3) hold, then

$$\begin{aligned} D_{\dot\gamma} V(t_0) &= \sum_{i=1}^{n} \left(\dot V^i(t_0) \frac{\partial}{\partial x_i} + V^i(t_0) \nabla_{\dot\gamma(t_0)} \frac{\partial}{\partial x_i} \right) \\ &= \sum_{i,j,k=1}^{n} \left(\dot V^k(t_0) + V^j(t_o) \dot\gamma^i(t_0) \Gamma_{ij}^k(\gamma(t_0)) \right) \frac{\partial}{\partial x_k}. \end{aligned} \tag{6.9}$$

This shows that such an operator is unique if it exists. For existence, we can define $D_{\dot\gamma}$ by the formula (6.9) and check that it satisfies the requisite properties. □

We are now in position to define acceleration and geodesics. Let M be a manifold endowed with a connection ∇, and let γ be a curve in M. The *acceleration* of γ is the vector field $D_{\dot\gamma}\dot\gamma$ along γ. A curve is called a *geodesic* with respect to ∇ if its acceleration is zero: $D_{\dot\gamma}\dot\gamma \equiv 0$.

Let $\{x^1, \ldots, x^n\}$ be a system of coordinates on a neighborhood U. From the formula (6.9), a curve $\gamma : I \longrightarrow U$ is a geodesic if and only if its component functions $\gamma(t) = (x^1(t), \ldots, x^n(t))$ satisfy the *the geodesic equations*

$$\ddot x^k(t) + \sum_{i,j=1}^{n} \dot x^i(t) \dot x^j(t) \Gamma_{ij}^k(x(t)) = 0, \qquad k = 1, \ldots, n. \tag{6.10}$$

This is a second-order system of ordinary differential equations for the functions $x^i(t)$. If we set $v^i = \dot x^i$, (6.10) is equivalent to the following first-order system in twice the number of variables:

$$\begin{aligned} \dot x^k(t) &= v^k(t), \\ \dot v^k(t) &= -\sum_{i,j=1}^{n} v^i(t) v^j(t) \Gamma_{ij}^k(x(t)). \end{aligned} \tag{6.11}$$

By applying the general existence theorem for solutions of a differential equation to (6.11) we get the following theorem.

Theorem 67 (Existence and uniqueness of geodesics) *Let M be a manifold endowed with a connection ∇. For any $p \in M$, any $V \in T_pM$, and any $t_0 \in \mathbb{R}$, there exists an open interval $I \subset \mathbb{R}$ containing t_0 and a geodesic $\gamma : I \longrightarrow M$ satisfying $\gamma(t_0) = p$, $\dot\gamma(t_0) = V$. Any two such geodesics agree on their common domain.*

It follows from the uniqueness statement in the preceding theorem that for any $p \in M$ and $V \in T_pM$, there is a unique *maximal geodesic* $\gamma : I \longrightarrow M$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = V$, defined on some open interval I. This maximal geodesic is often called simply *the geodesic with initial point p and initial velocity V*, and denoted γ_V.

A connection ∇ is called *geodesically complete* if every maximal geodesic of ∇ is defined for all $t \in \mathbb{R}$.

Remark. 1. Since the Christoffel symbols of the Euclidean connection on $\mathbb{R}^n$ vanish, it follows from the equation (6.10) that the geodesics of $\widetilde{\nabla}$ are straight lines and then $\widetilde{\nabla}$ is geodesically complete.

2. We will show later that the tangential connection of the sphere S^n is also geodesically complete.

We conclude this section by giving another interpretation of the symmetry of a connection which will be crucial later.

Let M be a manifold with a connection ∇. An *admissible family* of curves is a continuous map $\Gamma :]-\epsilon, \epsilon[\times[a, b] \longrightarrow M$ that is smooth on each rectangle of the form $]-\epsilon, \epsilon[\times[a_{i-1}, a_i]$ for some finite subdivision $a = a_0 < \ldots < a_k = b$, and such that $\Gamma_s(t) := \Gamma(s, t)$ is an admissible curve for each $s \in]-\epsilon, \epsilon[$. If Γ is an admissible family, a *vector field along* Γ is a continuous map $V :]-\epsilon, \epsilon[\times[a, b] \longrightarrow TM$ such that $V(s, t) \in T_{\Gamma(s,t)}M$ for each (s, t), and such that $V\,|_{]-\epsilon,\epsilon[\times[\bar{a}_{i-1},\bar{a}_i]}$ is smooth for some subdivision $a = \bar{a}_0 < \ldots < \bar{a}_m = b$.

Any admissible family Γ defines two collections of curves: the *main curves* $\Gamma_s(t) = \Gamma(s, t)$ defined on $[a, b]$ by setting $s = constant$, and the *transverse curves* $\Gamma^{(t)}(s) = \Gamma(s, t)$ defined on $]-\epsilon, \epsilon[$ by setting $t = constant$. If V is a vector field along Γ, we can compute the covariant derivative of V either along the main curves or along the transverses curves; the resulting vector fields along Γ are denoted by D_tV and D_sV.

Let Γ be an admissible family of curves. We put

$$\partial_t\Gamma(s, t) := \frac{d}{dt}\Gamma_s(t); \qquad \partial_t\Gamma(s, t) := \frac{d}{dt}\Gamma^{(t)}(s).$$

Lemma 41 (symmetry Lemma) *Let $\Gamma :]-\epsilon, \epsilon[\times[a, b] \longrightarrow M$ be an admissible family of curves in a manifold M with a symmetric connection ∇. On any rectangle $]-\epsilon, \epsilon[\times[a_{i-1}, a_i]$ where Γ is smooth,*

$$D_s\partial_t\Gamma = D_t\partial_s\Gamma.$$

Proof. By choosing a system of coordinates $\{x^1, \ldots, x^n\}$ around any point $\Gamma(s_0, t_0)$ and writing $\Gamma(s, t) = (x^1(s, t), \ldots, x^n(s, t))$, we have

$$\partial_t\Gamma = \sum_{k=1}^{n} \frac{\partial x^k}{\partial t}\frac{\partial}{\partial x^k}; \qquad \partial_s\Gamma = \sum_{k=1}^{n} \frac{\partial x^k}{\partial s}\frac{\partial}{\partial x^k}.$$

Then, using formula (6.9) for covariant derivatives along curves,

$$\begin{aligned} D_s\partial_t\Gamma &= \sum_{i,j,k=1}^{n}\left(\frac{\partial^2 x^k}{\partial s\partial t}+\frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial s}\Gamma_{ij}^k\right)\frac{\partial}{\partial x^k};\\ D_t\partial_s\Gamma &= \sum_{i,j,k=1}^{n}\left(\frac{\partial^2 x^k}{\partial t\partial s}+\frac{\partial x^i}{\partial s}\frac{\partial x^j}{\partial t}\Gamma_{ji}^k\right)\frac{\partial}{\partial x^k}. \end{aligned}$$

Reversing the roles of i and j in the second line above, and using the symmetry condition $\Gamma_{ij}^k = \Gamma_{ji}^k$, we see immediately that these two expressions are equal.□

6.2.2 The Levi-Civita Connection

The following result insures that on each Riemannian manifold there is a natural connection that is particulary suited to computations in Riemannian geometry.

Theorem 68 (T. Levi-Civita) *If (M,g) is a Riemannian manifold then there exists a unique connection ∇ (henceforth called the Levi-Civita connection or the Riemannian connection) for which*

$$\nabla_X Y = \nabla_Y X + [X,Y], \tag{6.12}$$

$$X.g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \tag{6.13}$$

for all vector fields $X,Y,Z \in \mathcal{X}(M)$.

Proof. By using (6.12) and (6.13), we have for any $X,Y,Z \in \mathcal{X}(M)$,

$$\begin{aligned} g(\nabla_X Y, Z) &= X.g(Y,Z) - g(Y,\nabla_X Z)\\ &= X.g(Y,Z) - g(Y,\nabla_Z X) - g(Y,[X,Z])\\ &= X.g(Y,Z) - Z.g(Y,X) + g(\nabla_Z Y, X) - g(Y,[X,Z])\\ &= X.g(Y,Z) - Z.g(Y,X) + g(\nabla_Y Z, X) + g([Z,Y],X)\\ &\quad -g(Y,[X,Z])\\ &= X.g(Y,Z) - Z.g(Y,X) + Y.g(Z,X) - g(Z,\nabla_Y X) + g([Z,Y],X)\\ &\quad -g(Y,[X,Z])\\ &= X.g(Y,Z) - Z.g(Y,X) + Y.g(Z,X) - g(Z,\nabla_X Y)\\ &\quad -g(Z,[Y,X]) + g([Z,Y],X) - g(Y,[X,Z]). \end{aligned}$$

Therefore, we have

$$\begin{aligned} g(\nabla_X Y, Z) &= \frac{1}{2}\{X.g(Y,Z) + Y.g(Z,X) - Z.g(Y,X)\\ &\quad +g([X,Y],Z) + g([Z,X],Y) + g([Z,Y],X)\}. \end{aligned} \tag{6.14}$$

Thus $\nabla_X Y$ is uniquely determined. To establish the existence of ∇, one takes the relation (6.14) to define ∇, and check directly that indeed defines a connection satisfying the requisite properties.□

Remarks. 1. It is an easy exercise to show that the Levi-Civita connection associated with the Euclidean metric on $\mathbb{R}^n$ is the Euclidean connection $\overline{\nabla}$ and, if $M \subset \mathbb{R}^n$ is a submanifold, the Levi-Civita connection associated with the induced Riemannian metric on M is the tangential connection $\nabla^{\perp}$.

2. Note that the relation (6.12) says that the torsion of the Levi-Civita connection vanishes identically on M.

On any Riemannian manifold, covariant derivatives along curves, geodesics and parallel translations will be defined with respect to the Levi-Civita connection.

6.3 Geodesics, Distance and Completeness

In this section, we will give the main results involving the relationships among geodesics, lengths and distance on a Riemannian manifold. All the results will be given without proof. The reader can see, for instance, [5], [2]...

Let (M, g) be a Riemannian manifold with its Levi-Civita connection ∇. For any initial point $p \in M$ and any $V \in T_pM$, there exists a maximal geodesic $\gamma_V : I \longrightarrow M$ defined on an open interval containing 0 and such that $\gamma_V(0) = p$ and $\dot{\gamma_V}(0) = V$.

Consider now

$$\mathcal{E} = \{V \in TM : \gamma_V \text{ is defined on an interval containing } [0,1]\},$$

and define the *exponential map* $exp : \mathcal{E} \longrightarrow M$ by

$$exp(V) := \gamma_V(1).$$

For each $p \in M$, the *restricted exponential map* $\exp_p$ is the restriction of exp to $\mathcal{E}_p := \mathcal{E} \cap T_pM$.

By using the standard theory of ordinary differential equations, one can show that $\mathcal{E}$ is open in TM and that $exp : \mathcal{E} \longrightarrow M$ is smooth. Moreover, each $\mathcal{M}_p$ is thus open in T_pM and $exp_p : \mathcal{E}_p \longrightarrow M$ is also smooth.

Let us compute the differential of $exp_p : \mathcal{E}_p \longrightarrow M$ at 0 i.e. $(exp_p)_* : T_0(\mathcal{E}_p) = T_pM \longrightarrow T_pM$. We have, for any $V \in T_pM$,

$$(exp_p)_*V = \frac{d}{dt}\mid_{t=0} exp_p(tV) = \frac{d}{dt}\mid_{t=0} \gamma_V(t) = V.$$

Hence $(exp_p)_* : T_0(\mathcal{E}_p) = T_pM \longrightarrow T_pM$ is the identity.

Let us enumerate some immediate consequences of those facts:

1. from the inverse function theorem, there is a neighborhood $\mathcal{V}$ of the origin in T_pM and a neighborhood $\mathcal{U}$ of p in M such that $exp_p : \mathcal{V} \longrightarrow \mathcal{U}$ is a diffeomorphism.
   Any open neighborhood $\mathcal{U}$ of $p \in M$ that is the diffeomorphic image under exp_p of a star-shaped open neighborhood of $0 \in T_pM$ is called a *normal neighborhood* of p.
   If $\epsilon > 0$ is such that exp_p is a diffeomorphism on the ball $B_\epsilon(0) \subset T_pM$ (where the radius of the ball is measured with respect to the norm defined by g), then the image set $exp_p(B_\epsilon(0))$ is called a *geodesic ball* in M. Also, if the closed ball $\overline{B}_\epsilon(0)$ is contained in an open set $\mathcal{V} \subset T_pM$ on which exp_p is a diffeomorphism, then $exp_p(\overline{B}_\epsilon(0))$ is called a *closed geodesic ball* , and $exp_p(\partial\overline{B}_\epsilon(0))$ is called a *geodesic sphere.*

2. if we choose an orthonormal basis $\{e_1, \ldots, e_n\}$ for T_pM and a normal neighborhood $\mathcal{U}$ of p, we get an isomorphism e from $\mathbb{R}^n$ to T_pM by $e(x_1, \ldots, x_n) = x_1e_1 + \ldots + x_ne_n$ and a coordinate chart
   $$\phi := e^{-1} \circ exp_p^{-1} : \mathcal{U} \longrightarrow \mathbb{R}^n.$$
   Any such coordinates are called *normal coordinates* centered at p;

3. the metric topology obtained from the Riemannian distance defined by (6.3) is the same as the manifold topology.

In any normal coordinate chart $\{x^1, \ldots, x^n\}$ centered at p, define the *radial distance* function r by

$$r(x) := \left(\sum_{i=1}^{n}(x^i)^2\right)^{\frac{1}{2}}, \tag{6.15}$$

and the unit *unit radial vector field* $\frac{\partial}{\partial r}$ by

$$\frac{\partial}{\partial r} := \sum_{i=1}^{n} \frac{x^i}{r}\frac{\partial}{\partial x^i}. \tag{6.16}$$

In Euclidean space, $r(x)$ is the distance to the origin, and $\frac{\partial}{\partial r}$ is the unit vector field tangent to straight lines through the origin.

In general, at any point $q \in \mathcal{U} - p$,

$$\frac{\partial}{\partial r} = \frac{d}{dt}\,|_{t=0}\, exp_p(t\frac{V}{|V|} + V) = \dot{\gamma}_{\frac{V}{|V|}}(|V|),$$

where $q = exp_p(V)$. Hence $\frac{\partial}{\partial r}$ is the velocity vector of a unit speed geodesic from p to q, and therefore has unit length with respect to g.

The unit radial vector field, actually, verifies a simple geometric property which is crucial in showing that the geodesics are locally minimizing. Indeed, if $\mathcal{U} = exp_p(B_\epsilon(0))$ is a geodesic ball centered at $p \in M$, the unit radial vector field $\dfrac{\partial}{\partial r}$ is g-orthogonal to

the geodesic spheres $exp_p(\partial\overline{B}_{\epsilon'}(0))$ $(\epsilon' \leq \epsilon))$ in $\mathcal{U}$. This simple geometric fact is known as *Gauss Lemma* . One can easily deduce from Gauss Lemma that the Riemannian gradient of the radial distance is the radial vector field, namely

$$dr(Y) = g(Y, \frac{\partial}{\partial r})$$

for any vector fields Y on $\mathcal{U} - p$.

Now, if q is point in $\mathcal{U} - p$ and $q = exp_p(V)$ for some $V \in T_pM$. The geodesic $\gamma_V : [0,1] \to M$, $t \mapsto exp_p(tV)$ is called the radial geodesic from p to q. The following proposition is a consequence of Gauss Lemma.

Proposition 25 *Suppose $p \in M$ and q is contained in a geodesic ball around p. Then (up to parametrization) the radial geodesic from p to q is the unique minimizing curve from p to q in M.*

The calculus of variations is a tool which will permit to show that all length-minimizing curves are geodesics and that all geodesics are length minimizing (at least locally). Let us explain this method.

If $\gamma : [a,b] \longrightarrow M$ is an admissible curve, a *variation of* γ is an admissible family Γ such that $\Gamma_0(t) = \gamma(t)$ for all $t \in [a,b]$. It is called a *proper variation* or a *fixed-endpoint variation* if in addition $\Gamma_s(a) = \gamma(a)$ and $\Gamma_s(b) = \gamma(b)$ for all s. If Γ is a variation of γ, *the variation field* of Γ is the vector field $V(t) = \partial_s\Gamma(t,0)$ along γ. A vector field V along γ is *proper* if $V(a) = V(b) = 0$. It is clear that the variation field of a proper variation is itself proper.

One can easily check that If γ is an admissible curve and V is a vector field along γ, then V is the variation field of some variation of γ. If V is proper, the variation can be taken to be proper as well.

Proposition 26 (First variation formula) *Let $\gamma : [a,b] \longrightarrow M$ be any unit speed admissible curve, Γ a proper variation of γ, and V its variation field. Then*

$$\frac{d}{ds}\,|_{s=0}\, l(\Gamma_s) = -\int_a^b g(V, D_t\dot{\gamma})dt - \sum_{i=1}^{k-1} g(V(a_i), \Delta_i\dot{\gamma}), \tag{6.17}$$

where $\Delta_i\dot{\gamma} = \dot{\gamma}(a_i^+) - \dot{\gamma}(a_i^-)$ is the "jump" in the tangent vector field $\dot{\gamma}$ at a_i.

The following theorem is a direct consequence of the First variation formula.

Theorem 69 *Every minimizing curve is a geodesic when its parametrization is unit speed.*

To get the converse of this theorem, we need the following refinement of the concept of normal neighborhood. An open set $\mathcal{V} \subset M$ is called *uniformly normal* if there exists some $\delta > 0$ such that $\mathcal{V}$ is contained in a geodesic ball of radius δ around each of its points.

Lemma 42 (Uniformly normal Neighborhood Lemma) *Let (M, g) be a Riemannian manifold. Given $p \in M$ and any neighborhood $\mathcal{U}$ of p, there exists a uniformly normal neighborhood $\mathcal{V}$ of p contained in $\mathcal{U}$.*

We say that a curve $\gamma : I \longrightarrow M$ is *locally minimizing* if any $t_0 \in I$ has a neighborhood $J \subset I$ such that $\gamma\,|_J$ is minimizing between each pair of its points. Note that a minimizing curve is automatically locally minimizing, because it is minimizing between any two of its points.

Theorem 70 *Every Riemannian geodesic is locally minimizing.*

We conclude this section by stating Hopf-Rinow theorem and two important corollaries. In all of these, M is assumed to be a connected Riemannian manifold.

Theorem 71 (Hopf-Rinow) *A connected Riemannian manifold is geodesically complete if and only if it is complete as a metric space.*

Corollary 17 *M is complete if and only if any two points in M can be joined by a minimizing geodesic segment.*

Corollary 18 *If M is compact, then every geodesic can be defined for all time.*

6.4 Curvature

In this section, we will introduce the Riemann curvature tensor of a Riemannian manifold. We will show that this tensor vanishes identically iff the Riemannian manifold is locally isometric to an Euclidean space. Then, we will quickly move on to submanifolds theory in order to give to the curvature tensor a concrete quantitative interpretation.
Let (M, g) be a Riemannian manifold with its Levi-Civita connection ∇. The *curvature endomorphism* is the map $R : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ defined by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Proposition 27 *The curvature endomorphism is a $(3, 1)$-tensor field.*

Proof. Left to the reader.□
A Riemannian manifold is said to be *flat* if it is locally isometric to an Euclidean space, that is, if every point has a neighborhood that is isometric to an open set in $\mathbb{R}^n$ with its Euclidean metric.

Theorem 72 *A Riemannian manifold is flat if and only if its curvature tensor vanishes identically.*

Proof. One can check immediately that the tensor curvature of the Euclidean metric vanishes identically. If (M, g) is flat, in a neighborhood of any point, there is an isometry ϕ to an open set in $(\mathbb{R}^n, \bar{g})$. It is then easy to check that the tensor curvature of g is the pullback of that of $\bar{g}$, and thus is zero.
Now suppose (M, g) has vanishing curvature tensor. One can check (see [5] pp. 120 or [6] Vol. II pp.241) that the vanishing of the curvature tensor is equivalent to the following important geometric property: For any point $p \in M$, there exists an open neighborhood $\mathcal{U}$ of p and a family $(X_1, \ldots, X_n)$ $(n = dimM)$ of vector fields on $\mathcal{U}$ such that:

1. $(X_1, \ldots, X_n)$ is a family of orthonormal vector fields i.e. $g(X_i, X_j) = \delta_{ij}$ for any i, j;
2. for any $i = 1, \ldots, n$, $\nabla X_i = 0$ which means that X_i is parallel along any curves lying in $\mathcal{U}$.

Now, let $p \in M$ and $(X_1, \ldots, X_n)$ as above. Since the Riemannian connection is symmetric, we have

$$[X_i, X_j] = \nabla_{X_i} X_j - \nabla_{X_j} X_i = 0.$$

Thus the vector fields $(X_1, \ldots, X_n)$ form a commuting frame on $\mathcal{U}$. By a classical result from elementary differential geometry (see [1] p.161), there are coordinates $\{x^1, \ldots, x^n\}$ on a (possibly smaller) neighborhood of p such that $X_i = \dfrac{\partial}{\partial x^i}$, so the map $x = (x^1, \ldots, x^n)$ is an isometry from a neighborhood of p to an open subset of Euclidean space.□

The curvature tensor on a Riemannian manifold has a number of symmetrics.

Proposition 28 (Symmetries of the curvature tensor) *The curvature tensor for a Riemannian manifold has the following symmetries for any vector fields X, Y, Z, W:*

1. $R(X,Y)Z + R(Y,X)Z = 0$.
2. $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$. *(algebraic Bianchi identity).*
3. $g(R(X,Y)Z, W) + g(R(X,Y)W, Z) = 0$.
4. $g(R(X,Y)Z, W) = g(R(Z,W)X, Y)$.

Proof. (1) is a trivial consequence of the definition of the curvature endomorphism, (2) follows from the symmetry of the connection and (3) is a consequence of the compatibility of the Riemannian connection with the metric expressed in (13). Finally (4) follows from the others. Writing the algebraic Bianchi identity four times:

$$\begin{aligned}
g(R(X,Y)Z,W) + g(R(Y,Z)X,W) + g(R(Z,X)Y,W) &= 0\\
g(R(Y,Z)W,X) + g(R(Z,W)Y,X) + g(R(W,Y)Z,X) &= 0\\
g(R(Z,W)X,Y) + g(R(W,X)Z,Y) + g(R(X,Z)W,Y) &= 0\\
g(R(W,X)Y,Z) + g(R(X,Y)W,Z) + g(R(Y,W)X,Z) &= 0.
\end{aligned}$$

Now add up all four equations and apply (1) and (3) to get $2g(R(X,Z)Y,W) - 2g(R(Y,W)X,Z) = 0$, which is equivalent to (4).□

6.5 Riemannian Submanifold and the Second Fundamental Form

Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold and suppose M is an embedded submanifold of $\widetilde{M}$ of dimension n. The Riemannian metric $\widetilde{g}$ induces a Riemannain metric on M which will be denoted by g. At each $p \in M$, the ambiant tangent space $T_p\widetilde{M}$ splits as an orthogonal direct sum $T_p\widetilde{M} = T_pM \oplus N_pM$, where $N_pM = (T_pM)^\perp$ is the normal space at p with respect to the inner product $\widetilde{g}$ on $T_p\widetilde{M}$. $NM := \coprod_{p\in M} N_pM$ is a smooth vector bundle over M called the *normal bundle* of M. We shall denote by $\mathcal{N}M$ the space of its smooth sections. Projecting orthogonally at each point $p \in M$ onto the subspaces T_pM and N_pM gives maps called the *tangential* and *normal projections*

$$\pi^\top : T\widetilde{M}\,|_M \longrightarrow TM, \qquad \pi^\perp : T\widetilde{M}\,|_M \longrightarrow NM.$$

Now, let us compare the Riemannian connection ∇ of M with the Riemannian connection $\widetilde{\nabla}$ of $\widetilde{M}$. If X, Y are vector fields in $\mathcal{X}(M)$, we can extend them to vector fields on $\widetilde{M}$, apply the Riemannian connection $\widetilde{\nabla}$, and then decompose at points of M to get

$$\widetilde{\nabla}_X Y = (\widetilde{\nabla}_X Y)^\top + (\widetilde{\nabla}_X Y)^\perp. \tag{6.18}$$

We define the *second fundamental form* of M to be the map $h : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{N}(M)$ given by

$$h(X, Y) := (\widetilde{\nabla}_X Y)^\perp,$$

where X and Y are extended arbitrarily to $\widetilde{M}$.

One can immediately check that the second fundamental form is independent of the extensions of X and Y, bilinear over $C^\infty(M)$ and symmetric in X and Y.

Theorem 73 (The Gauss formula) *If $X, Y \in \mathcal{X}(M)$ are extended arbitrarily to vector fields on $\widetilde{M}$, the following formula holds along M:*

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y).$$

Proof. Define a map $\nabla^\top : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$ by

$$\nabla^\top_X Y := (\widetilde{\nabla}_X Y)^\top.$$

It is easy to show that this defines a connection on M. On other hand, we have

$$\nabla^\top_X Y - \nabla^\top_Y X = (\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X)^\top = [X, Y]^\top = [X, Y]$$

where we have used (1) that $\widetilde{\nabla}$ is symmetric and (2) the bracket of two vector fields tangent to M is a vector field tangent to M. Moreover, for any $X, Y, Z \in \mathcal{X}(M)$ extended arbitrarily extended to $\widetilde{M}$, we have in restriction to M,

$$\begin{aligned} X.g(Y, Z) = X.\widetilde{g}(Y, Z) &= \widetilde{g}(\widetilde{\nabla}_X Y, Z) + \widetilde{g}(Y, \widetilde{\nabla}_X Z) \\ &= g((\widetilde{\nabla}_X Y)^\top, Z) + g(Y, (\widetilde{\nabla}_X Z)^\top) \\ &= g(\nabla^\top_X Y, Z) + g(Y, \nabla^\top_X Z). \end{aligned}$$

Therefore $\nabla^\top$ is a symmetric connection compatible with g, so $\nabla^\top = \nabla$ by the uniqueness of the Riemannian connection and the theorem follows from the equation (6.18).□

Lemma 43 (Weingarten equation) *Suppose $X, Y \in \mathcal{X}(M)$ and $N \in \mathcal{N}M$. When X, Y, N are extended arbitrarily to $\widetilde{M}$, the following equation holds at points of M:*

$$\widetilde{g}(\widetilde{\nabla}_X N, Y) = -\widetilde{g}(N, h(X, Y)).$$

Proof. Since $\widetilde{g}(N, Y)$ vanishes identically along M and X is tangent to M, we have

$$\begin{aligned} 0 &= X.\widetilde{g}(N, Y) = \widetilde{g}(\widetilde{\nabla}_X N, Y) + \widetilde{g}(N, \widetilde{\nabla}_X Y) \\ &= \widetilde{g}(\widetilde{\nabla}_X N, Y) + \widetilde{g}(N, \nabla_X Y + h(X, Y)) \\ &= \widetilde{g}(\widetilde{\nabla}_X N, Y) + \widetilde{g}(N, h(X, Y)). \end{aligned}$$

□

In addition to describe the difference between the intrinsic and extrinsic connections, the second fundamental form plays a significant role in describing the difference between the curvature tensors of $\widetilde{M}$ and M. The explicit formula, also due to Gauss, is given in the following theorem.

Theorem 74 (Gauss equation) *For any $X, Y, Z, W \in T_pM$, the following equation holds:*

$$\widetilde{g}(\widetilde{R}(X, Y)Z, W) = g(R(X, Y)Z, W) - \widetilde{g}(h(X, W), h(Y, Z)) + \widetilde{g}(h(X, Z), h(Y, W)).$$

Proof. Let X, Y, Z, W be extended arbitrarily to vector fields on M, and then to vector fields on $\widetilde{M}$ that are tangent to M at points of M. Along M, the Gauss formula gives

$$\begin{aligned} \widetilde{g}(\widetilde{R}(X, Y)Z, W) &= \widetilde{g}(\widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z, W) \\ &= \widetilde{g}(\widetilde{\nabla}_X (\nabla_Y Z + h(Y, Z)) - \widetilde{\nabla}_Y (\nabla_X Z + h(X, Z)), W) \\ &\quad -\widetilde{g}(\nabla_{[X,Y]} Z + h([X, Y], Z), W). \end{aligned}$$

Since the second fundamental form takes its values in the normal bundle and W is tangent to M, we have

$$\widetilde{g}(\widetilde{\nabla}_X (\nabla_Y Z), W) = g(\nabla_X \nabla_Y Z, W).$$

On the other hand, the Weingarten equation gives

$$\widetilde{g}(\widetilde{\nabla}_X (h(Y, Z)), W) = -\widetilde{g}(h(Y, Z), h(X, W)).$$

With this remark, we can transform the equation above to get the Gauss equation.□

We say that a Riemannian submanifold $M \subset \widetilde{M}$ is *totally geodesic* if for every $V \in TM$, the $\widetilde{g}$-geodesic γ_V lies entirely in M. It is easy to prove that the following assertions are equivalent:

1. M is totally geodesic.
2. Every g-geodesic in M is also a $\widetilde{g}$-geodesic in $\widetilde{M}$.
3. The second fundamental form of M vanishes identically.

Now, we consider the particular case of hypersurfaces in Euclidean space. This means that M is a n-manifold embedded in $\mathbb{R}^{n+1}$ endowed with his Euclidean metric $\bar{g}$. Covariant derivative and curvatures associated with $\bar{g}$ will be indicated by a bar.
Suppose M orientable. In this case the normal bundle NM is trivial and then there exists a global unit normal vector N. Given a unit normal vector field N, the *scalar second fundamental form* h_s is the symmetric 2-tensor on M defined by

$$h_s(X,Y) = \bar{g}(h(X,Y),N).$$

In a natural way, we can associate to h a field of endomorphism A of TM called the *shape operator* of M. It is characterized by

$$g(X,AY) = h_s(X,Y) \qquad \text{for all } X,Y \in \mathcal{X}(M).$$

Because h is symmetric, A is a self-adjoint endomorphism of TM, that is,

$$g(AX,Y) = g(X,AY) \qquad \text{for all } X,Y \in \mathcal{X}(M).$$

In terms of the tensor field h_s and A, the formulas above can be rewritten somewhat more simply. First, we have the *Gauss formula for Euclidean hypersurfaces* :

$$\bar{\nabla}_X Y = \nabla_X Y + h_s(X,Y)N. \tag{6.19}$$

Moreover, the Weingarten equation can be rewritten

$$\bar{g}(\bar{\nabla}_X N, Y) = -h_s(X,Y) = -g(AX,Y). \tag{6.20}$$

Since $2\bar{g}(\bar{\nabla}_X N, N) = X.|N|^2 = 0$, it follows that $\bar{\nabla}_X N$ is tangent to M, so the equation (6.20) is equivalent to the *Weingarten equation for Euclidean hypersurfaces* :

$$\bar{\nabla}_X N = -AX. \tag{6.21}$$

Finally, since $\bar{R} = 0$ on $\mathbb{R}^{n+1}$, the *Gauss equation for Euclidean hypersurfaces* is

$$g(R(X,Y)Z,W) = h_s(X,W)h_s(Y,Z) - h_s(X,Z)h_s(Y,W). \tag{6.22}$$

At any point $p \in M$, we have seen that the shape operator A is a selfadjoint linear transformation on the tangent space T_pM. From elementary linear algebra, any such operator has real eigenvalues $\kappa_1, \ldots, \kappa_n$. The eigenvalues of S are called the *principal curvatures* of M at p, and the corresponding eigenspaces are called the *principal directions*. The principal curvatures change sign if we change the normal vector. There are two combinations of the

principal curvatures that plays particularly significant roles for Euclidean hypersurfaces. The *Gaussian curvature* is defined as $K = detA$, and the *mean curvature* as $H = \frac{1}{n}trA$. In terms of the principal curvatures, they are

$$K = \kappa_1\kappa_2\ldots\kappa_n; \qquad H = \frac{1}{n}(\kappa_1 + \ldots + \kappa_n).$$

A hypersurface M with mean curvature identically equal to zero is called *minimal*. Clearly the mean curvature of any hypersurface changes sign if we change the sign of the normal vector field. If n is odd, the Gaussian curvature also changes sign, but if n is even (in particular for surfaces in $\mathbb{R}^3$), the Gaussian curvature is independent of the choice of N. In any case, both the Gaussian and mean curvatures are defined in terms of particular embedding of M into $\mathbb{R}^{n+1}$, and there are few reasons to suspect they have much to do with the intrinsic Riemannian geometry of M with its induced metric g. The amazing discovery made by Gauss was that the Gaussian curvature of a surface in $\mathbb{R}^3$ is an intrinsic invariant of the Riemannian manifold (M, g).

Theorem 75 (Gauss Theorema Egregium) *Let $M \subset \mathbb{R}^3$ be a 2-dimensional submanifold and g the induced metric on M. For any $p \in M$ and any basis (X, Y) for T_pM, the Gaussian curvature of M at p is given by*

$$K = \frac{g(R(X,Y)Y,X)}{|X|^2|Y|^2 - g(X,Y)^2}.$$

Therefore the Gaussian curvature of (M, g) is invariant by isometry.

Proof. If (X, Y) is an orthonormal basis for T_pM, we have

$$\begin{aligned} |X|^2|Y|^2 - g(X,Y)^2 &= 1, \\ g(R(X,Y)Y,X) &= h_s(X,X)h_s(Y,Y) - h_s(X,Y)h_s(Y,X) = detA = K, \end{aligned}$$

where we have used the Gauss equation. Now let (X, Y) be any basis of T_pM. The orthonormal basis (E_1, E_2) associated to (X, Y) given by the Gram-Schmidt algorithm is given by

$$\begin{aligned} E_1 &= \frac{X}{|X|}; \\ E_2 &= \frac{Y - \frac{g(Y,X)}{|X|^2}X}{\left|Y - \frac{g(Y,X)}{|X|^2}X\right|}. \end{aligned}$$

Then by the preceding computation, the Gaussian curvature at p is

$$\begin{aligned} K &= g(R(E_1,E_2)E_1,E_2) \\ &= \frac{g(X, Y-\frac{g(Y,X)}{|X|^2}X, Y-\frac{g(Y,X)}{|X|^2}X, X)}{|X|^2\left|Y-\frac{g(Y,X)}{|X|^2}X\right|^2} \\ &= \frac{g(R(X,Y)X,Y)}{|X|^2\left(|Y|^2-2\frac{g(Y,X)^2}{|X|^2}+\frac{g(Y,X)^2}{|X|^2}\right)} \\ &= \frac{g(R(X,Y)Y,X)}{|X|^2|Y|^2-g(X,Y)^2}. \end{aligned}$$

This proves the theorem.□

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Differential Geometry and Topology, Discrete and Computational Geometry
M. Boucetta and J.-M. Morvan (Eds.)
IOS Press, 2005
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Chapter 7

Holonomy Groups of Pseudo-Riemannian Manifolds

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7.1 Introduction

Let $(M^{p,q}, g)$ be a connected pseudo-Riemannian manifold of signature (p, q). There is a unique torsion-free metric connection D, called the Levi-Civita connection, giving rise to a parallel transport along each curve. Given a point x in M, the holonomy group H_x is the subgroup of $O(T_xM, g_x)$ generated by parallel transport along all the closed curves starting at x. If we only consider the closed curves starting at x which are null-homotopic, we define the restricted holonomy group H_x^0. The (restricted) holonomy group at points of M are all isometric, and we may talk about the (restricted) holonomy of M, up to conjugacy. If $(M^{p,q}, g)$ is simply connected, then the holonomy group is equal to the restricted holonomy group. To simplify our course, we only consider this case. The holonomy group H is one of the fundamental algebraic objects associated to a pseudo-Riemannian manifold $(M^{p,q}, g)$. It is a Lie subgroup of $O(p, q)$ measuring the parallel tensors on the manifolds. For example,
H is reduced to the identity $\Longleftrightarrow$ $(M^{p,q}, g)$ is flat
$H \subset U(p, q) \Longleftrightarrow (M^{p,q}, g)$ is a Kähler manifold
$H \subset SU(p, q) \Longleftrightarrow (M^{p,q}, g)$ is a special Kähler manifold
$H \subset Sp(p, q) \cdot Sp(1) \Longleftrightarrow (M^{p,q}, g)$ is a quaternionic Kähler manifold.
H is decomposable into the direct product of normal subgroups $\Longleftrightarrow$ $(M^{p,q}, g)$ is at least locally isometric to a product of pseudo-Riemannian manifolds.
Hence if we have some (algebraic) informations on the holonomy group, we can transform them into geometric informations on the manifold. From where the importance to study this group. Then we have the following natural problem known as **holonomy problem** and which is divided in two parts: an algebraic problem and a geometric problem :

(AP) What are all possible holonomy groups $H \subset O(p,q)$ (up to conjugacy) of some pseudo-Riemannian manifold $(M^{(p,q)}, g)$?
(GP) For every group classified H in **(AP)**, does there exist at least a pseudo-Riemannian manifold (complete, compact) for which the holonomy group is H ?
The key tool is the famous theorem of decomposition of G. De Rham theorem (for the Riemannian case see [25]) generalized by H. Wu ([28]) which asserts that every pseudo-Riemannian manifold is at least locally isometric to a product of indecomposable pseudo-Riemannian manifolds (the holonomy group does not leave any non-degenerate proper subspace of $R^{p,q}$). A first answer to the problem **(AP)** is given by Berger's works in ([7]) for the irreducible (the holonomy group does not leave any proper subspace of $R^{p,q}$) and non-locally symmetric case (see paragraph 8, for the definition of locally symmetric space). Later there were made some corrections and additions to Berger's list by Aleksevski ([1]), Brown and Gray ([5]) , McLean and Bryant ([12]). In this case, the efforts of several mathematicians ([1, 9, 5, 12, 17, 26, 29]) gave a complete answer to the problem **(GP)**. In 1957, Berger gave the complete list of the irreducible locally symmetric spaces [8]. Therefore, in the irreducible case, the problem of holonomy is solved. But it remains even open, because some indecomposable but non irreducible manifolds exist, (i.e. its holonomy group leaves a degenerate proper subspace of $R^{p,q}$ but it does not leave any non-degenerate proper subspace). We also signal that indecomposable pseudo-Riemannian symmetric spaces are not classified in the general case, except for Lorentzian ones and for the case of dimension ≤ 7 ([14, 15]). For the indecomposable-reducible general case, the complete list of the possible holonomy groups has been given by Bérard Bergery and Ikemakhen in [5, 19] and by Leistner in [23] for Lorentzian case. Also note that examples of Lorentzian metrics of the classified groups are given in [5, 19, 10]. The holonomy problem is partially solved for the case of signature $(2, n-2)$ in [20, 18]. For the case of neutral signature $(2,2)$, a complete classification is given in [6]. Note that, even though big progress have been made, the holonomy problem remains even open.
In this course we recall the main results on the holonomy problem. In paragraph 2, we recall the definition of holonomy and the one of restricted holonomy, and we study their properties. In paragraph 3, we look at the relation between the curvature and the holonomy which leads to the Ambrose-Singer's theorem. In paragraph 4, we show the holonomy principle and we study its applications. In paragraph 5, we state the De Rham-Wu's decomposition theorem and we give a sketch of proof. In paragraph 6, we state Berger's classification theorem for the irreducible and non locally symmetric case, and we indicate the corrections made to Berger's list. In paragraph 7, we give the last answers to the holonomy problem in the indecomposable-reducible case. In the last paragraph, we study the pseudo-Riemannian symmetric spaces.

7.2 Definitions and properties

Let $(M^{p,q}, g)$ be a connected pseudo-Riemannian manifold of signature (p,q) (where p is the number of $+$, and $q = n - p$ the number of $-$). There is a unique torsion-free

metric connection D, called the Levi-Civita connection, giving rise to a parallel transport $\tau_\gamma : T_xM \to T_yM$ along each curve γ on M with origin x and extremity y. τ_γ is an isometry with respect to the metric on T_xM and on T_yM. If α is another curve from y to z, we have $\tau_{\gamma*\alpha} = \tau_\alpha \circ \tau_\gamma$. This relation allows the following

Definition 84 *We call the holonomy group of $(M^{p,q}, g)$ at a point x, (or some times the holonomy representation of $(M^{p,q}, g)$ at x) and denote it by H_x, the subgroup of the orthogonal group $O(T_xM, g_x)$ generated by the set of all τ_γ where γ runs through the set of C^1-piecewise loops of M based at x. If we only consider the closed curves starting at x which are null-homotopic, we define the restricted holonomy group and we denote it by H_x^0.*

If we change the point x by the point y, and if γ is a C^1-piecewise curve of origin x and extremity y, then

$$H_y = \tau_\gamma H_x \tau_\gamma^{-1} \text{ and } H_y^0 = \tau_\gamma H_x^0 \tau_\gamma^{-1}. \tag{7.1}$$

As a consequence all the holonomy representations at various points of M are in fact isomorphic. So we can talk about the holonomy representation H of $(M^{p,q}, g)$ and the restricted one H^0.

Proposition 29 *a) H^0 is an path connected subgroup of $O(p,q)$.*
b) H^0 is a Lie subgroup of $O(p,q)$.
c) H^0 is a normal subgroup of H. Moreover the quotient H^0/H is countable.
d) H is a Lie subgroup of $O(p,q)$ whose identity component is H^0.

Proof a) Let β be a loop which is homotopic to zero. An easy result in approximation theory yields that our homotopy can be realized by intermediate loops β_t which are all C^1- piecewise. Since parallel transport is given by solutions of a differential equation, the classical theorem of continuity with respect to parameter solutions of such a differential equation will show that the map $t \to \tau_{\beta_t}$ with values in $O(p,q)$ is continuous in t. Then τ_β belongs to the path component of the identity element of $O(p,q)$.
b) is a consequence of a theorem of Yamabe which asserts that any subgroup H of a Lie group is a Lie subgroup of G, if every element of H can be joined to the identity by a C^1-piecewise curve which is contained in H. Note that in the proof, we can get easily not only a continuous arcwise connectedness but a C^1-piecewise connectedness in $O(p,q)$ ([22], appendix 4).
c) H^0 is a normal subgroup of H follows from 7.1. Moreover, let $\pi_1(M,x)$ be the fundamental group of M at x. Define a homomorphism $\pi : \pi_1(M,x) \to H_x/H_x^0$ as follows: for $a \in \pi_1(M,x)$, let γ the continuous loop at x which represents a. We may cover γ by a finite number of coordinate neighborhoods, modify γ within each neighborhood and obtain a C^1-piecewise curve at x which is homotopic to γ. If γ_1 and γ_2 are two such loops, then $\gamma_1 * \gamma_2^{-1}$ is homotopic to zero and defines an element of H_x/H_x^0. Since M is a manifold then $\pi_1(M,x)$ is countable, then also H^0/H.
d) follows from b) and c).
Remark : i) From proposition 1, if M is simply connected, then $H = H^0$.

ii) For a Riemannian manifold (M, g), the restricted holonomy group H^0 is closed in $O(n)$, and in particular is compact ([9], ch. 10). But it is not true in general (see for the Lorentzian case a counterexample in subparagraph 7.1, Types 3 and 4).

7.3 Holonomy and Curvature

In this paragraph, we will recall an interesting theorem that gives the relation between the holonomy and the curvature and with which we can calculate the restricted holonomy. We recall that the curvature of (M, g) is a $(3, 1)$-tensor R defined by

$$R(X, Y) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z,$$

where D is the Levi-Civita connection of (M, g). It satisfies the following proprieties:
(i) $R(a, b) \in so(T_x M, g_x)$, for $a, b \in T_x M$;
(ii) (first Bianchi-identity)

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0;$$

(iii)

$$g(R(X, Y)Z, U) = g(R(Z, U)X, Y);$$

(iv) (second Bianchi-identity)

$$(DR)(X, Y, Z) + (DR)(Y, Z, X) + (DR)(Z, X, Y) = 0.$$

Now, let $a, b \in T_x M$. We denote by γ_t the loop based at the point x and made up by the parallelogram formed by the following geodesics with the same length t:

$$exp(ta),\ exp(t\tau_{exp(ta)}(b)),\ exp(-t\tau_{[exp(ta)*exp\tau_{exp(ta)(b)}]}(a)),\ exp(-tb).$$

It is known that

$$R(a, b) = \frac{d}{dt}(\tau_{\gamma_t})(0).$$

Consequently $R(a, b)$ lies in the Lie algebra hol_x of the holonomy group of (M, g) at x. Now, if λ is a curve from x to y we have

$$\tau_\lambda^{-1} \circ R(\tau_\lambda(x), \tau_\lambda(y)) \circ \tau_\lambda = \frac{d}{dt}(\tau_{\lambda * \gamma_t * \lambda^{-1}})(0) \in hol_x,$$

where γ_t is the loop based at the point x and made up by the parallelogram formed as above by $\tau_\lambda(x)$ and $\tau_\lambda(y)$.
Now, we want to know if the holonomy algebra contains only the above endomorphisms. The answer is yes thanks to the Ambrose-Singer's theorem:

Theorem 76 *([2]) The holonomy algebra at a point x of a pseudo-Riemannian manifold (M, g) is exactly the subalgebra of $so(T_x M, g_x)$ generated by the element*

$$\tau_\lambda^{-1} \circ R(\tau_\lambda(a), \tau_\lambda(b)) \circ \tau_\lambda,$$

where a, b run through $T_x M$ and λ through all C^1- piecewise curves starting from x.

7.4 Fundamental holonomy principle

Proposition 30 *Let $(M^{p,q}, g)$ be a connected pseudo-Riemannian manifold. The following three properties are equivalent:*
(i) there exists on $(M^{p,q}, g)$ a tensor field w of type (r, s) which is invariant by parallel transport;
(ii) there exists on $(M^{p,q}, g)$ a tensor field w of type (r, s) which is D-parallel : $Dw = 0$;
(iii) there exists on $\mathbb{R}^n$ a tensor field w_0 of type (r, s) which is invariant by the holonomy representation.

Proof. Let us give a fixe a point x. w is invariant by parallel transport means that

$$w_x(X_1, ..., X_s) = w_y(\tau_\gamma(X_1), ..., \tau_\gamma(X_s)),$$

for all γ C^1- piecewise curve from x to y and $X_1, ..., X_s \in T_xM$. Therefore w_x is invariant by the holonomy representation H_x. Conversely if w_x is invariant by the holonomy representation H_x the above formula is well defined. So (i) and (iii) are equivalent. By definition

$$Dw(X_1, ..., X_s, X) = D_X(w(X_1, ..., X_s)) - \sum_{i=1}^{s} w(X_1, ..., D_X X_i, ..., X_s);$$

if we take a curve γ such that $\gamma' = X$ and $X_1, ..., X_s$ vectors fields which are parallel along γ, then $Dw(X_1, ..., X_s, X) = 0$ if and only if $D_X(w(X_1, ..., X_s)) = 0$, i.e. the value $w(X_1, ..., X_s)$ is constant along γ. Consequently (i) and (ii) are equivalent.
Remark : If we replace in (iii) the holonomy representation by the restricted one, we get of course the corresponding local result. Such a principle can be extended to geometric objets more general than tensor fields. The following particular case will be useful in the sequel:

Proposition 31 *Let k be an integer between 1 and $n-1$. There is an equivalence between the following assertions:*
(i) there exists on $(M^{p,q}, g)$ a k-dimensional distribution T which is invariant by parallel transport;
(ii) The holonomy representation leaves invariant a k-dimensional subspace of $\mathbb{R}^n$.
Moreover such a distribution is necessarily involutive. More precisely T verifies

$$D_X Y \in T \text{ for all tensor field } X \text{ on } M \text{ and } Y \in \Gamma(T). \tag{7.2}$$

Proof. T is invariant by parallel transport, then $T_y = \tau_\gamma(T_x)$ for all C^1- piecewise curve γ from x to y. Then H_x leaves invariant T_x. Conversely if H_x leaves invariant T_x, it is easy to have (ii). Moreover

$$D_X Y = \frac{d}{dt}[\tau_{\gamma_t}^{-1}(Y_{\gamma(t)}]_{|t=0},$$

where $\gamma' = X \in T_xM$. Since $\tau_{\gamma_t}^{-1}(Y_{\gamma(t)}) \in T_x$, then $D_XY \in T_x$. Consequently $[X,Y] = D_XY - D_YX \in \Gamma(T)$ for $X, Y \in \Gamma(T)$.
In the following, we will illustrate the holonomy principle by examples.

Example : Generic manifold. $H = SO(p,q)$ means that there are no parallel tensor fields except the metric and the orientation. Such a metric is called generic.

Example : Kähler manifold. $U(p,q)$ is the subgroup of $O(p,q)$ preserving a complex orthogonal structure J on $\mathbb{R}^{2m}$ $(2m = p+q)$ (that is $J \in O(p,q)$ and $J^2 = -I$). Therefore, by the holonomy principle, $H \subset U(p,q)$ if and only if (M,g) admits a complex orthogonal structure J which is parallel, i.e. (M,g) is a Kähler manifold.

Example : Flat manifold. (M,g) is a flat manifold if it admits, locally, a basis of parallel vector fields. By the holonomy principle, that means H^0 is reduced to the identity.

Among the applications of the holonomy principle, we present the De Rham-Wu decomposition theorem, in the next paragraph.

7.5 De Rham-Wu decomposition theorem

Let (M_1,g_1), (M_2,g_2) be two pseudo-Riemannian manifolds, and $(M,g) = (M_1,g_1) \times (M_2,g_2)$ their Riemannian product. Let $x = (x_1,x_2) \in M$, $T_xM = T_{x_1}M_1 \times T_{x_2}M_2$, which we can identified to the g_x-orthogonal sum $T_x^1 \oplus T_x^2$, where $T_x^1 = T_{x_1}M_1 \times \{0\}$ and $T_x^2 = \{0\} \times T_{x_2}M_2$. It is easy to see that the holonomy group H_x of (M,g) at x leaves invariant T_x^1 and T_x^2. More precisely, we have

$$H_x = H_{x_1}^1 \times H_{x_2}^2 \subset O(p_1,q_1) \times O(p_2,q_2) \subset O(p,q), \tag{7.3}$$

where H_x^i, $i = 1,2$ is the holonomy group of (M_i,g_i) at x_i and (p_1,q_1), (p_2,q_2) are the signatures of the metric g restricted respectively to T_x^1 and T_x^2. This means that H_x is decomposable.
Conversely, let (M,g) be a pseudo-Riemannian manifold such that, at a fixed point x, H_x is decomposable i.e. H_x satisfies (7.3). Therefore we have a g_x-orthogonal decomposition of the tangent space of M at x:

$$T_xM = T_x^1 \oplus T_x^2, \quad H_x(T_x^i) \subset T_x^i, \ \ for \ \ i = 1,2.$$

According to Proposition 3, we obtain two orthogonal involutive distributions. Moreover, by Frobenius theorem, they are integrable, and we have on $M,g)$ two orthogonal foliations M_1 and M_2. Then around x, there exists a system of coordinates $U = U_1 \times U_2$, where $U_1 = \{(x_1,...,x_{n_1},0,...,0)\} \subset M_1$ and $U_2 = \{(0,...,0,y_1,...,y_{n_2})\} \subset M_1$. We will show that the metric is locally a Riemannian product. We have:

$$\frac{\partial}{\partial x_k} g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}) = g(D_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}) + g(\frac{\partial}{\partial x_i}, D_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial y_j}).$$

$\frac{\partial}{\partial x_i} \in T^1$ and $\frac{\partial}{\partial y_j} \in T^2$. Then according to Proposition 3, $D_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_i} \in T^1$ and $D_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial y_j} \in T^2$. Since T^1 and T^2 are orthogonal, then

$$\frac{\partial}{\partial x_k} g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}) = 0.$$

In the same way, we show that $\frac{\partial}{\partial y_k} g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}) = 0$. But, at x, $g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}) = 0$, hence it is zero everywhere. With the same proof as above, we have $\frac{\partial}{\partial y_k}.g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = 0$ and $\frac{\partial}{\partial x_k}.g(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}) = 0$. This means that, on U, the metric g is the Riemannian product of its restrictions respectively to U_1 and U_2. So we have

Theorem 77 *([25, 28])*
a) Every pseudo-Riemannian manifold (M, g) is at least locally a Riemannian product of a flat pseudo-Riemannian manifold (M_0, g_0) eventually and indecomposable pseudo-Riemannian manifolds (M_i, g_i).
b) Moreover, if (M, g) is simply connected and complete, the product is global and the manifolds M_i are also simply connected and complete.

Remark : a) The proof of b) is delicate. The reader can consult [22] for the Riemannian case and [28] for the general case.
b) According to Theorem 2, the holonomy problem is reduced to the indecomposable case. In the following paragraph, we give solutions to this problem for the irreducible case which is a part of the indecomposable case.

7.6 Berger's Classification

In this section we will discuss the holonomy problem for irreducible non-locally symmetric pseudo-Riemannian manifolds. To simplify the problem, we suppose that the manifold is simply connected. We ask the following question:
Which subgroups of $O(p, q)$ can be the holonomy group of an irreducible non-symmetric pseudo-Riemannian simply connected manifold (M, g)?
In [7], Berger gave necessary conditions that a holonomy group must satisfy. Then he classified all groups satisfying these conditions. Recall that, according to Theorem 1, we can view the curvature tensor at a point x as $R_x \in \Lambda^2(T_x^*M) \otimes \mathcal{H}_x$, where $\mathcal{H}_x$ is the holonomy algebra at x.

Definition 85 *Let $V_{p,q}$ be a real vector space of dimension n endowed with the canonical scalar product of signature (p, q). Let $\mathcal{G} \subset so(V_{p,q})$ be a Lie subalgebra. The space of formal curvature maps of $\mathcal{G}$ is*

$$K(\mathcal{G}) := \{R \in \Lambda^2 V_{p,q}^* \otimes \mathcal{G};\ \ R(u, v)w + R(v, w)u + R(w, u)v = 0,\ \ \forall u, v, w \in V_{p,q}\}$$

The space of formal curvature derivatives is

$$K^1(\mathcal{G}) := \{\phi \in V_{p,q}^* \otimes K(\mathcal{G});\ \ \phi(u)(v,w) + \phi(v)(w,u) + \phi(w)(u,v) = 0,\ \ \forall u,v,w \in V_{p,q}\}.$$

An element $R \in K(\mathcal{G})$ has full curvature if $\mathcal{G}$ is generated by the image $R : \Lambda^2 V_{p,q} \to \mathcal{G}$. We also let

$$\underline{\mathcal{G}} = \{R(u,v);\ R \in K(\mathcal{G}),\ \ u,v \in V_{p,q}\} \subseteq \mathcal{G}.$$

According to the first and second Bianchi identities, $R_x \in K(\mathcal{H}_x)$ and $DR_x \in K^1(\mathcal{H}_x)$, where $\mathcal{H}_x$ is the holonomy algebra of a pseudo-Riemannian manifold (M,g).

Definition 86 *Let $\mathcal{G} \in so(V_{p,q})$ be a Lie subalgebra :*
$\mathcal{G}$ satisfies Berger's first criterion if $\underline{\mathcal{G}} = \mathcal{G}$.
$\mathcal{G}$ satisfies Berger's second criterion if $K^1(\mathcal{G}) \neq 0$.
$\mathcal{G}$ is a Berger algebra if it satisfies Berger's first and second criteria.
A connected Lie subgroup $G \subset SO(V_{p,q})$ is a Berger group if its algebra $\mathcal{G} \subset so(V_{p,q})$ is a Berger algebra.

If $\mathcal{G}' \subset \mathcal{G} \subset so(V_{p,q})$ is a pair of subalgebras of $so(V_{p,q})$, then $K(\mathcal{G}') \subset K(\mathcal{G})$ and $K^1(\mathcal{G}') \subset K^1(\mathcal{G})$. Moreover, if $\mathcal{G}' \subset so(V_{p,q})$ is any subalgebra which is $SO(V_{p,q})$-conjugate to $\mathcal{G}$, then there are isomorphisms $K(\mathcal{G}) \simeq K(\mathcal{G}')$ and $K^1(\mathcal{G}') \simeq K^1(\mathcal{G})$.
Using this with the Ambrose-Singer theorem, Berger derived the following result.

Theorem 78 *Let $V^{(p,q)}$ be a real vector space of dimension n endowed with the canonical scalar product of signature (p,q). Let $\mathcal{G} \subset so(V_{p,q})$ be the Lie algebra of a Lie subgroup $G \subset SO(V_{p,q})$.*
(i) If $K(\mathcal{G}) = K(\mathcal{G}')$ for any proper subalgebra $\mathcal{G}' \subset \mathcal{G}$, then G is not (conjugate to) the holonomy group of any pseudo-Riemannian manifold (M,g).
(ii) If G is an irreducible Lie subgroup which occurs as the holonomy of a non-locally symmetric pseudo-Riemannian manifold (M,g), then G is a Berger group.

Using this theorem, Berger proved the following result,

Proposition 32 *(Berger's original list)*
Let $(M^{p,q},g)$ be a simply connected irreducible non-symmetric pseudo-Riemannian manifold of signature (p,q). Then its holonomy group is (up to conjugacy in $O(p,q)$) one of the following: $SO(p,q)$, $SO(n,\mathbb{C})$, $U(p,q)$, $SU(p,q)$, $Sp(p,q)$, $Sp(p,q)\cdot Sp(1)$, $SO(n,\mathbb{H})$, G_2, G_2', $G_2^{\mathbb{C}}$, $Spin(7)$, $Spin(4,3)$, $Spin(7,\mathbb{C})$, $Spin(9)$, $Spin(8,1)$, $Spin(5,4)$.

This list of holonomy groups has been studied intensively. The group $SO(n,\mathbb{H})$ is eliminated by McLean [24], this group does not satisfy Berger's first criterion. The three groups of type $Spin(9-i,i)$ were eliminated independently by Alexeevski [1], Brown and Gray [5], who showed that these groups did not actually satisfy Berger's second criterion. Bryant found that there were two groups missing from Berger's list: $Sp(p,\mathbb{R})\cdot SL(2,\mathbb{R})$ and $Sp(p,\mathbb{C})\cdot SL(2,\mathbb{C})$. The efforts of several mathematicians ([1, 9, 5, 11, 12, 17, 26]) proved that all the other groups in Berger's list occurred and the final result of possible holonomy groups with their corresponding geometry was the following :

Theorem 79 *([7, 1, 24, 9, 5, 11, 12, 3, 27, 21, 17, 26, 29])*
Let $(M^{p,q}, g)$ be a simply connected irreducible non-symmetric pseudo-Riemannian manifold. Then its holonomy group is (up to conjugacy in $O(p,q)$) one of the following :

Holonomy group	*Geometry*
$SO(p,q)$	*Generic metric*
$SO(n,\mathbb{C})$	*Holomorphic metric*
$U(p,q)$	*Kähler*
$SU(p,q)$	*Special Kähler*
$Sp(p,q)$	*HyperKähler*
$Sp(p,q)\cdot Sp(1)$	*Quaternionic Kähler*
$Sp(p,\mathbb{R})\cdot SL(2,\mathbb{R})$	" "
$Sp(p,\mathbb{C})\cdot SL(2,\mathbb{C})$	" "
G_2	*associative*
G_2'	*split-associative*
$G_2^{\mathbb{C}}$	*holomorphic-associative*
$Spin(7)$	*Cayley*
$Spin(4,3)$	*split-Cayley*
$Spin(7,\mathbb{C})$	*holomorphic-Cayley*

Remark : In 1957, Berger gave a complete classification of the irreducible pseudo-Riemannian symmetric spaces (see paragraph 8). Then according to the above theorem and the De Rham- Wu's one, the holonomy problem is reduced to the indecomposable-reducible case.

7.7 Indecomposable-reducible Case

Now we suppose that (M,g) is indecomposable-reducible. Then its holonomy representation H preserves a totally isotropic subspace E of $T_\theta M$ of dimension $l \leq \min(p,q)$. At a fixed point θ, we can suppose that the metric g has the following form :

$$<,>= g_\theta \ = 2\sum_{i=1}^{l} dx_i dy_i + g'$$

where g' is the canonical scalar product of signature $(p-l, q-l)$ on $E^\perp/E$. If we denote by $\mathcal{H}$ the Lie algebra of H, then $\mathcal{H}$ is a subalgebra of $so(<,>)$. Let L_l be the connected subgroup of $SO_0(<,>)$ which is the full stabilizer of E. Then the Lie algebra $\mathcal{L}_l$ of L_l contains $\mathcal{H}$ and consists in the following "block" matrices :

$$(U,A,X,B) := \begin{pmatrix} U & {}^tX & B \\ 0 & A & -G'X \\ 0 & 0 & -{}^tU \end{pmatrix} \tag{7.4}$$

where $U \in \mathcal{M}(l)$ (i.e. the space of $l \times l$ real matrices) $A \in so(p-l, q-l)$, $X \in \mathcal{M}(l, m-2l)$ (i.e.the space of $l \times (m-2l)$ real matrices), and $B \in so(l)$.

$$G' = \begin{pmatrix} I_{p-l} & 0 \\ 0 & -I_{q-l} \end{pmatrix}.$$

We remark that
$\mathcal{L}_l = (\mathcal{A}_l \oplus \mathcal{T}_l) \ltimes (\mathcal{N}_l \ltimes \mathcal{B}_l) = (\mathcal{M}(l, m-2l) \oplus so(p-l, q-l)) \ltimes \left(\mathbb{R}^{(l(m-2l)} \ltimes \bigwedge^2 \mathbb{R}^l\right)$ where
$\mathcal{B}_l = \{(0,0,0,B) \ ; \ B \in so(l)\}$ is an abelian ideal of $\mathcal{L}_l$,
$\mathcal{N}_l \ltimes \mathcal{B}_l =$
$\{(0,0,X,B) \ ; \ X \in \mathcal{M}(l, m-2l), B \in so(l)\}$ is an ideal of $\mathcal{L}_l$,
$\mathcal{T}_l = \{(0,A,0,0) \ ; \ A \in so(g')\}$ is a subalgebra of $\mathcal{L}_l$ isomorphic to $so(p-l, q-l)$ and
$\mathcal{A}_l = \{(U,0,0,0) \ ; \ U \in \mathcal{M}(l)\}$ is a subalgebra of $\mathcal{L}_l$.
Now, let $T \subset TM$ be a l-dimensional parallel totally isotropic distribution on (M, g), then $T^\perp$ is also a parallel distribution. In particular the Levi-Civita connection D of g induces a connection D' on the bundle $T^\perp/T$ defined by:

$$D'[X] = [DX] \text{ , for } X \in \Gamma(T^\perp).$$

Furthermore the metric g' induced by g on $T^\perp/T$ is D'-parallel :

$$D'g' = 0.$$

From this we deduce

Proposition 33 *The holonomy algebra $\mathcal{H}'$ of the connection D' at the point θ is precisely the projection of $\mathcal{H}$ in the algebra $\mathcal{T}_l$.*

7.7.1 Lorentzian Case

In [5], Bérard Bergery and Ikemakhen described the possible holonomy groups of simply connected indecomposable-reducible Lorentzian manifolds and they showed the following two theorems :

Theorem 80 *([5]) Let (M, g) be a connected Lorentzian manifold of signature $(1, 1+n)$. Then (M, g) is indecomposable-reducible if, and only if, its holonomy algebra $\mathcal{H}$ is contained in $\mathcal{L}_1 = (\mathcal{A}_1 \oplus \mathcal{T}_1) \ltimes \mathcal{N}_1$ and either $\mathcal{H}$ contains $\mathcal{N}_1 = \{(0,0,X,0) \ / \ X \in \mathbb{R}^n\}$ or there exists a nontrivial decomposition $n = r + s$ et $\mathbb{R}^n = \mathbb{R}^r \oplus \mathbb{R}^s$, a nontrivial abelian subalgebra $\mathcal{C}$ of $so(r)$, a semi-simple subalgebra $\mathcal{D}$ of $so(r)$ (eventually 0), commuting with $\mathcal{C}$ and a surjective linear map $f : \mathcal{C} \to \mathbb{R}^s$ such that, up to conjugacy in $\mathcal{L}_1$, $\mathcal{H}$ is the subalgebra formed by the following matrices "block" :*

$$\left\{ \begin{pmatrix} 0 & {}^tX & {}^tf(A) & 0 \\ 0 & A+B & 0 & -X \\ 0 & 0 & 0 & -f(A) \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ A \in \mathcal{C}, \ \ B \in \mathcal{D}, \ \ X \in \mathbb{R}^r \right\}.$$

Theorem 81 *([5])*
With the hypothesis of Theorem 5, the $so(n)$-component of the holonomy algebra $\mathcal{H}$ of (M, g) is completely reducible.

From theorem 6 Leistner proved in [23] that

Theorem 82 *([23])*
With the hypothesis of Theorem 5, the $so(n)$-component $\mathcal{H}'$ of the holonomy algebra $\mathcal{H}$ of (M, g) is a Riemannian holonomy algebra, i.e. there exists a Riemannian metric with holonomy algebra $\mathcal{H}'$.

Bérard Bergery and Ikemakhen divided the indecomposable-reducible subalgebra $\mathcal{H}$ of $\mathcal{L}_1 = (\mathcal{A}_1 \oplus so(n)) \ltimes \mathcal{N}_1$ which satisfies the hypothesis of both theorem 5 and theorem 6 in four types :

- Type 1: $\mathcal{H}$ contains $\mathcal{N}_1$ and $\mathcal{A}_1$, then the group H generated by H is a product: $A_1 H' N_1$ where A_1 is the subgroup of $\mathcal{L}_1$ generated by $\mathcal{A}_1$, H' the subgroup of $SO(n)$ generated by $\mathcal{H}'$ and

$$N_1 := \left\{ \begin{pmatrix} 1 & {}^tX & -\frac{1}{2}{}^tX.X \\ 0 & I_n & -X \\ 0 & 0 & 1 \end{pmatrix} \quad X \in \mathbb{R}^n \right\}.$$

  So the restricted groups of Type 1 are closed. If g' is a Riemannian metric with holonomy group H' then for a sufficiently generic function f, H' is the one of the Lorentzian metrics $g = 2dx_0 dx_{n+1} - g' + f dx_{n+1}^2$.

- Type 2 : $\mathcal{H}$ contains $\mathcal{N}_1$ and the projection of $\mathcal{H}$ into $\mathcal{A}_1$ is zero, then $H = H'N_1$. So the restricted groups of Type 2 are closed. As examples of metric with holonomy H, we may take the same as above, but with f independent of x_0.

- Type 3 : $\mathcal{H}$ contains $\mathcal{N}_1$ but is neither of Type 1 nor Type 2: in this case there exists a decomposition $\mathcal{H}' = \mathcal{C} \oplus \mathcal{D}$ and a surjective linear map $\varphi : \mathcal{C} \to \mathcal{A}_1$ such that

$$\mathcal{H} = \mathcal{E} \oplus \mathcal{D} \oplus \mathcal{N}_1,$$

  where $\mathcal{E}$ is the subalgebra of $\mathcal{A}_1 \oplus so(n)$ defined by $\mathcal{E} = \{(\varphi(A), A), A \in \mathcal{C}\}$. So H' is closed in L_1 if and only if the subalgebra $Ker\varphi$ of $\mathcal{C}$ generates a compact torus (which is satisfied if φ is injective). So the first non-closed example appears in dimension 6. Examples of metric with holonomy H are given in [5] and [10].

- Type 4 : in this case there exists a nontrivial decomposition $\mathcal{N}_1 = \mathcal{N}_r \oplus \mathcal{N}_s$, a decomposition $\mathcal{H}' = \mathcal{C} \oplus \mathcal{D}$ and a surjective linear map $f : \mathcal{C} \to \mathcal{N}_s$ such that $\mathcal{H} = \mathcal{C}' \oplus \mathcal{D} \oplus \mathcal{N}_s$, where $\mathcal{C}'$ is the subalgebra of $so(n) \oplus \mathcal{N}_1$ defined by $\mathcal{C}' = \{(A, f(A)), A \in \mathcal{C}\}$. So H' is closed in L_1 if and only if the subalgebra $Kerf$ of $\mathcal{C}$ generates a compact torus (which is satisfied if f is injective). So the first non-closed example appears in dimension 7. Examples of metric with holonomy H are given in [5] and [10].

7.7.2 Neutral Case (signature (p,p))

Even for this case, the holonomy problem is not solved except in the 4-dimensional case made by Bérard Bergery and Ikemakhen in [6]. In the following step, we recall the results. We consider the subalgebra $\mathcal{L}_2$ de $so(<>)$ which leaves invariant a isotropic plan, with

$$<(x_1,x_2),(y_1,y_2)>=2(x_1y_2+x_2y_1).$$

Then

$$\mathcal{L}_2=\left\{(U,a):=\begin{pmatrix} U & aJ_0 \\ 0 & -{}^tU\end{pmatrix}, U\in\mathcal{M}(2,\mathbb{R}),\ \ a\in\mathbb{R}\right\}$$

where $J_0=\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$. We consider the following subalgebra of $\mathcal{L}_2$:

$\mathcal{K}_1=\{(\alpha N,0),;\ \ \alpha\in\mathbb{R}\}$ with $N=\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}$;

$\mathcal{K}_{2,\lambda}=\left\{(\begin{pmatrix} \alpha & \beta \\ 0 & \alpha\lambda\end{pmatrix},0)\ ;\ \alpha,\beta\in\mathbb{R}\right\}$, for $\lambda\in[-1,1]$;

$K_3=\left\{(\begin{pmatrix} \alpha & \gamma \\ 0 & \beta\end{pmatrix},0),\ \ \alpha,\beta,\gamma\in\mathbb{R}\right\}$;

$\mathcal{K}_4=\{(U,0),\ ;U\in sl(2,\mathbb{R}\}$;

$\mathcal{K}_5=\{(U,0),\ \ U\in gl(2,\mathbb{R}\}$;

$\mathcal{K}_6=\{(\alpha I_2+\beta J_0,0),\ \ \alpha\beta\in\mathbb{R}\}$; $\mathcal{A}$ (respectively $\mathcal{B}$) the ideal $\mathcal{L}_2$ generated by the matrix $(0,1)$ (resp. the subalgebra of $\mathcal{L}_2$ generated by the matrices $(N,0)$ and $(0,1)$); $\mathcal{U}_1$ (resp. $\mathcal{U}_{2,\lambda}$ for $\lambda\in\mathbb{R}$) the subalgebra of $\mathcal{L}_2$ generated by the matrix $(I_2+N,0)$ (resp. by the matrix $(\lambda I_2+J_0,0)$). If $\mathcal{P}$ is the subalgebra of $\mathcal{L}_2$, we will denote by P the connected subgroup of L_2 generated by $\mathcal{P}$.

Theorem 83 *([6])*
Let (M,g) be an indecomposable-reducible simply connected pseudo-Riemannian manifold of signature $(2,2)$. Then, up to an isomorphism, the holonomy group H is contained in L_2 and
Either H leaves invariant two isotropic supplementary plans, and (up to conjugacy in L_2) it can be only K_1, $K_{2,\lambda}$, K_3, K_4, K_5, or K_6.
Or H leaves invariant only one isotropic plan, and (up to conjugacy in L_2) it can be only B, $U_1\ltimes A$, $U_{2,\lambda}\ltimes A$, $K_{2,\lambda}\ltimes A$, $K_3\ltimes A$, $K_4\ltimes A$, $K_5\ltimes A$, or $K_6\ltimes A$.

Remark : In [6], we show that all groups of the first list of Theorem 8 are effectively the holonomy groups of metrics. For the second list, we don't know any examples up till now.

7.7.3 Indecomposable Completely Reducible Case

Theorem 84 *([4]) Let (M,g) be a simply connected indecomposable pseudo-Riemannian manifold of signature (p,q) and dimension $m=p+q$. We suppose that the holonomy representation H is completely reducible. Then*

1) either it is irreducible (and the possible cases of H are classified by Berger),
2) or the holonomy representation is the direct sum of two representations $\rho :\longrightarrow W$ and $\mu :\longrightarrow W'$ such that
i) $p = q$,
ii) W and W' are p-dimensional isotropic subspaces of V,
iii) $V = W \oplus W'$, ρ and μ are irreducible,
iv) the quadratic form $<,>$ identifies W' to the dual W^* of W and μ to the dual representation of ρ. Furthermore, W does not admit any $\rho(H)$-invariant quadratic form,
v) in the decomposition $V = W \oplus W^*$, H is a subgroup of the group

$$\{\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} ; A \in GL(p, \mathbb{R})\},$$

3) If in addition (M, g) is not a locally symmetric space, then, up to conjugacy, the couple (H, W) is one of the following :

H	W	*restrictions*
$GL(p,\mathbb{R})$	$\mathbb{R}^p$	$p \geq 1$
$SL(p,\mathbb{R})$	$\mathbb{R}^p$	$p \geq 2$
$Sp(p,\mathbb{R})$	$\mathbb{R}^{2p}$	$p \geq 2$
$GL(p,\mathbb{C})$	$\mathbb{C}^p$	$p \geq 1$
$SL(p,\mathbb{C})$	$\mathbb{C}^p$	$p \geq 2$
$Sp(p,\mathbb{C})$	$\mathbb{C}^{2p}$	$p \geq 2$

7.8 Pseudo-Riemannian Symmetric Spaces

Definition 87 *A pseudo-Riemannian manifold $(M^{(p,q)}, g)$ is called a symmetric space (resp. a locally symmetric space) if for all $x \in M$ there exists an involutive isometry (resp. local isometry) (i.e. $s_x^2 = -id_M$) with isolated fixed point x. s_x is called the symmetry of the point x.*
Any isometry of the form $s_x s_y$ is called a transvection of M. The subgroup of the isometry group formed by all transvections is called the group of transvections of M.

Definition 88 *A pseudo-Riemannian symmetric triple of signature (p, q) (in abbreviate form PST) is a triple $(\mathcal{G}, \sigma, \eta)$ with the following properties :*
(i) $\mathcal{G}$ is a finite dimensional real algebra.
(ii) $\sigma : \mathcal{G} \longrightarrow \mathcal{G}$ is an involutive automorphism of $\mathcal{G}$ and $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$, with $\mathcal{K}$ (resp. $\mathcal{P}$) the eigensubspace of σ associated to the eigenvalue 1 (resp. to the eigenvalue -1).
(iii) $\mathcal{K}$ is generated by $\mathcal{P}$: $\mathcal{K} = [\mathcal{P}, \mathcal{P}]$.
(iv) η is a non-degenerate symmetric bilinear-form on $\mathcal{P}$ of signature (p, q) which is invariant by σ and $ad(\mathcal{K})$.
$(\mathcal{G}, \sigma, \eta)$ is said to be indecomposable, if $\mathcal{P}$ doesn't contain any non-trivial non-degenerate subspace invariant under $ad(\mathcal{K})$.

We know that the data of a (indecomposable) simply connected pseudo-Riemannian symmetric space (M, g) is equivalent to the data of a (indecomposable) PST $(\mathcal{G}, \sigma, \eta)$, where $\mathcal{G}$ is the Lie algebra of the group of transvections of (M, g), σ the differential of the conjugation by the symmetry s_θ at a basis point $\theta \in M$, $\mathcal{P} = T_\theta M$ and $\eta = g_\theta$.

Theorem 85 *Let (M, g) be a simply connected indecomposable pseudo-Riemannian symmetric space and $(\mathcal{G}, \sigma, \eta)$ its corresponding PST. Then*
1) The holonomy representation $\mathcal{H}$ of (M, g) is the faithful representation $ad\mathcal{K} : \mathcal{P} \longrightarrow \mathcal{P}$;
2) the center of $\mathcal{G}$ is

$$\mathcal{Z} = \{X \in \mathcal{P}, [\mathcal{K}, X] = 0\} = [\mathcal{K}, \mathcal{P}] \cap [\mathcal{K}, \mathcal{P}]^{\perp},$$

then $\mathcal{Z}$ is isotropic.
3) $\mathcal{G}$ is semi-simple if and only if the holonomy representation is completely reducible;
4) $\mathcal{G}$ is solvable if and only if the holonomy representation is nilpotent.

Remark : The classification of simply connected indecomposable pseudo-Riemannian symmetric spaces is equivalent to the classification of the indecomposable PST. In 1957, M. Berger studied all PST in the case where the group of transvections G is semi-simple and found a long list (see [8]). In [15], Cahen and Parker studied the general case, but they could only classify the symmetric spaces of dimension ≤ 7. In [16], Cahen and Wallach studied the Lorentzian case, then Cahen and others gave the complete classification of these Lorentzian spaces that we give in the following.

Theorem 86 *([16]) Let (M, g) be a simply connected indecomposable Lorentzian symmetric space. Then its group of transvections G is either semi-simple or solvable.*

The semi-simple case is a part of Berger's classification. Now, we will recall the complete classification of Lorentzian solvable symmetric space. Let $\underline{\lambda} = (\lambda_1, ..., \lambda_n) \in (\mathbb{R} \setminus \{0\})^n$ such that $||\underline{\lambda}||^2 = \sum_{i=1}^n \lambda_i^2 = 1$. And let's note $M^m_{\underline{\lambda}}$ the Lorentzian space $M^m_{\underline{\lambda}} := (\mathbb{R}^m, g_{\underline{\lambda}})$, with

$$g_{\underline{\lambda}} = 2dx_0 dx_{n+1} - \sum_{i=1}^n dx_i^2 + \left(\sum_{i=1}^n \lambda_i x_i^2\right) dx_{n+1}^2.$$

Theorem 87 *([14]). Let $(M, <>)$ be a m-dimensional indecomposable-reducible simply connected Lorentzian space ($m = n + 2$). Then, $(M, <>)$ is isometric to some $M^m_{\underline{\lambda}}$.*

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Differential Geometry and Topology, Discrete and Computational Geometry
M. Boucetta and J.-M. Morvan (Eds.)
IOS Press, 2005
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Chapter 8

Curvature and Symmetries of Parallel Transport

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Introduction

Based on our human sensations and perceptions of "the physical geometry of what we experience as being our ambient world", and, after a long history as an experimental science in various cultures around the globe, finally geometry was established as the central field of mathematics in the world of the Old Greeks. And still today, as such formulated by Chern in his introduction to [53]: "*While algebra and analysis provide the foundations of mathematics, geometry is at the core*". From Serres' "*Les origines de la géométrie*" [123], in this respect we freely quote the following: "Ainsi l'élimination de ce qui cache la forme, ainsi l'exclusion de tout sujet, rendent possible une science pour l'Universel pour nous et, en rigueur et en vérité, dans l'Universel en soi. Les mathématiques vinrent d'identifier l'un à l'autre."

Thus being conceived in the human mind via psychologically created von Helmholtz-hypotheses about "our surrounding world and its contents", which were and are made at service to the evolution of our species and its individuals, abstract geometry was born and has grown ever since by brain-design [11, 76, 113].

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The Old Greeks's awareness of the sophistication of the nature of mathematics as abstract and deductive science should not be underestimated: in Euclid's Alexandria-school it may well have been similar to the position formulated in Hilbert's foundations of geometry. And yet, all along to the present times, exception made from the true geometers, this Euclidean geometry philosophy basically is regarded to be "the formal idealisation of the world in which we live". In this respect, concering the very meaning of seeing and of human perceiving in general, we refer to [39, 62, 72, 74, 75, 96, 97, 99, 124, 135, 136, 138], for obtaining an appreciation of how difficult it seems to be for non-geometers to be aware of the nature of observation.

The Greek "*mathema*" basically stands for knowledge and understanding. The essential purpose of all so-called exact and human sciences is to gain ever better knowledge and understanding of the operations of the forces of nature and of our mind. And "*mathematics*" refers to how, by which way, to achieve knowledge and understanding, or still, to achieve wisdom. And that it is really an art to do so is perfectly expressed in the Dutch term for mathematics: "*wiskunde*", or "Wijs-konst" in the elder spelling, "Wijs" and "wis" standing for "wise", and "konst" and "kunde" for "art". All this also reflects the saying of for instance Feynman according to which "Nature speaks to us in the language of mathematics", mathematics consisting in some sense of the formal and the universal aspects of the workings of nature and of the mind. Whereas before them, in different cultures, tables of data and recepees for their use had been elaborated by means of which for instance positions of celestial bodies and times of eclipses could be determined, as again Serres put it, "it were the Old Greeks who invented the theory, or, stated otherwise, who provided the scenes of vision, like the ones by Plutarchus and Diogenes, and probably it was there that they made the breack. And, at the same theatre of vision, in the shadows of the pyramides, also Thales staged his distinct inaugural lecture; they invented to represent" [123].

Euclidean geometry pretty well describes the objective, i.e., the common, or still, intelligably communicable perceptions that humans have in and of their worlds and accordingly shapes the observed objects and phenomena in the human minds. Euclidean spaces therefore are the most natural stages on which succesfully to perform the sciences, the exact as well as the human sciences, i.e., the stages on which some universal characters (including non-Euclidean ones!) can attempt to explain the often intriguing liaisons between these objects and phenomena. But one should be fully aware that the above only concerns, in some sense didactical, performing and attempting to explain.

The Egyptian "maât" basically stands for what is correct, precise, just, ..., as in the various senses of "being well-measured, well-balanced". Therefore, the field of mathematics which in most languages is called "geometry", i.e., "the measuring of the earth", however important this plain matter may be, was so much more wisely and accurately called "meetkunde" in Dutch, i.e., meaning literally: "the art to achieve correctness, preciseness, justice, ...", and which also well expresses the crucial importance of geometry in the global process of going for knowledge and understanding. Let it be completely clear at this point that there is no doubt at all concerning the importance of "literal" geometry, in itself and in its outer repercursions and in its mathematical sophistications, as for instance is shown by the studies related to Frisius' works on triangulations and to Mercator's atlasses

giving straight lines in their planar charts for the loxodromes on the real surface of the earth. But to cover "geometry" as the field of mathematics that it was in the times of the Old Greeks and into which it continuously further developed from thereon untill the very present times, the Dutch term "meetkunde" for the art to achieve the Egyptian "maât" is beyond any doubt so much more correct, precise, just, ...

And, as in some sense the term "geometry" itself reflects a rather primitive antropomorphism, the same holds for all literal references to any "physical reality". Such latter views on "physical reality" namely express the wide-spread illusion, which at first was highlighted in the works of Newton and which on a larger scale continued untill the works of Helmholtz, Mach, Poincaré, ..., and finally the general scientific acceptance of Einstein's theory of relativity, that men could analyse nature fully neutrally and objectively, as "from outside", and so could get complete access to "the truths" governing "the cosmos" from the small to the large. To write this down or say this is easy enough, but to be fully aware of the fact that we, as observers, are essentially an integral part of the natural reality which we try to understand via our observations, seems to be a matter of a completely different nature. And so in particular, one can observe a not decreasing number of people who continue to debit, admittidly sometimes smart, formulations of notions of time, mass, ..., but inevitably do so in words that are so vague for those with an extremely positive disposition, or incredibly empty for those who don't beat around the bush.

As expressed by Teillard de Chardin: "The history of the living can be summarised as the elaboration of ever more perfect eyes within a cosmos in which there is always more to be seen". So, as a small example of improving eyes: recently in stead of sometimes weird but always interesting theories of "explaining" visual illusions, a geometrical description of visual perceptions was given, which takes away their mistery. It might be considered as one more token that, paraphrasing Feynman: "Nature likes to be looked at with geometer's eyes and brains" [136].

Anyway, the origin of the field of mathematics called geometry and its main source of intuition is our human visual perception. Now, reformulating somewhat freely Gegory along the appreciations of von Helmholtz: "Perceptions are the constructions, registered by early vision, of hypotheses which produce visual images that model physical reality in accordance with our previously acquired experiences of this reality". But however really great this formulation is, also here the term "reality" must be treated with care, and by in particular taking into account, as Simon Stevin put it that "we should always be fully aware that we can never see things as they really are" or as Huygens continued in this spirit, that "all we can see are shadows (i.e., "shows") ...". By the way, also the term "curvature" as far as we know was introduced by Huygens in the "low countries", but this time not in our language but in Latin.

Along the lines of then again truly deep studies concerning most in particular Euclidean geometry and its applications, the worries about the eventuallity of being fooled by intuition in connection with the parallel postulate and, technically, by the just acquired ability to calculate with algebraic forms and equations basically like with the real numbers as exposed in Simon Stevin's booklet "De Thiende" of 1585, the co-ordinate geometry was developed systematically. In particular, and most notably, this was done by Descartes (what's in a

name?) after contacts with Stevin. How this work of Descartes, Fermat and others paved the way to the infinitesimal calculus and via the applications of this calculus in the geometry of curves and surfaces in Euclidean 2D- and 3D-spaces to the Riemann-Finsler geometry is well enough known [28], and was indeed in some sense already prophetically formulated in the very first sentence of Descartes' first book on geometry: "Tous les Problesmes de Geometrie se peuvent facilement reduire a tels termes, qu'il n'est besoin par aprés que de connoistre la longeur de quelques lignes droites pour les construire" [40]. It seems hard to give more glory to "the line element".

As expressed by Berger: "*La courbure est l'invariant riemanniens numéro 1 et le plus naturel*" [7]. In the following we intend to present some well-known and also some new views on curvature in geometry and in some applications of geometry. In the following sections, we thus consider curvature and symmetries of parallel transport in (semi-)Riemannian spaces. In chapter 8.1, in particular, the development from the Euclidean spaces over the spaces of constant curvature and over the locally and the semi-symmetric spaces to the pseudo-symmetric spaces of Deszcz is given in the context of Schouten's holonomy of directions which we extend to holonomy of sectional curvatures and isometries occuring in parallely transported geodesic triangles and parallelogramoïds of Levi-Civita. And the occurence of this pseudo-symmetry in the relativistic space-times is discussed. We will also briefly indicate how similarly the basic notions of the differential geometry of submanifolds, being the second fundamental form and the curvature tensor of the normal bundle, can be defined themselves as well as their basic properties like, in particular, their extrinsic symmetries, in terms of parallel transport along the submanifolds. This essentially concerns joint work with Franki Dillen, Joeri Van der Veken and Johan Fastenakels [55].

The epilogue treats two concepts: 1) ideal submanifolds in geometry, following up the pioneering works of Bang-yen Chen [23] on his so-called δ-curvatures and other new intrinsic curvatures and the general inequalities that relate them to the main extrinsic curvatures, and 2) geometrical shapes realised by natural forms in plants, animals, cells, molecules, crystalls, etc., as described by Johan Gielis [70] in co-operation with whom we wrote a more technical report on these matters [71]. We would like to point out here that this work of Chen a.o. gives ample solutions to the question of Chern [27] to find intrinsic obstructions for Riemannian manifolds to allow a minimal isometric immersion into some Euclidean space. In this context, as well as related to some other points we tried to make before, we would like to refer here to the 1900-Hilbert problems [90] of which the above one of Chern well satisfies the high criteria. This work of Gielis on the other hand, fits in well in the purpose of the celebrated "On Growth and Form" of d'Arcy Tompson [33]. From this book we quote as follows: " The search for differences or fundamental contrasts between the phenomena of organic or inorganic, of animate or inanimate things, has occupied many men's minds, while the search for community of principles or essential similitudes has been pursuid by few". Gielis might be one of these few, looking for years and eventually finding a formula which describes at once so many forms occuring in nature either globally or in their essential parts. The appropriate shapes realised by this formula, given one of the forms to which it applies, finally correspond to the choice of a few natural or real numbers to be plugged in into the formula. Pythagoras' saying that there is a harmony in nature,

a unity in its variety, which is expressed by numbers, (cfr. e.g. [11]), could hardly be illustrated more strongly. And, as our human perceptions seem to be best understood in terms of the curvature of surfaces in a 3D-Euclidean space, or, for that matter, even more efficiently as done by J. Koenderink in his lecture "Image processing done right" [98], in a 3D-isotropic space, equally so, the description of the natural priciples which to a large extent determine the shapes which are assumed in nature, from proteins over snowflakes and bamboo, etc., to the relativistic univers itself, are well done in terms of the intrinsic and the extrinsic curvatures of the "physical submanifolds" concerned.

One of the exponents of Einstein's general theory of relativity is the Friedmann-Lemaître theory of the "big bang", as it got widely known (a.o. see [103] for historical comments). Lemaître so much more appropriately, it seems to us, spoke in this respect of the "principle atom", or, still of course merely presenting an image, but this one having a most natural ring to it, of "the cosmic egg". The symbolic value of this expression as referring to a natural conception of "our" cosmos by a fertilisation of "the cosmic egg" may eventually have had some special appeal to Mgr. Georges Lemaître, in one or other way, but no doubt in his days at least it created extra attention for the purely scientific interests that he always tried to raise. Next follows a quotation from one of his popularising talks on his big bang at Antwerpen in 1949: "Birth of space and origin of multiplicity: all the familiar notions on which we reflect and base our knowledge loose their meaning and disappear; space vanishes to a point and muliplicity reduces to unity ... Let us modestly admit our essential limitations". And, precisely, this latter awareness might well be the foundation of our scientific strength, as expressed as follows by Freudenthal: "To detect antropomorphisms and, where they appear to be inevitable, to accept them conciously, so as to properly emerge them into our natural philosophy, such was the attitude probably taken at first by Einstein".

Acknowledgment

The authors enjoyed support of the K.U. Brussel-K.U. Leuven Center PADGE (Pure and Applied Differential GEometry) and of the K.U. Leuven section on Geometry for research visits at the Agricultural and Technical University of Wroclaw (Poland), at the University of Granada (Spain) and at the University of Kragujevac (Serbia and Montenegro) during which a substantial part of the research on which we report was done. The discussions with our colleagues and friends at these universities certainly have significantly contributed to the work. Moreover the first author would like to thank T. Moons and I. Van de Woestyne of the K.U. Brussel for their permanent encourgagement during his doctoral work, R. Deszcz of the Agricultural University of Wroclaw for his continuing interest and valuable suggestions and A. Romero of the University of Granada for the hospitality and colaboration during the final stages of this work. Finally, the second named author would like to add special thanks to the IAMIT (Institute for Applied Mathematics and Information Technologies) near Prague (Tchechia) for its frequent hospitality allowing study in excellent conditions. Both autors thank M. Boucetta and J.-M Morvan for including this somewhat informal text on several aspects of their joint work on which the first author reported at the meeting.

8.1 Parallel Translation of Vectors Along Curves

We begin with the case of parallel translation of a given vector along a geodesic in a two-dimensional Riemannian surface [4]. Let γ be a geodesic in Σ^2 and $\vec{v}$ a vector at $\gamma(0)$. We define the parallel translate of $\vec{v}$ in $\gamma(t)$ by continuously moving $\vec{v}$ from $\gamma(0)$ to $\gamma(t)$ such that its angle with the geodesic and its length remain constant. In this way we obtain a linear and isometric map from the tangent plane to the surface at $\gamma(0)$ to the tangent plane at $\gamma(t)$. If γ is a broken line consisting of several geodesics, a vector is parallel translated by repeating the above procedure for every geodesic arc separately. Finally, parallel translation of a vector along any smooth curve on a surface is defined by a limiting procedure whereby the curve is approximated by broken lines consisting of geodesics. The construction of parallel translation in higher-dimensional Riemannian manifolds is more complicated because the direction of the translated vector can no longer be uniquely determined by the condition that the angle with respect to the curve be invariant. In fact, the vector could rotate around the direction of the curve while preserving the angle.

To be able to give a consistent definition we reduce the problem to parallel translation in two-dimensional surfaces. We again start with parallel translation along a geodesic γ of a given vector $\vec{v}$ at $\gamma(0)$. At the starting point $\gamma(0)$ we consider the plane spanned by $\vec{v}$ and the tangent to the geodesic $\gamma'(0)$. Then look at all geodesics going through $\gamma(0)$, tangent to this plane. The set of all these geodesics forms a smooth surface which contains the given geodesic γ. Consider a point $\gamma(\varepsilon)$ at a small distance from the starting point. The tangent plane at this new point to the geodesic surface introduced above contains the direction of the geodesic. We take this new point as the initial point and use its tangent plane to construct a new geodesic surface. This new surface again contains the original geodesic γ. We move along γ with steps ε and repeat the above constructions. As a result we have, at every point of the geodesic, a tangent plane containing the direction of the geodesic. As $\varepsilon \to 0$ we obtain a field of two-dimensional planes along γ containing the direction of the geodesic and being determined in an intrinsic manner by the metric of the manifold. Parallel translation of a vector along the geodesic γ is then defined as in the two-dimensional case: the translation of the vector $\vec{v}$ must keep $\vec{v}$ in the planes and must preserve its length and its angle with the direction of γ. Parallel translation along a general smooth curve is again defined by approximating with geodesics.

Using the notion of parallel transport along curves in a Riemannian manifold we can introduce the concept of covariant differentiation. Let $\vec{v}$ be a vector tangent to a Riemannian manifold $\mathcal{M}$ at a point p and X a vector field given on $\mathcal{M}$ in a neighbourhood of p. Take any curve through p with tangent $\vec{v}$. Consider the vector of the field X at a point on the curve a small distance ε from p, $X(\varepsilon)$, and parallel translate it back to p. We obtain a vector depending on ε in the tangent space to $\mathcal{M}$ at p, denoted by $P_\varepsilon X$. If $\varepsilon = 0$, this vector is $X(p)$, for other ε it changes according to the non-parallelness of the vector field

X along the curve in the direction of $\vec{v}$. Consider the derivative of the resulting vector with respect to ε, evaluated at $\varepsilon = 0$. This derivative is a vector in the tangent space to $\mathcal{M}$ at p and is called the Levi-Civita covariant derivative of the field X with respect to $\vec{v}$,

$$\nabla_{\vec{v}} X := \lim_{\varepsilon \to 0} \frac{P_\varepsilon X - X(p)}{\varepsilon} .$$

The vector $\nabla_{\vec{v}} X$ does not depend on the choice of curve, but only on $\vec{v}$ and X. From the construction it is obvious that a parallel vector field X along a curve with tangent $\vec{v}$ satisfies

$$\nabla_{\vec{v}} X = 0 .$$

Furthermore, a geodesic is defined as a curve γ for which the tangent vector field is parallel, i.e. $\nabla_{\gamma'} \gamma' = 0$.

Let $\mathcal{M}$ be a n-dimensonal *(semi-)Riemannian manifold*, i.e., an n-dimensional differential manifold equiped with a (0,2)-metric tensor field g. Then, according to the so-called *fundamental theorem* of Riemannian geometry, $\mathcal{M}$ admits a unique *metric symmetric affine connection* ∇, i.e., a derivation which associates a tangent vector field $\nabla_X Y$ to any two tangent vector fields X and Y on $\mathcal{M}$ such that $\nabla g = 0$ ($\forall X, Y, Z : (\nabla_X g)(Y, Z) := X[g(Y, Z)] - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$, or still, such that g is parallel) and such that $[X, Y] = \nabla_X Y - \nabla_Y X$ whereby $[.,.]$ denotes the Lie-bracket on $\mathcal{M}$. This connection ∇ is called the Riemannian or the *Levi-Civita connection* of $\mathcal{M}$; for its general study as well as for some interesting related observations, see e.g. [1, 8, 26, 29, 57, 100, 122, 125, 141].

Let $(x^1, x^2, \ldots, x^n)$ be *local co-ordinates* on $\mathcal{M}$ and $g_{\alpha\beta}$ the corresponding local components of the metric tensor g, $(\alpha, \beta, \gamma, \mu, \ldots \in \{1, \ldots, n\})$. Then, the corresponding *Christoffel-symbols* $\Gamma^\mu_{\alpha\beta} : \mathcal{M} \to \Re$ are defined by

$$\nabla_{\partial_\alpha} \partial_\beta = \sum_\mu \Gamma^\mu_{\alpha\beta} \partial_\mu = \Gamma^\mu_{\alpha\beta} \partial_\mu ,$$

whereby $(\partial_1, \ldots, \partial_n)$ denotes the natural local basis of tangent vector fields, such that

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu} \right) ,$$

whereby $g^{\mu\nu}$ are the local components of the inverse of g; (the symmetric property of ∇ amounts to $\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}$). For arbitrary local vector fields $X = \sum_\alpha X^\alpha \partial_\alpha = X^\alpha \partial_\alpha$ and $Y = \sum_\beta Y^\beta \partial_\beta = Y^\beta \partial_\beta$, the covariant derivative $\nabla_X Y$ of Y with respect to X consequently is given by the *formula of Koszul* as

$$\nabla_X Y = \sum_\alpha \left\{ \sum_\beta X^\beta \left[\frac{\partial Y^\alpha}{\partial x^\beta} + \sum_\gamma Y^\gamma \Gamma^\alpha_{\beta\gamma} \right] \right\} \partial_\alpha = X^\beta \left[Y^\alpha_{,\beta} + Y^\gamma \Gamma^\alpha_{\beta\gamma} \right] \partial_\alpha ,$$

which clearly manifests how Christoffel's symbols measure how the natural derivations of vector fields on general Riemannian manifolds $\mathcal{M}$ differ from their directional derivations

in the standard Euclidean spaces E^n. Concerning Levi-Civita's 1917-conception of his connection ∇ via parallel transport while anticipating the imbedding theorems of Janet-Cartan-Nash and on the 1952-Koszul approach to affine connections and on other historic comments, in particular on connections in some sense taking over the group-rôle in the Klein-Lie perspective on geometry, see e.g. [9, 16, 53, 59, 61, 101, 107]. In these respects, at this stage we only mention the following. In the Klein-Lie view on geometry, as formulated in the so-called *Erlangen Programme* of 1872, its purpose is the following: "*Es ist eine Mannigfaltigkeit und in derselben eine Tranformationsgruppe gegeben; man entwickle die auf die Gruppe bezugliche Invariantentheorie*". Via the parallel transport of Levi-Civita [101], which was obtained independently by Schouten [120] who rather preferred to call it *pseudo-parallel transport*, this Erlanger-concept of geometry was properly generalised, essentially by the works of Hessenberg, König, Wirtingen, Noether and Cartan, as follows: "*Jedem Element einer (mannigfaltigkeit)* X_n *sei eine* X_N *zugeordant, die dieses Element enthält. Benachbarte* X_N *werden auf einander afgebildet durch Transformationen, die einer gegebene kontinuierlichen Transformationsgruppe angehörn. Es ist die Theorie des Invarianten zu entwicklen*" [121].

8.2 The Curvature Tensor

In the following $\mathcal{M}$ always denotes an n-dimensional (semi-)Riemannian manifold with metric tensor g and Levi-Civita connection ∇. The *Riemann-Christoffel (1,3)-curvature tensor* R of $\mathcal{M}$ is defined by associating the vector field

$$R(X,Y)Z := (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z$$

to any three given vector fields X, Y and Z on $\mathcal{M}$. So, in analytical terms, R measures how much $\mathcal{M}$'s natural differentiation ∇ defects from Schwarz-Young's theorem of permutability of second order partial differentiation (say, $\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$) in $\Re^n$. It's (0,4)-version, which hereafter we will also denote by R, is defined by associating the real functions

$$R(X,Y;Z,W) := g(R(X,Y)Z,W)$$

to any four given vector fields X, Y, Z and W on $\mathcal{M}$. The (1,1)- tensor $R(X,Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ is called the *curvature operator* of $\mathcal{M}$. For a general study of curvature, as well for curves and surfaces as for abstract manifolds, and some historical comments, (and of course besides also other preceeding and following references), see e.g. [5, 7, 20, 21, 32, 56, 58, 60, 68, 73, 88, 106, 109, 110, 111, 114, 127, 131, 140, 144] .

Next, we recall Schouten's 1918-view on the curvature R of a Riemannian manifold $\mathcal{M}$ [120], at first following the procedure presented by Wald in [139]. Let p be any point in $\mathcal{M}$ and let Σ be a two-dimensional surface in $\mathcal{M}$ through p with local co-ordinates x and y. Consider a parallelogram in Σ constructed from the co-ordinate curves with tangent vector fields X and Y. Let $\vec{u}$ be a vector tangent to $\mathcal{M}$ in p, not necessarily tangent to Σ. We then perform the parallel transport of $\vec{u}$ along such parallelogram and compute the change

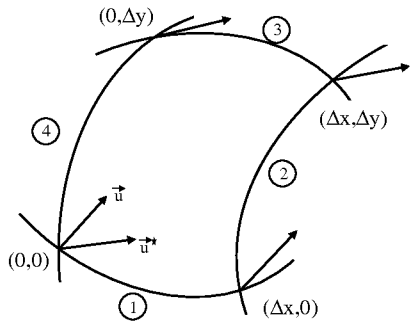

Figure 8.1: Characterisation of curvature by Schouten(1918).

in the vector $\vec{u}$ as we traverse the parallelogram by considering the change in the scalar $u^\alpha v_\alpha$, with $\mathbf{v}$ an arbitrary one-form; $\vec{u} = (u^\alpha \partial_\alpha)(p)$ and $\mathbf{v} = (v_\alpha dx^\alpha)(p)$, $\alpha \in \{1, \ldots, n\}$. For later convenience we will evaluate in the midpoints of the curves of the parallelogram. For small Δx, the change δ_1 of $u^\alpha v_\alpha$ is given by

$$\delta_1 = \left[\partial_x (u^\alpha v_\alpha) \mid_{(\Delta x/2,0)}\right] \Delta x \ .$$

Then, putting $X = X^\beta \partial_\beta$ and $Y = Y^\beta \partial_\beta$,

$$\begin{aligned} \delta_1 &= \left[X^\beta \nabla_\beta (u^\alpha v_\alpha) \mid_{(\Delta x/2,0)}\right] \Delta x \\ &= \left[u^\alpha X^\beta \nabla_\beta v_\alpha \mid_{(\Delta x/2,0)}\right] \Delta x \ , \end{aligned}$$

because $\vec{u}$ is parallely transported along $\gamma(x)$ and analogously,

$$\begin{aligned} \delta_2 &= \left[u^\alpha Y^\beta \nabla_\beta v_\alpha \mid_{(\Delta x,\Delta y/2)}\right] \Delta y \ , \\ \delta_3 &= \left[u^\alpha X^\beta \nabla_\beta v_\alpha \mid_{(\Delta x/2,\Delta y)}\right] \Delta x \ , \\ \delta_4 &= -\left[u^\alpha Y^\beta \nabla_\beta v_\alpha \mid_{(0,\Delta y/2)}\right] \Delta y \ . \end{aligned}$$

So, the change along the x-curves is

$$\delta_1 + \delta_3 = \left\{u^\alpha X^\beta \nabla_\beta v_\alpha \mid_{(\Delta x/2,0)} - u^\alpha X^\beta \nabla_\beta v_\alpha \mid_{(\Delta x/2,\Delta y)}\right\} \Delta x \ . \tag{8.1}$$

Because the term between brackets goes to zero as $\Delta y \to 0$, we have that up to first order in Δx and Δy the total change in $u^\alpha v_\alpha$ vanishes, which implies that the total change in $\vec{u}$ vanishes because $\mathbf{v}$ was arbitrary. We thus find that parallel transport is path-independent up to first order in Δx and Δy.

To calculate the second order change in $u^\alpha v_\alpha$ we proceed as follows. Consider the curve $x = \frac{\Delta x}{2}$ and suppose that we transport $\vec{u}$ and $X^\beta \nabla_\beta v_\alpha$ parallely along this curve from $(\Delta x/2, 0)$ to $(\Delta x/2, \Delta y)$. Up to first order in Δy, $\vec{u}$ in $(\Delta x/2, 0)$ is equal to the parallel transport of $\vec{u}$ in $(\Delta x/2, \Delta y)$ because parallel transport is path-independent up to first order. We further have that, up to first order, $X^\beta \nabla_\beta v_\alpha$ in $(\Delta x/2, \Delta y)$ is different from the parallel transport of this quantity in $(\Delta x/2, 0)$ by

$$\left[Y^\gamma \nabla_\gamma (X^\beta \nabla_\beta v_\alpha) \mid_{(\Delta x/2, \Delta y/2)}\right] \Delta y \ .$$

Equation (8.1) becomes, up to second order in Δx and Δy,

$$\delta_1 + \delta_3 = - \left[u^\alpha Y^\gamma \nabla_\gamma (X^\beta \nabla_\beta v_\alpha) \mid_{(\Delta x/2, \Delta y/2)}\right] \Delta x \Delta y \ .$$

So, the total change in $u^\alpha v_\alpha$ is

$$\begin{aligned} \delta(u^\alpha v_\alpha) &= \left[u^\alpha \{X^\gamma \nabla_\gamma (Y^\beta \nabla_\beta v_\alpha) - Y^\gamma \nabla_\gamma (X^\beta \nabla_\beta v_\alpha)\} \mid_{(\Delta x/2, \Delta y/2)}\right] \Delta x \Delta y \\ &= \left[u^\alpha \{X^\beta Y^\gamma (\nabla_\beta \nabla_\gamma - \nabla_\gamma \nabla_\beta) v_\alpha + [X, Y]^\beta \nabla_\beta v_\alpha\} \mid_{(\Delta x/2, \Delta y/2)}\right] \Delta x \Delta y \ . \end{aligned}$$

Because the (x, y) co-ordinate fields X and Y commute, we have $[X, Y] = 0$. We further recall that $R_{\alpha\beta\gamma}{}^\mu v_\mu = (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) v_\gamma$. The total change in $u^\alpha v_\alpha$ up to second order in Δx and Δy can then be written as

$$\delta(u^\alpha v_\alpha) = \left[u^\alpha X^\beta Y^\gamma R_{\alpha\beta\gamma}{}^\mu v_\mu \mid_{(\Delta x/2, \Delta y/2)}\right] \Delta x \Delta y \ . \tag{8.2}$$

This holds for all $\mathbf{v}$ if and only if the total change in $\vec{u}$, up to second order in Δx and Δy, equals

$$\delta u^\alpha = \left[u^\mu X^\beta Y^\gamma R_{\beta\gamma\mu}{}^\alpha\right] \Delta x \Delta y \ .$$

Similarly, but more in the original spirit of Schouten's work in this direction, one has the following. Consider as before an infinitesimal co-ordinate parallelogram $\mathcal{P}_{\alpha\beta}$ in $\mathcal{M}$, say spanned by local co-ordinate lines x^α and x^β, such that the co-ordinates of its vertices are

$$\begin{aligned} p &= (x^1, \ldots, x^\alpha, \ldots, x^\beta, \ldots, x^n) \ , \\ q &= (x^1, \ldots, x^\alpha + \Delta x^\alpha, \ldots, x^\beta, \ldots, x^n) \ , \\ r &= (x^1, \ldots, x^\alpha + \Delta x^\alpha, \ldots, x^\beta + \Delta x^\beta, \ldots, x^n) \ , \\ s &= (x^1, \ldots, x^\alpha, \ldots, x^\beta + \Delta x^\beta, \ldots, x^n) \ . \end{aligned}$$

Now, we parallel transport a vector $\vec{v}$ completely around $\mathcal{P}_{\alpha\beta}$. Then there results a vector $\vec{v}^\star$ which in general differs from $\vec{v}$, and which, more precisely, up to second order in Δx^α and Δx^β (the "sides" of the parallelogram $\mathcal{P}_{\alpha\beta}$) is given by

$$\vec{v}^\star = \vec{v} + [R(\partial_\alpha(p), \partial_\beta(p))\, \vec{v}]\, \Delta x^\alpha \Delta x^\beta \ , \tag{8.3}$$

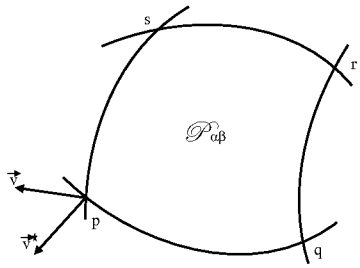

Figure 8.2: Schouten's holonomy.

whereby $R\left(\partial_\alpha(p), \partial_\beta(p)\right)\vec{v} = \left(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha\right)\vec{v}$.
By the tensorial character of R, it suffices to observe the changes in direction resulting by parallely translating the natural basic vectors ∂_γ at p completely around the infinitesimal co-ordinate parallelogram $\mathcal{P}_{\alpha\beta}$. As shown before, these are, up to the second order in Δx^α and Δx^β, given by

$$R(\partial_\alpha, \partial_\beta)\partial_\gamma = \sum_\mu R_{\alpha\beta\gamma}{}^\mu \partial_\mu = R_{\alpha\beta\gamma}{}^\mu \partial_\mu \ ,$$

whereby $R_{\alpha\beta\gamma}{}^\mu$ denote the local components of the (1,3)-tensor R, given in terms of the Christoffel symbols as

$$R_{\alpha\beta\gamma}{}^\mu = \partial_\beta \Gamma^\mu_{\alpha\gamma} - \partial_\alpha \Gamma^\mu_{\beta\gamma} + \Gamma^\nu_{\alpha\gamma}\Gamma^\mu_{\nu\beta} - \Gamma^\nu_{\beta\gamma}\Gamma^\mu_{\nu\alpha} \ .$$

Correspondingly, for the (0,4)-curvature tensor, the local components are given by

$$R_{\alpha\beta\gamma\delta} = \sum_\mu R_{\alpha\beta\gamma}{}^\mu g_{\mu\delta} = R_{\alpha\beta\gamma}{}^\mu g_{\mu\delta} \ ;$$

(from here onwards, we will systematically use Einstein's summation convention, i.e., use the notation of the right-side as to mean the middle-side without expliciting the Σ in case when the summation index occurs "below" and "on top").

In terms of symmetry, with Schouten, the curvature of a Riemannian manifold can thus be looked at as follows. When moving a line parallely along a rectangle in a Euclidean plane, described with Cartesian co-ordinates (x, y), from p over q, r and s back to p, it there returns to its original position.
In a Riemannian manifold $\mathcal{M}$ this in general is no longer the case. Given a tangent vector $\vec{v}$ at a point p in $\mathcal{M}$, by ∇ one can still transport $\vec{v}$ parallely along any curve α in $\mathcal{M}$ through $p = \alpha(0)$, in the sense that there exists a unique vector field V along α such that

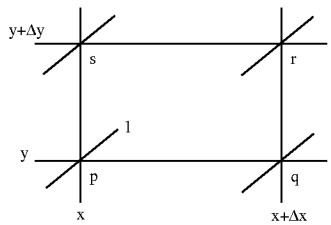

Figure 8.3: Parallel transport of directions in Euclidean space around a rectangle.

$V(0) = \vec{v}$ and which is parallel, i.e. such that $\nabla_{\alpha'} V = 0$, its value $V(t)$ at $q = \alpha(t)$ then being called *the parallel translate of* $\vec{v}$ *from* p *to* q *along* α.

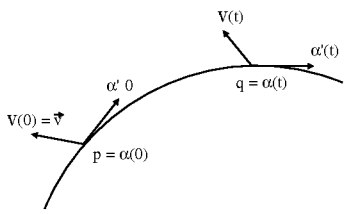

Figure 8.4: Parallel transport of a vector along a curve in a Riemannian space.

But in contrast to the situation in Euclidean geometry, where this parallel transport of directions or vectors is "*absolute*", in Riemannian geometry it is such that whereas *parallel transport of vectors* still *preserves their lengths*, in general *it changes their directions*. In particular, when moving a vector $\vec{v}$ at p parallely all along a closed curve back to p, in general its direction will have changed; Cartan called the phenomenon "*holonomy*", from "whole" and "law or rule" referring to *completely* going around a closed curve and the resulting changes in directions satisfying a precise "*law*". Accordingly, following Schouten's formula (8.3), the curvature of a Riemannian manifold can be defined as the measure for *this change of direction* when parallely moving fully around a co-ordinate parallelogram p, q, r, s. *The fact that directions on Riemannian spaces remain invariant under parallel transport around such parallelograms if and only if these spaces are locally flat or locally Euclidean, i.e., have vanishing curvature* $R = 0$, *was already known by Levi-Civita.*

8.3 Spaces of Constant Curvature

In the works of Riemann(1854) and Christoffel(1869), their curvature tensor R originated in the study of the applicability of one Riemannian manifold onto another one, i.e., in the study of the conditions for the changes $x^\alpha = x^\alpha(y^\beta)$ of local co-ordinates to transform a metric $\mathrm{ds}^2 = h_{\alpha\beta}(y)\mathrm{d}y^\alpha \mathrm{d}y^\beta$ into a metric $\mathrm{ds}^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu$. These works moreover formed the basis for the development by Ricci(1887) and Levi-Civita of the tensor calculus and of the covariant differentiation ∇, which Levi-Civita could later give its geometrical interpretation in terms of parallel transport.

The Nomizu-Kulkarni square of the metric tensor may well define in $G = \frac{1}{2} g \wedge g$ the most simple and natural $(0,4)$-tensor on $\mathcal{M}$ which shares the algebraic properties of the $(0,4)$-curvature tensor R:

$$G(X_1, X_2, X_3, X_4) = g(X_1, X_4)g(X_2, X_3) - g(X_1, X_3)g(X_2, X_4) \ .$$

For any pair of linearly independent vectors $\vec{v}$ and $\vec{w}$ at any point p of $\mathcal{M}$,

$$K(p, \pi) = \frac{R(\vec{v}, \vec{w}, \vec{w}, \vec{v})}{G(\vec{v}, \vec{w}, \vec{w}, \vec{v})}$$

is a real number which depends only on the point and on the tangent 2-plane π spanned by $\vec{v}$ and $\vec{w}$, namely the *sectional curvature* of $\mathcal{M}$ at p for π. $\mathcal{M}$ is said to be *isotropic* when for every point p this number $K(p, \pi)$ is the same for all 2-planes π. By *Schur's theorem*(1886), for $n > 2$, then $K(p, \pi)$ is moreover independent of p: thus $K(p, \pi)$ is *a constant* c. The Riemannian manifolds $\mathcal{M}^n$, $n \geq 2$, characterised by $R = c\,G$, i.e., for which $R(X_1, X_2, X_3, X_4) = c\,G(X_1, X_2, X_3, X_4)$, are called *the spaces* $\mathcal{M}^n(c)$ *of constant curvature*. For $c \neq 0$, they constitute the most immediate notable generalisation of the Euclidean spaces E^n to spaces of non-vanishing curvature. Historically, the spaces $\mathcal{M}^2(c)$ originated in the speculations of Gauss and Beltrami concerning Euclid's parallel postulate and the spaces $\mathcal{M}^n(c)$ in the works of Riemann [116] and von Helmholtz [89] concerning the geometrical worlds in which the local perceptions of bodies are independent of their location. The simplest non-flat spaces $\mathcal{M}^n(c)$ are the spheres S^n of radius r in Euclidean spaces E^{n+m}, their curvature being $c = \frac{1}{r^2}$. From the viewpoint of the extrinsic geometry of submanifolds, these spheres S^n in E^{n+m} are the totally umbilical submanifolds whose mean curvature vector field is parallel in the normal bundle; therefore, in general, a submanifold $\mathcal{M}^n$ having these two properties in a space $\tilde{\mathcal{M}}^{n+m}$ is called an *extrinsic sphere* of $\tilde{\mathcal{M}}^{n+m}$.

As shown already by E. Cartan, the Riemann-Christoffel curvature tensor R of a Riemannian manifold $\mathcal{M}$ is completely determined by the knowledge of its sectional curvatures $K(p, \pi)$. The sectional curvature of a tangent plane π to a Riemannian manifold $\mathcal{M}$ at a point p can be geometrically interpreted as the Gauss curvature at p of the Riemannian surface G^2, formed locally by the geodesics of $\mathcal{M}$ going through p tangent to π at p; this was first shown by Riemann [116], thus in the spirit of Gauss' theorema egregium creating abstract intrinsic geometry and in particular extending the Gauss curvature K of surfaces $\mathcal{M}^2$ in E^3 to the curvature tensor R of Riemannian spaces $\mathcal{M}$. Therefore, the sectional curvatures are also called *the Riemannian curvatures of* $\mathcal{M}$.

To give another geometrical interpretation of the sectional curvature of a plane π at a point p in $\mathcal{M}$, we construct a *parallelogramoïd* of Levi-Civita in the point p [65]. The metric tensor $g_{\alpha\beta}$ can be expressed in a neighbourhood of the origin p of normal co-ordinates x^α, up to second order, as

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \frac{1}{3} R_{\gamma\alpha\beta\mu} x^\gamma x^\mu \ , \tag{8.4}$$

where the curvature tensor in the remainder of this section is understood to be evaluated at p. The Christoffel symbols of the Levi-Civita connection can also be expressed in the neighbourhood of p in terms of normal co-ordinates as

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{3}(R^\alpha{}_{\beta\mu\gamma} + R^\alpha{}_{\gamma\mu\beta}) x^\mu \ . \tag{8.5}$$

The geodesic through p with any given tangent vector $\vec{u}$ in normal co-ordinates is given by the parametrisation

$$x^\alpha = s u^\alpha \ .$$

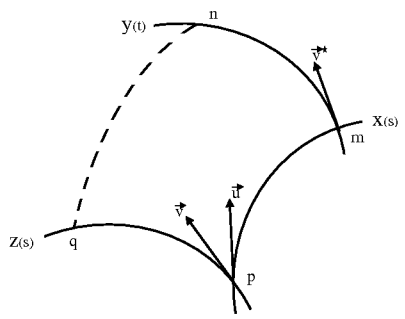

Figure 8.5: The parallelogramoïd of Levi-Civita.

Let m be the point on this geodesic for which $s = a$. We then parallely translate another given vector $\vec{v}$ from p to m. In normal co-ordinates the parallely transported vector $\vec{v}^\star$ is given by

$$v^{\star\alpha} = v^\alpha + \frac{1}{6} R^\alpha{}_{\beta\gamma\mu} u^\beta v^\gamma u^\mu a^2 \ .$$

We construct through m the unique geodesic $y(t)$ with tangent vector $\vec{v}^\star$. After solving the geodesic equations

$$\frac{\mathrm{d}^2 y^\alpha}{\mathrm{d}t^2} + \Gamma^\alpha{}_{\beta\gamma} \frac{\mathrm{d}y^\beta}{\mathrm{d}t} \frac{\mathrm{d}y^\gamma}{\mathrm{d}t} = 0 \ ,$$

the point n on a distance b from m has co-ordinates, to within fourth order of a and b,

$$y^\alpha = au^\alpha + b\left\{v^\alpha + \frac{1}{6}R^\alpha{}_{\beta\gamma\mu}u^\beta v^\gamma u^\mu a^2\right\} + \frac{1}{3}R^\alpha{}_{\beta\gamma\mu}v^\beta v^\gamma u^\mu ab^2 \ .$$

Construct the geodesic $z(s)$ through p with tangent $\vec{v}$ and let q be the point on this geodesic at a distance b from p. The normal co-ordinates of this point are

$$z^\alpha = b\,v^\alpha \ .$$

If $b_\circ^2$ is equal to the Euclidean distance $\eta_{\alpha\beta}(y_n^\alpha - a\,u^\alpha)(y_n^\beta - a\,u^\beta)$ between the points m and n, then, to within terms of sixth order in a and b, we have

$$b_\circ^2 = b^2\left\{1 + \frac{1}{3}R_{\alpha\beta\gamma\mu}u^\alpha v^\beta u^\gamma v^\mu a^2\right\} \ .$$

Using the definition of the sectional curvature and denoting $B^2 = (g_{\alpha\gamma}g_{\beta\mu} - g_{\alpha\mu}g_{\beta\gamma})u^\alpha v^\beta u^\gamma v^\mu$, we have

$$b_\circ^2 = b^2\left\{1 + \frac{1}{3}K(p,\pi)B^2a^2\right\} \ ,$$

whereby $\pi = \vec{u} \wedge \vec{v}$. To within the same accuracy, we thus obtain the following relation between the geodesic and Euclidean distance between m and n:

$$b^2 = b_\circ^2\left\{1 - \frac{1}{3}K(p,\pi)B^2a^2\right\} \ .$$

If we denote with $a_\circ'^2$ the Euclidean distance $\eta_{\alpha\beta}(y_n^\alpha - z_q^\alpha)(y_n^\beta - z_q^\beta)$ between the points q and n, we have

$$a_\circ'^2 = a^2\left\{1 - \frac{2}{3}K(p,\pi)B^2b^2\right\} \ .$$

The geodesic distance a' between both points can then, like above, be written as

$$a'^2 = a_\circ'^2\left\{1 - \frac{1}{3}K(p,\pi)B^2b^2\right\} \ ,$$

or, using the expression for $a'_\circ$,

$$a'^2 = a^2\{1 - K(p,\pi)B^2b^2\} \ .$$

In the Riemannian case, B^2 is the sine of the angle between the two vectors $\vec{u}$ and $\vec{v}$ at p. Then $a\,b\,B$ can be considered as the area of the parallelogramoïd and therefore the sectional curvature $K(p,\pi)$ of the plane π at the point p can be geometrically interpreted as the difference in geodesic distance between the points (p,m) and (q,n), divided by the area of the parallelogramoïd:

$$K(p,\pi) = \frac{a^2 - a'^2}{(\text{area})^2} \ .$$

This geometric interpretation of the sectional curvature K was first obtained by Levi-Civita.

By the theorem of Nash, every Riemannian manifold $\mathcal{M}^n$ can be isometrically imbedded in Euclidean spaces E^{n+m}. Consider a plane section π at a point p of, in this case, a Riemannian submanifold $\mathcal{M}^n$ in E^{n+m}. Then the *two-dimensional normal section* of $\mathcal{M}^n$ at p in the direction π, i.e., the local intersection of $\mathcal{M}^n$ with the $(m+2)$-plane through p spanned by π and $T_p^{\perp}\mathcal{M}^n$, the normal m-plane to $\mathcal{M}^n$ at p in E^{n+m}, is a two-dimensional surface Σ^2 in E^{n+m}, whose tangent plane at p is $T_p\Sigma^2 = \pi$. And, clearly [81], the sectional curvature $K(p,\pi)$ of $\mathcal{M}^n$ at p for the plane π equals the Gauss curvature $K_{\Sigma^2}(p)$ of this two-dimensional normal section Σ^2 of $\mathcal{M}^n$ at p for π: $K(p,\pi) = K_{\Sigma^2}(p)$. Moreover, as shown in [81], considering also projections of this surface Σ^2 on well choosen three-dimensional Euclidean spaces containing π, *all Riemannian curvatures* $K(p,\pi)$ of any Riemannian manifold $\mathcal{M}^n$ *can further be completely determined by the knowledge of the curvatures of at most two planar Euclidean curves.* After Newton determined (~ 1670) the curvature of all Euclidean planar curves, Euler, Meusnier, Monge, Dupin, Rodrigue, ... exploited this knowledge of the curvatures of planar curves to controle the curvature behaviour of all surfaces $\mathcal{M}^2$ in E^3 ($\sim 1760-1815$). The theorema egregium of Gauss (1827) showed that the Gauss curvature K equals the product of *the principal curvatures* k_1 and k_2, $K = k_1k_2$, and that K is invariant under the isometric deformations of the surface $\mathcal{M}^2$, i.e., K belongs to the *intrinsic geometry* of $\mathcal{M}^2$ in E^3. The extension of this intrinsic geometry on surfaces $\mathcal{M}^2$ in E^3 to *the intrinsic geometry on n-dimensional abstract manifolds* $\mathcal{M}^n$ was anticipated by Gauss and carried out by Riemann and von Helmholtz (~ 1860). In particular, the Gauss curvature of surfaces $\mathcal{M}^2$ in E^3 was extended to the *sectional curvature* K of Riemannian manifolds. Via Nash's imbedding theorem, conversely in some sense, the *abstract* sectional curvatures K of Riemannian manifolds can now be determined by the knowledge of the curvatures of two associated curves in Euclidean planes (~ 2000, cfr. also [81]). Conceptually, this result might be interpreted as asserting that all intrinsic curvatures of Riemannian geometry basically reduce to the curvatures of associated Euclidean planar curves, or still, to the *area's* of associated *curvature-* or, in Huygens terminology, *osculating circles.*

A manifold is called *isotropic* or a *space of constant curvature* or a real space form when the sectional curvature does not depend on the plane. From the theorem of Schur(1886) it follows that then $K(p) = K$, a constant, if the dimension ≥ 3. Remark that the definition for sectional curvature given above only holds for non-degenerate planes. In [86, 87] a definition for the sectional curvature of a degenerate plane in a semi-Riemannian manifold is proposed as follows. Let π be a degenerate plane containing a null geodesic γ with tangent vector field V. The null sectional curvature of π with respect to V is

$$K_V(p,\pi) = \frac{R_{\alpha\beta\gamma\mu}u^\alpha v^\beta v^\gamma u^\mu}{g_{\alpha\beta}u^\alpha u^\beta} \ , \tag{8.6}$$

with $\vec{u}$ an arbitrary vector at p in π, not parallel to V. Remark that $K_V(p,\pi)$ is independent of the choice of $\vec{u}$, but depends quadratically on V. In the case of a Lorentzian manifold, where there is a preferred time-orientation, a null sectional curvature can be defined independent of the choice of null vector field in the plane [112]. The above "isotropy" concerning curvature amounts to the so-called *axiom of free mobility*, i.e., freely speaking, to the lengths of rigid measuring rods being independent of their locations in *space*, which was probably first carefully studied as such by Helmholtz, in particular for three-dimensional Riemannian spaces. When this "Tatsache" is quite reasonably imposed to hold for "the physical space" at any given time in Einstein's relativistic space-times, then in particular the Friedmann-Lemaître-Robertson-Walker models for such space-times naturally appear. Their metrics are essentially of the form

$$ds^2 = -dt^2 + R(t)^2 \frac{dx^2 + dy^2 + dz^2}{\left[1 + \frac{k}{4}(x^2 + y^2 + z^2)\right]^2} \ ,$$

whereby $k = 0, 1$ or -1 according to our world instantaneously would be either *Euclidean, spherical (or elliptical) or hyperbolic.* Concerning the case of constant curvature "spaces", in particular Lemaître's special interest in the elliptical case might not be surprising in view of the fact that the elliptical geometry so naturally combines what is most beautiful in metrical geometry (constant curvature) with what is most beautiful in projective geometry (duality), see e.g. [17, 117].

The Ricci tensor is defined as the (0,2)-tensor obtained by the contraction of the Riemann tensor, $S_{\alpha\beta} = g^{\gamma\mu} R_{\alpha\gamma\beta\mu}$, and the Ricci curvature $S(X)$ is defined as

$$S(X) = S(X,X) = g^{\gamma\mu} < R(X, \partial_\gamma)\partial_\mu, X > \ .$$

Thus, the Ricci curvature $S(X)$ is a kind of average of the sectional curvatures of the planes containing X [92]. The scalar curvature τ is defined as the contraction of the Ricci tensor,

$$\tau = g^{\alpha\beta} S_{\alpha\beta} \ ,$$

and can thus be seen as an average of the sectional curvatures of the tangent two-planes of $\mathcal{M}$ [92]. A (semi-)Riemannian manifold $\mathcal{M}$ is said to be an *Einstein space* if $S = \lambda g$ for some $\lambda : \mathcal{M} \to \Re$. For $n > 2$, Einstein spaces have constant scalar curvature τ. By the theorem of Schouten-Struik the 3-dimensional Einstein spaces are the 3-dimensional real space forms [38].

8.4 Locally Symmetric Spaces

Let $p \in \mathcal{M}$, $\gamma(x)$ a curve in $\mathcal{M}$ with $\gamma(0) = p$ and X the tangent vector field of $\gamma(x)$. Consider the plane π in p spannend by the basis $\{\vec{u}, \vec{v}\}$ (from now on all the planes are understood to be non-degenerate). The sectional curvature of this plane is defined as above. Because the sectional curvature is independent of the basis of the plane, we can

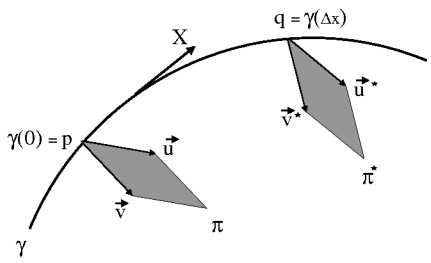

Figure 8.6: Characterisation of locally symmetric spaces by Levy(1925).

take $\{\vec{u}, \vec{v}\}$ orthonormal at p. We then consider the change of the sectional curvature after parallel transport of the basis along $\gamma(x)$ to a point q,

$$\begin{aligned} \delta K(p,\pi) &= \left[X^\sigma \nabla_\sigma (R_{\alpha\beta\gamma\mu} u^\alpha v^\beta v^\gamma u^\mu)\right] \Delta x \\ &= \left[u^\alpha v^\beta v^\gamma u^\mu X^\sigma \nabla_\sigma R_{\alpha\beta\gamma\mu}\right] \Delta x \ . \end{aligned}$$

So, the sectional curvature of any plane is unchanged after parallel transport along $\gamma(x)$ if and only if $X^\sigma \nabla_\sigma R_{\alpha\beta\gamma\mu} = 0$. Hence in $\mathcal{M}$ the sectional curvature of any plane is preserved after parallel transport along any curve if and only if $\nabla_\sigma R_{\alpha\beta\gamma\mu} = 0$, i.e., if and only if *the curvature tensor R of $\mathcal{M}$ is parallel*, $\nabla R = 0$. In invariant notation we have for the above situation that, up to first order:

$$K(q,\pi^\star) = K(p,\pi) + \left[\left(\nabla_{\gamma'(0)} R\right)(\vec{u},\vec{v},\vec{v},\vec{u})\right] \Delta x \ .$$

This characterisation of $\nabla R = 0$ is done by Levy(1925) [65]. The Levy-property of preservation of sectional curvature under parallel transport of two-planes along curves can geometrically also be interpreted as *preservation of sectional curvature under mirror symmetry* . Hereby we mean the following: planes π and $\pi^\star$ related to each other by parallel transport along a curve γ from p to q can be considered as each other's images in the *hyperplane mirror* $\gamma^\perp$ perpendicular to γ at the midpoint of p and q. In this sense, $\nabla R = 0$ amounts to *the preservation of all sectional curvatures under all reflections in hyperplane mirrors.*

E. Cartan (1926-1927) [17] showed that the Riemannian manifolds $\mathcal{M}$ whose curvature tensor R is parallel, i.e., for which $\nabla R = 0$, are the so-called *locally symmetric spaces* , i.e., have the symmetry property that *all their point-reflections σ_p, reversing the geodesics of $\mathcal{M}^n$ through p, are local isometries.*

In two-dimensional spaces, being locally symmetric and being of constant curvature are equivalent. For dimension $n > 2$, the locally symmetric spaces form a proper extension of

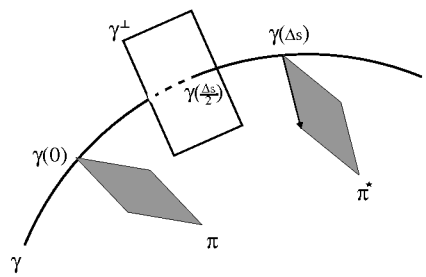

Figure 8.7: π and $\pi^\star$ as mirror-images in hyperplanes $\gamma^\perp(\Delta s/2)$.

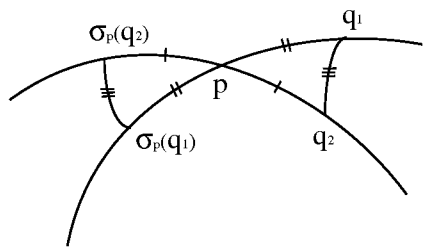

Figure 8.8: Geodesic reflections in locally symmetric spaces.

the real space forms. For the classification of the locally symmetric spaces, see [17] for the riemannian case and [12, 13] for the indefinite case.

8.5 Semi-symmetric Spaces

Let $p \in \mathcal{M}$ and Σ a two-dimensional surface through p in $\mathcal{M}$ with local co-ordinates x and y.

Consider the parallelogram in Σ constructed from the co-ordinate curves with tangent vector fields X and Y. Let π be a plane in p,not necessarily tangent to Σ, spanned by a basis $\{\vec{u}, \vec{v}\}$. We then perform the parallel transport of the basic vectors along the parallelogram. The plane $\pi^\star$ which we obtain in p after parallel transport is in general different from π. We are now interested in the difference of the sectional curvature of both

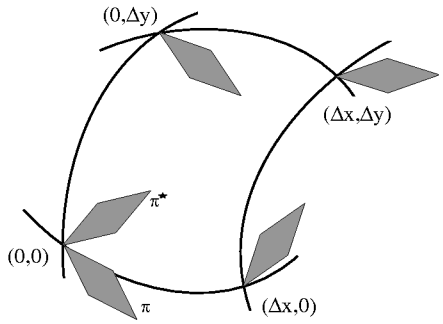

Figure 8.9: Characterisation of semi-symmetric spaces.

planes in p. The procedure we follow next is similar to Wald's recalled before. We again take the basic vectors orthonormal. The sectional curvature of π in p is given by

$$K(p,\pi)\mid_p = R_{\alpha\beta\gamma\mu}u^\alpha v^\beta v^\gamma u^\mu \mid_p \ .$$

For a small Δx, the change δ_1 in $K(p,\pi)$ is given by

$$\delta_1 = \left[u^\alpha v^\beta v^\gamma u^\mu X^\sigma \nabla_\sigma R_{\alpha\beta\gamma\mu}\mid_{(\Delta x/2,0)}\right]\ \Delta x \ ,$$

because $\vec{u}$ and $\vec{v}$ are parallely transported along $\gamma(x)$. We find analogous expressions for δ_2, δ_3 and δ_4. The change after transport along the x-curves is given by

$$\delta_1+\delta_3 = \left\{u^\alpha v^\beta v^\gamma u^\mu X^\sigma \nabla_\sigma R_{\alpha\beta\gamma\mu}\mid_{(\Delta x/2,0)} - u^\alpha v^\beta v^\gamma u^\mu X^\sigma \nabla_\sigma R_{\alpha\beta\gamma\mu}\mid_{(\Delta x/2,\Delta y)}\right\}\ \Delta x \ .$$

Because the term between brackets goes to zero as $\Delta y \to 0$ it follows that the change in $K(p,\pi)$ vanishes up to first order. Now we compute the second order change in $K(p,\pi)$. Consider the curve $x=\frac{\Delta x}{2}$ and assume that we perform the parallel transport of u^α, v^α and $X^\sigma\nabla_\sigma R_{\alpha\beta\gamma\mu}$ from $(\Delta x/2,0)$ to $(\Delta x/2,\Delta y)$ along this curve. Because the parallel transport of the basis vectors is path-independent up to first order, we find that up to first order in Δy,

$$\begin{aligned} &u^\alpha v^\beta v^\gamma u^\mu X^\sigma \nabla_\sigma R_{\alpha\beta\gamma\mu}\mid_{(\Delta x/2,\Delta y)} \\ &\quad = u^\alpha v^\beta v^\gamma u^\mu X^\sigma \nabla_\sigma R_{\alpha\beta\gamma\mu}\mid_{(\Delta x/2,0)} + \left[u^\alpha v^\beta v^\gamma u^\mu Y^\tau \nabla_\tau (X^\sigma \nabla_\sigma R_{\alpha\beta\gamma\mu})\mid_{(\Delta x/2,\Delta y/2)}\right]\ \Delta x \ . \end{aligned}$$

Thus, we have up to second order in Δx and Δy,

$$\delta_1+\delta_3 = -\left[u^\alpha v^\beta v^\gamma u^\mu Y^\tau \nabla_\tau (X^\sigma \nabla_\sigma R_{\alpha\beta\gamma\mu})\mid_{(\Delta x/2,\Delta y/2)}\right]\ \Delta x \Delta y \ .$$

The total change in $K(p,\pi)$ is

$$\begin{aligned}\delta K(p,\pi) &= \left[u^\alpha v^\beta v^\gamma u^\mu \{X^\sigma \nabla_\sigma (Y^\tau \nabla_\tau R_{\alpha\beta\gamma\mu}) - Y^\sigma \nabla_\sigma (X^\tau \nabla_\tau R_{\alpha\beta\gamma\mu})\} \,|_{(\Delta x/2,\Delta y/2)}\right] \Delta x \Delta y \\ &= \left[u^\alpha v^\beta v^\gamma u^\mu \{X^\sigma Y^\tau (\nabla_\sigma \nabla_\tau - \nabla_\tau \nabla_\sigma) R_{\alpha\beta\gamma\mu} + [X,Y]^\sigma \nabla_\sigma R_{\alpha\beta\gamma\mu}\} \,|_{(\Delta x/2,\Delta y/2)}\right] \Delta x \Delta y \ .\end{aligned}$$

The commutator $[X,Y]$ vanishes because X and Y are tangent to co-ordinate curves and the first term between brackets can be written as

$$\begin{aligned}(\nabla_\sigma \nabla_\tau - \nabla_\tau \nabla_\sigma) R_{\alpha\beta\gamma\mu} &= 2 R_{\tau\sigma\nu[\alpha} R^\nu{}_{\beta]\gamma\mu} + 2 R_{\tau\sigma\nu[\gamma} R^\nu{}_{\mu]\alpha\beta} \\ &=: (R\cdot R)_{\alpha\beta\gamma\mu\sigma\tau} \ . \end{aligned} \tag{8.7}$$

So, the total change in the sectional cuvature $K(p,\pi)$ up to second order in Δx and Δy can then be written as

$$\delta K(p,\pi) = \left[u^\alpha v^\beta v^\gamma u^\mu (R\cdot R)_{\alpha\beta\gamma\mu\sigma\tau} X^\sigma Y^\tau\right] \Delta x \Delta y \ . \tag{8.8}$$

The (0,6)-tensor $R\cdot R$ resulting from the action of the (1,1)-curvature operator R as a derivation on the (0,4)-curvature tensor R is defined by

$$\begin{aligned}(R\cdot R)(X_1,X_2,X_3,X_4;X,Y) &= (R(X,Y)\cdot R)(X_1,X_2,X_3,X_4) \\ &= -R(R(X,Y)X_1,X_2,X_3,X_4) - R(X_1,R(X,Y)X_2,X_3,X_4) \\ &\quad -R(X_1,X_2,R(X,Y)X_3,X_4) - R(X_1,X_2,X_3,R(X,Y)X_4) \ ,\end{aligned}$$

for arbitrary tangent vector fields X,Y,X_1,X_2,X_3 and X_4. And, in invariant notation, the above then reads as follows. When moving a tangent two-plane π at a point p in a Riemannian manifold $\mathcal{M}$ parallely along an infinitesimal co-ordinate parallelogram $\mathcal{P}_{\alpha\beta}$ from p over q, r and s back to p, in general it does not return to the original *position* π, but gives at p another tangent two-plane $\pi^\star$.

By formula (8.3), the natural comparison of the measures of the sectional curvatures $K(p,\pi)$ and $K(p,\pi^\star)$ after such parallel displacement of π, up to terms of order two in Δx^α and Δx^β, gives:

$$K(p,\pi^\star) = K(p,\pi) - [(R\cdot R)(\vec{u},\vec{v},\vec{v},\vec{w};\partial_\alpha(p),\partial_\beta(p)]\,\Delta x^\alpha \Delta x^\beta \ .$$

Szabó called a space $\mathcal{M}^n$ *semi-symmetric* when

$$R\cdot R = 0 \ ,$$

and he classified the Riemannian semi-symmetric spaces [128, 129]. From the above, *the characterising symmetry property of semi-symmetric spaces is the preservation of sectional curvature under parallel transport around infinitesimal co-ordinate parallelograms.* Hence, as immediately follows from the analytical point of view as well as from the point of view of

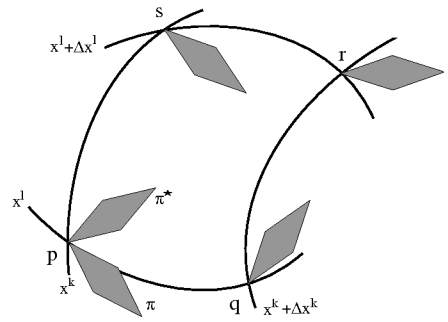

Figure 8.10: Characterisation of semi-symmetric spaces.

parallel translation, all locally symmetrical spaces are also automatically semi-symmetric. Also, we note that every (semi-)Riemannian surface $\mathcal{M}^2$ is automatically semi-symmetric, the plane $\pi^\star$ after parallel transport along a parallelogram as above of course coinciding with the original plane $\pi = T_p\mathcal{M}^2$, or, put otherwise, having $\frac{\partial^2 K}{\partial x \partial y} = \frac{\partial^2 K}{\partial y \partial x}$ whereby K is the Gauss curvature of the surface $\mathcal{M}^2$ and (x, y) are local co-ordinates on $\mathcal{M}^2$. So, the property of being semi-symmetric essentially assumes its meaning only for spaces $\mathcal{M}^n$ of dimension $n > 2$, and there it is a proper extension of local symmetry.

As was recalled above, locally symmetric spaces were characterised by Cartan as those for which the geodesic reflection in every point is a local isometry. In analogy with this, we now give an interpretation of semi-symmetry in terms of geodesics. Let us therefore again take an infinitesimal co-ordinate parallelogram with tangent vector fields X and Y, and in the origin p of the co-ordinate system we take two vectors $\vec{u}$ and $\vec{v}$. After parallel transport of these vectors around the parallelogram we have two, in general distinct, vectors,

$$\begin{aligned} u^{\star\alpha} &= u^\alpha + \left[R^\alpha{}_{\beta\gamma\mu} u^\beta Y^\gamma X^\mu\right] \Delta x \Delta y + \dots \\ v^{\star\alpha} &= v^\alpha + \left[R^\alpha{}_{\beta\gamma\mu} v^\beta Y^\gamma X^\mu\right] \Delta x \Delta y + \dots \end{aligned}$$

with $R^\alpha{}_{\beta\gamma\mu}$ the Riemann curvature tensor evaluated at p. The geodesics through p with these two vectors as tangents are given in normal co-ordinates by

$$x^{\star\alpha}(s) = s u^{\star\alpha} \quad , \quad y^{\star\alpha}(s) = s v^{\star\alpha} \ .$$

We then move a parameter distance a along the first geodesic and a distance b along the second, and construct the geodesic connecting the points $x^{\star\alpha}(a) = a\, u^{\star\alpha}$ and $y^{\star\alpha}(b) = b\, v^{\star\alpha}$. An expansion of this geodesic with respect to an a priori arbitrary parameter t can be written as

$$z^{\star\alpha} = \phi^{\star\alpha} + t\,\psi^{\star\alpha} + \frac{1}{2}t^2\gamma^{\star\alpha} + \dots$$

with boundary conditions $z^{\star\alpha}(-\varepsilon) = a\,u^{\star\alpha}$ and $z^{\star\alpha}(\varepsilon) = b\,v^{\star\alpha}$. Putting this in the geodesic equation,

$$\frac{\mathrm{d}^2 z^{\star\alpha}}{\mathrm{d}t^2} + \Gamma^{\alpha}{}_{\beta\gamma}(z^\star(t))\frac{\mathrm{d}z^{\star\beta}}{\mathrm{d}t}\frac{\mathrm{d}z^{\star\gamma}}{\mathrm{d}t} = \beta\frac{\mathrm{d}z^{\star\alpha}}{\mathrm{d}t} ,$$

and using (8.5), we find

$$\phi^{\star\alpha} = \frac{1}{2}a(1+\frac{1}{2}\beta\varepsilon)u^{\star\alpha} + \frac{1}{2}b(1-\frac{1}{2}\beta\varepsilon)v^{\star\alpha} + \frac{1}{12}ab\,R^{\alpha}{}_{\beta\gamma\mu}u^{\star\gamma}v^{\star\mu}(bv^{\star\beta} - au^{\star\beta}) + \dots$$

and

$$\psi^{\star\alpha} = \frac{1}{2\varepsilon}(bv^{\star\alpha} - au^{\star\alpha}) .$$

The length of this geodesic is given by the formula

$$\int_{-\varepsilon}^{\varepsilon}\sqrt{g_{\alpha\beta}(z^\star(t))\frac{\mathrm{d}z^{\star\alpha}}{\mathrm{d}t}\frac{\mathrm{d}z^{\star\beta}}{\mathrm{d}t}}\mathrm{d}t .$$

If we use the expression (8.4) for the metric, the integrand can be expanded in powers of ε, a and b, Δx and Δy. To lowest order we have

$$\begin{aligned} & g_{\alpha\beta}(z^\star(t))\frac{\mathrm{d}z^{\star\alpha}}{\mathrm{d}t}\frac{\mathrm{d}z^{\star\beta}}{\mathrm{d}t} \\ = \; & \frac{1}{4\varepsilon^2}\left\{a^2u^\alpha u_\alpha + b^2v^\alpha v_\alpha - 2ab\,v^\alpha u_\alpha - \frac{1}{3}R_{\alpha\beta\gamma\mu}u^\alpha v^\beta u^\gamma v^\mu\right\} \\ & -\frac{1}{12\varepsilon^2}\left[(R\cdot R)(\vec{u},\vec{v},\vec{u},\vec{v};X,Y)\right]\,\Delta x\Delta y\,a^2b^2 \\ & + O(\Delta x^2\Delta y^2) + O(a^3b^2) + O(a^2b^3) . \end{aligned}$$

Thus

$$(R\cdot R)(\vec{u},\vec{v},\vec{u},\vec{v};X,Y) = \lim 12\varepsilon^2\,\frac{g_{\alpha\beta}(z(t))\frac{\mathrm{d}z^\alpha}{\mathrm{d}t}\frac{\mathrm{d}z^\beta}{\mathrm{d}t} - g_{\alpha\beta}(z^\star(t))\frac{\mathrm{d}z^{\star\alpha}}{\mathrm{d}t}\frac{\mathrm{d}z^{\star\beta}}{\mathrm{d}t}}{a^2b^2\Delta x\Delta y} ,$$

where the limit is understood for $(a,b)\to(0,0)$, $(\Delta x,\Delta y)\to(0,0)$ and $\varepsilon\to 0$. Hence, somewhat roughly speaking, *semi-symmetry amounts to the preservation of the lengths of the bases of geodesic triangles under parallel translation all around infinitesimal co-ordinate parallelograms.*

By the way, these *semi-symmetry properties* can geometrically equivalently be interpreted as follows in terms of *commuting double-mirror symmetries* in the sense given above. After mirror-imaging in mirrors M_1 (in the middle of "the side p,q"), M_2 (in the middle of "the side q,s") and $\bar{M}_1$ (in the middle of "the side p,s"), $\bar{M}_2$ (in the middle of "the side s,r"), the resulting planes $\pi_{1,2}$ and $\pi_{\bar{1},\bar{2}}$ in general (at the point r opposite to p in the parallelogram $\mathcal{P}$) are different. *Semi-symmetry means* that the sectional curvatures $K(r, \pi_{1,2})$ and $K(r, \pi_{\bar{1},\bar{2}})$ are equal, or, put otherwise, *that the sectional curvatures obtained after double mirror reflections in the sides of infinitesimal parallelograms in both opposite directions from a vertex to the opposite vertex are the same.*

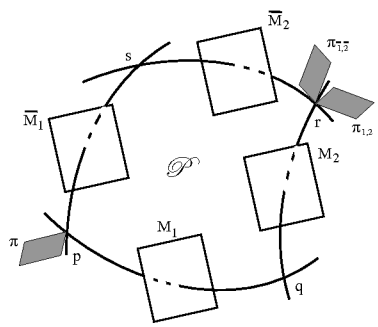

Figure 8.11: Characterisation of semi-symmetric spaces by means of double-mirror symmetries.

8.6 Pseudo-symmetric Spaces

Let $(\mathcal{M}, g)$ be a (semi-)Riemannian manifold with dimension $m \geq 3$. At a point $p \in \mathcal{M}$ we consider a plane π spanned by a basis $\{\vec{u}, \vec{v}\}$ and a surface S through p with co-ordinate curves X and Y. We can define at $p \in U_R = \{x \in \mathcal{M} \mid Q(g, R) \neq 0 \text{ at } x\}$ the scalar

$$L(p, \pi, S) = \frac{(R \cdot R)(\vec{u}, \vec{v}, \vec{u}, \vec{v}; X, Y)}{Q(g, R)(\vec{u}, \vec{v}, \vec{u}, \vec{v}; X, Y)} , \tag{8.9}$$

with $Q(g, R)(\vec{u}, \vec{v}, \vec{u}, \vec{v}; X, Y) = ((X \wedge_g Y) \cdot R)(\vec{u}, \vec{v}, \vec{u}, \vec{v})$ the *Tachibana tensor* given in local components by

$$Q(g, R)_{\alpha\beta\gamma\mu\sigma\tau} = 2g_{\sigma[\beta}R_{\alpha]\tau\gamma\mu} - 2g_{\tau[\beta}R_{\alpha]\sigma\gamma\mu} + 2g_{\sigma[\mu}R_{\gamma]\tau\alpha\beta} - 2g_{\tau[\mu}R_{\gamma]\sigma\alpha\beta} . \tag{8.10}$$

The Tachibana tensor may well be the simplest $(0, 6)$-tensor on a (semi-)Riemannian manifold which has the same algebraic properties as the $(0, 6)$-tensor $R \cdot R$. Remark that $Q(g, R) = 0$ is a necessary and sufficient condition for $\mathcal{M}$ to be of *constant curvature* [65].

The scalar $L(p,\pi,\mathcal{S})$ is independent of the choice of basis for the plane π and the generators of the parallelogram in the surface $\mathcal{S}$. We can give a *geometrical interpretation of this scalar* in terms of geodesics in analogy with the interpretation of the sectional curvature. Consider again two vectors parallely transported along an infinitesimal co-ordinate parallelogram and construct from both sets of vectors the parallelogramoïd of Levi-Civita with the same lengths a and b. The length of the *closing* geodesic of the parallelogramoïd based on $\vec{u}$ and $\vec{v}$ is

$$a'^2 = a^2\left[1 - K(p,\pi)B_\pi^2 b^2\right] ,$$

and the length of the closing geodesic of the parallelogramoïd based on $\vec{u}^\star$ and $\vec{v}^\star$ is

$$a^{\star\prime 2} = a^2\left[1 - K(p,\pi^\star)B_{\pi^\star}^2 b^2\right] .$$

Up to the considered orders in a, b, Δx and Δy we have that $B_\pi = B_{\pi^\star}$. Thus

$$K(p,\pi^\star) - K(p,\pi) = \frac{a'^2 - a^{\star\prime 2}}{a^2 b^2 B_\pi^2} .$$

If we now use (8.8) and the definition of the scalar $L(p,\pi,\mathcal{S})$ in (8.9) we have that, to second order in Δx and Δy,

$$L(p,\pi,\mathcal{S}) = \frac{a'^2 - a^{\star\prime 2}}{(\text{area})^2 Q(g,R)(\vec{u},\vec{v},\vec{u},\vec{v};X,Y)} ,$$

whereby *area* is the area of *the original parallelogramoïd.* Thus, *the scalar $L(p,\pi,\mathcal{S})$ measures the difference in length between the* closing *geodesics of two parallelogramoïds, which are related by parallel transport of their generating vectors around an infinitesimal parallelogram.*

From the above observations we also may conclude the following: *a space $\mathcal{M}^n$ is semi-symmetric if and only if under parallel transport all around infinitesimal co-ordinate parallelograms the lengths of the closing geodesics of Levi-Civita's parallelogramoïds are preserved.* And, also analogously as several previous observations, *a space $\mathcal{M}^n$ is semi-symmetric if and only if the lengths of the closing geodesics of Levi-Civita's parallelogramoïds are the same after commuting mirror-symmetries associated with infinitesimal co-ordinate parallelograms.*

In analogy with spaces of constant curvature one can prove that if $L(p,\pi,S)$ is *isotropic*, i.e., only depends on the point, but not the choice of plane and two-surface, $R\cdot R$ and $Q(g,R)$ are linearly dependent on the set U_R [84]. The manifolds $\mathcal{M}$ for which $R\cdot R = L\,Q(g,R)$ for some $L:\mathcal{M}\to\Re$ are the *pseudo-symmetric manifolds introduced by Deszcz.* Remark that we have no theorem of Schur in this case because there exist many examples of manifolds which are pseudo-symmetric but *not* pseudo-symmetric of *constant type*, i.e., L a constant.

Recalling some of the above in invariant notation now, we mention the following. Similar to the determination of the spaces of constant curvature as a natural extension of the locally

Euclidean spaces, *the pseudo-symmetric spaces of Deszcz* can be seen as a natural extension of the semi-symmetric spaces. Denoting by $X \wedge_g Y$ the $(1,1)$-tensor given by $(X \wedge_g Y)Z = g(Z,Y)X - g(Z,X)Y$. This vector can be geometrically interpreted in a point p as follows: assume $\vec{x}$ and $\vec{y}$ orthonormal and choose $\{\vec{e}_3, \ldots, \vec{e}_n\}$ such that $\{\vec{x}, \vec{y}, \vec{e}_3, \ldots, \vec{e}_n\}$ forms an orthonormal basis of $T_p\mathcal{M}$. A vector $\vec{z} \in T_p\mathcal{M}^n$ can then be decomposed as

$$\vec{z} = g(\vec{x}, \vec{z})\vec{x} + g(\vec{y}, \vec{z})\vec{y} + \sum_{i=3}^{n} g(\vec{e}_i, \vec{z})\vec{e}_i \ .$$

Denote by $\vec{z}^\star$ the vector obtained after an infinitesimal rotation with angle ε of the projection of $\vec{z}$ in the plane $\tilde{\pi}$ spanned by $\{\vec{x}, \vec{y}\}$. This vector reads, up to first order in ε,

$$\vec{z}^\star = \vec{z} + \varepsilon\{g(\vec{y}, \vec{z})\vec{x} - g(\vec{x}, \vec{z})\vec{y}\} + O(\varepsilon^2) \ .$$

Thus, while $R(\vec{x}, \vec{y})\vec{z}$ measures the difference in direction of a vector $\vec{z}$ after a movement of $\vec{z}$ *away* from the point p and around an infinitesimal parallelogram, the vector $(\vec{x} \wedge_g \vec{y})\vec{z}$ measures the difference in direction of $\vec{z}$ after an infinitesimal rotation of the $\{\vec{x}, \vec{y}\}$-projection of $\vec{z}$ *in* the point p. The action of $X \wedge_g Y$ as a derivation on the $(0,4)$-curvature tensor R defines *the* $(0,6)$*-Tachibana tensor* $Q(g,R)$,

$$\begin{aligned}
&(Q(g,R))(X_1, X_2, X_3, X_4; X, Y) \\
&= -((X \wedge_g Y) \cdot R)(X_1, X_2, X_3, X_4) \\
&= R((X \wedge_g Y)X_1, X_2, X_3, X_4) + R(X_1, (X \wedge_g Y)X_2, X_3, X_4) \\
&\quad + R(X_1, X_2, (X \wedge_g Y)X_3, X_4) + R(X_1, X_2, X_3, (X \wedge Y)_g X_4) \ ,
\end{aligned}$$

whose vanishing characterises the spaces of constant curvature; $Q(g,R)$ may well be the simplest $(0,6)$-tensor on a Riemannian manifold which has the same algebraic properties as the $(0,6)$-tensor $R \cdot R$. In light of the previous geometrical interpretation of $(X \wedge_g Y)Z$, the components $(Q(g,R))(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y})$ of the Tachibana tensor can geometrically be seen as follows: consider the vectors $\vec{v}^\star$ and $\vec{w}^\star$ obtained after an infinitesimal rotation of $\vec{v}$ and $\vec{w}$ in the plane $\tilde{\pi}$ spanned by $\{\vec{x}, \vec{y}\}$, i.e.,

$$\begin{aligned}
\vec{v}^\star &= \vec{v} + \varepsilon(\vec{x} \wedge_g \vec{y})\vec{v} + O(\varepsilon^2) \\
\vec{w}^\star &= \vec{w} + \varepsilon(\vec{x} \wedge_g \vec{y})\vec{w} + O(\varepsilon^2) \ .
\end{aligned}$$

Comparing the sectional curvature of the planes π spanned by $\{\vec{v}, \vec{w}\}$ and $\pi^\star$ spanned by $\{\vec{v}^\star, \vec{w}^\star\}$ in p gives, up to first order in ε,

$$K(p, \pi^\star) = K(p, \pi) + \varepsilon(Q(g,R))(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y}) + O(\varepsilon^2) \ .$$

Thus, the components $(Q(g,R))(\vec{v},\vec{w},\vec{w},\vec{v};\vec{x},\vec{y})$ measure the first order difference of the sectional curvature of the plane π after an infinitesimal rotation of this plane in the plane $\tilde{\pi}$. Again the difference with $(R\cdot R)(\vec{v},\vec{w},\vec{w},\vec{v};\vec{x},\vec{y})$ is clear: $(R\cdot R)(\vec{v},\vec{w},\vec{w},\vec{v};\vec{x},\vec{y})$ measures the difference of the sectional curvature of π after the *movement of the plane around an infinitesimal parallelogram* tangent to the plane $\tilde{\pi}$, while $(Q(g,R))(\vec{v},\vec{w},\vec{w},\vec{v};\vec{x},\vec{y})$ measures the first order difference of the sectional curvature of π after an infinitesimal rotation of the $\tilde{\pi}$-projection of the plane π *in the point* p itself.
For all tangent vectors $\vec{v}$, $\vec{w}$, $\vec{x}$, $\vec{y}$ at all points p in $\mathcal{M}$ such that $\vec{v}$ and $\vec{w}$ span a plane π and $\vec{x}$ and $\vec{y}$ span a plane $\tilde{\pi}$, on every Riemannian manifold $\mathcal{M}$ which is not of constant curvature, the ratio

$$L(p,\pi,\tilde{\pi}) = \frac{(R\cdot R)(\vec{v},\vec{w},\vec{w},\vec{v};\vec{x},\vec{y})}{(Q(g,R))(\vec{v},\vec{w},\vec{w},\vec{v};\vec{x},\vec{y})}$$

defines a real number that only depends on the point p and on the planes π and $\tilde{\pi}$. The pseudo-symmetric spaces are the Riemannian manifolds $\mathcal{M}$ for which this number $L(p,\pi,\tilde{\pi})$ at every point p has the same value regardless the planes π and $\tilde{\pi}$, i.e., depends only on the point p, $L(p,\pi,\tilde{\pi}) = L(p)$, or still are the spaces $\mathcal{M}$ for which the ratio of $R\cdot R$ over $Q(g,R)$ is *isotropic*. By the algebraic properties of the tensors $R\cdot R$ and $Q(g,R)$ this is equivalent to the curvature condition $R\cdot R = L\,Q(g,R)$ whereby L is a real-valued function on $\mathcal{M}$.

The *spheres* S^m of radius r in a Euclidean space E^{n+m} are probably the simplest spaces of constant non-zero curvature, with $K = \frac{1}{r^2}$. From the point of view of submanifold theory these spheres are totally umbilical submanifolds whose mean curvature vector field is parallel. Therefore, a general submanifold $\mathcal{M}^m$ in a space $\tilde{\mathcal{M}}^{n+m}$ is called an *extrinsic sphere* of $\tilde{\mathcal{M}}^{n+m}$ if it is totally umbilical with parallel mean curvature vector field. A first indication of the relevance of the pseudo-symmetry condition arose in submanifold theory with the result of Deszcz [43] that the extrinsic spheres $\mathcal{M}^m$ of semi-symmetric spaces $\tilde{\mathcal{M}}^{n+m}$ are pseudo-symmetric. This extends the result of ordinary spheres being of constant curvature in Euclidean space. And similar to the fact that the extrinsic spheres $\mathcal{M}^m$ in spheres S^{n+m} are themselves of constant curvature, we have that the extrinsic spheres $\mathcal{M}^m$ in pseudo-symmetric spaces $\tilde{\mathcal{M}}^{n+m}$ are themselves pseudo-symmetric. From an intrinsic point of view pseudo-symmetry appeared in the study of *geodesic mappings*. For spaces of constant curvature we have that if a (semi-)Riemannian manifold $\mathcal{M}^m$ admits a geodesic mapping onto a (semi-)Riemannian flat space $\tilde{\mathcal{M}}^m$, then $\mathcal{M}^n$ must be a space of constant curvature. Further we have that if a space $\mathcal{M}^m$ admits such a geodesic mapping onto a space of constant curvature, then $\mathcal{M}^m$ must itself have constant curvature. Accordingly, results of Mikesh and Venzi [132] and Defever and Deszcz [34] learn that when a (semi-)Riemannian manifold $\mathcal{M}^m$ admits a geodesic mapping onto a semi-symmetric manifold $\tilde{\mathcal{M}}^m$, then $\mathcal{M}^m$ must be pseudo-symmetric, and when a manifold $\mathcal{M}^m$ admits a geodesic mapping onto a pseudo-symmetric manifold, then $\mathcal{M}^m$ is also pseudo-symmetric. Thus the pseudo-symmetric spaces are also in this sense a natural generalisation of the spaces of constant curvature. The relation of pseudo-symmetry with geodesic mappings was fur-

ther investigated in [47, 48], while further such results in relation with submanifolds were discussed in [42, 133].

Several other conditions of pseudo-symmetry type were introduced during the last decades. We recall that an m-dimensional (semi-)Riemannian manifold ($m \geq 3$) is said to be *pseudo-symmetric* if and only if the relation

$$R \cdot R = L_R \, Q(g, R) \; , \tag{8.11}$$

holds on the set $U_R = \{x \in \mathcal{M} \mid R_{\alpha\beta\gamma\mu} - 2\frac{\tau}{m(m-1)} g_{\alpha[\gamma} g_{\mu]\beta} \neq 0 \text{ at } x\} = \{x \in \mathcal{M} \mid Q(g, R)(x) \neq 0\}$, with L_R a real scalar function on $\mathcal{M}$. Analogously, for $m \geq 4$, $\mathcal{M}^m$ is said to be *Weyl pseudo-symmetric* if and only if the relation

$$R \cdot C = L_C \, Q(g, C) \; , \tag{8.12}$$

holds on the set $U_C = \{x \in \mathcal{M} \mid C \neq 0 \text{ at } x\}$. Evidently, every pseudo-symmetric manifold is Weyl pseudo-symmetric. The converse is not true in general. It is known that (8.11) and (8.12) are equivalent on the subset U_C of every semi-Riemannian manifold with dimension $m \geq 5$. Also, both conditions are equivalent on the subset U_C of every four-dimensional *warped product manifold.* We will show further on that both conditions are also equivalent on the subset U_C of every Lorentzian four-dimensional manifold. We moreover recall that a manifold $\mathcal{M}^m$ of dimension $m \geq 4$ is said to be *Weyl semi-symmetric* if and only if

$$R \cdot C = 0 \; . \tag{8.13}$$

The conditions $R \cdot R = 0$ and $R \cdot C = 0$ are equivalent on the subset U_C of every manifold with dimension $m \geq 5$. We will show further on that in the four-dimensional Lorentzian case this equivalence also holds on the subset U_C. Additionally, a manifold $\mathcal{M}^m$ of dimension $m \geq 4$ is said to have a *pseudo-symmetric Weyl tensor* if and only if the relation

$$C \cdot C = L \, Q(g, C) \tag{8.14}$$

holds on U_C. It is known that every warped product of two 2-dimensional manifolds has a pseudo-symmetric Weyl tensor. We will further on also consider space-times satisfying the condition

$$R \cdot C = L \, Q(S, C) \; , \tag{8.15}$$

on the subset $U = \{x \in \mathcal{M} \mid Q(S, C) \neq 0 \text{ at } x\}$ with S the Ricci tensor. We note that every semi-symmetric manifold, as well as every Weyl-semi-symmetric manifold, satisfies (8.15) trivially. The converse is not true, as follows from our classification of the space-times satisfying (8.15). For other generalisations of Deszcz-pseudo-symmetry and for supplementary information concerning the statements made above, we refer to [6, 35, 36, 43, 50, 51, 52, 80].

In the next section we classify the space-times that satisfy one of the earlier mentioned pseudo-symmetry type conditions. These results were first presented in [45, 80, 79]. The

classification is based on the algebraic properties of the Weyl tensor, known as *the Petrov classification* , and of the Ricci tensor, known as the *Segré type* . We will first give a brief summary of these classifications and of the relationship between the Petrov type and the properties of the sectional curvature function as presented by Thorpe.

8.7 Algebraic Classifications

8.7.1 The Newman-Penrose Formalism

Instead of working with an orthonormal basis it is appropiate here to work in a complex null tetrad (also known as a Dewitt frame) consisting of two real null vectors $\{\vec{l}, \vec{n}\}$ and two complex conjugate null vectors $\{\vec{m}, \bar{\vec{m}}\}$ such that the metric can be written as

$$g_{\alpha\beta} = 2l_{(\alpha}n_{\beta)} - 2m_{(\alpha}\bar{m}_{\beta)} \ ,$$

with $l_\alpha n^\alpha = -m_\alpha \bar{m}^\alpha = 1$. If we denote the tetrad with $\{e_a\} = \{\vec{l}, \vec{n}, \vec{m}, \bar{\vec{m}}\}$ we can use the definition of the connection coefficients,

$$\nabla_b e_a = \Gamma^c{}_{ab} e_c \ ,$$

to define the so called spin coefficients, i.e., 12 independent complex linear combinations of the connection coefficients,

$$\{\kappa, \sigma, \epsilon, \rho, \mu, \nu, \alpha, \beta, \pi, \tau, \gamma, \lambda\} \ .$$

For a physical interpretation of these spin-coefficients see [18, 126]. The projection of the Weyl curvature tensor on the null tetrad gives 5 independent scalars Ψ_i,

$$\begin{aligned}
C_{1212} &= -(\Psi_2 + \bar{\Psi}_2) \ , & C_{1324} &= \Psi_2 \ , \\
C_{1213} &= -\Psi_1 \ , & C_{1334} &= \Psi_1 \ , \\
C_{1223} &= \bar{\Psi}_3 \ , & C_{2323} &= -\bar{\Psi}_4 \ , \\
C_{1234} &= \Psi_2 - \bar{\Psi}_2 \ , & C_{2334} &= \bar{\Psi}_3 \ , \\
C_{1313} &= -\Psi_0 \ , & C_{3434} &= -(\Psi_2 + \bar{\Psi}_2) \ ,
\end{aligned}$$

and the projection of the Ricci tensor gives the 7 scalars $\{\Phi_{ij}, \Lambda\}$,

$$\begin{aligned}
S_{11} &= -2\Phi_{00} \ , & S_{24} &= -2\Phi_{21} \ , \\
S_{12} &= -2\Phi_{11} + 6\Lambda \ , & S_{33} &= -2\Phi_{02} \ , \\
S_{13} &= -2\Phi_{01} \ , & S_{34} &= -2\Phi_{11} - 6\Lambda \ , \\
S_{14} &= -2\Phi_{10} \ , & S_{44} &= -2\Phi_{20} \ , \\
S_{22} &= -2\Phi_{22} \ , & \kappa &= 24\Lambda \ , \\
S_{23} &= -2\Phi_{12} \ .
\end{aligned}$$

If we denote the differential operators as $l^\alpha \nabla_\alpha = D$, $n^\alpha \nabla_\alpha = \Delta$ and $m^\alpha \nabla_\alpha = \delta$, the Ricci identities, Bianchi identities and commutation relations give the complete set of Newman-Penrose equations. The Einstein equations then give a relation between the Ricci scalars Φ_{ij} and Λ, e.g., in vacuum we have $\Phi_{ij} = \Lambda = 0$, $\forall i, j$.

8.7.2 Lorentz Transformations

After a choice of tetrad, be it an orthonormal or null frame, we can subject it to a Lorentz transformation in a point and continuously expand it through the whole of space-time. Corresponding with the 6 parameters of the Lorentz group we have 6 degrees of freedom to rotate a chosen frame. We consider a general Lorentz transformation of the basis vectors $\{\vec{l}, \vec{n}, \vec{m}, \bar{\vec{m}}\}$ made up from the following three classes of rotations:
a) rotations of class I which leave $\vec{l}$ unchanged,

$$\vec{l} \to \vec{l}\ , \quad \vec{n} \to \vec{n} + \bar{a}\vec{m} + a\bar{\vec{m}} + a\bar{a}\vec{l}\ , \quad \vec{m} \to \vec{m} + a\vec{l}\ ,$$

b) rotations of class II which leave $\vec{n}$ unchanged,

$$\vec{n} \to \vec{n}\ , \quad \vec{l} \to \vec{l} + \bar{b}\vec{m} + b\bar{\vec{m}} + b\bar{b}\vec{n}\ , \quad \vec{m} \to \vec{m} + b\vec{n}\ ,$$

c) rotations of class III which leave the directions of $\vec{l}$ and $\vec{n}$ unchanged and which rotate $\vec{m}$ and $\bar{\vec{m}}$ over an angle θ in the $(\vec{m}, \bar{\vec{m}})$-plane,

$$\vec{l} \to B^{-1}\vec{l}\ , \quad \vec{n} \to B\vec{n}\ , \quad \vec{m} \to e^{i\theta}\vec{m}\ .$$

Under these transformations the spin coefficients and Weyl scalars change accordingly.

8.7.3 The Petrov Classification of the Weyl Tensor

We mentioned before that the 5 complex scalars Ψ_i completely determine the Weyl tensor after a null tetrad has been chosen. The values which these scalars obtain depend on the choice of frame and on the 6-parameter group of Lorentz transformations which works on these scalars. Depending on the number of scalars we can make to vanish by an appropriate rotation, we can set up an algebraic classification of the Weyl tensor, the so called Petrov classification.
Let $\Psi_4 \neq 0$ (if $\Psi_4 = 0$ we can with the aid of a type I rotation make it different from zero, as long as the space is not conformally flat). Then, by means of a class II rotation, we can set $\Psi_0 = 0$ if b is a root of the equation

$$\Psi_4 b^4 + 4\Psi_3 b^3 + 6\Psi_2 b^2 + 4\Psi_1 b + \Psi_0 = 0\ .$$

This equation always has four roots and the corresponding new direction $\hat{\vec{l}} = \vec{l} + b\bar{\vec{m}} + \bar{b}\vec{m} + b\bar{b}\vec{n}$ is called a principal null direction of the Weyl tensor. If one or more roots coincide we call $C_{\alpha\beta\gamma\delta}$ algebraically special, otherwise algebraically general.
We then have the following classification of the Weyl tensor:
a) Petrov type I: all four roots are distinct. We can, by choosing the correct rotations, make sure that

$$\Psi_0 = \Psi_4 = 0 \text{ and } \Psi_1 \neq 0\,,\ \Psi_2 \neq 0\,,\ \Psi_3 \neq 0\ .$$

b) Petrov type II: two roots coincide. We have

$$\Psi_0 = \Psi_1 = \Psi_4 = 0 \text{ and } \Psi_2 \neq 0\ ,\ \Psi_3 \neq 0\ .$$

c) Petrov type D: two distinct double roots. We have

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \text{ and } \Psi_2 \neq 0\ .$$

d) Petrov type III: three roots coincide. We have

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0 \text{ and } \Psi_3 \neq 0\ .$$

e) Petrov type N: all four roots coincide. We have

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0 \text{ and } \Psi_4 \neq 0\ .$$

f) Petrov type O: conformally flat space. All Weyl scalars vanish.

The above algebraic classification of the Weyl tensor has a nice geometrical interpretation in the case of Einstein space-times as given by Thorpe [130]. In this case the Petrov classification of the Weyl tensor is equal to the classification of the Riemann-Christoffel curvature tensor. Let $G_+(p)$, respectively $G_-(p)$, be the set of all unoriented non-degenerate spacelike, respectively timelike, two-planes tangent to $\mathcal{M}$ in p, and $G(p) = G_+(p) \cup G_-(p)$. Define the sectional curvature function σ_p on $G(p)$ as $\sigma_p(\pi) = g(\mathcal{R}(\pi), \pi)$ with $g(\mathcal{R}(e_\alpha \wedge e_\beta), e_\gamma \wedge e_\mu) = R_{\alpha\beta\gamma\mu}$ and $\{e_\alpha\}$ an orthonormal basis for $T_p\mathcal{M}$. Then, the Petrov type of R is determined by the number n of spacelike critical points of σ_p at p: for type I, $n = 3$ or ∞, for type II, $n = 1$ and for type III, $n = 0$. Further, Thorpe determined the curvature invariants of each Petrov type from the critical values of the sectional curvature function and the Hessian σ_p at the critical point π.

8.7.4 Classification of the Ricci Tensor

In a Riemmanian space, every second-order symmetric tensor defines a linear mapping which takes a vector $\vec{u}$ into a vector $\vec{v}$. To classify the Ricci tensor $S_{\alpha\beta}$ we consider the eigenvalue equation $S^\alpha{}_\beta u^\beta = \lambda u^\alpha$. If the metric is positive definite, every real symmetric matrix can be diagonalised by a suitable real orthogonal transformation. However, the Lorentz metric $g_{\alpha\beta}$ leads to a more complicated algebraic structure: the elementary divisors can be non-simple and the eigenvalues can be complex. The eigenvalue equation

determines the orders of the elementary divisors belonging to the various eigenvalues. The Segré notation (1884) gives these orders and round brackets indicate that the corresponding eigenvalues coincide [85, 126]. If two eigenvalues are complex conjugates they are symbolised by Z and $\bar{Z}$.

The different cases in the Segré classification are based on the (non-)existence of an invariant two-plane, i.e., a two-dimensional subspace of the tangent space which is mapped into itself by the Ricci tensor $S^{\alpha}{}_{\beta}$. In the table below we give the complete list of the possible types for the Ricci tensor and give a possible physical interpretation of the different types. The eigenvalues are presented in such an order that those whose corresponding eigenvectors are null or timelike appear first and we use a comma to separate these from the eigenvalues with corresponding spacelike eigenvectors.

The Segré types , and the corresponding canonical form of the Ricci tensor, are

A: *Timelike invariant two-plane*

A1: Two real orthogonal eigenvectors exist in this plane
$S_{\alpha\beta} = -\lambda_1 u_\alpha u_\beta + \lambda_2 x_\alpha x_\beta + \lambda_3 y_\alpha y_\beta + \lambda_4 z_\alpha z_\beta$,

A2: No real eigenvectors exist in this plane
$S_{\alpha\beta} = \lambda_1 l_{(\alpha} n_{\beta)} + \lambda_2 (l_\alpha l_\beta - n_\alpha n_\beta) + \lambda_3 x_\alpha x_\beta + \lambda_4 y_\alpha y_\beta$,

A3: One double null real eigenvector exists in this plane
$S_{\alpha\beta} = -2\lambda_1 l_{(\alpha} n_{\beta)} \pm l_\alpha l_\beta + \lambda_2 x_\alpha x_\beta + \lambda_3 y_\alpha y_\beta$.

B: *Null invariant two-plane*

One triple real null eigenvector exists
$S_{\alpha\beta} = \lambda_1 l_{(\alpha} n_{\beta)} + l_{(\alpha} x_{\beta)} + \lambda_2 x_\alpha x_\beta + \lambda_3 y_\alpha y_\beta$.

Segré notation		Physical interpretations
A1	[1,111]	Anisotropic perfect fluid
	[(1,1)11]	
	[1,1(11)]	
	[(1,1)(11)]	Non-null Maxwell field
	[(1,11)1]	
	[1,(111)]	Perfect fluid
	[(1,111)]	Cosmological constant
A2	[$Z\bar{Z}$,11]	
	[$Z\bar{Z}$,(11)]	
A3	[2,11]	
	[(2,1)1]	
	[2,(11)]	
	[(2,11)]	Null Maxwell field, pure radiation
B	[3,1]	
	[(3,1)]	

8.8 Classification of the Pseudo-symmetric Space-times

Let $\{l^\alpha, n^\alpha, m^\alpha, \bar{m}^\alpha\}$ be a general Newman-Penrose null tetrad as in section 8.7.1. We project the pseudo-symmetry condition (8.11) on the general null tetrad and obtain 34 algebraic relations of which only the 16 presented below will be used in the classification:

$$2\Psi_0(\Psi_2 - \Lambda) - 2(\Psi_1)^2 = L\Psi_0 \ , \tag{8.16}$$

$$2\Psi_1\Psi_3 + \Psi_0\Psi_4 - 3\Psi_2(\Psi_2 + 2\Lambda) = 3L\Psi_2 \ , \tag{8.17}$$

$$2\Psi_4(\Psi_2 - \Lambda) - 2(\Psi_3)^2 = L\Psi_4 \ , \tag{8.18}$$

$$\bar{\Psi}_0\Phi_{02} - 2\Phi_{01}\Phi_{10} - \Phi_{00}(\bar{\Psi}_2 - 2\Phi_{11} + 2\Lambda) = L\Phi_{00} \ , \tag{8.19}$$

$$\Phi_{02}(\bar{\Psi}_1 - \Phi_{10}) - \Phi_{01}(\bar{\Psi}_2 + 2\Lambda) + \Phi_{00}\Phi_{12} = L\Phi_{01} \ , \tag{8.20}$$

$$\bar{\Psi}_4\Phi_{00} + 2\Phi_{01}\Phi_{12} - \Phi_{02}(\bar{\Psi}_2 + 2\Phi_{11} + 2\Lambda) = L\Phi_{02} \ , \tag{8.21}$$

$$2\Psi_3\Phi_{01} - 2\Phi_{11}(\Psi_{22} + 2\Lambda) - \Phi_{02}\Phi_{20} + \Phi_{00}\Phi_{22} = 2L\Phi_{11} \ , \tag{8.22}$$

$$\Phi_{01}\Phi_{22} + \Phi_{02}(\Psi_3 - \Phi_{21}) - \Phi_{12}(\Psi_2 + 2\Lambda) = L\Phi_{12} \ , \tag{8.23}$$

$$\Psi_4\Phi_{02} - 2\Phi_{12}\Phi_{21} - \Phi_{22}(\Psi_2 - 2\Phi_{11} + 2\Lambda) = L\Phi_{22} \ . \tag{8.24}$$

$$\Psi_3\Phi_{20} - \Psi_4\Phi_{10} = 0 \ , \tag{8.25}$$

$$\Psi_3\Phi_{21} - \Psi_4\Phi_{11} = 0 \ , \tag{8.26}$$

$$\Psi_3\Phi_{22} - \Psi_4\Phi_{12} = 0 \ , \tag{8.27}$$

$$\Psi_4\Phi_{02} + 2\Psi_3\Phi_{12} - 3\Psi_2\Phi_{22} = 0 \ , \tag{8.28}$$

$$\Psi_0\Phi_{20} + 2\Psi_1\Phi_{10} - 3\Psi_2\Phi_{00} = 0 \ , \tag{8.29}$$

$$\Psi_0\Phi_{22} + 2\Psi_1\Phi_{12} - 3\Psi_2\Phi_{02} = 0 \ , \tag{8.30}$$

and

$$\Psi_0\Phi_{22} - \Psi_1\Phi_{12} + \bar{\Psi}_3\Phi_{01} - \bar{\Psi}_4\Phi_{00} = 0 \ . \tag{8.31}$$

If the above equations are satisfied, the remaining 18 equations are automaticaly fulfilled. The above relations hold for a general null tetrad. However we have seen that for each Petrov type there exists a preferred tetrad, adapted to the principal null directions of the Weyl tensor, such that a maximum number of Weyl scalars vanish.
If the space-time is of Petrov type I, we can find a null tetrad such that $\Psi_0 = \Psi_4 = 0$ and $\Psi_1 \neq 0$, $\Psi_2 \neq 0$ and $\Psi_3 \neq 0$. But from (8.16) and (8.18) we then find a contradiction because $\Psi_1 = \Psi_3 = 0$.
In a Petrov type II space-time we can construct a null tetrad such that $\Psi_0 = \Psi_1 = \Psi_4 = 0$, $\Psi_2 \neq 0$ and $\Psi_3 \neq 0$. Again we find a contradiction with (8.18).
If the space-time is of Petrov type D we can choose l^α and n^α along the two double degenerate null directions such that $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ and $\Psi_2 \neq 0$. Equations (8.16)-(8.24) are satisfied if $\Phi_{0i} = \Phi_{2i} = 0$ and $L = -(\Psi_2 + 2\Lambda)$. Remark that this is only possible if Ψ_2 is real.
In a Petrov type III space-time we can always construct a null tetrad such that $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0$ and $\Psi_3 \neq 0$. Equation (8.18) gives again a contradiction.
If the space-time is of Petrov type N we can find a null tetrad such that $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$ and $\Psi_4 \neq 0$. The pseudo-symmetry relations (8.16)-(8.24) are satisfied if and only if $\Phi_{0i} = \Phi_{1i} = 0$ and $L = -2\Lambda$.
A space-time is conformally flat, i.e., of Petrov type O, if $\Psi_i = 0$, $\forall i = 0, \ldots, 4$. The equations (8.16) to (8.24) reduce to

$$2\Phi_{00}(\Phi_{11} - \Lambda) - 2\Phi_{01}\Phi_{10} = L\Phi_{00} \ , \tag{8.32}$$

$$\Phi_{00}\Phi_{12} - \Phi_{02}\Phi_{10} - 2\Phi_{01}\Lambda = L\Phi_{01} , \tag{8.33}$$

$$2\Phi_{01}\Phi_{12} - 2\Phi_{02}(\Phi_{11} + \Lambda) = L\Phi_{02} , \tag{8.34}$$

$$\Phi_{00}\Phi_{22} - \Phi_{02}\Phi_{20} - 4\Phi_{11}\Lambda = 2L\Phi_{11} , \tag{8.35}$$

$$\Phi_{01}\Phi_{22} - \Phi_{02}\Phi_{21} - 2\Phi_{12}\Lambda = L\Phi_{12} , \tag{8.36}$$

$$2\Phi_{22}(\Phi_{11} - \Lambda) - 2\Phi_{12}\Phi_{21} = L\Phi_{22} . \tag{8.37}$$

We collect the results in the following.

Theorem 88 ([83]) *i) Petrov type I, II and III space-times are not pseudo-symmetric.*
ii) A Petrov type D space-time is pseudo-symmetric iff w.r.t. the principal null tetrad the Ricci tensor has the form $S_{\alpha\beta} = 2S_{12}l_{(\alpha}n_{\beta)} + 2S_{34}m_{(\alpha}\bar{m}_{\beta)}$ *(i.e. the Segré type is* $[(1,1)(11)]$*) and* $L = -(\Psi_2 + 2\Lambda)$ *with* Ψ_2 *real.*
iii) A Petrov type N space-time is pseudo-symmetric iff w.r.t. the principal null tetrad the Ricci tensor has the form $S_{\alpha\beta} = S_{22}l_\alpha l_\beta + S_{12}g_{\alpha\beta}$ *(i.e. the Segré type is* $[2,(11)]$*) and* $L = -2\Lambda$.
iv) A Petrov type O space-time is pseudo-symmetric iff the Ricci tensor satisfies the relations (8.32)-(8.37).

If we take $L = 0$ the above theorem gives a classification of the semi-symmetric space-times (see e.g. [115] for an alternative way of classifying the semi-symmetric space-times).

Theorem 89 ([83]) *i) Petrov type I, II, III space-times are not semi-symmetric.*
ii) A Petrov D space-time is semi-symmetric iff w.r.t. the principal null tetrad the Ricci tensor has the form $S_{\alpha\beta} = 2S_{12}l_{(\alpha}n_{\beta)} + 2S_{34}m_{(\alpha}\bar{m}_{\beta)}$ *(i.e. the Segré type is* $[(1,1)(11)]$*) and* $\Psi_2 = -2\Lambda$ *real.*
iii) A Petrov type N space-time is semi-symmetric iff w.r.t. the principal null tetrad the Ricci tensor has the form $S_{\alpha\beta} = S_{22}l_\alpha l_\beta$ *(i.e. the Segré type is* $[(1,111)]$*).*
iv) A Petrov type O space-time is semi-symmetric iff the Ricci tensor satisfies the relations (8.32)-(8.37) with $L = 0$.

Using the Segré type of the Ricci tensor we have information about the possible matter content of the pseudo-symmetric space-times. We consider only the physically most relevant cases of vacuum, Einstein, perfect fluid and (non)-null electromagnetic Maxwell fields.

Corollary 19 ([83]) *i) Every vacuum Petrov type D space-time with real* Ψ_2 *w.r.t. the principal null tetrad is pseudo-symmetric (e.g. the Schwarzschild and Kantowski-Sachs metrics). Every Petrov type D non-null Maxwell field with real* Ψ_2 *is pseudo-symmetric*

($S_{12} = S_{34}$) and every Petrov type D Einstein space with real Ψ_2 is pseudo-symmetric ($S_{12} = -S_{34}$). Petrov type D perfect fluids and null electromagnetic fields are not pseudo-symmetric.
ii) A Petrov type N Einstein space-time is pseudo-symmetric ($R_{22} = 0$). Because $\Lambda \neq 0$ we have that a Petrov type N vacuum, perfect fluid or (non)-null electromagnetic field are not pseudo-symmetric.
iii) Einstein and perfect fluid (e.g. Robertson-Walker) conformally flat space-times are pseudo-symmetric. In the latter case the proportionality factor equals $L_R = -\frac{1}{3}\rho$, with ρ the energy density. The (non)-null electromagnetic Petrov type O space-times are not pseudo-symmetric.

Because a Petrov type D vacuum space-time is only pseudo-symmetric if Ψ_2 is real this excludes Kinnersley class II.E (for an overview of the different Kinnersley classes see [93] and an application in the theory of Kerr-Schild transformations see [79]). The metrics in the classes I (i.e. the NUT space-times), II.F, III.A and IV.A are all pseudo-symmetric. In the other classes only those with real non-vanishing Weyl scalar are pseudo-symmetric. For example, the Kerr metric, which belongs to Kinnersley class II.A, is not pseudo-symmetric (see [37]).

8.9 Classification of the Weyl Pseudo-symmetric Space-times

In order to classify the Weyl pseudo-symmetric space-times, we project the condition

$$C_{\mu\nu\gamma[\sigma}R^{\gamma}{}_{\tau]\alpha\beta} + C_{\sigma\tau\gamma[\mu}R^{\gamma}{}_{\nu]\alpha\beta} - L_C\{g_{\beta[\mu}C_{\nu]\alpha\sigma\tau} - g_{\alpha[\mu}C_{\nu]\beta\sigma\tau} + g_{\beta[\sigma}C_{\tau]\alpha\mu\nu} - g_{\alpha[\sigma}C_{\tau]\beta\mu\nu}\} = 0 \ ,$$

with L_C a scalar valued function on $\mathcal{U}_C$, on a Newman-Penrose tetrad. We then find a set of algebraic relations between the Ricci and Weyl scalars which, after rotating the frame to the principal tetrad, give the following.

Theorem 90 ([45]) *i) Petrov type I, II and III space-times are not Weyl pseudo-symmetric.*
ii) A Petrov type D space-time is Weyl pseudo-symmetric if and only if with respect to the principal null tetrad the Ricci tensor has the form $S_{\alpha\beta} = 2S_{12}l_{(\alpha}n_{\beta)} + 2S_{34}m_{(\alpha}\bar{m}_{\beta)}$, i.e., the Segré type is $[(1,1)(11)]$, and $L_C = -(\Psi_2 + 2\Lambda)$ with Ψ_2 real.
iii) A Petrov type N space-time is Weyl pseudo-symmetric if and only if with respect to the principal null tetrad the Ricci tensor has the form $S_{\alpha\beta} = S_{22}l_\alpha l_\beta + S_{12}g_{\alpha\beta}$, i.e., the Segré type is $[2,(11)]$, and $L_C = -2\Lambda$.

Remark that the conformally flat space-times are excluded from the definition of Weyl pseudo-symmetry.
If we compare theorem 90 with theorem 88 we find the following.

Theorem 91 ([45]) *A non-conformally flat space-time is pseudo-symmetric if and only if it is Weyl pseudo-symmetric on $\mathcal{U}_R \cap U_C$. We further have that $L_R = L_C$.*

8.10 Classification of Space-times

We project the condition (8.14) on a general NP null tetrad. This gives the following set of algebraic relations between the Weyl scalars:

$$2\Psi_0\Psi_2 - 2(\Psi_1)^2 = L\Psi_0 \ , \tag{8.38}$$

$$\Psi_0\Psi_3 - \Psi_1\Psi_2 = L\Psi_1 \ ,$$

$$2\Psi_1\Psi_3 + \Psi_0\Psi_4 - 3(\Psi_2)^2 = 3L\Psi_2 \ , \tag{8.39}$$

$$\Psi_1\Psi_4 - \Psi_2\Psi_3 = L\Psi_3 \ ,$$

$$2\Psi_2\Psi_4 - 2(\Psi_3)^2 = L\Psi_4 \ . \tag{8.40}$$

We again rotate the NP basis to the principal null tetrad in each case and find the following.

Theorem 92 ([45]) *The only space-times with pseudo-symmetric Weyl tensor are the Petrov type D space-times with real Ψ_2 w.r.t. the principal null tetrad.*

In the derivation of the equations above we nowhere used the fact that $L \neq 0$.

Theorem 93 ([45]) *The only space-times which satisfy $C \cdot C = 0$ are the Petrov type N space-times.*

Remark that the condition (8.14) puts no restriction on the Ricci tensor or equivalently on the type of matter.

8.11 Space-times Satisfying $R \cdot C = L\,Q(S, C)$

We project the condition

$$C_{\alpha\beta\gamma[\tau}R^{\gamma}{}_{\sigma]\mu\nu} + C_{\sigma\tau\gamma[\beta}R^{\gamma}{}_{\alpha]\mu\nu} = L\{S_{\mu[\beta}C_{\alpha]\nu\sigma\tau} + S_{\mu[\tau}C_{\sigma]\nu\alpha\beta} + S_{\nu[\alpha}C_{\beta]\mu\sigma\tau} + S_{\nu[\sigma}C_{\tau]\mu\alpha\beta}\} \ ,$$

on a general Newman-Penrose null tetrad and find a set of 48 algebraic relations between the Ricci and Weyl scalars. In the classification we only need the results of the following 20 equations:

$$\Psi_1\Phi_{22} + \bar{\Psi}_1\bar{\Psi}_4 - \Psi_3\Phi_{02} - \bar{\Psi}_3(\bar{\Psi}_2 + 2\Lambda) = 2L\{\Psi_1\Phi_{22} + (\Phi_{11} - 3\Lambda)\bar{\Psi}_3 - \Phi_{12}(\Psi_2 + \bar{\Psi}_2)\} \ ,$$

$$\Psi_0(\Psi_3+\Phi_{21})+2\Psi_1\Phi_{11}-3\Psi_2\Phi_{01}+\bar{\Psi}_1\Phi_{02}-(\Psi_3+\bar{\Psi}_3)\Phi_{00}=0\ ,$$

$$\Psi_0(\Psi_2-\Lambda)-\Psi_1^2=L\{(\Phi_{11}-3\Lambda)\Psi_0-\Phi_{01}\Psi_1\}\ ,$$

$$3\Psi_2\Phi_{00}-2\Psi_1\Phi_{10}-\Psi_0\Phi_{20}=2L\{\Phi_{00}(2\Psi_2-\bar{\Psi}_2)+\Phi_{01}\bar{\Psi}_1-2\Phi_{10}\Psi_1\}\ ,$$

$$-3\Psi_2\Phi_{02}+2\Psi_1\Phi_{12}+\Psi_0\Phi_{22}=2L\{\Phi_{22}\Psi_0+\Phi_{01}\bar{\Psi}_3-\Phi_{02}(\Psi_2+\bar{\Psi}_2)\}\ ,$$

$$3\bar{\Psi}_2\Phi_{02}-2\bar{\Psi}_3\Phi_{01}-\bar{\Psi}_4\Phi_{00}=2L\{\Phi_{02}(\Psi_2+\bar{\Psi}_2)-\Phi_{12}\Psi_1-\Phi_{00}\bar{\Psi}_4\}\ ,$$

$$\bar{\Psi}_3^2-\bar{\Psi}_4(\Psi_2-\Lambda)=L\{\Psi_{12}\bar{\Psi}_3-(\Phi_{11}-3\Lambda)\bar{\Psi}_4\}\ ,$$

$$-3\bar{\Psi}_2\Phi_{22}+2\bar{\Psi}_3\Phi_{21}+\bar{\Psi}_4\Phi_{20}=2L\{\Phi_{22}(\Psi_2-2\bar{\Psi}_2)+2\Phi_{21}\bar{\Psi}_3-\Phi_{12}\Psi_3\}\ ,$$

$$\begin{aligned}&\bar{\Psi}_2\bar{\Psi}_3+2\bar{\Psi}_3\Lambda-\bar{\Psi}_1\bar{\Psi}_4\\&\quad=L\{\Phi_{12}(2\Psi_2-\bar{\Psi}_2)-\Phi_{02}\Psi_3+\Phi_{10}\bar{\Psi}_4-\Phi_{22}\Psi_1+6\Lambda\bar{\Psi}_3\}\ ,\end{aligned}$$

$$\Psi_0\Phi_{11}-\Psi_1\Phi_{01}=0\ ,$$

$$L\{\Phi_{22}\Psi_0-\Phi_{12}\Psi_1+\Phi_{01}\bar{\Psi}_3-\Phi_{00}\bar{\Psi}_4\}=0\ ,$$

$$L\{\Phi_{12}\Psi_1-\Phi_{02}(\Psi_2+\bar{\Psi}_2)+\Phi_{01}\bar{\Psi}_3\}=0\ ,$$

$$L\{\Phi_{00}(\Psi_2-2\bar{\Psi}_2)+2\Phi_{01}\bar{\Psi}_1-2\Phi_{10}\Psi_1+\Phi_{20}\Psi_0\}=0\ ,$$

$$L\{\Phi_{12}\bar{\Psi}_1-\Phi_{21}\Psi_1+\Phi_{10}\bar{\Psi}_3+\Phi_{11}(\Psi_2-2\bar{\Psi}_2)\}=0\ ,$$

$$3\Psi_2(\Psi_2+2\Lambda)-2\Psi_1\Psi_3-\Psi_0\Psi_4=18L\,\Lambda\Psi_2\ ,$$

$$\Psi_4(\Psi_2-\Lambda)-\Psi_3^2=-3L\,\Lambda\Psi_4\ ,$$

$$\Phi_{02}\bar{\Psi}_3-\Phi_{01}\bar{\Psi}_4=0\ ,$$

$$\Phi_{21}\bar{\Psi}_4-\Phi_{22}\bar{\Psi}_3=0\ ,$$

$$\Phi_{11}\bar{\Psi}_4 - \Phi_{12}\bar{\Psi}_3 = 0 \ ,$$

and

$$L\{\Phi_{22}(2\Psi_2 - \bar{\Psi}_2) + 2\Phi_{21}\bar{\Psi}_3 - 2\Phi_{12}\Psi_3 - \Phi_{20}\bar{\Psi}_4\} = 0 \ .$$

If these equations are satisfied, the remaining 28 equations are automatically fulfilled.
We first consider the case $L \neq 0$, i.e., the non Weyl semi-symmetric space-times satisfying (8.15). If we specialise the null tetrad to the principal tetrad for each Petrov type we find the following.

Theorem 94 ([45]) *i) Petrov types I, II and III do not satisfy the pseudo-symmetry type condition (8.15).*
ii) A Petrov type D space-time satisfies the condition (8.15) if and only if w.r.t. the principal null tetrad the Ricci tensor has the form

$$S_{\alpha\beta} = 6\Lambda g_{\alpha\beta} \ ,$$

i.e., the Segré type is $[(1,111)]$, *and* $L = \frac{1}{6\Lambda}(\Psi_2 + 2\Lambda) \neq 0$ *with* Ψ_2 *real and* $\Lambda \neq 0$.
iii) A Petrov type N space-time satisfies the condition (8.15) if and only if w.r.t. the principal null tetrad the Ricci tensor has the form

$$S_{\alpha\beta} = 6\Lambda\, g_{\alpha\beta} - 2\Phi_{11} l_\alpha l_\beta \ ,$$

i.e., the Segré type is $[2,(11)]$, *and* $(3L-1)\Lambda = 0$ *with* $\vec{l}$ *is the degenerate null direction with multiplicity four.*

Using the Segré type we can determine the matter content of the space-times that satisfy condition (8.15).

Corollary 20 ([45]) *i) A Petrov type D space-time satisfies condition (8.15) if and only if it is a non-vacuum Einstein space-time.*
ii) Petrov type N space-times which satisfy condition (8.15) are either vacuum, Einstein or are filled with a null electromagnetic field or with pure radiation.

The case $L = 0$ gives a classification of the Weyl semi-symmetric space-times.

Theorem 95 ([45]) *i) The Petrov types I, II and III space-times are not Weyl pseudo-symmetric.*
ii) A Petrov type D space-time is Weyl pseudo-symmetric if and only if w.r.t. the principal null tetrad the Ricci tensor has the form

$$S_{\alpha\beta} = -4\Phi_{11}[l_{(\alpha}n_{\beta)} + m_{(\alpha}\bar{m}_{\beta)}] + 6\Lambda g_{\alpha\beta} \ ,$$

with $\Psi_2 = -2\Lambda \neq 0$.

iii) A Petrov type N space-time is Weyl pseudo-symmetric if and only if w.r.t. the principal null tetrad the Ricci tensor has the from

$$S_{\alpha\beta} = -2\Phi_{22} l_\alpha l_\beta \ ,$$

with $\vec{l}$ the degenerate principal null direction.

8.12 Roter-type Space-times

A four-dimensional manifold $\mathcal{M}$ with Lorentzian metric g is called a *Roter-type space-time* if and only if the Riemann-Christoffel curvature tensor satisfies the condition,

$$R = \phi \frac{1}{2} S \wedge S + \mu g \wedge S + \eta \frac{1}{2} g \wedge g \ , \tag{8.41}$$

with S the Ricci tensor and ϕ, μ and η real functions on $\mathcal{U}_C \cap \mathcal{U}_S$, with $\mathcal{U}_C = \{x \in \mathcal{M} \mid C_x \neq 0\}$ and $\mathcal{U}_S = \{x \in \mathcal{M} \mid S \neq \frac{\kappa}{4} g \text{ at } x\}$. For background on general Roter-type manifolds see [44, 51].
We can write (8.41) with respect to some basis as

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= \phi\{S_{\alpha\delta}S_{\beta\gamma} - S_{\alpha\gamma}S_{\beta\delta}\} \\ &\quad +\mu\{g_{\alpha\delta}S_{\beta\gamma} + g_{\beta\gamma}S_{\alpha\delta} - g_{\alpha\gamma}S_{\beta\delta} - g_{\beta\delta}S_{\alpha\gamma}\} \\ &\quad +\eta\{g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta}\} \ . \end{aligned} \tag{8.42}$$

If we take the components with respect to a Newman-Penrose null basis, we obtain the following set of 13 algebraic relations between the Ricci and Weyl scalars:
1)

$$-\Psi_2 - \bar{\Psi}_2 - 2\Phi_{11} + 2\Lambda = 4\phi\{(\Phi_{11} - 3\Lambda)^2 - \Phi_{00}\Phi_{22}\} - 4\mu(\Phi_{11} - 3\Lambda) + \eta \ ,$$

2)

$$-\Psi_1 - \Phi_{01} = 4\phi\{\Phi_{01}(\Phi_{11} - 3\Lambda) - \Phi_{00}\Phi_{12}\} - 2\mu\Phi_{01} \ ,$$

3)

$$\bar{\Psi}_3 + \Phi_{12} = 4\phi\{\Phi_{01}\Phi_{22} - \Phi_{12}(\Phi_{11} - 3\Lambda)\} + 2\mu\Phi_{12} \ ,$$

4)

$$\Psi_2 - \bar{\Psi}_2 = 4\phi\{\Phi_{10}\Phi_{12} - \Phi_{01}\Phi_{21}\} \ ,$$

5)

$$\Psi_0 = 4\phi\{\Phi_{00}\Phi_{02} - \Phi_{01}\Phi_{01}\} \ ,$$

6)

$$\Phi_{00} = 4\phi\{\Phi_{00}(\Phi_{11} + 3\Lambda) - \Phi_{01}\Phi_{10}\} + 2\mu\Phi_{00} \ ,$$

7)

$$\Phi_{02} = 4\phi\{\Phi_{01}\Phi_{12} - \Phi_{02}(\Phi_{11} - 3\Lambda)\} + 2\mu\Phi_{02} \ ,$$

8)

$$\Psi_2 + 2\Lambda = 4\phi\{\Phi_{10}\Phi_{12} - (\Phi_{11} - 3\Lambda)(\Phi_{11} + 3\Lambda)\} + 12\mu\Lambda + \eta \ ,$$

9)

$$\Psi_1 - \Phi_{01} = 4\phi\{\Phi_{10}\Phi_{02} - \Phi_{01}(\Phi_{11} + 3\Lambda)\} - 2\mu\Phi_{01} \ ,$$

10)

$$\bar{\Psi}_4 = 4\phi\{\Phi_{22}\Phi_{02} - \Phi_{12}\Phi_{12}\} \ ,$$

11)

$$\Phi_{22} = 4\phi\{\Phi_{22}(\Phi_{11} + 3\Lambda) - \Phi_{12}\Phi_{21}\} + 2\mu\Phi_{22} \ ,$$

12)

$$\bar{\Psi}_3 - \Phi_{12} = 4\phi\{\Phi_{02}\Phi_{21} - \Phi_{12}(\Phi_{11} + 3\Lambda)\} - 2\mu\Phi_{12} \ ,$$

13)

$$-\Psi_2 - \bar{\Psi}_2 + 2\Phi_{11} + 2\Lambda = 4\phi\{(\Phi_{11} + 3\Lambda)^2 - \Phi_{02}\Phi_{20}\} + 4\mu(\Phi_{11} + 3\Lambda) + \eta \ .$$

Based on the algebraic classification of the Ricci tensor there exist four canonical forms of S:
A1:

$$S_{\alpha\beta} = \lambda_1 x_\alpha x_\beta + \lambda_2 y_\alpha y_\beta + \lambda_3 z_\alpha z_\beta - \lambda_4 u_\alpha u_\beta \ ,$$

with $\{x, y, z, u\}$ an orthonormal basis of $(\mathcal{M}, g)$.
A2:

$$S_{\alpha\beta} = \lambda_1 x_\alpha x_\beta + \lambda_2 y_\alpha y_\beta + \frac{1}{2}\lambda_3(l_\alpha n_\beta + n_\alpha l_\beta) + \lambda_4(n_\alpha n_\beta - l_\alpha l_\beta) \ .$$

A3:

$$S_{\alpha\beta} = \lambda_1 x_\alpha x_\beta + \lambda_2 y_\alpha y_\beta - \lambda_3(l_\alpha n_\beta + n_\alpha l_\beta) \pm n_\alpha n_\beta \ .$$

B:

$$S_{\alpha\beta} = \lambda_1 x_\alpha x_\beta + \lambda_2 y_\alpha y_\beta + \frac{1}{2}\lambda_3(l_\alpha n_\beta + n_\alpha l_\beta) + \frac{1}{2}(n_\alpha x_\beta + x_\alpha n_\beta) \ .$$

After inserting each canonical form of the Ricci tensor in the above relations which define a Roter-type space-time we have the following theorem:

Theorem 96 ([46]) *A space-time is Roter-type if and only if it is a Petrov type D, Segré type A1, [(11)(1,1)], four-dimensional Lorentzian manifold with*

$$\begin{aligned}
\mu &= \frac{1}{2}\phi(\lambda_1 - \lambda_3) + \frac{1}{2} \ , \\
\eta &= \frac{1}{3}\phi(\lambda_1^2 + \lambda_3^2 - \lambda_1\lambda_3) + \frac{1}{3}(\lambda_1 - \lambda_3) \ , \\
\Psi_0 &= \Psi_1 = \Psi_3 = \Psi_4 = 0 \ , \\
\Psi_2 &= -\frac{1}{6}\phi(\lambda_1 + \lambda_3)^2 \neq 0 \ .
\end{aligned}$$

We have the following observations:
1) The Ricci tensor of a Roter-type space-time is diagonalisable and has one timelike and three spacelike eigenvectors.
2) The Segré type A1, $[(11)(1,1)]$ space-times can be physically interpreted as filled with a non-null Maxwell field. Thus, every Petrov type D non-null Maxwell field is Roter-type. For example, the Reissner-Nordström metric and the charged C-metric are Roter-type. The general non-null Einstein-Maxwell type D metric (with non-null double aligned electromagnetic field) is known (see (21.11) in [126]).
3) There exists the theorem that every Roter-type manifold is pseudo-symmetric [49]. Looking at the classification theorem of the pseudo-symmetric space-times it follows that the converse is not true. For example, every Petrov type N pseudo-symmetric space-time with Segré type A3, $[(11),2]$, is not Roter-type. These spaces (which are quasi-Einstein) can be considered as filled with a null electromagnetic field (or pure radiation) and a cosmological constant.

Differential Geometry and Topology, Discrete and Computational Geometry
M. Boucetta and J.-M. Morvan (Eds.)
IOS Press, 2005
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Chapter 9

Extrinsic Symmetries of Parallel Transport

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Let $(\mathcal{M}^n, g)$ and $(\tilde{\mathcal{M}}^{n+m}, \tilde{g})$ be two (semi-)Riemannian manifolds with dimension n and $n+m$ and with respective Levi-Civita connections ∇ and $\tilde{\nabla}$. Let $f : (\mathcal{M}, g) \to (\tilde{\mathcal{M}}, \tilde{g})$ be an isometric immersion, i.e., let $\mathcal{M}^n$ be a submanifold of $\tilde{\mathcal{M}}^{n+m}$. We can decompose the covariant derivative in $\tilde{\mathcal{M}}$ of two tangent vector fields X and Y to $\mathcal{M}$ into its tangential and normal part as follows,

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \ , \tag{9.1}$$

whereby $h(X, Y)$ is normal to $\mathcal{M}$ and h is called the *second fundamental form.* Equation (9.1) is called the Gauss formula. We can further define the *normal connection* $\nabla^\perp$ through the decomposition of the covariant derivative in $\tilde{\mathcal{M}}$ of a normal vector field ξ of $\mathcal{M}$ in $\tilde{\mathcal{M}}$ with respect to a tangent vector field X of $\mathcal{M}$ into its tangential and normal parts:

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi \ , \tag{9.2}$$

whereby A is called the *shape operator* with respect to ξ, which is related to the second fundamental form by

$$g(A_\xi X, Y) = \tilde{g}(h(X, Y), \xi) \ .$$

(9.2) is called *the Weingarten formula.* The *Bortolotti - Van der Waerden connection* of $\mathcal{M}$ in $\tilde{\mathcal{M}}$ is given by

$$(\bar{\nabla}h)(X,Y,Z) = \nabla^{\perp}_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \ ,$$

with X, Y, Z tangent vector fields to $\mathcal{M}$.

Like the intrinsic characterisations of a manifold based on the properties of the Riemann curvature tensor from part 8.1, we can now consider analogous extrinsic characterisations based for instance on the properties of the second fundamental form and symmetries of parallel transport. More details of what follows will appear in [55].

Let γ be a curve in $\mathcal{M}$, $\vec{v}$ tangent to $\mathcal{M}$ in $p = \gamma(0)$. We can then parallely propagate $\vec{v}$ along γ to $q = \gamma(t_0)$ using both connections ∇ and $\tilde{\nabla}$.

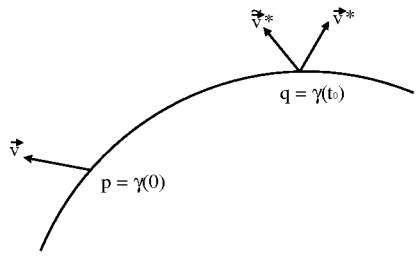

Figure 9.1: Totally geodesic submanifolds.

First using ∇ we find

$$\begin{aligned} v^{\star\alpha}(q) &= v^{\alpha}(p) + (\partial_\beta v^\alpha)_p \Delta x^\beta + \dots \\ &= v^{\alpha}(p) - (\Gamma^\alpha_{\beta\mu} v^\mu)_p \Delta x^\beta + \dots \end{aligned}$$

because $\nabla_\beta v^\alpha = \partial_\beta v^\alpha + \Gamma^\alpha_{\beta\mu} v^\mu = 0$ due to parallel transport. We have

$$\begin{aligned} \vec{v}^\star(q) &= \vec{v}(p) - \left[(v^\mu \Gamma^\alpha_{\beta\mu} \partial_\alpha)_p\right] \Delta x^\beta + \dots \\ &= \vec{v}(p) - \left[v^\mu(p)(\nabla_{\partial_\beta}\partial_\mu)_p\right] \Delta x^\beta + \dots \end{aligned}$$

Analogously, using $\tilde{\nabla}$, we have

$$\tilde{\vec{v}}^\star(q) = \vec{v}(p) - \left[v^\mu(p)(\tilde{\nabla}_{\partial_\beta}\partial_\mu)_p\right] \Delta x^\beta + \dots$$

Using the Gauss formula we have

$$\vec{v}^\star - \tilde{\vec{v}}^\star = [h(\partial_\beta, \vec{v})] \Delta x^\beta + \dots$$

Thus, the second fundamental form gives the first order difference of a vector after parallel transport with both connections along a curve in the submanifold. A submanifold being called *totally geodesic* when $h = 0$, we conclude that *a submanifold $\mathcal{M}$ in $\tilde{\mathcal{M}}$ is totally geodesic if and only if the parallel transports of tangent vector fields to $\mathcal{M}$ with ∇ and $\tilde{\nabla}$ are equivalent.*

A point p of $\mathcal{M}$ is called *umbilic* is there exists a normal vector $\vec{\xi}$ such that

$$h(\vec{u}, \vec{v}) = g(\vec{u}, \vec{v})\vec{\xi}\,,$$

for all $\vec{u}, \vec{v} \in T_p\mathcal{M}$. A semi-Riemannian submanifold $\mathcal{M}$ of $\tilde{\mathcal{M}}$ is called *totally umbilical* if every point of $\mathcal{M}$ is umbilic. Then accordingly there exists a smooth normal vector field ξ on $\mathcal{M}$, called the *mean curvature vector field* H or also called *the Bompiani vector field* of the submanifold $\mathcal{M}$ in $\tilde{\mathcal{M}}$, namely $\vec{H} = \frac{1}{n}\mathrm{tr}h$. For example, a hypersphere $S^n(r)$ in a Euclidean space E^{n+1} is totally umbilic with $\xi = -N/r$ where N is the outward unit normal.

Given a curve γ in $\mathcal{M}$ and two vectors $\vec{u}$ and $\vec{v}$ at $p = \gamma(0)$. In the normal space of $\mathcal{M}$, $N(\mathcal{M})$, at the point p, we have the vector $h(\vec{u}, \vec{v})$. At the point $q = \gamma(t_0)$, we can consider two normal vectors: the parallel translate of $h(\vec{u}, \vec{v})$ by $\nabla^{\perp}$ and the vector $h(\vec{u}^{\star}, \vec{v}^{\star})$ obtained after parallel translating $\vec{u}$ and $\vec{v}$ by ∇.

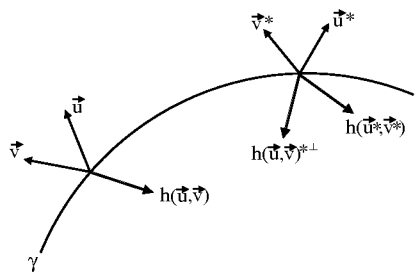

Figure 9.2: Parallel submanifolds.

Using the expression of the parallel transport of a vector along a curve, i.e.,

$$\vec{v}^{\star} = \vec{v} - \nabla_{\gamma'}\vec{v} + \ldots$$

it follows that

$$\begin{aligned} h(\vec{u}^{\star}, \vec{v}^{\star}) - h(\vec{u}, \vec{v})^{\star\perp} &= \nabla^{\perp}_{\gamma'}(\vec{u}, \vec{v}) - h(\nabla_{\gamma'}\vec{u}, \vec{v}) - h(\vec{u}, \nabla_{\gamma'}\vec{v}) + \ldots \\ &=: (\tilde{\nabla}h)(\gamma', \vec{u}, \vec{v}) + \ldots \end{aligned}$$

A submanifold is called *parallel* or *extrinsic symmetric* when $\bar{\nabla} h = 0$. For example, the only surfaces in E^3 which are parallel are the planes, spheres and cylinders [66, 104].

Given a co-ordinate parallelogram with tangents X and Y in $\mathcal{M}$, the parallel translate of a tangent vector $\vec{v}$ to $\mathcal{M}$ with ∇ is given by

$$\vec{v}^\star = \vec{v} - R(\partial_\alpha, \partial_\beta)\vec{v}\Delta x^\alpha \Delta x^\beta + \dots$$

while the parallel translate of a normal vector $\vec{\xi}$ with $\nabla^\perp$ is given by

$$\vec{\xi}^\star = \vec{\xi} - R^\perp(\partial_\alpha, \partial_\beta)\vec{\xi}\Delta x^\alpha \Delta x^\beta + \dots$$

whereby the *normal curvature tensor* $R^\perp$ is defined by

$$R^\perp(\vec{u}, \vec{v})\vec{\xi} = (\nabla^\perp_{\vec{u}}\nabla^\perp_{\vec{v}} - \nabla^\perp_{\vec{v}}\nabla^\perp_{\vec{u}} - \nabla^\perp_{[\vec{u},\vec{v}]})\vec{\xi}\,.$$

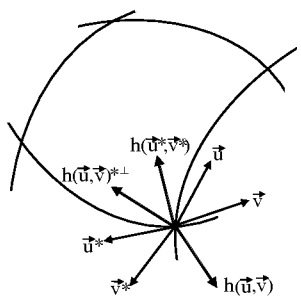

Figure 9.3: Semi-parallel submanifolds.

We can then consider the second fundamental form after parallel transport of $\vec{u}$ and $\vec{v}$ around the parallelogram,

$$h(\vec{u}^\star, \vec{v}^\star) = h(\vec{u}, \vec{v}) - h(R(\partial_\alpha, \partial_\beta)\vec{u}, \vec{v})\Delta x^\alpha \Delta x^\beta - h(\vec{u}, R(\partial_\alpha, \partial_\beta)\vec{v})\Delta x^\alpha \Delta x^\beta + \dots$$

and the parallel translate of the second fundamental form around the parallelogram with $\nabla^\perp$,

$$h(\vec{u}, \vec{v})^{\star\perp} = h(\vec{u}, \vec{v}) - R^\perp(\partial_\alpha, \partial_\beta)h(\vec{u}, \vec{v})\Delta x^\alpha \Delta x^\beta + \dots$$

Substracting both expressions gives

$$h(\vec{u}^\star, \vec{v}^\star) - h(\vec{u}, \vec{v})^{\star\perp} = (\bar{R} \cdot h)(\partial_\alpha, \partial_\beta; \vec{u}, \vec{v})\Delta x^\alpha \Delta x^\beta + \dots$$

whereby

$$(\bar{R}\cdot h)(X,Y;\vec{u},\vec{v}) = (\bar{R}(X,Y)\cdot h)(\vec{u},\vec{v}) = R^{\perp}(X,Y)h(\vec{u},\vec{v}) - h(R(X,Y)\vec{u},\vec{v}) - h(\vec{u},R(X,Y)\vec{v}).$$

A submanifold is called *semi-parallel* when $\bar{R}\cdot h = 0$, i.e., if and only if the normal vectors $h(\vec{u},\vec{v})^{\star\perp}$ and $h(\vec{u}^{\star},\vec{v}^{\star})$ coincide for all $\vec{u},\vec{v} \in T_p\mathcal{M}$ and for every co-ordinate parallelogram in $\mathcal{M}$. Likely, the most simple normal (0,4)-tensor acting on tangent vectors and having the same symmetry properties as $\bar{R}\cdot h$, is defined as

$$Q(g,h)(X,Y;\vec{u},\vec{v}) := -((X\wedge Y)\cdot h)(\vec{u},\vec{v}) = h((X\wedge Y)\vec{u},\vec{v}) + h(\vec{u},(X\wedge Y)\vec{v}) \ .$$

A submanifold is called *pseudo-parallel* when it satisfies the condition

$$\bar{R}\cdot h = L\,Q(g,h) \ ,$$

with L a real scalar function [133]. From the Gauss equation it follows that a hypersurface in a space with constant curvature c is pseudo-parallel if and only if the shape operator A has *at most two distinct eigenvalues* λ and μ. If $\lambda \neq \mu$, then $L = \lambda\mu + c$. From the next observation it follows that the concept of pseudo-parallel submanifolds is the extrinsic analogue of a pseudo-symmetric manifold: if $f : \mathcal{M}^n \to \tilde{\mathcal{M}}^{n+m}(c)$ is a L-pseudo-parallel immersion, then $\mathcal{M}^n$ is L-pseudo-symmetric [3, 133].

The following diagram illustrates some of the basic analogues between intrinsic and extrinsic notions occurring in parts 8.1 and 9.

Intrinsic		**Extrinsic**	
Flat space	$R = 0$	Totally geodesic	$h = 0$
Space of constant curvature	$R = c\,G$	Totally umbilical	$h = g\,\xi$
Locally symmetric	$\nabla R = 0$	Parallel	$\bar{\nabla} h = 0$
Semi-symmetric	$R\cdot R = 0$	Semi-parallel	$\bar{R}\cdot h = 0$
Pseudo-symmetric	$R\cdot R = L\,Q(g,R)$	Pseudo-parallel	$\bar{R}\cdot h = L\,Q(g,h)$

Epilogue: Optimal Shapes in Geometry and Nature

Let $\mathcal{M}^n$ be a submanifold of a Euclidean space E^{n+m}. Then from [81] we recall that for every tangent plane section π at any point p of $\mathcal{M}^n$, the sectional curvature $K(p,\pi)$ equals the Gauss curvature at p of the corresponding two-dimensional normal section Σ^2_π of $\mathcal{M}^n$ in E^{n+m}. The two-dimensional normal section Σ^2_π is defined as the intersection of $\mathcal{M}^n$ with the affine $(2+m)$-dimensional subspace of E^{n+m} spanned by π and $T^\perp_p\mathcal{M}^n$, the normal space at p.

The idea of using the extrinsic space as a tool to describe the intrinsic manifold has a long history and goes back to the surface theory of Gauss. After Riemann (1854) introduced his manifolds, one looked for theorems to imbed these manifolds in some higher-dimensional (Euclidean) spaces. Schläfli (1873) conjectured that every n-dimensional Riemannian manifold could be locally and isometrically imbedded in an m-dimensional Euclidean space with $m \leq n(n+1)/2$ [65]. This was later proven by Janet(1926), Cartan(1927) and Burstin(1931), and extended to semi-Riemannian manifolds by Friedman [67]. A lesser known theorem by Campbell and Magaard [14] states that every analytical (semi-)Riemannian space $\mathcal{M}^n(s,t)$ with s the number of spacelike dimensions and t the number of timelike directions ($n = s+t$), can be locally and isometrically imbedded as a hypersurface in a Ricci flat (semi-)Riemannian space $\mathcal{M}^{n+1}(\tilde{s},\tilde{t})$, with $\tilde{s} = s+1, \tilde{t} = t$ or $\tilde{s} = s, \tilde{t} = t+1$. For further generalisations of the Campbell-Magaard theorem to embedding spaces which are Einstein, scalar field sourced or have non-degenerate Ricci tensor, see [2, 30, 31]. In the global case, Nash(1954) showed that every Riemannian manifold can be globally and isometrically imbedded in a Euclidean space with sufficiently high codimension. For further results on the imbedding problem, see [94]. As expressed for instance by Yau [145], a major problem in the study of imbeddings is the lack of *control of the extrinsic quantities of the imbedded manifold in relation to the intrinsic quantities.* Recent work, originated by Chen [22, 23, 24, 25], deals with this problem by obtaining optimal general inequalities between intrinsic and extrinsic curvature invariants. The new intrinsic curvature invariants and their inequalities were extended to the indefinite case in [82], where the four-dimensional Lorentzian manifolds satisfying the equality were studied.

The Chen inequalities generalise Euler's inequality $K \leq H^2$ for surfaces $\mathcal{M}^2$ in E^3, whereby equality holds if and only if $\mathcal{M}^2$ is planar or spherical. Further inequalities, going back to Wintgen, state for general submanifolds $\mathcal{M}^n$ in E^{n+m} that

$$\rho \leq H^2 - \rho^\perp \ ,$$

whereby ρ is the normalised intrinsic scalar curvature, $H = \|\vec{H}\|$ the mean curvature and $\rho^\perp$ the normalised extrinsic scalar curvature. This inequality is known to be true in the case $n = 2$ and arbitrary m [78, 118, 119] and for arbitrary n and $m = 2$ [41], and conjectured to be true in general. In [54] the space-times satisfying this equality were obtained. Generalisations of the above inequality using other normal curvatures based on normal sections of submanifolds and their projections, were obtained in [81].

The optimal inequalities introduced by Chen mentioned above compare some newly

defined intrinsic curvature invariants based on the scalar curvature of the manifold and sectional curvatures of subspaces, the so-called *Chen curvatures*, and the extrinsic squared mean curvature. Equality is satisfied if and only if the mean curvature is minimal, i.e., if and only if the manifold receives the *least amount of tension from the surrounding space.* Therefore it comes as no surprise that among the space-times satisfying the equality studied in [82] is one belonging to the class of Friedmann-Lemaître models, which are generally considered to model our universe in first approximation.

In analogy with the universe as a whole being an optimum of some simple equation, many creatures living in this universe assume their shapes following similar principles. Together with Johan Gielis, we expand on this in our paper [71]: "*Universal natural shapes: from the supereggs of Piet Hein to the cosmic egg of Georges Lemaître*". At this stage, we restrict ourselves in this respect to the following few comments. Concerning many shapes occuring in nature as being imposed by various kinds of optimalisation criteria involving physical, chemical, biological, geological, ... conditions, see amongst many others [19, 11, 33, 69, 70, 91, 137, 143]. And, so very often indeed, such kind of natural conditions amount to various kinds of conditions on curvatures, when expressed in the language of mathematics. In particular, all kinds of "visual" symmetries reflect geometrically in quite simple curvature conditions. As far as we know, above are, in terms of parallel translation, a few small additions made to our understanding of such relations. In some sense, at least qualitatively, such relations seem inherent to the nature of human-made sensations, perceptions and measurements, together with, in the words of Penrose [114]: "the profound interplay between the laws and sensitivities of thought and the workings of the natural world". In this respect, it is also quite natural that the first papers of Schouten and Levi-Civita concernig (pseudo-)parallel transport, to a large extent originated in close connexion with Einstein's new phenomenal concept of the physical space-time. In his 1918-Amsterdam article, Schouten put the theory of relativity even in the title, and he went deeper into the phenomenon of holonomy of directions than Levi-Civita did and in this way came to the modern description of the curvature tensor of Riemann-Christoffel. In his 1917-Palermo article, Levi-Civita geometrically expressed the Riemannian curvature in terms of his parallelogramoïds, of which the importance is clear from the physical dimension which from then onwards was attributed to the curvature in the theory of relativity as was expressed in this paper as follows: " *La teoria della gravitazione di Einstein considera la struttura geometrica dello spazio ambiente come tenuissimamente, ma pur intimamente, dipente dai fenomeni fisici che vi svolqone; a differenza delle teorie classiche, che assumono tutte lo spazio fisico quale dato a priori. Lo suolgimento matematico della grandiosa concezione di Einstein fa intervenire come elemento essenziale la curvatura di una certa varietà a quattro dimensioni e i relativi simboli di Riemann. L'incontro, anzi il maneggio continuativo di tali simboli in questioni di cosi alto interesse generale mi ha condotto a ricercare se non sia possibile ridurre alquanto l'apparato formale che serve abitualmente ad introdurli e a stabilirne il comportamento covariante"* [10]. For basic reading and rereading on geometry and relativity in our opinion, Einstein's [63] and [64] are and will remain so rewarding. It appears that only sometimes a small detail has been or can been added. The above observations concerning the intrinsic symmetries of space-times,

as now interpretable in terms of parallel transport, might be seen as such.

Finally, we'd like to make the following comment, for the technical account of which we refer to [71]. Formally, the deformations performed on the equations of Lamé's super-ellipses or super-ellipsoïds [77, 102, 146] by *Gielis-transformations* to arrive at *Gielis curves or surfaces* are basically the same as the deformations of the Minkowskian theorem of Pythagoras which lead to the metrics of the relativistic space-times. As super-ellipses very well model for instance square bamboo [105], the Gielis curves and surfaces very well model the shapes of honeybee-cells and shells, crystals and galaxies, DNA-molecules and flowers, stems, tissues and pollen of plants, etc. [33, 69, 70, 91]. Amongst Chen's general curvature conditions which optimally link the intrinsic and the extrinsic geometrical structures of submanifolds to each other, in the setting of space-time submanifolds, the simplest such conditions give an inequality between the Λ-*curvatures* (the Lorentz-version of Chen's Riemannian δ-curvatures [54, 82, 134]) of the space-time and the squared mean curvature, i.e., physically speaking the *surface-tension*, of this space-time as a causal-type preserving hypersurface in a 5-dimensional Minkowski space. In geometrical terms, when a submanifold actually realises, at everyone of its points, the equality in such inequality, then it is said to be an *ideal* submanifold. Namely, given its intrinsic geometrical (semi-)Riemannian structure, such Chen-type inequality can be considered as imposing a definite bound (in the Lorentz case from above or from below depending on the character of the mean curvature normal direction $\vec{H}$) to the possible surface-tension which results, in correspondence with the kind of shape that the submanifold takes in the given ambient space. So, roughly speaking, a submanifold is ideal in a given ambient space when it adapts its form in such a way that it experiences the least amount of surface-tension which is at all possible given the intrinsic structure of the submanifold. These shapes therefore perfectly fit into the general schemes which usually are taken into consideration when trying to describe the forms and the patterns that appear in the natural world as we see it. As stated before, therefore it comes as no suprise that Friedmann-Lemaître space-times readily show up as ideal submanifolds in the above context. Hereby, strong further evidence is given for the principle of natural philosophy which was so nicely worded by d'Arcy Thompson in his 1917-book "On Geometry and Form": "..., *things animate and inanimate, we dwellers in the world and this world wherein we dwell are bound alike by physical and mathematical law*".

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Differential Geometry and Topology, Discrete and Computational Geometry
M. Boucetta and J.-M. Morvan (Eds.)
IOS Press, 2005
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Chapter 10

Characteristic Classes

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The aim of this course is to provide an introduction to characteristic classes for post graduate students who had no previous specialised knowledge (nevertheless, the group who faithfully attended the course was quite heteogeneous including beginners and experts). It was of course impossible to give a whorthwhile elementary survey of this very large area of mathematics in six hours'lecturing time. I thus decided to focus on some particular aspects of the theory and to leave a lot out. To choose what to put in and what to leave out I added a (somewhat artificial but) motivating extra goal to these lectures: the construction of the Milnor exotic sphere. The discovery of such a sphere marked a milestone in the history of differential topology and the key tool in this discovery was the Hirzebruch signature theorem which expresses the signature of a manifold as a combination of characteristic classes. The statement of this result will provide us our target and will guide us in our choices.

I also included an appendix on Chern-Weil theory since this theory is needed in the sequel of this book.

In preparing these notes, I have relied on the books of Atiyah, Hatcher and Milnor-Stasheff (see the bibliography).

10.1 Grassmannians

10.1.1 Fiber Bundles Classification

We denote by $G_n(\mathbb{C}^k)$ the Grassmannian of complex n-planes of $\mathbb{C}^k$ and we put

$$G_n(\mathbb{C}^\infty) = \bigcup_{k\in\mathbb{N}} G_n(\mathbb{C}^k).$$

The topology on $G_n(\mathbb{C}^\infty)$ is CW-topology: a subset of $G_n(\mathbb{C}^\infty)$ is open (resp. closed) if and only if its intersection with every $G_n(\mathbb{C}^k)$, $k \in \mathbb{N}$ is open (resp. closed). The *universal bundle* $EU(n)$ is the bundle over $G_n(\mathbb{C}^\infty)$ whose fiber over each $P \in G_n(\mathbb{C}^\infty)$ is the k-plane P itself. It turns out that this bundle is the key object to classify complex vector bundles over a given compact manifold M. We denote by $Vect_n^{\mathbb{C}}(M)$ the set of isomorphism classes of complex bundles of rank n. We have the following :

Fiber bundles classification theorem. – *The map*

$$\begin{array}{rccc} \beta: & [M, G_n(\mathbb{C}^\infty)] & \longrightarrow & Vect_n^{\mathbb{C}}(M) \\ & [f] & \longmapsto & [f^*EU(n)] \end{array}$$

is a bijection.

In this theorem, as usual, the symbol $[X, Y]$ denotes the set of homotopy classes of continuous maps from X into Y. There is a similar theorem for real vector bundles (resp. oriented real vector bundles) whose universal bundle $EO(n)$ is the infinite Grassmannian $G_n(\mathbb{R}^\infty)$ (resp. universal bundle $ESO(n)$ of oriente infinite Grassmannian $G_n^{or}(\mathbb{R}^\infty)$). The proof of this theorem can be divided in four lemmas.

Lemma 1. – *Let $E \to M$ be a complex bundle of rank n. There exists a finite number of sections of E which span the fiber at each point.*

Proof. – Let $\{U_i\}_{i\in I}$ be a finite covering of M where each U_i is a contractible open set and let $\{V_i\}$ be an another open covering such that $\overline{V_i} \subset U_i$[1]. Since $E_{|U_i}$ is trivial, we can find n sections $\sigma_1^i, \cdots, \sigma_n^i$ which define a basis of the fiber over any point of U_i. Let $f_i : M \to \mathbb{R}$ be a map such that $f_i \equiv 1$ on V_i and $f_i \equiv 0$ on $M \setminus U_i$. Then $\{f_i\sigma_1^i, \cdots, f_i\sigma_n^i\}_{i\in I}$ are global sections of E which span the fiber over any point of M. □

Let $\mathcal{S}_N(E)$ be the set of all familly $S = \{s_1, \cdots, s_N\}$ of N sections of E which, at each point, span the fiber and let $\mathcal{S}(E)$ be the union $\cup_N \mathcal{S}_N(E)$. Let $S = \{s_1, \cdots, s_N\} \in \mathcal{S}(E)$ and $V \simeq \mathbb{C}^N$ be the vector space spanned by linear combinations of these sections. On every point $p \in M$, the evaluation map:

$$\begin{array}{rccc} ev_{S,p}: & V & \longrightarrow & E_p \\ & (v_1, \cdots, v_N) & \longmapsto & \sum_{i=1}^N v_i s_i(p) \end{array}$$

[1] The shrinking lemma states that such a covering exists.

is onto and $ker\ ev_{S,p}$ is a codimension n subspace of V. Let us define a map:

$$\begin{array}{ccc} \mathcal{S}(E) & \longrightarrow & C^0(M, G_n(\mathbb{C}^\infty)) \\ S & \longmapsto & f_S, \end{array}$$

by $f_S(p) = (\ker ev_{S,p})^\perp$.

Lemma 2. – *The homotopy class of f_S does not depend on $S \in \mathcal{S}(E)$. It is denoted by $f_E = [f_S]$.*

Proof. – We are going to prove that $f_{S_0} : M \to G_n(\mathbb{C}^{N_0})$ and $f_{S_1} : M \to G_n(\mathbb{C}^{N_1})$ are homotopic in $C^0(M, G_n(\mathbb{C}^{N_0+N_1}))$. We denote by:

$$\begin{array}{rccc} \Psi_i = ev_{S_i} : & M \times \mathbb{C}^{N_i} & \longrightarrow & E \\ & (p, v) & \longmapsto & ev_{S,p}(v) \end{array}$$

with $i = 0, 1$, the evaluation maps and we put:

$$\begin{array}{rccc} \Phi_t : & M \times \mathbb{C}^{N_0} \times \mathbb{C}^{N_1} & \longrightarrow & E \\ & (p, v_0, v_1) & \longmapsto & (1-t)\Psi_0(p, v_0) + t\Psi_1(p, v_1). \end{array}$$

Since Ψ_0 and Ψ_1 are epimorphisms, Φ_t is an epimorphism for any $t \in [0, 1]$. Let:

$$\begin{array}{rccc} g_t : & M & \longrightarrow & G_n(\mathbb{C}^{N_0+N_1}) \\ & p & \longmapsto & (\ker \Phi_t(p))^\perp. \end{array}$$

We have:

$$g_0 = j_0 \circ f_{S_0}, \quad g_1 = T_\sigma \circ j_1 \circ f_{S_1},$$

where $j_i : G_n(\mathbb{C}^{N_i}) \longrightarrow G_n(\mathbb{C}^{N_0+N_1})$, $i = 0, 1$ are the natural inclusions and $T_\sigma : G_n(\mathbb{C}^{N_0+N_1}) \longrightarrow G_n(\mathbb{C}^{N_0+N_1})$ is the map induced by the permutation $\sigma : \mathbb{C}^{N_0} \oplus \mathbb{C}^{N_1} \to \mathbb{C}^{N_1} \oplus \mathbb{C}^{N_0}$. Since $Gl(N_0 + N_1, \mathbb{C})$ is connected, σ is homotopic to the identity and thus T_σ is homotopic to $id_{G_n(\mathbb{C}^{N_0+N_1})}$. The conclusion follows. □

Lemma 3. – *If E_1 and E_2 are isomorphic then $f_{E_1} = f_{E_2}$.*

Proof. – Let $\Phi : E_1 \longrightarrow E_2$ be an isomorphism and f_{S_1} with $S_1 = \{s_1, \cdots, s_{N_1}\}$ an element in the class f_{E_1}. Then $S_2 = \{\Phi \circ s_1, \cdots, \Phi \circ s_{N_1}\} \in \mathcal{S}(E_2)$ and by construction $f_{S_1} = f_{S_2}$. □

Therefore, we can define a map:

$$\begin{array}{rccc} \alpha : & Vect_n^{\mathbb{C}}(M) & \longrightarrow & [M, G_n(\mathbb{C}^\infty)] \\ & [E] & \longmapsto & f_E. \end{array}$$

Lemma 4. – *The map α is the inverse of β.*

Proof. – Let $S \in \mathcal{S}_N(E)$ and $f_S : M \to G_n(\mathbb{C}^N) \subset G_n(\mathbb{C}^\infty)$. There is a commutative diagram:

$$\begin{array}{ccc} f_S^* EU(n) & \longrightarrow & EU(n) \\ \downarrow & & \downarrow \\ M & \xrightarrow{f_S} & G_n(\mathbb{C}^\infty) \end{array}$$

where $f_S^* EU(n) = \{(p, P_P) \in M \times EU(n) \mid f_S(p) = P\}$. Now $f_S(p) = (\ker\ ev_{S,p})^\perp$ with $ev_S : M \times \mathbb{C}^\infty \to E$, so $f_S^* EU(n) = (\ker ev_S)^\perp$. The map ev_S induces a bundle isomorphism between $(\ker ev_S)^\perp$ and E. Finally: $f_S^* EU(n) \simeq E$, which can be written: $\beta \circ \alpha([E]) = [E]$.

Let $f \in C^0(M, G_n(\mathbb{C}^\infty))$, there exists N such that $f \in C^0(M, G_n(\mathbb{C}^N))$, ($M$ is compact). We put: $F = f^* EU(n) = \{(p, P_P) \in M \times EU(n) \mid f(x) = P\}$, the fibre over $p \in M$ is thus $F_p = f(p)$. Let $S \in \mathcal{S}_N(F)$ and $ev_{S,p} : \mathbb{C}^N \to F_p = f(p) \subset \mathbb{C}^N$, by definition: $f_S(p) = (\ker\ ev_S(p))^\perp$. We denote by $\pi_{f,p} : \mathbb{C}^n = f(p) \oplus f(p)^\perp \to f(p)$ the orthogonal projection, we thus have: $f(p) = (\ker\ \pi_{f,p})^\perp$. The argument used in the proof of Lemma 2, can be used again with the epimorphisms $\Psi_0 = ev_S$ and $\Psi_1 = \pi_f$ to obtain that f and f_S are homotopic in $C^0(M, G_n(\mathbb{C}^\infty))$. Since $[f_S] = f_F = f_{f^* EU(n)}$, we thus have: $[f] = f_{f^* EU(n)} = \alpha \circ \beta[f]$. □

Observations. – 1) Let $j : M \longrightarrow \mathbb{R}^n$ (resp. $\mathbb{C}^N$) be an immersion (resp. complex immersion), the Gauss map $G : M \to G_n(\mathbb{R}^N)$ (resp. $G_n(\mathbb{C}^N)$) is a classifying map for the tangent bundle, i. e.

$$G^* EO(n) = TM \quad (\text{ resp. } \ G^* EU(n) = TM).$$

2) Recall that a manifold M^m is said to be *stably parallelizable* if $TM \oplus \epsilon^1$ is trivial. This means that there exists a classifying map g taking its values in $G_m^{or}(\mathbb{R}^{m+1})$. In other words, there exists $g : M \longrightarrow \mathbb{S}^m$ such that $TM = g^* T\mathbb{S}^m$. The Smale-Hirsch theorem of classification of immersions claims that M admits an immersion in $\mathbb{R}^{m+1}$ if and only if M is stably parallelizable.

Complex lines bundles on the sphere. – As an application, let us compute $Vect_1^{\mathbb{C}}(\mathbb{S}^n)$. We have $G_1(\mathbb{C}^\infty) = \mathbb{C}P^\infty$ so:

$$Vect_1^{\mathbb{C}}(\mathbb{S}^n) = [\mathbb{S}^n, \mathbb{C}P^\infty] = \pi_n(\mathbb{C}P^\infty).$$

The long exact sequence of homotopy groups of the fibration $\mathbb{S}^1 \to \mathbb{S}^\infty \to \mathbb{C}P^\infty$ shows that $\pi_i(\mathbb{C}P^\infty) = \pi_i(\mathbb{S}^1)$ for any $i > 0$ (which means that $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$), and therefore:

$$Vect_1^{\mathbb{C}}(\mathbb{S}^n) \simeq \begin{cases} \{0\} & \text{si } n \neq 2, \\ \mathbb{Z} & \text{si } n = 2. \end{cases}$$

10.1.2 Cohomology of Grassmannians

Leray-Hirsch Theorem

Theorem. – *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a vector bundle such that:*
(a) $H^n(F,\mathbb{R})$ is finitely generated,
(b) there exist classes $c_j \in H^{k_j}(E,\mathbb{R})$ whose restrictions i^*c_j define a basis of $H^*(F,\mathbb{R})$.
Then the map:

$$\begin{array}{rccc} \Phi: & H^*(B,\mathbb{R}) \otimes H^*(F,\mathbb{R}) & \longrightarrow & H^*(E,\mathbb{R}) \\ & \sum_{ij} b_i \otimes i^*c_j & \longmapsto & \sum_{ij} p^*b_i \cup c_j \end{array}$$

is an isomorphism.

Therefore, $H^*(E,\mathbb{R})$ is a free $H^*(B,\mathbb{R})$-module with basis c_j, the product being defined by $bc = p^*c \cup c$ with $b \in H^*(B,\mathbb{R})$ and $c \in H^*(E,\mathbb{R})$. The isomorphism Φ preserves the additive and the modular structures on $H^*(B,\mathbb{R})$. On the other hand, it is not a ring isomorphism, the cup-product is not preserved in general.

The *Poincaré's series* of a graduated vector space $A = \bigoplus_i A_i$ is the formal series $p_A(t) = \sum_i a_i t^i$ where a_i is the dimension of A_i, which is assumed to be finite. If A and B are two graduated vector spaces, then the Poincaré's series of the graduated product $A \otimes B$ is given by:

$$p_{A\otimes B} = p_A p_B.$$

In particular, under the assumptions of Leray-Hirsch theorem:

$$p_{H^*(E,\mathbb{R})} = p_{H^*(B,\mathbb{R})} p_{H^*(F,\mathbb{R})}.$$

Cohomology of Flag Manifolds

We put:

$$\begin{array}{rcl} F_n(\mathbb{C}^k) & = & \{(d_1, \cdots, d_n) \text{ of complex lines two-by-two orthogonal}\} \\ & = & \{\text{chains of subspaces } V_1 \subset \cdots \subset V_n \text{ of } \mathbb{C}^n \text{ with } \dim_{\mathbb{C}} V_i = i\}. \end{array}$$

It is easily checked that the $F_n(\mathbb{C}^k)$ are manifolds, let:

$$F_n(\mathbb{C}^\infty) = \bigcup_{k\in N} F_n(\mathbb{C}^k)$$

endowed with the CW-topology. We denote by:

$$\begin{array}{rccc} \pi_i: & F_n(\mathbb{C}^\infty) & \longrightarrow & \mathbb{C}P^\infty \\ & (d_1, \cdots, d_n) & \longmapsto & d_i \end{array}$$

be the i-th projection. Recall that $H^*(\mathbb{C}P^\infty,\mathbb{Z}) = \mathbb{Z}[x]$ with x the generator of $H^*(\mathbb{C}P^\infty,\mathbb{Z})$ such that $< x, [\mathbb{C}P^1] >= 1$. Let $i : \mathbb{Z} \subset \mathbb{R}$ be the natural inclusion and $i^* : H^*(\mathbb{C}P^\infty,\mathbb{Z}) \to$

$H^*(\mathbb{C}P^\infty, \mathbb{R})$ the induced monomorphism, we still denote by x the image i^*x and by $x_i = \pi_i^* x$ the pull-back of a generator x is of $H^2(\mathbb{C}P^\infty, \mathbb{R})$ (the reason for such a precise choice for the generator is to eventually obtain integral classes).

Proposition. –
$H^*(F_n(\mathbb{C}^\infty), \mathbb{R}) = \mathbb{R}\,[x_1, \cdots, x_n]$.

The Poincaré's series of $\mathbb{R}[x_1]$ is $\sum_i t^{2i} = (1-t^2)^{-1}$, since $\mathbb{R}[x_1, \cdots, x_n] = \mathbb{R}[x_1] \otimes \cdots \otimes \mathbb{R}[x_n]$ the Poincaré's series of $H^*(F_n(\mathbb{C}^\infty), \mathbb{R})$ is $p(t) = (1-t^2)^{-n}$.

Proof. – By induction on n. Since $F_1(\mathbb{C}^\infty) = \mathbb{C}P^\infty$ the proposition is true for $n = 1$. To pass from $n-1$ to n, we apply the Leray-Hirsch theorem to the fibration:

$$P(V) \xrightarrow{i} F_n(\mathbb{C}^\infty) \xrightarrow{p} F_{n-1}(\mathbb{C}^\infty).$$

We denote by $(e_1, \cdots)$ an orthonormal frame of the Hilbert space $\mathbb{C}^\infty$, the fibre of p over $(d_1, \cdots d_n) = (Span_{\mathbb{C}}(e_1), \cdots, Span_{\mathbb{C}}(e_n))$ is the infinite projective space $P(V)$ with $V = Span_{\mathbb{C}}(e_n, \cdots)$. The natural inclusion $j : P(V) \to \mathbb{C}P^\infty$ is induced by the shift operator: $s(e_i) = e_{i+n}$ i. e.

$$j(P(V)) = s(\mathbb{C}P^\infty) \subset \mathbb{C}P^\infty.$$

It turns out that $Id_{\mathbb{C}^\infty}$ and s are homotopic among isometries[2]. Indeed, we just have to apply the Gram-Schmidt process to the homotopy $(1-t)Id_{\mathbb{C}^\infty} + ts$. The resulted homotopy gives a retraction of $\mathbb{C}P^\infty$ on $s(\mathbb{C}P^\infty)$. Thus, $P(V)$ and $\mathbb{C}P^\infty$ are homotopically equivalent and we can write this fibration as:

$$\mathbb{C}P^\infty \xrightarrow{i} F_n(\mathbb{C}^\infty) \xrightarrow{p} F_{n-1}(\mathbb{C}^\infty).$$

We then notice that:

1) $H^n(\mathbb{C}P^\infty, \mathbb{R}) = \mathbb{R} < x^n >$ is finitely generated,

2) The composition $\mathbb{C}P^\infty \xrightarrow{i} F_n(\mathbb{C}^\infty) \xrightarrow{\pi_n} \mathbb{C}P^\infty$ is precisely the shift s, so $(\pi_n \circ i)^*$ is the identity in cohomology and $i^* x_n^k = i^*(\pi_n^* x^k) = x^k$ is a basis of $H^{2k}(\mathbb{C}P^\infty, \mathbb{R})$. The Leray-Hirsch theorem gives the result. □

Similar arguments lead to the following proposition:

Proposition. – $H^*(F_n(\mathbb{C}^k), \mathbb{R}) = \mathbb{R} < x_1^{i_1}, \cdots, x_n^{i_n};\ i_j \leq k - j >$.

Therefore the cohomology of $H^*(F_n(\mathbb{C}^k), \mathbb{R})$ is the one of $\mathbb{C}P^{k-1} \times \cdots \times \mathbb{C}P^{k-n}$. Hence the Poincaré's series of $H^*(F_n(\mathbb{C}^k), \mathbb{R})$ has the following expression:

$$(1 + t^2 + \cdots + t^{2(k-n)}) \cdots (1 + t^2 + \cdots + t^{2(k-1)}) = \prod_{i=k-n}^{k} \frac{1 - t^{2i}}{1 - t^2}.$$

[2] That is, linear maps preserving the inner product, they are not necessarily isomorphisms.

Cohomology of the Complex Grassmannian

Proposition. –
$H^*(G_n(\mathbb{C}^\infty),\mathbb{R}) = \mathbb{R}\,[c_1,\cdots,c_n]$ *where* $\deg c_i = 2i$.

Therefore the Poincaré's series of $H^*(G_n(\mathbb{C}^\infty),\mathbb{R})$ is $p(t) = \prod_{i=1}^n (1-t^{2i})^{-1}$.

Proof. – Let:

$$F_n(\mathbb{C}^n) \longrightarrow F_n(\mathbb{C}^\infty) \xrightarrow{p} G_n(\mathbb{C}^\infty)$$

be the obvious fibration. The previous results show that the assumptions of the Leray-Hirsch theorem are fulfilled, thus $H^*(F_n(\mathbb{C}^\infty),\mathbb{R})$ is a free module on $H^*(G_n(\mathbb{C}^\infty),\mathbb{R})$, a basis being given by the set of monomials $\{x_1^{i_1}...x_n^{i_n},\ i_j \leq n-j\}$. We are going to prove that the image of the monomorphism

$$p^* : H^*(G_n(\mathbb{C}^\infty),\mathbb{R}) \longrightarrow H^*(F_n(\mathbb{C}^\infty),\mathbb{R}) = \mathbb{R}[x_1,\cdots,x_n]$$

is the set of symmetric polynomials of $\mathbb{R}[x_1,\cdots,x_n]$, That is, the polynomial ring $\mathbb{R}[\sigma_1,\cdots,\sigma_n]$ where the σ_i are the elementary symmetric polynomials[3]. Putting $c_i = p^*\sigma_i$ we shall obtain the proposition.

Let σ be a permutation of the integers $1,\cdots,n$. It induces a "permutation" $\sigma : F_n(\mathbb{C}^\infty) \to F_n(\mathbb{C}^\infty)$ and σ^* permutes the indeterminates of $H^*(F_n(\mathbb{C}^\infty),\mathbb{R}) = \mathbb{R}[x_1,\cdots,x_n]$. Since $p\circ\sigma = p$, we have $\sigma^*\circ p^* = p^*$ which means that $Im\ p^* \subset \mathbb{R}[\sigma_1,\cdots,\sigma_n]$. We notice that the Poincaré's series of $\mathbb{R}[\sigma_1,\cdots,\sigma_n]$ is $\prod_{i=1}^n (1-t^{2i})^{-1}$ since $deg\ \sigma_i = 2i$. This series is precisely the one of $H^*(G_n(\mathbb{C}^\infty),\mathbb{R})$, and thus, for dimensional reasons: $Im\ p^* = \mathbb{R}[\sigma_1,\cdots,\sigma_n]$. □

Some minor changes in this proof allow to show that:

$$H^*(G_n(\mathbb{C}^\infty),\mathbb{Z}) = \mathbb{Z}\,[c_1,\cdots,c_n] \text{ with } \deg c_i = 2i.$$

Other Grassmannians

Similar approaches lead to the determination of the cohomology groups of the other Grassmannians. We obtain:

$$\begin{array}{lcll}
H^*(G_n(\mathbb{R}^\infty),\mathbb{Z}_2) & = & \mathbb{Z}_2[w_1,\cdots,w_n] & \text{where } \deg w_i = i. \\
H^*(G_n(\mathbb{R}^\infty),\mathbb{R}) & = & \mathbb{R}\,[p_1,\cdots,p_{[n/2]}] & \text{where } \deg p_i = 4i. \\
H^*(G^{or}_{2m+1}(\mathbb{R}^\infty),\mathbb{R}) & = & \mathbb{R}\,[p_1,\cdots,p_m] & \text{where } \deg p_i = 4i. \\
H^*(G^{or}_{2m}(\mathbb{R}^\infty),\mathbb{R}) & = & \mathbb{R}\,[p_1,\cdots,p_{m-1},e] & \text{where } \deg p_i = 4i,\ \deg e = 2m.
\end{array}$$

For these results, we can substitute $\mathbb{R}$ by any integral domain [4] containing $1/2$ (for example $\mathbb{Z}[\frac{1}{2}]$ or $\mathbb{Q}$) but not by $\mathbb{Z}$. In fact, a precise computation of $H^*(G_n(\mathbb{R}^\infty),\mathbb{Z})$ and

[3] As a polynomial, $deg\ \sigma_i = i$ but as a form, $deg\ \sigma_i = 2i$.
[4] A commutative ring with identity and no zero divisors.

$H^*(G_n^{or}(\mathbb{R}^\infty), \mathbb{Z})$ shows that all torsion elements have order 2 and modulo this torsion, the integral cohomology is again a polynomial ring on the generator p_i (and e for $G_{2m}^{or}(\mathbb{R}^\infty)$).

Relations between different classes are the following. If $\pi : G_n^{or}(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty)$ denotes the natural projection, the last three lines means in particular that:

$$\pi^* p_i = p_i.$$

Let $i : G_n(\mathbb{C}^\infty) \longrightarrow G_{2n}^{or}(\mathbb{R}^\infty)$ be the natural inclusion and $\mu^* : H^*(X, \mathbb{R}) \longrightarrow H^*(X, \mathbb{Z}_2)$ the coefficient homomorphism (with $X = G_{2n}^{or}(\mathbb{R}^\infty)$ or $X = G_n(\mathbb{C}^\infty)$), we have:

$$i^* e = c_n, \quad i^* w_{2i} = \mu^* c_i, \quad p_i = \mu^* w_{2i}^2.$$

Let $j : G_n(\mathbb{R}^\infty) \to G_n(\mathbb{C}^\infty)$ be the map $V \mapsto V \otimes \mathbb{C}$, we also have:

$$p_i = (-1)^i j^* c_{2i}.$$

10.2 Characteristic Classes of a Vector Bundle

10.2.1 Definitions and First Properties

Let E be a complex vector bundle of rank n over M, the *k-th Chern class* of E is the class:

$$c_k(E) = f_E^* c_k \in H^{2k}(M, \mathbb{Z})$$

where $f_E : M \to G_n(\mathbb{C}^\infty)$ is the classifying map of E and c_k is the k-th generator of $H^*(G_n(\mathbb{C}^\infty), \mathbb{Z}) = \mathbb{Z}[c_1, ..., c_n]$, (deg $c_k = 2k$). The *total Chern class* is the element:

$$c(E) = 1 + c_1(E) + \cdots + c_n(E) \in H^*(M, \mathbb{Z}).$$

Similarly $H^*(G_n(\mathbb{R}^\infty), \mathbb{Z}_2) = \mathbb{Z}_2[w_1, ..., w_n]$, with deg $w_k = k$ and the *k-th Stiefel-Whitney class* of a real vector bundle of rank n over M is the class:

$$w_k(E) = f_E^* w_k \in H^k(M, \mathbb{Z}_2),$$

where $f_E : M \to G_n(\mathbb{R}^\infty)$ is the classifying map. The *total Stiefel-Whitney class* is the element:

$$w(E) = 1 + w_1(E) + \cdots + w_n(E) \in H^*(M, \mathbb{Z}_2).$$

The *k-th Pontrjagin class* of E is the class:

$$p_k(E) = f_E^* p_k \in H^{4k}(M, \mathbb{Z}).$$

The *total Pontrjagin class* is the element :

$$p(E) = 1 + p_1(E) + \cdots + p_{[n/2]}(E) \in H^*(M, \mathbb{Z}).$$

If E is a oriented real bundle of rank n, its *Euler class* is the class:

$$e(E) = f_E^* e \in H^n(M, \mathbb{Z})$$

where $f_E : M \to G_n^{or}(\mathbb{R}^\infty)$ is the classifying map.

In order to lighten our lecture, we shall be interested in Chern classes, but the following properties are also satisfied *mutatis mutandis* for the other characteristic classes.

Naturality property. – *Let $E \to M$ and $H \to N$ be two complex bundles with the same rank, $\Phi : E \to H$ be a bundle map, i. e. such that for any $x \in M$, the linear map $\Phi_x : E_x \to H_{\varphi(x)}$ is a $\mathbb{C}$-isomorphism (we denote by $\varphi = bs(\Phi) : M \to N$), then: $c_k(E) = \varphi^* c_k(H)$.*

In particular, a diffeomorphism $\varphi : M \to N$ induces a bundle map $d\varphi : TM \to TN$ and therefore $c_k(TM) = \varphi^* c_k(TN)$. More generally, any map φ between M and N induces a bundle map $\varphi^* H \to H$ and thus $c_k(\varphi^* H) = \varphi^* c_k(H)$.

This property also shows the non triviality of the Chern classes. If N is large enough, the natural inclusion $G_n(\mathbb{C}^N) \xrightarrow{i} G_n(\mathbb{C}^\infty)$ induces an isomorphism in cohomology:

$$H^k(G_n(\mathbb{C}^\infty), \mathbb{Z}) \xrightarrow{i^*} H^k(G_n(\mathbb{C}^N), \mathbb{Z}),$$

where $k \leq n$. If $E = i^* EU(n)$ then $c_k(E) = i^* c_k(EU(n)) = i^* c_k$ does not vanish.

The naturality property for the Euler class sets out that if $E \to M$ and $H \to N$ are two oriented real bundles with same rank and $\Phi : E \to H$ is a bundle map which is *orientation preserving*, then $\varphi^* e(H) = e(E)$ where $\varphi = bs(\Phi)$.

Proof. – Let $S = \{s_1, ..., s_N\} \in \mathcal{S}(H)$, for any $1 \leq i \leq N$ we set:

$$\begin{array}{rccl} \sigma_i : & M & \longrightarrow & E \\ & x & \longmapsto & \Phi^{-1}_{\varphi(x)}(s_i(\varphi(x))), \end{array}$$

thus $\Sigma = \{\sigma_1, ..., \sigma_N\}$ belongs to $\mathcal{S}(E)$. We have:

$$\begin{array}{rccl} ev_{S,y} : & (\alpha_1, ..., \alpha_N) & \longmapsto & \sum_{i=1}^N \alpha_i s_i(y) \\ ev_{\Sigma,x} : & (\alpha_1, ..., \alpha_N) & \longmapsto & \sum_{i=1}^N \alpha_i \sigma_i(x) = \Phi^{-1}_{\varphi(x)}(\sum_{i=1}^n \alpha_i s_i(\varphi(x))), \end{array}$$

and so, $ev_{\Sigma,x} = \Phi^{-1}_{\varphi(x)}(ev_{S,\varphi(x)})$. In one hand $f_\Sigma(x) = (\ker ev_{\Sigma,x})^\perp$, and in the other

$$(\alpha_1, ..., \alpha_N) \in \ker ev_{\Sigma,x} \iff (\alpha_1, ..., \alpha_N) \in \ker ev_{S,\varphi(x)}.$$

Thus $f_\Sigma(x) = f_S(\varphi(x))$ i. e. $f_\Sigma^* = \varphi^* f_S^*$, which proves the proposition. $\square$

Proposition. – *Let ϵ^k be a trivial bundle of rank k, then $c_i(E \oplus \epsilon^k) = c_i(E)$.*

Proof. – Let $S = \{s_1, ..., s_N\} \in \mathcal{S}(E)$, since $\epsilon^k \to M$ is trivial there exists k sections $\Sigma = \{\sigma_1, ..., \sigma_k\}$ such that $\Sigma \in \mathcal{S}(\epsilon^k)$. Thus $S' = S \cup \Sigma \in \mathcal{S}(E \oplus \epsilon^k)$. We denote by $i : \mathbb{C}^N \subset \mathbb{C}^{N+k}$ the natural inclusion, we have:

$$\forall x \in M, \quad i(\ker ev_{S,x}) = \ker ev_{S',x}$$

since the sections $\sigma_1, ..., \sigma_k$ never vanish. In particular:

$$\begin{array}{rcl} f_{S'}(x) & = & (\ker ev_{S',x})^{\perp} \\ & = & i(\ker ev_{S,x})^{\perp} \oplus \mathbb{C}^k \\ & = & i(f_S(x)) \oplus \mathbb{C}^k. \end{array}$$

If $j : G_n(\mathbb{C}^N) \to G_{n+k}(\mathbb{C}^n)$ is the natural inclusion, we thus have: $f_{S'} = j \circ f_S$, from which the result follows. □

10.2.2 The Product Formula

There exists a product formula for the Euler class, for the total Chern class, for the total Stiefel-Whitney class and for the total Pontrjagin class. To save place, we admit the product formula for the Euler class. The reader could find a proof in the book of Milnor and Stasheff.

Proposition. – *Let E and H be two oriented real bundles, then:*

$$e(E \oplus H) = e(E)e(H).$$

The following lemma will be useful to prove the product formula for the total Chern class.

Lemma. – *Let E and H be two complex bundles over M of rank n and m. There exists a unique polynomial $p_{n,m}$ with integer coefficients in $n + m$ indeterminates such that:*

$$c(E \oplus H) = p_{n,m}(c_1(E), ..., c_n(E), c_1(H), ..., c_m(H)).$$

Proof. – Let $\pi_1 : G_n(\mathbb{C}^\infty) \times G_m(\mathbb{C}^\infty) \to G_n(\mathbb{C}^\infty)$ and $\pi_2 : G_n(\mathbb{C}^\infty) \times G_m(\mathbb{C}^\infty) \to G_m(\mathbb{C}^\infty)$ be the natural projections, we put: $EU(n)_1 = \pi_1^* EU(n)$ and $EU(m)_2 = \pi_2^* EU(m)$, and we have:

$$EU(n)_1 \oplus EU(m)_2 = EU(n) \times EU(m).$$

On the other hand, the Künneth isomorphism

$$\begin{array}{ccc} H^*(G_n(\mathbb{C}^\infty), \mathbb{Z}) \otimes H^*(G_m(\mathbb{C}^\infty), \mathbb{Z}) & \longrightarrow & H^*(G_n(\mathbb{C}^\infty) \times G_m(\mathbb{C}^\infty), \mathbb{Z}) \\ a \otimes b & \longmapsto & \pi_1^* a \cup \pi_1^* b \end{array}$$

shows that $H^*(G_n(\mathbb{C}^\infty) \times G_m(\mathbb{C}^\infty), \mathbb{Z})$ is generated by the independent generators:

$$\pi_1^* c_i(EU(n)) = c_i(EU(n)_1), \quad \pi_2^* c_i(EU(m)) = c_i(EU(m)_2).$$

Thus, there exists a unique polynomial $p_{n,m}$ such that:

$$c(EU(n)_1 \oplus EU(m)_2) = p_{n,m}(c_1(EU_1(n)), ..., c_N(EU(m)_2)).$$

Using the naturality property, this relation can be pulled-back on $H^*(M, \mathbb{Z})$ *via* the classifying maps. Indeed, let f and h such that:

$$E = f^* EU(n), \quad H = h^* EU(m).$$

There is a commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{f} & G_n(\mathbb{C}^\infty) \\ h \downarrow & \searrow g & \uparrow \pi_1 \\ G_m(\mathbb{C}^\infty) & \xleftarrow{\pi_2} & G_n(\mathbb{C}^\infty) \times G_m(\mathbb{C}^\infty), \end{array}$$

where g denotes the pair (f, h). We thus have:

$$\begin{aligned} c(E \oplus H) &= c(f^* EU(n) \oplus h^* EU(m)) \\ &= c(g^* EU(n)_1 \oplus EU(m)_2) \\ &= g^* c(EU(n)_1 \oplus EU(m)_2) \\ &= p_{n,m}(c_1(EU(n)), ..., c_N(EU(m)_2)). \end{aligned}$$

□

Proposition. – *Let E and H be two complex bundles over the same base, then:*

$$c(E \oplus H) = c(E)c(H).$$

Proof. – By using classifying maps, we just have to show that the relation holds for the universal bundles $EU(n)_1$ and $EU(m)_2$. We shall prove it by induction. If $m+n = 0$ then:

$$c(EU(n)_1) = 1, \quad c(EU(m)_2) = 1, \quad \text{and} \quad c(EU(n)_1 \oplus EU(m)_2) = 1,$$

and so the relation holds. We now suppose that:

$$c(EU(n)_1 \oplus EU(m-1)_2) = c(EU(n)_1)c(EU(m-1)_2).$$

According to the previous lemma:

$$\begin{aligned} c(EU(n)_1 \oplus EU(m-1)_2) &= c(EU(n)_1 \oplus (EU(m-1)_2 \oplus \epsilon^1)) \\ &= p_{n,m}(c_1, ..., c_n, c'_1, ..., c'_{m-1}, 0), \end{aligned}$$

where we have written $c_i = c_i(EU(n)_1)$ and $c'_i = c'_i(EU(m-1)_2)$. According to the induction assumption:

$$p_{n,m}(c_1, ..., c_n, c'_1, ..., c'_{m-1}, 0) = (1 + c_1 + ... + c_n)(1 + c'_1 + ... + c'_{m-1})$$

so:

$$p_{n,m}(c_1, ..., c_n, c'_1, ..., c'_m) = (1+c_1+...+c_n)(1+c'_1+...+c'_{m-1})+c_m R_1(c_1+...+c_n, c'_1, ..., c'_{m-1}).$$

Symmetrically, the relation:

$$c(EU(n-1)_1 \oplus EU(m)_2) = c(EU(n-1)_1)c(EU(m)_2)$$

gives:

$$p_{n,m}(c_1, ..., c_n, c'_1, ..., c'_m) = (1+c_1+...+c_{n-1})(1+c'_1+...+c'_m)+c_n R_2(c_1+...+c_{n-1}, c'_1, ..., c'_m).$$

The comparison between the two relations implies:

$$p_{n,m}(c_1, ..., c_n, c'_1, ..., c'_m) = (1 + c_1 + ... + c_n)(1 + c'_1 + ... + c'_m) + \lambda c_n c_m,$$

where $\lambda \in \mathbb{Z}$. In particular:

$$c_{n+m}(EU(n)_1 \oplus EU(m)_2) = (1 + \lambda)c_n(EU(n)_1)c_m(EU(m)_2).$$

Since $e(E) = c_n(E)$ and $e(E \oplus H) = e(E)e(H)$ we obtain $\lambda = 0$. □

Remark. – For the total Stiefel-Whitney class, the product theorem states that: $w(E \oplus H) = w(E)w(H)$. For the total Pontrjagin class, $p(E \oplus H)$ is congruent to $p(E)p(H)$ modulo elements of order 2, that is: $2p(E \oplus H) = 2p(E)p(H)$.

10.2.3 Applications

Chern Classes of $\mathbb{C}P^n$

Let $\mathbb{C}^* \to \mathbb{C}^{2n+2} \setminus \{0\} \xrightarrow{f} \mathbb{C}P^n$ be the Hopf fibration. The differential of the Hopf fibration at $x \in \mathbb{C}^{2n+2} \setminus \{0\}$ induces a $\mathbb{C}$-isomorphism:

$$df_x : [x]^\perp \longrightarrow T_{[x]}\mathbb{C}P^n$$

where $[x]^\perp = (Span_{\mathbb{C}}(x))^\perp$. The relation $f(\lambda x) = f(x)$, $\lambda \in \mathbb{C}^*$, implies:

$$df_{\lambda x}(\lambda v) = df_x(v).$$

Therefore the map:

$$\begin{array}{ccc} Hom([x], [x]^\perp) & \longrightarrow & T_{[x]}\mathbb{C}P^n \\ L & \longmapsto & df_x(L(x)) \end{array}$$

is a well-defined $\mathbb{C}$-isomorphism. We denote by

$$\gamma^1 = \{([x], v) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} \mid v \in [x]\} \longrightarrow \mathbb{C}P^n$$

the tautological bundle and by

$$\gamma^\perp = \{([x], v) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} \mid (v, x) = 0\}$$

the orthogonal bundle. The previous map provides an isomorphism between $T\mathbb{C}P^n$ and $Hom(\gamma^1, \gamma^\perp)$. Since $\epsilon^1 = Hom(\gamma^1, \gamma^1)$ we have:

$$\begin{aligned} T\mathbb{C}P^n \oplus \epsilon^1 &= Hom(\gamma^1, \gamma^\perp) \oplus Hom(\gamma^1, \gamma^1) \\ &= Hom(\gamma^1, \gamma^\perp \oplus \gamma^1) \\ &= Hom(\gamma^1, \epsilon^{n+1}) \\ &= Hom(\gamma^1, \epsilon^1) \oplus ... \oplus Hom(\gamma^1, \epsilon^1). \end{aligned}$$

Since $Hom(\gamma^1, \epsilon^1)$ is isomorphic to γ^1, we obtain:

$$T\mathbb{C}P^n \oplus \epsilon^1 = \gamma^1 \oplus ... \oplus \gamma^1, \quad (n+1 \text{ times}).$$

thus:

$$\begin{aligned} c(T\mathbb{C}P^n) &= c(T\mathbb{C}P^n \oplus \epsilon^1) \\ &= c(\gamma^1 \oplus ... \oplus \gamma^1) \\ &= c(\gamma^1)^{n+1}. \end{aligned}$$

On the other hand, the natural inclusion $i : \mathbb{C}P^n \subset \mathbb{C}P^\infty$ induces an isomorphism $\mathbb{Z} = H^2(\mathbb{C}P^\infty, \mathbb{Z}) \xrightarrow{i^*} H^2(\mathbb{C}P^n, \mathbb{Z}) = \mathbb{Z}$ and $\gamma^1 = i^*EU(1)$, thus:

$$c_1(\gamma^1) = i^*c_1(EU(1))$$

is a generator of $H^2(\mathbb{C}P^n, \mathbb{Z})$, we denote it by a. We have $c(\gamma^1) = 1 + a$ and

$$c(T\mathbb{C}P^n) = (1 + a)^{n+1}.$$

Similarly, it is easy to show that $w(T\mathbb{R}P^n) = (1 + a)^{n+1}$ where a is a generator of $H^1(T\mathbb{R}P^n, \mathbb{Z}_2) = \mathbb{Z}_2$.

Existence of Immersions

If $f : M \longrightarrow W$ is an immersion, its normal bundle is defined to be $N_fM = f^*TW/TM$, thus we have a splitting:

$$f^*TW = TM \oplus N_fM.$$

By the product formula we have:

$$w(f^*TW) = w(TM)w(N_fM).$$

Thus the Stiefel-Whitney classes provide obstructions to the existence of immersions. For example, if we choose $W = \mathbb{R}^6$, $M = \mathbb{R}P^4$, the above relation gives:

$$1 = w(\mathbb{R}P^4)w(N_f\mathbb{R}P^4)$$

so $w(\mathbb{R}P^4)$ is invertible and its inverse $w(N_f\mathbb{R}P^4)$ is in $H^{*\leq 2}(\mathbb{R}P^4, \mathbb{Z}_2)$. Let

$$w(N_f\mathbb{R}P^4) = 1 + u_1 + u_2$$

with $u_i \in H^i(\mathbb{R}P^4, \mathbb{Z}_2)$. Since $w(\mathbb{R}P^4) = 1 + a + a^4$, we must get:

$$1 = (1 + a + a^4)(1 + u_1 + u_2).$$

It is easily checked that this equation have no solution. Thus, there is no immersion of $\mathbb{R}P^4$ into $\mathbb{R}^6$. (On the other hand, there exists immersions of $\mathbb{R}P^4$ into $\mathbb{R}^7$).

Classification of Vector Bundles

Characteristic classes allow to distinguish isomorphism classes of fiber bundles. We give here some results without proof (there is no assumption on the dimension of the base manifold).

Complex line bundles. They are classified by their first Chern class, precisely, the map $c_1 : Vect_1^{\mathbb{C}}(M) \longrightarrow H^2(M, \mathbb{Z})$ is a bijection.

Complex plane bundles. – Such two fiber bundles E_1 and E_2 are isomorphic if and only if $c_1(E_1) = c_1(E_2)$ and $c_2(E_1) = c_2(E_2)$. In fact, the map $(c_1, c_2) : Vect_2^{\mathbb{C}}(M) \longrightarrow H^2(M, \mathbb{Z}) \times H^4(M, \mathbb{Z})$ is a bijection.

Real line bundles. – They are ranked by the first class of Stiefel-Whitney, precisely, $w_1 : Vect_1^{\mathbb{R}}(M) \longrightarrow H^1(M, \mathbb{Z}_2)$ is a bijection.

Real oriented plane bundles. – The orientation gives a complex structure on each fibre, so we can consider them as complex line bundles. Thus their classification follows from the complex line bundles one. This reflects the isomorphism between $SO(2)$ and $U(1)$.

Real oriented 3-plane bundles. – Such two fiber bundles E_1 and E_2 are isomorphic if and only if $w_2(E_1) = w_2(E_2)$ and $p_1(E_1) = p_1(E_2)$. Moreover $p_1(E_1) = \mathcal{P}(w_2(E_1))$ mod 4 where $\mathcal{P} : H^2(M, \mathbb{Z}_2) \to H^4(M, \mathbb{Z}_4)$ is the Pontrjagin square. Conversely, for any pair (p_1, w_2) such that $p_1 = \mathcal{P}(w_2)$ mod 4, there exists an oriented real 3-plane bundle $E \to M$ such that $p_1(E) = p_1$ and $w_2(E) = w_2$.

Real oriented 4-plane bundles. – Such two fiber bundles E_1 and E_2 are isomorphic if and only if $w_2(E_1) = w_2(E_2)$, $p_1(E_1) = p_1(E_2)$ and $e(E_1) = e(E_2)$.

10.3 The Cobordism Ring Ω_*

10.3.1 Chern Numbers and Pontrjagin Numbers

Definition. – A *partition* I of an integer $n \in \mathbb{N}$ is an ordered family of positive integers $i_1, ..., i_r$ such that $i_1 + ... + i_r = n$.

We denote by $\mathcal{P}(n)$ the partition set of n and by $p(n)$ its cardinal.

Chern Numbers of a Complex Manifold

Let M^n be a compact complex manifold and I be a partition of $n = \dim_{\mathbb{C}} M$, the Ith *Chern number* of M is the integer:

$$c_I(M) =< c_{i_1} \cup ... \cup c_{i_r}, [M] > .$$

Example. – Let $I = \{i_1, ..., i_r\}$, since $c_i(\mathbb{C}P^n) = C^i_{n+1} a^i$, the Ith Chern number of $\mathbb{C}P^n$ is: $c_I(\mathbb{C}P^n) =< c_{i_1} \cup ... \cup c_{i_r}, [\mathbb{C}P^n] >= C^{i_1}_{n+1} ... C^{i_r}_{n+1}$.

If $\dim_{\mathbb{C}} M$=1, there is a unique Chern number which is the first Chern class $c_1(M) =< c_1(TM), [M] >$, or Euler characteristic. If $\dim_{\mathbb{C}} M$=2, there are two Chern numbers which are $c_1^2(M)$ and the Euler characteristic $c_2(M)$. In general, if $\dim_{\mathbb{C}} M = n$, there are $p(n)$ Chern numbers. The first values of $p(n)$ are listed in the following table

n	1	2	3	4	5	6	7	8	9	10	...	100	...	1000
$p(n)$	1	2	3	5	7	11	15	22	30	42	...	190569292	...	$\sim 2.10^{31}$

The number $p(n)$ grows quickly with n, a well-known result asserts that:

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}}.$$

Note that $H^{2n}(G_n(\mathbb{C}^\infty), \mathbb{Z})$ is a free Abelian group of rank $p(n)$ on $\mathbb{Z}$ which is generated by the products $c_{i_1} \cup ... \cup c_{i_r}$, consequently, if $f : M \longrightarrow G_n(\mathbb{C}^\infty)$ is the classifying map of TM, the $c_I(M)$ determine the image $f_*[M]$ of the fundamental class in $H^{2n}(G_n(\mathbb{C}^\infty), \mathbb{Z})$.

Pontrjagin Numbers of an Oriented Manifold

Let M^n be an oriented compact manifold of real dimension $n = 4m$ and I a partition of m, the Ith*Pontrjagin number* of M is the integer:

$$p_I(M) =< p_{i_1} \cup ... \cup p_{i_r}, [M] > .$$

Example. – Let $I = \{i_1, ..., i_r\}$, the Ith Pontrjagin numbers of $\mathbb{C}P^{2n}$ are: $p_I(\mathbb{C}P^{2n}) =< p_{i_1} \cup ... \cup p_{i_r}, [\mathbb{C}P^{2n}] >= C^{i_1}_{2n+1} ... C^{i_r}_{2n+1}$.

Convention. – To avoid any confusion, we denote by $p_I(TM)$ the class $p_{i_1}(TM) \cup ... \cup p_{i_r}(TM)$ and by $p_I(M)$ the number $< p_I(TM), [M] >$.

Proposition. – *If M^{4m} admits an orientation reversing diffeomorphism then $p_I(M) = 0$ for all partitions I of m.*

Example. – According to the above example, the complex projective space $\mathbb{C}P^{2n}$ does not admit any orientation reversing diffeomorphism. Complex conjugation in $\mathbb{C}^{N+1} \setminus \{0\}$ induces a diffeomorphism of $\mathbb{C}P^N$ but this diffeomorphism is orientation reversing if and only if N is odd.

Proof. – Let $\varphi : M \longrightarrow M$ be an orientation reversing diffeomorphism, i. e. $\varphi_*[M] = -[M]$. By naturality, $p_i(TM) = \varphi^* p_i(TM)$, thus:

$$\begin{aligned} < p_I(TM), [M] > &= < \varphi^* p_I(TM), [M] > \\ &= < p_I(TM), \varphi_*[M] > \\ &= < p_I(TM), -[M] > \end{aligned}$$

so: $< p_I(TM), [M] >= 0$. □

Remark. – This proposition does not generalize to the Euler characteristic. For example, on $\mathbb{S}^2 = \mathbb{C}P^1$ the equatorial reflection is orientation reversing, yet $< e(T\mathbb{S}^2), [\mathbb{S}^2] >= 2$. The previous proof cannot be applied to e since the naturality property of the Euler class holds for maps which preserve the orientation of the fibers.

Proposition. – *If M^{4m} is the boundary of an oriented compact manifold, then $p_I(M) = 0$ for any partition I of m.*

Proof. – Let M be a boundary of an oriented compact manifold W and let $i : M = \partial W \longrightarrow W$ be the natural inclusion, we have:

$$i^* TW = TM \oplus \epsilon^1$$

where ϵ^1 is the trivial fiber bundle of rank 1 on M, so:

$$i^* p_I(TW) = p_I(i^* TW) = p_I(TM \oplus \epsilon^1) = p_I(TM).$$

Thus:

$$p_I(M) =< p_I(TM), [M] >=< i^* p_I(TW), [M] >=< p_I(TW), i_*[M] >,$$

but $i_*[M] = 0$ since M is a boundary, so $p_I(M) = 0$. □

10.3.2 Oriented Cobordism

We denote by $+$ the disjoint union of two manifolds and by $-M$ the manifold M endowed with the opposite orientation. We say that two oriented C^∞ compact and without boundary manifolds M and M' are cobordant if there exists a C^∞ compact oriented manifold W such that:

$$\partial W = M + (-M').$$

It is easily checked that the oriented cobordism relation is an equivalence relation and we denote by Ω_n the set of oriented cobordism classes of manifolds of dimension n. This set is an Abelian group whose additive law is given by the disjoint union $[M_1] + [M_2] := [M_1 + M_2]$. Of course if $[M_1] = [N_1]$ and $[M_2] = [N_2]$ then $[M_1 + M_2] = [N_1 + N_2]$ and thus this sum is well-defined. Zero is the cobordism class $[\emptyset]$ of the empty set $\emptyset$, the opposite of $[M]$ is $[-M]$, the manifold W being the cylinder $M \times [0,1]$. The Cartesian product $(M_1^{n_1}, M_2^{n_2}) \longrightarrow M_1^{n_1} \times M_2^{n_2}$ induces a product:

$$\Omega_{n_1} \times \Omega_{n_2} \longrightarrow \Omega_{n_1+n_2}.$$

Indeed, if $[M_1] = [N_1]$ and $[M_2] = [N_2]$ then: $[M_1 \times M_2] = [N_1 \times M_2] = [N_1 \times N_2]$, which precisely means that the Cartesian product defined a product in

$$\Omega_* = \bigoplus_{k=0}^{\infty} \Omega_k.$$

Therefore Ω_* is a graduate ring . The identity element is given by the class $[*]$ of the point in Ω_0 and the product is commutative in the graduate sense since:

$$\begin{array}{ccc} M_1^{n_1} \times M_2^{n_2} & \longrightarrow & M_2^{n_2} \times M_1^{n_1} \\ (x,y) & \longmapsto & (y,x) \end{array}$$

preserves or reverses the orientation according to the sign of $(-1)^{n_1 n_2}$.

Proposition. – *Let I be a partition of m, then:*

$$p_I : \begin{array}{ccc} \Omega_{4m} & \longrightarrow & \mathbb{Z} \\ [M^{4m}] & \longmapsto & p_I(M^{4m}) \end{array}$$

is a group homomorphism.

Proof. – Let us prove first that this map is well-defined. Let M_1^{4m} and M_2^{4m} be such that $[M_1^{4m}] = [M_2^{4m}]$, so there exists an oriented manifold with boundary W^{4m+1} such that: $\partial W^{4m+1} = M_1^{4m} - M_2^{4m}$. We denote by $i_1 : M_1 \to W$ and $i_2 : M_2 \to W$ the natural inclusions, we have:

$$i_1^* TW = TM_1 \oplus \epsilon, \quad i_2^* TW = TM_2 \oplus \epsilon.$$

On the other hand, $i_{1*}[M_1] = i_{2*}[M_2]$ in $H_{4m}(W, \mathbb{Z})$ so:

$$< p_I(TW), i_{1*}[M_1] > = < p_I(TW), i_{2*}[M_2] > .$$

By naturality, this implies:

$$< p_I(i_1^*TW), [M_1] >=< p_I(i_2^*TW), [M_2] > .$$

Since $p_I(TM \oplus \epsilon) = p_I(TM)$, we obtain:

$$< p_I(TM_1), [M_1] >=< p_I(TM_2), [M_2] >,$$

i. e. $p_I(M_1) = p_I(M_2)$, which means that $p_I : \Omega_{4m} \to \mathbb{Z}$ is well-defined. Moreover: $p_I([M_1 + M_2]) = p_I(M_1 + M_2) = p_I(M_1) + p_I(M_2) = p_I([M_1]) + p_I([M_2])$. □

If $I = \{i_1, ..., i_r\} \in \mathcal{P}(m)$, we denote by $\mathbb{C}P^I$ the Cartesian product $\mathbb{C}P^{2i_1} \times ... \times \mathbb{C}P^{2i_r}$.

Proposition. – *The family* $\{[\mathbb{C}P^I]\}_{I\in\mathcal{P}(m)}$ *is linearly independent in* Ω_{4m}, *in particular: rank* $\Omega_{4m} \geq p(m)$.

Proof. – Suppose that there exists a non trivial linear combination such that:

$$\sum_{J\in\mathcal{P}(m)} \alpha_J[\mathbb{C}P^J] = 0, \quad \alpha_J \in \mathbb{Z}.$$

Then:

$$\forall I \in \mathcal{P}(m), \quad \sum_{J\in\mathcal{P}(m)} \alpha_J p_I[\mathbb{C}P^J] = 0,$$

i. e. the matrix $p(n) \times p(n)$: $P = (p_I[\mathbb{C}P^J])_{I,J}$ is singular: $det\ P = 0$. It remains to compute this determinant explicitly in order to check that $det\ P \neq 0$. To perform this computation, the idea is to simplify the matrix P by judicious linear combinations of the lines. We do the job for $m = 2$ only. The matrix P is then the 2×2 matrix:

$$P = \begin{pmatrix} p_1^2(\mathbb{C}P^2 \times \mathbb{C}P^2) & p_1^2(\mathbb{C}P^4) \\ p_2(\mathbb{C}P^2 \times \mathbb{C}P^2) & p_2(\mathbb{C}P^4) \end{pmatrix}.$$

If M and N are two manifolds, then:

$$T(M \times N) = i^*TM \times j^*TN$$

where $i : M \longrightarrow M \times N$ and $j : N \longrightarrow M \times N$ are natural inclusions. We have:

$$p(M \times N) = i^*p(M) \cup j^*p(N),$$

and in particular, if $\dim M = \dim N = 4$:

$$\begin{array}{rcl} p_1(M \times N) & = & i^*p_1(M) + j^*p_1(N) \\ p_1^2(M \times N) & = & 2i^*p_1(M) \cup j^*p_1(N) \\ p_2(M \times N) & = & i^*p_1(M) \cup j^*p_1(N). \end{array}$$

modulo elements of order 2 (which do not exist in $\mathbb{C}P^J$). Thus, the transformation: $L_2' \longleftarrow L_2 - \frac{1}{2}L_1$ gives a triangular matrix:

$$P' = \begin{pmatrix} p_1^2(\mathbb{C}P^2 \times \mathbb{C}P^2) & p_1^2(\mathbb{C}P^4) \\ 0 & p_2(\mathbb{C}P^4) - \frac{1}{2}p_1^2(\mathbb{C}P^4) \end{pmatrix}.$$

The Pontrjagin numbers of the manifolds $\mathbb{C}P^J$ are known, in our case:

$$P' = \begin{pmatrix} 12 & 25 \\ 0 & 10 - \frac{1}{2}25 \end{pmatrix}$$

and $det\ P = -30 \neq 0$. □

In fact, R. Thom showed that the group Ω_{4m} is a free $\mathbb{Z}$-module of rank $p(m)$ and generated by the $[\mathbb{C}P^I]$, $I \in \mathcal{P}(m)$. Here are the first cobordism groups:

Ω_0	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_7	Ω_8	Ω_9	...
$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	0	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	...

The generator of Ω_0 is the point $[*]$, the one of Ω_4 is $[\mathbb{C}P^2]$ and those of Ω_8 are $[\mathbb{C}P^4]$ and $[\mathbb{C}P^2 \times \mathbb{C}P^2]$. The triviality of a group Ω_3 means that any oriented manifold of dimension 3 is a boundary – not an obvious result – and even in dimension 6 and 7, any oriented manifold is a boundary. In particular, $\mathbb{R}P^3$ and $\mathbb{C}P^3$ are boundaries... and we are going to see why in some lines with a handy argument. The examination of this table reveals another non trivial phenomenon: the product of two manifolds without boundary may have a boundary. Indeed, *a contrario*, the Cartesian product should induce an injective map:

$$\Omega_n \times \Omega_m \longrightarrow \Omega_{n+m},$$

which is clearly not the case for the pair $(n, m) = (4, 5)$.

Proposition. – *Real projective spaces $\mathbb{R}P^{2k+1}$ $(k \geq 0)$ and complex projective spaces $\mathbb{C}P^{2k+1}$ $(k \geq 0)$ are boundaries.*

Proof. – We have the following commutative diagrams which mean that the Hopf fibration on odd dimensional spheres induces a quotient map on real projective spaces:

$$\begin{array}{ccccc} & & \mathbb{S}^1 & & \mathbb{R}P^1 = \mathbb{S}^1/\mathbb{Z}_2 \\ & & \downarrow & & \downarrow \\ \mathbb{Z}_2 & \longrightarrow & \mathbb{S}^{2k+1} & \longrightarrow & \mathbb{R}P^{2k+1} \\ & & \downarrow & & \downarrow \\ & & \mathbb{C}P^k & = & \mathbb{C}P^k \end{array}$$

and that the quaternionic Hopf fibration on spheres of dimention $4k+3$ induces a quotient

map on the complex projective spaces :

$$\begin{array}{ccccc} & & \mathbb{S}^3 & & \mathbb{C}P^1 = \mathbb{S}^3/\mathbb{S}^1 \\ & & \downarrow & & \downarrow \\ \mathbb{S}^1 & \longrightarrow & \mathbb{S}^{4k+3} & \longrightarrow & \mathbb{C}P^{2k+1} \\ & & \downarrow & & \downarrow \\ & & \mathbb{H}P^k & = & \mathbb{H}P^k. \end{array}$$

Thus, $\mathbb{R}P^{2k+1}$ is the total space of a bundle with fiber $\mathbb{S}^1 \simeq \mathbb{R}P^1$ and $\mathbb{C}P^{2k+1}$ the one of a bundle with fiber $\mathbb{S}^2 \simeq \mathbb{C}P^1$. In particular, we have the following fibrations:

$$\mathbb{S}^1 \longrightarrow \mathbb{R}P^3 \longrightarrow \mathbb{S}^2, \qquad \mathbb{S}^2 \longrightarrow \mathbb{C}P^3 \longrightarrow \mathbb{S}^4.$$

So the manifolds $\mathbb{R}P^{2k+1}$ and $\mathbb{C}P^{2k+1}$ are boundaries of the corresponding ball bundles. □

10.4 Milnor Exotic Sphere

10.4.1 Signature and Hirzebruch Formula

Signature of a Manifold

Let M be an oriented compact manifold without boundary of dimension $n = 4m$. As usual we denote by $[M] \in H_{4m}(M, \mathbb{Z})$ its fundamental class. The intersection form of M is the symmetric bilinear form :

$$\begin{array}{rccc} Q_M : & H^{2m}(M,\mathbb{Z}) \times H^{2m}(M,\mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ & (\alpha, \beta) & \longmapsto & < \alpha \cup \beta, [M] > . \end{array}$$

Examples. – $Q_{\mathbb{S}^{2m}\times\mathbb{S}^{2m}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Q_{\mathbb{C}P^{2m}} = (1)$.

Using the Poincaré duality $D : H^{2m}(M,\mathbb{Z}) \to H_{2m}(M,\mathbb{Z})$, $\alpha \mapsto \alpha \cap [M]$, we indifferently can write this form on $H_{2m}(M,\mathbb{Z})$:

$$\begin{array}{rccc} Q_M : & H_{2m}(M,\mathbb{Z}) \times H_{2m}(M,\mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ & (a, b) & \longmapsto & a.b := < D^{-1}(a) \cup D^{-1}(b), [M] > . \end{array}$$

Of course, if $a \in Tor(H_{2m}(M,\mathbb{Z}))$ or $b \in Tor(H_{2m}(M,\mathbb{Z}))$ then $Q_M(a,b) = 0$ and then Q_M defines a form on the quotient $H_{2m}(M,\mathbb{Z})/Tor$.

Definition. – The *signature* $\sigma(M)$ of an oriented compact manifold M without boundary of dimension $4m$ is the signature of Q_M on $H_{2m}(M,\mathbb{R}) \simeq (H_{2m}(M,\mathbb{Z})/Tor) \otimes \mathbb{R}$. If the dimension of M is not a multiple of 4, we put $\sigma(M) = 0$.

The quotient $H_{2m}(M,\mathbb{Z})/Tor$ is a free $\mathbb{Z}$-module of finite rank N. Let A be a matrix representing Q_M on a given basis. The *determinant* of Q_M is the determinant of A. Of course

any other choice of a basis will lead to another matrix PA^tP with $P \in Gl(N,\mathbb{Z})$. Note that $P \in Gl(N,\mathbb{Z}) \iff det\ P \in \mathbb{Z}^* = \{-1,1\}$ and so $det\ PA^tP = det\ A$. [5] If $-M$ is the manifold M with the opposite orientation, we have $Q_{-M} = -Q_M$, then $\sigma(-M) = -\sigma(M)$ and $\det\ Q_{-M} = (-1)^{rank\ H_{2m}(M,\mathbb{Z})} \det\ Q_M$.

Proposition. – *The intersection form Q_M is unimodular, that is,*

$$\det\ Q_M = \pm 1.$$

In particular if the rank of $H^{2m}(M,\mathbb{Z})$ is odd, then $\sigma(M) \neq 0$.

Proof. – Let $\mathcal{B} = (e_1, \cdots, e_N)$ be a $\mathbb{Z}$-basis of $H^{2m}(M,\mathbb{Z})$ and $\lambda_i : H_{2m}(M,\mathbb{Z}) \to \mathbb{Z}$, $1 \leq i \leq N$ such that:

$$\lambda_i(D(e_j)) = \delta_{ij}.$$

The universal coefficient theorem induces an isomorphism:

$$\beta : H^{2m}(M,\mathbb{Z})/Tor \longrightarrow Hom(H_{2m}(M,\mathbb{Z}),\mathbb{Z})$$

and putting $e_i^* = \beta^{-1}(\lambda_i) \in H^{2m}(M,\mathbb{Z})/Tor$, we obtain:

$$\begin{array}{rclcl} Q_M(e_i^*, e_j) & = & < e_i^* \cup e_j, [M] > & = & < e_i^*, e_j \cap [M] > \\ & = & < e_i^*, D(e_j) > & = & \lambda_i(D(e_j)) \\ & = & \delta_{ij}. & & \end{array}$$

Thus $(e_1^*, \cdots, e_N^*)$ is a dual Q_M-basis which is a $\mathbb{Z}$-basis of $H^{2m}(M,\mathbb{Z})/Tor$. We put:

$$A = Mat_{\mathcal{B}}\ Q_M = (Q_M(e_i, e_j))_{i,j}\,,$$

of course: $Ae_i^* = e_i$. Let $B : H^{2m}(M,\mathbb{Z})/Tor \longrightarrow H^{2m}(M,\mathbb{Z})/Tor$ be the map defined by $B(e_i) = e_i^*$, we immediately have: $AB = BA = Id$ and thus $det\ A = \pm 1$. □

Remark. – We can define an intersection form for M^{4m+2}:

$$\begin{array}{rccc} Q_M : & H^{2m+1}(M,\mathbb{Z}) \times H^{2m+1}(M,\mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ & (\alpha, \beta) & \longmapsto & < \alpha \cup \beta, [M] > . \end{array}$$

and obtain an unimodular skewsymmetric form on $H^{2m+1}(M,\mathbb{Z})$. However, skewsymmetric forms are classified by their rank, therefore intersection forms on M^{4m+2} do not provide more information than the rank of $H^{2m+1}(M,\mathbb{Z})$.

[5] However, $P \in Gl(N,\mathbb{Z})$ does not imply $P^tP = Id$ and this does not give the possibility to define a trace for Q_M.

Proposition. – *We denote by $M_1 + M_2$ the disjoint union of M_1 and M_2, we have:*

$$\begin{array}{ll} (1) & \sigma(M_1 + M_2) = \sigma(M_1) + \sigma(M_2), \\ (2) & \sigma(M_1 \times M_2) = \sigma(M_1)\sigma(M_2), \\ (3) & \textit{If } M = \partial W \textit{ then } \sigma(M) = 0. \end{array}$$

We refer to the books of Hirzebruch (p. 85) and/or Stong (p. 221) for a proof of this proposition.

Corollary. – $\sigma : \Omega_* \otimes \mathbb{R} \longrightarrow \mathbb{R}$ *is an algebra morphism.*

Hirzebruch formula

Let

$$A^* = \prod_{k=0}^{\infty} A^k$$

be a commutative graduate $\mathbb{R}$-algebra[6]. An element of A is identified with a formal sum:

$$a = a_0 + a_1 + \cdots + a_k + \cdots$$

and the product of two elements with first terms 1 is given by the formula:

$$(1 + a_1 + a_2 + \cdots)(1 + b_1 + b_2 + \cdots) = 1 + (a_1 + b_1) + (a_2 + a_1 b_1 + b_2) + \cdots$$

In particular $a = 1 + a_1 + a_2 + \cdots$ is invertible and the inverse is obtained by resolving an infinite triangular system:

$$\begin{array}{rcl} b_1 & = & -a_1 \\ b_2 & = & -a_2 - a_1 b_1 \\ b_3 & = & -a_3 - a_2 b_1 - a_1 b_2. \\ & \vdots & \end{array}$$

We thus obtain:

$$\begin{array}{rcl} b_1 & = & -a_1 \\ b_2 & = & a_1^2 - a_2 \\ b_3 & = & -a_1^3 + 2a_1 a_2 - a_3 \\ & \vdots & \end{array}$$

therefore:

$$a^{-1} = 1 + K_1(a_1) + K_2(a_1, a_2) + K_3(a_1, a_2, a_3) + \cdots$$

where $K_1(a_1) = -a_1$, $K_2(a_1, a_2) = a_1^2 - a_2$, $K_3(a_1, a_2, a_3) = -a_1^3 + 2a_1 a_2 - a_3$.

[6]The commutativity law has to be understood in the usual sense –not the graded one– i. e. $xy = yx$ whatever the degrees of x and y.

Definition. – A sequence of polynomials $K_1(a_1)$, $K_2(a_1, a_2)$, $K_3(a_1, a_2, a_3)$,..., with real coefficients such that every polynomial K_n is homogenous of degree n, is *a multiplicative sequence* if the polynomial:

$$K(a) = 1 + K_1(a_1) + K_2(a_1, a_2) + K_3(a_1, a_2, a_3) + ...$$

satisfies the equality $K(ab) = K(a)K(b)$ for any pair $(a, b) \in A^* \times A^*$ with $a_0 = 1$, $b_0 = 1$.

The sequence of polynomials previously defined by $K(a) = a^{-1}$ is mutiplicative because $(ab)^{-1} = a^{-1}b^{-1}$. We admit the following result due to Hirzebruch (for a proof, see the book of Milnor-Stasheff for instance).

Proposition. – *Let $f(t) = 1 + \lambda_1 t + \cdots + \lambda_k t^k + \cdots$ be a formal series whose coefficients are in $\mathbb{R}$, then there exists an unique multiplicative sequence such that the coefficient of a_1^n in $K_n(a_1, ..., a_n)$ is λ_n.*

Definition. – Let $K = \{K_n\}$ be a multiplicative sequence, then the K-genus of M^{4m} is the number:

$$K[M] =< K_m(p_1, ..., p_m), [M] >=< K(p(M)), [M] > .$$

If the dimension of M is not a multiple of 4, we put $K[M] = 0$.

Property. – *Let $K = \{K_n\}$ be a multiplicative sequence, then $M \mapsto K[M]$ defines an algebra homomorphism from $\Omega_* \otimes \mathbb{R}$ into $\mathbb{R}$.*

Proof. – Since $K[M]$ is a linear combination of the Pontrjagin numbers, it is clear that $K[M]$ is additive and that $K[M] = 0$ if M is a boundary. Since the coefficient ring is $\mathbb{R}$ and thanks to the product formula for the total Pontrjagin class, we have:

$$p(M \times M') = p(M)p(M')$$

(no element of order 2). We put $p = p(M)$ and $p' = p(M')$, since $K = \{K_n\}$ is multiplicative:

$$K(pp') = K(p)K(p')$$

then:

$$K(M \times M') =< K(pp'), [M \times M'] >= (-1)^{dim\, M dim\, M'} < K(p), [M] >< K(p), [M'] > .$$

The sign is 1 if the dimensions are multiple of 4, otherwise this sign does not matter. □

Theorem. – *Let $\{L_k(p_1, ..., p_k)\}$ be the multiplicative sequence corresponding to the series:*

$$\frac{\sqrt{t}}{th(\sqrt{t})} = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + ... + (-1)^{k-1}\frac{2^{2k}}{(2k)!}B_k t^k + ...$$

where B_k is the k-th Bernoulli number. Then:

$$\sigma(M^{4k}) =< L_k(p_1, ..., p_k), [M^{4k}] > .$$

The first three elements of $\{L_k\}$ are:

$$\begin{array}{rcl} L_1 & = & \frac{1}{3}p_1 \\ L_2 & = & \frac{1}{45}(7p_2 - p_1^2) \\ L_3 & = & \frac{1}{945}(62p_3 - 13p_2p_1 + 2p_1^3). \end{array}$$

In particular, on any compact oriented manifold without boundary, the Pontrjagin class p_1 is a multiple of 3 if the dimension is 4, $7p_2 - p_1^2$ is a multiple of 45 if the dimension is 8, etc.

Proof. – L and σ are both algebra homomorphisms from $\Omega_* \otimes \mathbb{R}$ into $\mathbb{R}$, it is enough to check that they coincide on the basis $(\mathbb{C}P^{2k})_{k\in\mathbb{N}^*}$ which is a mere computation. □

10.4.2 Milnor Sphere

Recall that the data of an oriented real vector bundle of rank k on a sphere $\mathbb{S}^n$, $n > 1$, is equivalent to the data of a transition map $t : \mathbb{S}^{n-1} \to Gl_+(n, \mathbb{R})$. Let $\xi_{k,j}$ be the bundle of rank 4 on $\mathbb{S}^4$ determined by the transition map:

$$\begin{array}{rccl} t_{k,j} : & \mathbb{S}^3 & \longrightarrow & SO(4) \subset Gl_+(4, \mathbb{R}) \\ & u & \longmapsto & t_{k,j}(u) : v \mapsto u^k v u^j. \end{array}$$

Here we have identified $\mathbb{R}^4$ with the quaternion field $\mathbb{H}$. We suppose that $k + j = 1$ and we denote by Σ_k^7 the total space of the unit bundle of $\xi_{k,1-k}$:

$$\mathbb{S}^3 \longrightarrow \Sigma_k^7 \longrightarrow \mathbb{S}^4.$$

Proposition. – *Σ_k^7 is homeomorphic to $\mathbb{S}^7$.*

Proof. – We shall build on Σ_k^7 a Morse function f with only two critical points. Let $\mathcal{U}_1 = \mathbb{S}^4 \setminus \{\text{North pole}\}$ and $\mathcal{U}_2 \setminus \{\text{South pole}\}$, we identify these two open sets to $\mathbb{H}^*$ *via* the stereographic projection. The transition map between $\mathcal{U}_1$ and $\mathcal{U}_2$ is given by $u \mapsto \frac{u}{|u|^2}$. Finally we denote by $r(u) = \frac{u}{|u|}$ the radial retraction. The space Σ_k^7 is obtained from the two copies W_1 and W_2 of $\mathbb{H}\times\mathbb{S}^3$ by identifying the subsets $\mathbb{H}^*\times\mathbb{S}^3$ with the diffeomorphism:

$$(u, v) \overset{\Psi}{\longmapsto} (u', v') = \left(\frac{u}{|u|^2}, t_{k,1-k}(r(u))v\right),$$

that is:

$$\Psi(u, v) = \left(\frac{u}{|u|^2}, \frac{u^k v u^{1-k}}{|u|}\right).$$

We define $f : \Sigma_k^7 \to \mathbb{R}$ by:

$$\begin{cases} f(u,v) = \dfrac{Re(v)}{(1+|u|^2)^{1/2}} & \text{if } (u,v) \in W_1, \\ f(u',v') = \dfrac{Re(v'\bar{u}')}{(1+|u'|^2)^{1/2}} & \text{if } (u',v') \in W_2. \end{cases}$$

It is easy to check that f has two non-degenerated critical points on W_1 which are $(u,v) = (0, \pm 1)$ and that it does not admit any critical point on W_2. □

We denote by B_k^8 the total space of the ball bundle of $\xi_{k,1-k}$. It is a connected differential manifold with boundary and $\Sigma_k^7 = \partial B_k^8$ is homeomorphic as a differential manifold to $\mathbb{S}^7$. We set the following assumption (H):

ASSUMPTION. – *There exists a diffeomorphism between Σ_k^7 and the standard sphere $\mathbb{S}^7$.*

Under this assumption, the manifold M_k^8 obtained by identifying, with this diffeomorphism, the space ∂B_k^8 with the boundary ∂D^8 of the standard ball of dimension 8 is a connected *differential* manifold without boundary.

Proposition. – *The manifold M_k^8 is orientable.*

Proof. – The space B_k^8 retracts on $\mathbb{S}^4$, hence, it is connected. The Van Kampen theorem[7] shows that M_k^8 is simply connected and thus orientable. □

Proposition. – *The signature $\sigma(M_k^8)$ is ± 1.*

Proof. – The Mayer-Vietoris theorem[8] shows that $H^4(M_k^8, \mathbb{Z}) = H^4(B_k^8, \mathbb{Z})$ and, since B_k^8 retracts on $\mathbb{S}^4$: $H^4(M_k^8, \mathbb{Z}) = \mathbb{Z}$. On the other hand $det\ Q_{M_k^8} = \pm 1$, thus $\sigma(M_k^8) = \pm 1$. □

We choose on M_k^8 the orientation for which $\sigma(M_k^8) = 1$, we also choose a generator $\iota \in H^4(M_k^8, \mathbb{Z}) = \mathbb{Z}$, that is, an orientation on $\mathbb{S}^4 \subset B_k^8 \subset M_k^8$, of course $\iota \cup \iota = [M_k^8]$.

Proposition. – *The first Pontrjagin class $p_1(M_k^8)$ equals $\pm 2(2k-1)\iota$*

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[7] *Van Kampen Theorem.* – If $\mathcal{U}_1, \mathcal{U}_2$ and $\mathcal{U}_1 \cap \mathcal{U}_2$ are arcwise connected, then the fundamental group $\pi_1(\mathcal{U}_1 \cup \mathcal{U}_2)$ is isomorphic to the amalgamated product: $\pi_1(\mathcal{U}_1) *_{\pi_1(\mathcal{U}_1 \cap \mathcal{U}_2)} \pi_1(\mathcal{U}_2)$. In other words, $\pi_1(\mathcal{U}_1 \cup \mathcal{U}_2)$ is obtained from the free product $\pi_1(\mathcal{U}_1) * \pi_1(\mathcal{U}_2)$ and the relations which identify the images of the elements of $\pi_1(\mathcal{U}_1 \cap \mathcal{U}_2)$ in $\pi_1(\mathcal{U}_1)$ and in $\pi_1(\mathcal{U}_2)$.

[8] *Mayer-Vietoris Theorem.* – There exists a long exact sequence relating the cohomology of $\mathcal{U}_1$, the one of $\mathcal{U}_2$, the one of $\mathcal{U}_1 \cap \mathcal{U}_2$ and the one of $\mathcal{U}_1 \cup \mathcal{U}_2$. Precisely:

$$... \to H^k(\mathcal{U}_1 \cup \mathcal{U}_2) \to H^k(\mathcal{U}_1) \oplus H^k(\mathcal{U}_2) \to H^k(\mathcal{U}_1 \cap \mathcal{U}_2) \to H^{k+1}(\mathcal{U}_1 \cup \mathcal{U}_2) \to ..$$

Proof. – The inclusion $i : B_k^8 \longrightarrow M_k^8$ induces an isomorphism $i^* : H^4(M_k^8, \mathbb{Z}) \to H^4(B_k^8, \mathbb{Z})$ and by naturality: $i^* p_1(M_k^8) = p_1(B_k^8)$. Let $h : \mathbb{S}^4 \longrightarrow B_k^8$ be the inclusion given by the zero section, it induces an isomorphism between the cohomology groups of degree 4 and moreover: $h^* p_1(B_k^8) = h^* p_1(TB_k^8) = p_1(h^* TB_k^8)$. On the other hand:

$$h^* TB_k^8 = T\mathbb{S}^4 \oplus \xi_{k,1-k},$$

thus applying the product theorem for the total Pontrjagin class, we get:

$$p_1(TB_k^8) = p_1(T\mathbb{S}^4) + p_1(\xi_{k,1-k})$$

Since $T\mathbb{S}^4 \oplus \epsilon^1 = \epsilon^5$, we have:

$$p(T\mathbb{S}^4) = p(T\mathbb{S}^4 \oplus \epsilon^1) = p(\epsilon^5) = 1.$$

Moreover, the classification of oriented bundles of rank 4 over $\mathbb{S}^4$ shows that $p_1(\xi_{k,j}) = \pm 2(k-j)[\mathbb{S}^4]$. □

Theorem. – Σ_2^7 *is homeomorphic but not diffeomorphic to the standard sphere* $\mathbb{S}^7$.

Proof. – Let us compute the second Pontrjagin number of M_k^8 by using the Hirzebruch formula:

$$< p_2, [M_k^8] >= \frac{< p_1^2, [M_k^8] > + 45\sigma(M_k^8)}{7} = \frac{4(2k-1)^2 + 45}{7}.$$

If $k = 2$ we obtain $< p_2, [M_k^8] >= \frac{81}{7} \notin \mathbb{Z}$ which is impossible.[9] So the assumption (H) is wrong for this value of k, this means that there does not exist any diffeomorphism of Σ_2^7 into $\mathbb{S}^7$. □

10.5 Appendix: Chern-Weil Theory

Let $\mathbb{Z} \subset \mathbb{R}$ be the natural inclusion and $i : H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{R}) \simeq H^*_{dR}(X, \mathbb{R})$ be the induced homomorphism. The aim of Chern-Weil theory is to build real valued closed forms which represent the images $i(c_k)$ of the Chern classes. In what follows, we shall forget about i and we shall write c_k instead of $i(c_k)$ but we have to keep in mind that $c_k \in H^{2k}_{dR}(X, \mathbb{R})$ only represents the torsion free part of $c_k \in H^{2k}(X, \mathbb{Z})$.

Curvature operator. – Let $E \to X$ be a real or complex vector bundle, a connection ∇ on $E \to X$ can be thought as a linear map:

$$\begin{array}{rccc} d_\nabla : & \Gamma(E) & \longrightarrow & \Gamma(T^*X \otimes E) \\ & \sigma & \longmapsto & \nabla\sigma \end{array}$$

[9] If $k = 1$ this gives $< p_2, [M_k^8] >= 7$, in fact in this case, M_1^8 is diffeomorphic to the quaternionic projective space $\mathbb{H}P^2$.

satisfying the Leibniz's rule $d_\nabla(f\sigma) = df \otimes \sigma + f d_\nabla \sigma$ ($f \in C^\infty(X)$ and d is the exterior derivative). We write

$$\Omega^k(E) = \Gamma(\Lambda^k(T^*X) \otimes E) = \{E - \text{valued } k - \text{form on } X\}.^{10}$$

With this notation $d_\nabla : \Omega^0(E) \to \Omega^1(E)$ formally appears as an exterior derivative. It is quite natural to define d_∇ on $\Omega^k(E)$, $k > 0$ by requiring:

$$d_\nabla(\omega \otimes \sigma) = d\omega \otimes \sigma + (-1)^k \omega \wedge d_\nabla \sigma$$

with $\omega \in \Omega^k(X)$ and $\sigma \in \Gamma(E)$. A mere computation shows that: $d_\nabla \circ d_\nabla : \Omega^k(E) \to \Omega^{k+2}(E)$ is a $C^\infty(X)$-linear map which does not vanish in general. The *curvature tensor* K of the connection ∇ is the operator

$$K = d_\nabla \circ d_\nabla : \Omega^0(E) \to \Omega^2(E).$$

and it defines a section of

$$Hom(E, \Lambda^2 T^*X \otimes E) = \Lambda^2 T^*X \otimes Hom(E, E) = \Lambda^2 T^*X \otimes E^* \otimes E.$$

If $E \to X$ is a (real or complex) line bundle then $E^* \otimes E$ is a trivial line bundle. Thus, K defines a section of $\Lambda^2 T^*X$, that is, a 2-form. Now, if $E \to X$ is a bundle with rank $n > 1$, K does not define a global 2-form in general. Let σ be a section of E and (Z, T) sections of TX, by the very definition of K, $K(\sigma)(Z, T)$ is a section of E. Let $\varphi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times \mathbb{R}^n$ (or $\mathbb{C}^n$) be a trivialization of $E \to X$, there exists a matrix $\Omega_\alpha(Z, T) \in \mathcal{M}_n(\mathbb{R}$ or $\mathbb{C})$ such that

$$(K(\sigma)(Z, T))_\alpha = \Omega_\alpha(Z, T)\sigma_\alpha,$$

(we have written $\sigma_\alpha = \varphi_\alpha \circ \sigma$). Obviously, we can forget about Z, T and consider Ω_α as a $n \times n$ matrix of 2-forms on $\mathcal{U}_\alpha$. Matrices Ω_α only depend on a choice of a trivialization of $E \to X$, a trivialization of $\Lambda^2 T^*X$ is not needed to define them. They will match together to form a global object if and only if $\Omega_\alpha = \Omega_\beta$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$. If we denote by $t_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to Gl(n, \mathbb{R}$ or $\mathbb{C})$ the transition maps of $E \to X$, we have

$$\Omega_\alpha = t_{\alpha\beta} \Omega_\beta t_{\alpha\beta}^{-1}.$$

The matrices Ω_α, known as *curvature forms*, do not match together in general to build a globally defined closed forms. Nevertheless, there is a way to actually build global objects from the Ω_α.

UNITARY CONNECTIONS. – From now on, we assume that $E \to X$ is a Hermitian bundle and ∇ is a connection on $E \to X$. As above, if $(\mathcal{U}_\alpha, \varphi_\alpha)$ is a trivialization of $E \to X$ and $\sigma \in \Gamma(E)$, then there exists a $n \times n$ matrix ω_α of 1-forms such that

$$(d_\nabla \sigma)_\alpha = d\sigma_\alpha + \omega_\alpha \sigma_\alpha.$$

[10] If $E = TX$, we write $\Omega^k(X)$ rather than $\Omega^k(TX)$.

(Matrices ω_α do not match together on a global object in general). The connection ∇ is said to be *unitary* if the ω_α are Hermitian matrices. This is equivalent to the condition:

$$\nabla(\sigma_1, \sigma_2) = (\nabla\sigma_1, \sigma_2) + (\sigma_1, \nabla\sigma_2) \quad \forall \sigma_1, \sigma_2 \in \Gamma(E),$$

where $(\ ,\ )$ is the Hermitian product on $E \to X$. From $K = d_\nabla \circ d_\nabla$, we easily deduce that:

$$\Omega_\alpha = d\omega_\alpha + \omega_\alpha \wedge \omega_\alpha.$$

Thus if ∇ is a unitary connection, matrices Ω_α are skew-Hermitian.

INVARIANT POLYNOMIALS. – An invariant polynomial on $\mathcal{M}_n(\mathbb{C})$ is a polynomial function with n^2 indeterminates $P : \mathcal{M}_n(\mathbb{C}) \to \mathbb{C}$ satisfying:

$$\forall A, B \in \mathcal{M}_n(\mathbb{C}), \quad P(AB) = P(BA),$$

or, equivalently:

$$\forall T \in Gl(n, \mathbb{C}),\ \forall A \in \mathcal{M}_n(\mathbb{C}),\ \ P(TAT^{-1}) = P(A).$$

The set of invariant polynomials on $\mathcal{M}_n(\mathbb{C})$ is a ring spanned by $Id, \sigma_1, ..., \sigma_n$ where the σ_k are defined by the equation

$$\forall A \in \mathcal{M}_n(\mathbb{C}),\ \forall t \in \mathbb{C},\ \ \det(Id + tA) = 1 + t\sigma_1(A) + ... + t^n\sigma_n(A).$$

Lemma. – *Let P be an invariant polynomial, we denote by P' the matrix $(\frac{\partial P}{\partial a_{ij}})_{ij}$. Then $P'(A)$ commute with A.*

Proof. – Let E_{ij} denotes the matrix with entry 1 at the (i, j)-th place and zeros elsewhere. By differentiating the relation $P((Id+tE_{ij})A) = P(A(Id+tE_{ij}))$ and setting $t = 0$ we obtain the desired commutativity result. □

THE EXTERIOR FORM $P(\Omega)$. – Let P be an invariant polynomial and Ω_α a curvature form, to each pair (P, Ω_α), we associate the element of $\Omega^*(TX_{|\mathcal{U}_\alpha})$ defined by $P(\Omega_\alpha)$. Since P is invariant, the $P(\Omega_\alpha)$ match together on a global object. Indeed, on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$,

$$P(\Omega_\alpha) = P(t_{\alpha\beta}\Omega_\beta t_{\alpha\beta}^{-1}) = P(\Omega_\beta).$$

We denote by $P(\Omega)$ the resulting element of $\Omega^*(TX)$. The key fact of this construction is the following:

Proposition. – *The exterior form $P(\Omega)$ is closed thus $[P(\Omega)] \in H^*(X, \mathbb{C})$.*

Proof. – It is easily seen that $dP(\Omega) = trace(P'(\Omega) \wedge d\Omega)$. By differentiating the relation $\Omega_\alpha = d\omega_\alpha + \omega_\alpha \wedge \omega_\alpha$ we obtain $d\Omega_\alpha = \omega_\alpha \wedge \Omega_\alpha - \Omega_\alpha \wedge \omega_\alpha$, thus

$$\begin{aligned} dP(\Omega) &= trace(P'(\Omega_\alpha) \wedge \omega_\alpha \wedge \Omega_\alpha - P'(\Omega_\alpha) \wedge \Omega_\alpha \wedge \omega_\alpha) \\ &= trace(P'(\Omega_\alpha) \wedge \omega_\alpha \wedge \Omega_\alpha - \Omega_\alpha \wedge P'(\Omega_\alpha) \wedge \omega_\alpha) \end{aligned}$$

according to the preceding lemma. We set $X_\alpha = P'(\Omega_\alpha) \wedge \omega_\alpha \in \Omega^{odd}(TX_{|\mathcal{U}_\alpha})$. We have

$$dP(\Omega) = trace(X_\alpha \wedge \Omega_\alpha - \Omega_\alpha \wedge X_\alpha) = 0.$$

□

Remark. – Note that in this proof, elements of the matrix Ω_α are 2-forms and thus they commute with every k-form since $\alpha \wedge \beta = (-1)^{\deg \alpha deg \beta} \beta \wedge \alpha$.

Here are some properties of $P(\Omega)$.

Proposition. – $[P(\Omega)]$ *does not depend on the connection. We write* $P(E) = [P(\Omega)]$.

Proof. – Let ∇_0 and ∇_1 be two connections on $E \to X$, $\pi^*E \to X$ be the pull-back of $\pi : X \times [0,1] \to X$ and ∇'_0, ∇'_1 be the induced connections. Then $\nabla = t\nabla'_1 + (1-t)\nabla'_0$ is a connection on π^*E and $P(\Omega_\nabla)$ is a cocycle. We write $i_t : x \mapsto (x,t)$. With obvious identifications $i_0^*\nabla'_0 = \nabla_0$ and $i_1^*\nabla'_1 = \nabla_1$. Thus

$$i_0^*P(\Omega_\nabla) = P(\Omega_{\nabla_0}) \qquad i_1^*P(\Omega_\nabla) = P(\Omega_{\nabla_1})$$

and since i_0 and i_1 are homotopic, $[P(\Omega_{\nabla_0})] = [P(\Omega_{\nabla_1})]$. □

Naturality properties of the construction of $P(E)$ obviously imply the following proposition.

Proposition. – *Let* $f : Y \to X$, *then* $P(f^*E) = f^*P(E)$.

Theorem. – *The* k*-th Chern class of* $E \to X$ *is given by*

$$c_k(E) = (\frac{1}{2i\pi})^k \sigma_k(E) \in H^{2k}(X, \mathbb{C}).$$

Our goal is to obtain a class $c_k(E) \in H^{2k}(X, \mathbb{R})$ but up to now, we only obtain $c_k(E) \in H^{2k}(X, \mathbb{C})$. In fact $c_k(E)$ belongs to $H^{2k}(X, \mathbb{R})$. Indeed, let us denote by S_j the invariant polynomial defined by $S_j(A) = trace(A^j)$. We can write σ_k as a linear combination with real coefficients of the S_j, $j = 1...k$. We endow $E \to X$ with an Hermitian metric and we choose a unitary connection. The Ω_α are skew-Hermitian, thus $\frac{1}{2i\pi}\Omega_\alpha$ are Hermitian and so $S_j(\Omega_\alpha) = trace((\frac{1}{2i\pi}\Omega_\alpha)^j) \in \mathbb{R}$.

Sketch of the proof of the theorem. – 1) If $E \to X$ is a complex line bundle then $\sigma_1(E) = 2i\pi c_1(E)$.
2) Let $\overline{c} = \sum_k (\frac{1}{2i\pi})^k \sigma_k$ and $E \to X$ be a bundle that splits in a direct sum of complex line bundles $E_1 \oplus ... \oplus E_n$. Point 1) implies that $\overline{c}(E) = c(E)$.
3) A standard argument (known as the *Splitting Principle*, see the book of Bott and Tu p.273 for instance) gives a way to handle with the case of an arbitrary complex bundle

$E \to X$. $\square$

CHERN CHARACTER. – Rather than invariant polynomials $P : \mathcal{M}_n(\mathbb{C}) \to \mathbb{C}$, we can consider formal power series

$$P = P_0 + P_1 + P_2 + ...$$

where each P_j is a homogeneous invariant polynomial of degree j. If we choose

$$P(A) = trace(e^{\frac{1}{2\imath\pi}A}),$$

the corresponding class $[P(\Omega)]$ is called the *Chern character* and is denoted by $ch(E) \in H^*(X, \mathbb{R})$. It turns out that this character is additive and multiplicative.

Proposition. – *Let $E \to X$ and $E' \to X$ be two complex vector bundles, then*

$$ch(E \oplus E') = ch(E) + ch(E'), \quad ch(E \otimes E') = ch(E) \wedge ch(E').$$

We refer the reader to the book of Milnor-Stasheff for a proof of this proposition.

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Differential Geometry and Topology, Discrete and Computational Geometry
M. Boucetta and J.-M. Morvan (Eds.)
IOS Press, 2005
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Chapter 11

On Stiefel-Whithney Classes

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Introduction

Characteristic classes are certain cohomology classes associated to manifolds and vector bundles. Their construction can be fulfilled explicitly in several branch of mathematics : the connection theory on vector bundles, the classification of vector bundles, the obstruction theory... In this note, we present three constructions of Stiefel-Whitney characteristic classes which will be constructed, supposing the reader has a good knowledge of homological algebra. In order to help the reader to understand the sense of the ideas that I will develop here, in spite of the number of pages of this note, I will survey on the notions of vector bundles, Poincaré duality, Thom isomorphism and obstruction theory.

11.1 Generalities on Real Vector Bundles

11.1.1 Definitions and Examples

Definition 11.1.1 *A differentiable vector bundle of rank $n \geq 0$ consists in two manifolds E, M and a surjective differentiable map, $\pi : E \to M$, such that for each $x \in M$ there is an open neighborhood U_x and a diffeomorphism $\varphi_x : U_x \times \mathbb{R}^n \to \pi^{-1}(U_x)$ which satisfies the two following conditions :*

1. $\forall y \in U_x, \forall v \in \mathbb{R}^n, \pi \circ \varphi_x(y, v) = y$;
2. *for each $y \in U_x$ the partial map $\varphi_x(y, -) : \mathbb{R}^m \to p^{-1}(\{y\})$ is a linear isomorphism.*

The space E is called the total space, M the base space, π the projection map and φ_x local trivialization map at $x \in M$. A vector bundle of rank $n \geq 0$ will be denoted by $\zeta^n = (E, \pi, M)$.

Let $\zeta_i = (E_i, \pi_i, M_i)$ be a vector bundle, i=1, 2. A bundle morphism $(F, f) : \zeta_1 \to \zeta_2$ is a couple of smooth functions $F : E_1 \to E_2$ and $f : M_1 \to M_2$ such that $\pi_2 \circ F = f \circ \pi_1$. If for each $x \in M_1$ the restriction $F_| : \pi_1^{-1}(\{x\}) \to \pi_2^{-1}(\{f(x)\})$ is an isomorphism and f is a homeomorphism, we will say that (F, f) an *equivalence*. In particular, when $M_1 = M_2 = M$ an equivalence (F, id_M) will be called *isomorphism*.

Proposition 11.1.2 *Let $\zeta_i = (E_i, \pi_i, M)$ be a vector bundle, i=1, 2. A morphism $(F, f) : \zeta_1 \to \zeta_2$ is an isomorphism if and only if for each $x \in M$ the restriction $F_| : \pi_1^{-1}(\{x\}) \to \pi_2^{-1}(\{f(x)\})$ is a linear isomorphism.*

The proof of this proposition can be found in J. Milnor and J. Stasheef [4] p. 19 or in D. Husemoller [3] p. 27.

11.1.2 Operations on real vector bundles and examples

In this paragraph we recall some classical algebraic operations on real vector bundles which will be useful for the sequel of this note.

Induced vector bundles. Let $\zeta^n = (E, \pi, M)$ be a real vector bundle and $f : N \to M$ a smooth function. On the subset

$$f^*(E) := \{(x, v) \in N \times E / f(x) = \pi(v)\}$$

there is a natural structure of vector bundle of rank n over the manifold N with the projection map $\pi'(x, v) = x$. The vector bundle $\pi' : f^*(E) \to N$ will be called *the induced vector bundle or the pullback of* ζ^n and will be denoted by $f^*(\zeta)$.

Note that the couple (pr_2, f) formed by the second projection $pr_2 : f^*(E) \to E$ and the function $f : N \to M$ defines a bundle morphism $(pr_2, f) : f^*(\zeta) \to \zeta$.

Cartesian products. Let $\zeta_i = (E_i, \pi_i, M_i)$ be a vector bundle, i=1, 2. The Cartesian product of ζ_1 and ζ_2 is a vector bundle with total space $E = E_1 \times E_2$, base space $M = M_1 \times M_2$ and projection map $\pi = \pi_1 \times \pi_2$. We denote this vector bundle by $\zeta_1 \times \zeta_2$.

Whitney sums. Let $\zeta_i = (E_i, \pi_i, M_i)$ be a vector bundle, $i = 1, 2$. The induced vector bundle by the cartesian product $\zeta_1 \times \zeta_2$ via the diagonal map $d : M \to M \times M$ (i.e. $d(x, x) = x$) is called the Whitney sum. We denote the Whitney sum of ζ_1 and ζ_2 by $\zeta_1 \oplus \zeta_2$.

The proof of the following theorem is not difficult, it can found in [3] p; 38.

Theorem 11.1.3 *Let ζ^n and η^m be vector bundles over the same base space M. For each bundle map $(F, f) : \zeta^n \to \eta^m$ which is injective over the fibres there is a vector bundle ν^{m-n} over M such that $\zeta^n \oplus \nu^{m-n}$ is isomorphic to η^m.*

Now we will give some examples of vector bundles whose Stiefel-Whithney characteristic classes will be calculated subsequently.

Example *Trivial vector bundle $\varepsilon^n(M)$*

The vector bundle with base space a manifold M and with total space $M \times \mathbb{R}^n$ is called a trivial vector bundle of rank n and is noted $\varepsilon^n(M)$.

Example *Tangent vector bundle TM*

Let M^n be a manifold. On the tangent bundle $\pi : TM \to M$ we will define a structure of vector bundle in the sense of definition 1.1.

Suppose that on the manifold M we have a smooth structure defined by an atlas of charts $\mathcal{A} = \{(U_i, \varphi_i)/i \in I\}$. For each chart (U_i, φ_i) we define a local trivialization map $\Phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$ by sending a tangent vector $X_x \in p^{-1}(U_i) \subset TM$ on the couple $(x, T_x(\varphi_i))$.

Example *Canonical line bundle λ_n^1*

We denote by $\mathbb{RP}^n$ the real projective space of dimension n defined as the set of all lines of the vector space $\mathbb{R}^{n+1}$ passing through the origin. We note that the real projective space $\mathbb{RP}^n$ can be obtained from the sphere S^n by identifying x to $-x$.

It is easy to verify that on the subset $E(\lambda_n^1) = \{(v, L) \in \mathbb{R}^{n+1} \times \mathbb{RP}^n / v \in L\}$ there is a structure of vector bundle of rank $n = 1$ with projection $\pi(v, L) = L$. Moreover, we may verify that the inclusion $j : \mathbb{RP}^n \hookrightarrow \mathbb{RP}^{n+1}$ implies $j^*(\lambda_{n+1}^1) = \lambda_n^1$.

Note that similarly, over the infinite real projective space $\mathbb{PR}^\infty := \bigcup_{n \geq 1} \mathbb{PR}^n$ we may define a canonical line bundle which will be noted $\lambda^1 = (E(\lambda^1), \pi, \mathbb{PR}^\infty)$. Equally, we verify that for each integer $n \geq 1$ the inclusion $j_n : \mathbb{RP}^n \hookrightarrow \mathbb{RP}^\infty$ induces the canonical line bundle λ_n^1.

Example *Normal bundle $\nu(M)$*

Let $i : M^n \hookrightarrow \mathbb{R}^{n+k}$ be an embedded submanifold. On the subset $\nu(M) = \{(x, v) \in M \times \mathbb{R}^{n+k} / v \perp T_x M\}$ we define a structure of vector bundle of rank k with projection $\pi(x, v) = x$ as follows.

Since M is embedded, then for each $x_0 \in M$ there is an open set $U \subset \mathbb{R}^{n+k}$ and a diffeomorphism $\varphi : U \to \mathbb{R}^k \times \mathbb{R}^n$ such that, $M \cap U = \varphi^{-1}(\{0\} \times \mathbb{R}^n)$. Thus, if for each

$1 \leq i \leq n$ we put $f_i = pr_i \circ \varphi : U \to \mathbb{R}$ we verify easily that the following map

$$\begin{array}{llll} \Phi : & U \times \mathbb{R}^n & \to \pi^{-1}(U) & \\ & (x, (a_1, \cdots, a_n) & \to & (x, \sum_{i=1}^{i=n} a_i \mathbf{grad} f_i(x)) \end{array}$$

defines a local trivialization map over x_0.

Note that if we consider the natural map $F : TM \oplus \nu(M) \to i^*(T\mathbb{R}^{n+k})$ which sends a couple of tangent vectors (v_x, w_x) on $v_x + w_x \in T_x\mathbb{R}^{n+k}$ we obtain in this way an isomorphism,

$$TM \oplus \nu(M) \simeq i^*(T\mathbb{R}^{n+k}) \simeq \varepsilon^{n+k}(M).$$

In the following, we will introduce the notion of cross-sections for bundles. We will use this notion to solve the problem of comparison of vector bundles.

Definition 11.1.4 *Let $\zeta^n = (E, \pi, M)$ be a real vector bundle of rank $n \geq 0$.*

1. *A cross-section of π is a smooth function $s : M \to E$ such that for each $x \in M$ we have $\pi \circ s(x) = x$.*

2. *We say that the family of cross-sections $s_1, \cdots, s_n : M \to E$ are nowhere dependent, if for each $x \in M$ the family of vectors $s_1(x), \cdots, s_n(x)$ are linearly nowhere dependent in the vector space $\pi^{-1}(\{x\}$.*

Proposition 11.1.5 *A real vector bundle $\zeta^n = (E, \pi, M)$ admits $1 \leq k \leq n$ cross-sections nowhere dependent if and only if there is a real vector bundle η^{n-k} over M such that the Whitney sum $\eta^{n-k} \oplus \varepsilon^k(M)$ is isomorphic to ζ^n. In particular, the vector bundle ζ^n is trivial if and only if ζ^n admits n cross-sections nowhere dependent.*

Proof. Assume that the bundle ζ^n admits k nowhere dependent cross-sections $s_1, \cdots, s_k : M \to E$, and observe that, if we consider the following bundle map

$$\begin{array}{llll} \psi : & M \times \mathbb{R}^k & \to & E \\ & (x, (a_1, \cdots, a_k)) & \to & \sum_{i=1}^{i=k} a_i s_i(x) \end{array}$$

then Theorem 1.3 allows us to find a vector bundle η^{n-k} over M such that $\eta^{n-k} \oplus \varepsilon^k(M) \simeq \zeta^n$.

Corollary 11.1.6 *A line vector bundle ζ^1 is trivial if and only if ζ^1 admits a non zero cross-section.*

Example In this example we shall show that each cross-section of the canonical line bundle $\lambda_n^1 = (E(\lambda_n^1), \pi, \mathbb{PR}^n)$ vanishes at some point $x_0 \in \mathbb{PR}^n$. Consequently, we conclude that the line bundle λ_n^1 is non trivial.

Let $s : \mathbb{PR}^n \to E(\lambda_n^1) \subset \mathbb{R}^{n+1} \times \mathbb{PR}^n$ be a cross-section. Observe that if we compose s with the canonical projection map $p : S^n \to \mathbb{PR}^n$ we can find a real smooth function $S : S^n \to \mathbb{R}$ such that

$$s \circ p(x) = (S(x) \cdot x, \{\pm x\}), \forall x \in S^n.$$

Thus, since we have $p(-x) = p(x)$ this implies that $S(-x) = -S(x)$. Consequently by the connectivity of the sphere S^n there is some x_0 such that $s \circ p(x_0) = (0, \{\pm x_0\})$.

11.1.3 Reduction of structural group

Let $\zeta^n = (E, \pi, M)$ be a real vector bundle. Let $\varphi_i : U_i \times \mathbb{R}^m \to \pi^{-1}(U_i)$ and $\varphi_j : U_j \times \mathbb{R}^m \to \pi^{-1}(U_j)$ be a couple of trivialization maps. Note that if $U_i \cap U_j \neq \emptyset$ then we can find a smooth function $g_{i,j} : U_i \cap U_j \to GL(m, \mathbb{R})$ called *transition map* and which is determined by the expression:

$$\varphi_j^{-1} \circ \varphi_i(x, v) = (x, g_{i,j}(x)(v)), \qquad \forall x \in U_i \cap U_j.$$

It is easily verified that on an open set $U_i \cap U_j \cap U_k \neq \emptyset$ we have the following relation:

$$g_{j,k} \circ g_{i,j} = g_{i,k}$$

called *the cocycle condition.*

Conversely, following D. Husmoller [3] Ch. 5, one can associate to any open covering U_i of a manifold M and any cocycle $g_{i,j} : U_i \cap U_j \to GL(m, \mathbb{R})$. Consequently, every preferred system $(U_i, g_{i,j})$ (or *atlas*) determines a real vector bundle $\zeta^n = (E, \pi, M)$. Indeed, in [3], it is shown that if two systems of cocycles $g_{i,j}, \widetilde{g}_{i,j} : U_i \cap U_j \to GL(m, \mathbb{R})$ are *equivalent*, that is there are maps $f_i : U_i \to GL(m, \mathbb{R})$ such that,

$$\widetilde{g}_{i,j} = f_j g_{i,j} f_i,$$

then $g_{i,j}$ and $\widetilde{g}_{i,j}$ define the same vector bundle over the manifold M.

Definition 89 *Let $G \subset GL(n, \mathbb{R})$ be a Lie group. If a vector bundle $\zeta = (E, \pi, M)$ can be defined by a cocycle $g_{i,j} : U_i \cap U_j \to G$ we say that ζ has a structural group G or that the structural group of ζ is reducible to G.*

Definition 90 *The structural group of every real vector bundle $\zeta^n = (E, \pi, M)$ can be reduced to the Euclidian orthogonal Lie group $O(n) \subset GL(n, \mathbb{R})$.*

Theorem 97 *For each vector bundle ζ^n the following propositions are equivalent :*

1. *ζ^n admis $1 \leq k \leq n$ nowhere dependent sections ;*

2. *ζ^n is isomorphic to the Whitney sum $\varepsilon^k(M^n) \oplus \zeta^{n-k}$;*

3. *the structural group $O(n)$ of ζ^n can be reduced to the subgroup $O(n-k) \subset O(n)$.*

In section 4, we will use the machinery of obstruction theory to solve the problem of reduction of structural group $O(n)$ to $O(n-k)$. In fact, the study of this question which will drive us to the constructing of the Stiefle-Whitney characteristic classes.

11.1.4 The Classification of Real Vector Bundles

In this paragraph, we generalize the construction of the canonical line bundle λ_n^1 for constructing a new vector bundle which will play a crucial role in the classification vector bundles on a manifold M.

We denote by $G_{n,k}$ the manifold of all vectorial k-planes $P \subset \mathbb{R}^{n+k}$. In [4], the smooth structure of $G_{n,k}$ is well detailed as well as the one of the infinite Grassmannian manifold $G_k = \bigcup_{n \geq k+1} G_{n,k}$. The manifold G_k is called sometimes *classifying space* for the group $O(k)$ and denoted by $BO(k)$. Observe that we have $G_{n,1} = \mathbb{PR}^n$ and $BO(1) = G_1 = \mathbb{PR}^\infty$.

Let $\gamma_{n,k} = (E(\gamma_{n,k}), \pi, G_{n,k})$ be the vector bundle with total space the space $E(\gamma_{n,k}) \subset G_{n,k} \times \mathbb{R}^{n+k}$ of all pairs (V, v) such that $v \in V$ and projection $\pi(V, v) = V$. We call it the Grassmann bundle or the universal bundle over $G_{n,k}$ [4].

Theorem 98 *If $n \geq k + n$ then for each vector bundle ζ of rank k over a manifold M there is a classifying map $f_\zeta : M \to G_{n,k}$ such that $(f_\zeta)^*(\gamma_{n,k}) \simeq \zeta$. In fact, the classifying map f_ζ is unique up homotopy and satisfies the property : $\zeta \simeq \eta$ if and only if $f_\zeta \simeq f_\eta$.*

To end this paragraph, we note that the result of this theorem is still true if we pass to the direct limit, $\varinjlim_{n \geq k} \gamma_{n,k} = \gamma_k = (E(\gamma_k), \pi, G_k)$. Consequently, the set of k-vector bundles on a manifold M is isomorphic to set of all homotopy classes $[M, BO(k)]$.

11.2 The First Construction of Stiefel-Whitney Classes

In this section, our purpose is to describe an historic formula which presents the Stiefel-Whitney characteristic classes as a dual class of a singular cycle associated to a family of special vector fields. To get this formula, we recall briefly the notion of orientation for maps as well as for real vector bundles, Poincaré duality and Thom isomorphism. Next, we recall the notion of index of vector field, the Euler characteristic number and its relation with the Euler class. We finish this section by an axiomatic characterization of the Stiefel-Whitney characteristic classes.

11.2.1 Orientable Manifolds and Poincaré Duality

Definition 11.2.1 *We denote by* $\mathbf{R}$ *the ring* $\mathbb{Z}$ *or* $\mathbb{Z}_2$. *Let* M^n *be a compact topological* n*-manifold and let* $L \subset M^n$ *be a closed subset.*

1. *We say that* M *is* $\mathbf{R}$*-orientable along* L *if we have* $H_n(M, M - L; \mathbf{R}) = \mathbf{R}$. *The generator of the group* $H_n(M, M - L; \mathbf{R}) = \mathbf{R}$ *is called the orientation class of* M *along* L *and is noted,* $[M]_L =: 1 \in \mathbf{R}$. *When* $M = L$ *we set* $[M] := [M]_M$ *and this class is called the* $\mathbf{R}$*-orientation class or fundamental class of* M.

2. *When the manifold* M *is with boundary, we will say that* M *is* $\mathbf{R}$*-orientable if we have* $H_n(M, M - \partial M; \mathbf{R}) = \mathbf{R}$. *The orientation class of* M*, in this case, is denoted by* $[M, \partial M]$ *or* $[M] := 1 \in H_n(M, \partial M; \mathbf{R})$.

Suppose that M^n is $\mathbf{R}$-orientable. For any compact subsets $L \subset K \subset M$ and for any open subsets $V \subset U \subset M$ such that $(K, L) \subset (U, V)$ the orientation class of $[M]$ allows us to define a natural morphism by the mean of the cap product,

$$\cap[M] : H^p(U, V; \mathbf{R}) \longrightarrow H_{n-p}(M - L, M - K; \mathbf{R}).$$

If we use the direct limit on open sets $V \subset U \subset M$, we obtain clearly a new cap product,

$$\cap[M] : \check{\mathrm{H}}^p(K, L; \mathbf{R}) \longrightarrow H_{n-p}(M - L, M - K; \mathbf{R})$$

where $\check{\mathrm{H}}^p(K, L; \mathbf{R}) := \varinjlim_{i \geq 1} H^p(U, V; \mathbf{R})$ defines the *Čech cohomology group* of degree $p \in \mathbb{N}$.

Note that the Čech cohomology is isomorphic to the singular cohomology in the case of spaces homotopy equivalent to CW complexes. That is to say, for any pair of suitable compact subsets $K \subset L$ there is a canonical isomorphism,

$$\check{\mathrm{H}}^p(K, L; \mathbf{R}) \simeq H^p(K, L; \mathbf{R}), \qquad \forall p \in \mathbb{N}.$$

The reader can find a proof of this result in [2].

Theorem 99 *Let* M^n *be a topological compact oriented* n*-manifold. Then for any compact subsets* $L \subset K$ *of* M *the cap product with the orientation class*

$$\cap[M] : H^p(K, L; \mathbf{R}) \longrightarrow H_{n-p}(M - L, M - K; \mathbf{R})$$

is an isomorphism.

If the boundary $\partial M \neq \emptyset$ *defines two isomorphisms,*

$$\cap[M] : H^p(M, \partial M; \mathbf{R}) \longrightarrow H_{n-p}(M, \mathbf{R}) \quad \textit{and} \quad \cap[M] : H^p(M; \mathbf{R}) \longrightarrow H_{n-p}(M, \partial M; \mathbf{R}).$$

In the following of this chapter, we denote by $D_{K,L}$ or $D : H_p(M-L, M-K; \mathbf{R}) \longrightarrow H^{n-p}(K, L; \mathbf{R})$ the inverse of the Poincaré duality isomorphism. That is,

$$D(x) \cap [M] = x, \qquad \forall x \in H_p(M-L, M-K; \mathbf{R}).$$

Let $N^p \subset M^n$ be a closed submanifold. Remark that if we denote the inclusion of N in M by $i : N \hookrightarrow M$ and if $[N] \in H_p(N, \mathbf{R})$ denotes the orientation class of N then we can define a cohomology class as follows :

$$D(i_*([N])) \in H^{n-p}(M, \mathbf{R}).$$

Definition 11.2.2 *Let $N^p \subset M^n$ be a closed submanifold. The dual class of N is the cohomology class $x \in H^{n-p}(M, \mathbf{R})$ given by*

$$x \cap [M] = i_*([N]).$$

In particular, the dual class of any point $\{*\} \subset M$ is called the dual cohomology class of M and is denoted by $\mu \in H^n(M, \mathbf{R})$. Thus we have $\mu \cap [M] = 1$.

11.2.2 Orientable Real Vector Bundles and Thom Class

In this paragraph, we define the notion of orientation of vector bundles. As in the case of manifolds, the notion of orientation will be defined by a cohomology class called *the Thom class*. In the sequel, the Thom class will play a fundamental role to construct explicitly the Stiefel-Whitney cohomology classes.

Giving a real vector bundle $\zeta^n = (E, \pi, M)$, for each subset $A \subset M$ we denote by $\pi^{-1}(A)_0$ the subset equals to the total space E without its zero section.

Definition 11.2.3 *We say that a real vector bundle ζ^n is $\mathbf{R}$-orientable if for each $x \in M$ there is a preferred generator (local orientation class)*

$$u_x \in H^n(\pi^{-1}(\{x\}), \pi^{-1}(\{x\})_0; \mathbf{R}),$$

such that the following local compatibility condition is satisfied :

For each $x \in M$ there is a neighborhood V and a cohomology class

$$u_V \in H^n(\pi^{-1}(V), \pi^{-1}(\{V\})_0; \mathbf{R})$$

such that the inclusion map $j_x : (\pi^{-1}(\{x\}), \pi^{-1}(\{x\})_0) \hookrightarrow (\pi^{-1}(V), \pi^{-1}(\{V\})_0)$ satisfies

$$(j_x)^*(u_V) = u_x$$

Now, we give without proof the Thom isomorphism which is fundamental in algebraic topology. A proof of this theorem can be found in [4] or in [1].

Theorem 100 *Let $\zeta^n = (E, \pi, M)$ be a real vector bundle $\mathbf{R}$-oriented. Then the cohomology group $H^k(E, E_0, \mathbf{R}) = 0$ for $1 \leq k \leq n-1$, and $H^n(E, E_0, \mathbf{R})$ is generated by a unique cohomology class $u \in H^n(E, E_0; \mathbf{R})$ which satisfies the two following propositions :*

1. *For each $x \in M$ the inclusion map $j_x : (\pi^{-1}(\{x\}), \pi^{-1}(\{x\})_0) \hookrightarrow (E, E_0)$ sends the class u on the local orientation class u_x.*
2. *The morphism $\Phi : H^p(M; \mathbf{R}) \xrightarrow{\pi^*} H^p(E, \mathbf{R}) \xrightarrow{\cup u} H^{p+n}(E, E_0; \mathbf{R})$ is an isomorphism.*

The cohomology class $\Phi(1) = u \in H^n(E, E_0; \mathbf{R}) \simeq \mathbf{R}$ is called the *Thom class* of the $\mathbf{R}$-oriented vector bundle ζ^n. As a consequence of Thom isomorphism, we conclude that the orientation of the bundle can be defined by the choice of a generator $u \in H^n(E, E_0, \mathbf{R})$.

Below, we will describe a method in which the Thom class is defined explicitly. This method comes from Bredon's book [1].

Let $\zeta^n = (E, \pi, M)$ be a real vector bundle such that its base M is a connected closed (i.e. compact without boundary) $\mathbf{R}$-oriented manifold and for which the total space E is $\mathbf{R}$-oriented. We denote by $\pi : \mathbf{D}(\pi) \to M$ the n-disk bundle associated to E and by $s : M \to \mathbf{D}(\pi)$ the zero section of π.

Definition 11.2.4 *With the above notations, the Thom class of the n-disk bundle $\pi : \mathbf{D} \to M$ is the class $\tau \in H^m(\mathbf{D}(\pi), \partial\mathbf{D}(\pi); \mathbf{R})$ given by*

$$\tau := D(s_*[M]).$$

That is, we have, $\tau \cap [\mathbf{D}(\pi)] = s_*([M])$.

Note that, in [1] p. 386, Bredon shows that the composed morphism

$$\Phi : H^p(M; \mathbf{R}) \xrightarrow{\pi^*} H^p(\mathbf{D}(\pi); \mathbf{R}) \xrightarrow{\cup\tau} H^{p+m}(\mathbf{D}(\pi), \partial\mathbf{D}(\pi); \mathbf{R})$$

is an isomorphism called Thom isomorphism. Besides, he showed that the restriction of τ at every point $x \in M$ induces a local orientation class $\tau_x \in H^n(\pi^{-1}(\{x\}), \pi^{-1}(\{x\})_0; \mathbf{R})$. In other words, the class τ is equal to the Thom class u defined in the Thom theorem.

11.2.3 Euler Class

To finish this section dedicated to the notions of orientation, for any real vector bundle $\mathbf{R}$-oriented $\zeta^n = (E, \pi, M)$ we associate our first characteristic cohomology class : the *Euler class* in the cohomology group $H^n(M; \mathbf{R})$.

Note that, since all continuous sections $s : M \to E$ are homotopic, the cohomology class given by the formula

$$\chi(\zeta^n) := s^*(\tau)$$

is well defined. It is called : the Euler class of the vector bundle ζ^n.

OrProposition

{ **1** *If a map bundle $(F, f) : \zeta \to \eta$ preserves the orientation, then, we have the naturality property $f^*(\chi(\eta)) = \chi(\zeta)$.*

The following proposition gives us a topological interpretation of the Euler class. In the next section we will give other topological properties of the Euler class.

Proposition 11.2.5 *If the vector bundle $\zeta^n = (E, \pi, M)$ has a non-zero section $s : M \to E$ then the Euler class satisfies :*

$$\chi(\zeta^n) = s^*(\tau) = 0 \in H^m(M; \mathbb{Z}).$$

We admit the proof of this proposition which can be found in Bredon [1]or in Spivak [6]. In particular, we have :

Proposition 11.2.6 *Let M be a differential manifold. If M has a vector field without singularity then the Euler class of the tangent bundle TM is zero.*

11.2.4 Index of Vector Field

Let X be a vector field defined on an open set $U \subset \mathbb{R}^n$ with an isolated zero $0 \in U$. For a small real $\varepsilon > 0$ we consider the function

$$\begin{array}{rccl} \overline{X}_\varepsilon : & S^{n-1}(O, \varepsilon) \subset U & \to & S^{n-1}(O, 1), \quad \varepsilon > 0 \\ & x & \to & \dfrac{X_x}{\| X_x \|} \end{array}$$

Note that, since for $\varepsilon > 0$ and $\varepsilon' > 0$ the function $\overline{X}_\varepsilon$ is homotopic to $\overline{X}_{\varepsilon'}$ then they have the same degree which only depends on the vector field X. We denote it by

$$ind(X, O) := deg(\overline{X}).$$

For any manifold M we can define the notion of index for vector fields at an isolated zero $m \in M$. And, for all vector field with finite number of isolated zeros $\{x_1, x_2, \cdots, x_k\} \subset M$ we associate the sum of indices,

$$\sigma(X) = \sum_{i=1}^{i=m} ind(X, x_i).$$

The next theorem gives us an elegant formula which allows us to calculate explicitly the integer $\sigma(M)$. Its proof can be found in Bredon and Spivak books.

Theorem 11.2.7 (Spivak [6]) *Let M^n be a compact closed manifold with an orientation $\mu \in H^n(M, \mathbb{Z})$. Let $X : M \to TM$ be a vector field with a finite number of zeros $\{x_1, \cdots, x_m\} \subset M^n$. Then the Euler class of the tangent bundle TM is given by*

$$\chi(TM) = \sigma(X)\mu.$$

Corollary 11.2.8 *If X and Y are vector fields with finitely many zeros then we have, $\sigma(X) = \sigma(Y)$.*

According to the result of this corollary we deduce that the sum of indices $\sigma(X)$ of any vector field X which have a finite number of zeros on a manifold M is a differential invariant of M. Furthermore, by the Poincaré-Hopf theorem (see [1] and [6]) we know that the sum of indices $\sigma(X)$ is in fact equal to the Euler characteristic

$$\chi(M) = \sum_{i \geq 0} \text{rank } (-1)^i H^i(M^n, \mathbb{R}).$$

Then we conclude that

$$\chi(TM) = \chi(M)\mu.$$

Now, according to this formula we deduce that the number $\sigma(X)$ is in fact a topological invariant of M.

Proposition 11.2.9 *The following propositions are equivalent :*

1. *the manifold M^n has a vector field without zero ;*
2. *the Euler-Poincaré characteristic is zero $\chi(M) = 0$;*
3. *the Euler class of M is zero, $\chi(TM) = 0$.*

11.2.5 The Construction of Stiefel-Whitney Classes

In this paragraph, we start with a new construction of the Euler class. As we will see below, this idea can be generalized easily to the constructing of Siefel-Whitney classes.

Let X be a vector field with a finite number of isolated zeros $\{x_1, x_2, \cdots, x_k\} \subset M^n$. If we consider the singular 0-cycle of M given by,

$$z(X) = \text{Ind } (X, x_1)x_1 + \text{Ind } (X, x_2)x_2 + \cdots + \text{Ind } (X, x_k)x_k \in Z_0(M, \mathbb{Z}),$$

then using the Poincré-Hopf theorem and Theorem 2.8 we see that,

$$\chi(TM) \cap [M] = \sigma(X).1 = \chi(M).1 = [z(X)] \in H_0(M, \mathbb{Z}).$$

Thus, as a consequence of this relation we conclude that the Euler class of the tangent vector bundle TM is equal to the dual class of the homology class $[z(X)] \in H_0(M, \mathbb{Z})$. Hence, we can write :

$$\chi(TM) = D([z(X)]). \tag{11.1}$$

Now we look at the generalization of formula (1). Let $\zeta^n = (E, \pi, M)$ be a smooth real vector bundle and let $s_1, s_2(x), \cdots, s_k : M \to E$ be a family of $k \leq m$ smooth sections such that the subset of zeros

$$\Sigma = \{x \in M / s_1(x) \wedge s_2(x) \wedge \cdots \wedge s_k(x) = 0\} \overset{i}{\hookrightarrow} M$$

forms a singular $(k-1)$-cycle in the manifold M. Thus, if we consider the dual class of the homology class $i_*([\Sigma]) \in H_{k-1}(M, \mathbb{Z}_2)$, we obtain a cohomology class,

$$\omega_{n-k+1}(\zeta^n) := D(i_*([\Sigma])) \in H^{n-k+1}(M, \mathbb{Z}_2), \qquad \forall 1 \leq k \leq m, \tag{11.2}$$

called *the* $(n-k+1)$*-th Siefel-Whitney characteristic classes* of the vector bundle E. In particular, if we consider a manifold M^n on which we apply Formula (2) to any family of k-fields $X_1, X_2, \cdots, X_k : M^n \to TM^n$ such that the set of zeros

$$\Sigma = \{x \in M / X_1(x) \wedge X_2(x) \wedge \cdots \wedge X_k(x) = 0\} \overset{i}{\hookrightarrow} M$$

is a $(k-1)$-cycle we will obtain the family of Siefel-Whitney characteristic classes of the tangent bundle TM, $\omega_k(M) \in H^{n-k+1}(M, \mathbb{Z}_2)$.

In this chapter, we do not show that the Siefel-Whitney characteristic classes defined by (2) depends only on the vector bundle ζ^n. However in the next sections, we shall see by other direct methods that this fact is true.

11.2.6 Axiomatic Characterization of Stiefel-Whitney Classes

In this paragraph, we shall give a new definition of the Stiefel-Whitney classes. Precisely, we shall use the axiomatic method introduced by Hirzebruch. As we shall see, the advantage of the axiomatic method is an easy computation of the characteristis classes for many vector bundles.

Axiom 1: For each real vector bundle $\zeta^n = (E, \pi, M)$ there is a family of cohomology classes

$$\omega_k(\zeta^n) \in H^k(M, \mathbb{Z}_2), \qquad k = 0, 1, 2, \cdots,$$

called the Stiefel-Whitney classes of ζ^n satisfying

$$\omega_0(\zeta^n) = 1 \in H^0(M, \mathbb{Z}_2) \qquad \text{and} \qquad \omega_k(\zeta^n) = 0, \forall k > n.$$

Axiom 2: Naturality. If $(F, f) : \zeta \to \eta$ is a bundle morphism then,

$$\omega_k(\zeta) = f^*(\omega_k(\eta)).$$

Axiom 3: The Whitney product theorem If ζ and η are vector bundles over the same base space, then

$$\omega_k(\zeta \oplus \eta) = \sum_{i+j=k} \omega_i(\zeta) \cup \omega_j(\eta).$$

Observe that if we define the total Stiefel-Whitney class $\omega(\zeta^n) = 1+\omega_1(\zeta^n)+\cdots+\omega_n(\zeta^n)$, then, we can write

$$\omega(\zeta \oplus \eta) = \omega(\zeta) \cup \omega(\eta).$$

Axiom 4: For the canonical line bundle $\lambda_1^1 = (E(\lambda_1^1), \pi, S^1)$ the Stiefle-Whitney class $\omega_1(\lambda_1^1)$ is non zero.

Example 1) The Stiefel-Whitney characteristic classes of a point $\{\star\}$ are zero.

2) As the trivial vector bundle $\varepsilon^n(M) = 1$ can be induced *via* a constant map $f : M \to \{\star\}$, then, according to the naturality axiom we see that $\omega(\varepsilon^n(M) = 1$.

Proposition 11.2.10 *If a real vector bundle ζ^n admits $1 \leq k \leq n$ linearly independent sections then*

$$\omega_{n-j+1}(\zeta^n) = 0, \qquad \forall 1 \leq j \leq k.$$

Proof. With the hypothesis of this proposition we can find a vector bundle η^{n-k} such that the Whitney sum $\eta^{n-k} \oplus \varepsilon^k(M)$ is isomorphic to ζ^n. Hence, by the Whitney product theorem (axiom 3) we deduce the result.

Section 4 will be dedicated to obstruction theory, we shall see that the necessary conditions expressed in the proposition 2.11 are also sufficient.

Theorem 101 *For each real vector bundle $\zeta^n = (E, \pi, M)$ there is an unique system of cohomology classes $\omega_k(\zeta^n) \in H^k(M, \mathbb{Z}_2)$, $k = 0, 1, \cdots, n$, which depends only on the equivalence type of ζ^n and which verify the four axioms above.*

Here I do not develop the proof of this theorem. Indeed, the reader can find several proofs of it in the literature. For example, [4] where uses the splitting principle is used, or in [3] where the Leray-Hirsh theorem to projective vector bundle is applied.

Note that this theorem can be used to get the Stiefel-Whitney characteristic classes

$$\omega_k \in H^k(BO(n), \mathbb{Z}_2)$$

for the universal bundle $\gamma_n = (E(\gamma_n), \pi, BO(n))$ which will be *universal* in the following sense : if we consider vector bundle $\zeta^n = (E, \pi, M)$ and a classifying map $f_\zeta : \zeta =$

$(E(\zeta), \pi, M) \to \gamma_n$ (i.e. ${f_\zeta}^*(\gamma_n) = \zeta^n$), by the axiom 2 of naturality, we obtain a relation of the form,

$$(f_\zeta)^*(\omega_k) = \omega_k(\zeta^n) \in H^k(M, \mathbb{Z}_2),$$

from which we see that the cohomology classes $\omega_k \in H^k(BO(n), \mathbb{Z}_2)$ induce the Stiefel-Whitney characteristic classes of any vector bundle.

We finish this paragraph by some interesting examples of computation of Stiefel-Whitney classes.

Example If we apply the result of the example 4 of the sphere $S^n \hookrightarrow \mathbb{R}^{n+1}$, we obtain $TS^n \oplus \nu(S^n) \simeq \varepsilon^{n+1}(S^{n+1})$. Thus, by the Whitney product theorem,

$$\omega(TS^n \oplus \nu(S^n)) = \omega(TS^n) \cup \omega(\nu(S^n)) = 1$$

.

Hence, since the normal vector bundle $\nu(S^n)$ is trivial, we see that $\omega(S^n) = 1$.

Example Since we know that the inclusion map $j_n : \mathbb{PR}^1 \hookrightarrow \mathbb{PR}^n$ induces an isomorphism in singular cohomology $(j_n)^* : H^1(\mathbb{PR}^n, \mathbb{Z}_2) \to H^1(\mathbb{PR}^1, \mathbb{Z}_2) = \mathbb{Z}_2$, then the axiom 4 implies that $\omega(\lambda_n^1) = 1 + a$, where $a \in H^1(\mathbb{PR}^n, \mathbb{Z}_2)$ is the generator of $\mathbb{Z}_2$.

Example Recall that in [4], Milnor showed that the tangent bundle of the real projective space $\mathbb{PR}^n$ satisfies the isomorphism

$$T\mathbb{PR}^n \oplus \varepsilon^1(\mathbb{PR}^n) \simeq \underbrace{\lambda_n^1 \oplus \cdots \oplus \lambda_n^1}_{(n+1)-fold}.$$

Consequently, if we apply axiom 3 we conclude that the total Stiefel-Whitney class of the real projective space $\mathbb{PR}^n$ is equal to $\omega(\mathbb{PR}^n) = (1+a)^{n+1}$.

11.3 The Second Construction of Stiefel-Whitney Classes

In this section, we will construct the Stiefel-Whitney classes by using Steenrod cohomological operations in singular cohomology with coefficient in the field $\mathbb{Z}_2$. Consequently, we will see that the Stiefel-Whitney classes are *topological invariants.* That is to say, they are independent of the differential structure.

11.3.1 Steenrod Cohomological Operations

Theorem 11.3.1 *For any pair of topological spaces (X, A) there is a family of additive homomorphisms*

$$Sq_n^k : H^n(X, A; \mathbb{Z}_2) \to H^{n+k}(X, A; \mathbb{Z}_2),$$

called the "ith Steenrod square" such that for all $x \in H^n(X, A; \mathbb{Z}_2)$ we have

1. $Sq_n^0(x) = x$.

2. $Sq_n^n(x) = x^2 =: x \cup x$.

3. $Sq_n^k(x) = 0$ *for* $k > n$.

4. *The Cartan formula : for* $x \in H^n(X, A; \mathbb{Z}_2)$ *and* $y \in H^m(X, A; \mathbb{Z}_2)$,

$$Sq_{n+m}^k(x \cup y) = \sum_{i+j=k} Sq^i(x) \cup Sq_m^j(y).$$

An elementary proof of this theorem can be found in Bredon [1].

11.3.2 Stiefel-Whitney Classes

Let $\zeta^n = (E, \pi, M)$ be a real vector bundle and $\pi : \mathbf{D} \to M$ be the n-disk bundle associated to ζ^n. Recall that in Definition 2.4 we have associated E a Thom class $\tau \in H^n(\mathbf{D}, \partial\mathbf{D}; \mathbb{Z}_2)$ such that the homomorphism

$$\Phi : H^k(M, \mathbb{Z}_2) \xrightarrow{\pi^*} H^k(\mathbf{D}; \mathbb{Z}_2) \xrightarrow{\cup\tau} H^{k+m}(\mathbf{D}, \partial\mathbf{D}; \mathbb{Z}_2)$$

is an isomorphism.

Now, observe that since Φ is an isomorphism then for the kth Steenrod square Sq_n^k there is a unique cohomology class $\omega_k \in H^k(M, \mathbb{Z}_2)$ such that

$$\Phi(\omega_k) = Sq_n^k(\tau) \in H^{k+n}(\mathbf{D}, \partial\mathbf{D}; \mathbb{Z}_2) \tag{11.3}$$

Proposition 11.3.2 *The family of cohomology classes* $\omega_k \in H^k(M, \mathbb{Z}_2)$ *defined by* (3) *are equal to the Stiefel-Whitney characteristic classes of the real vector bundle* $\zeta^n = (E, \pi, M)$.

Proof. To see this, we verify that the cohomology classes ω_k satisfy the four axioms. Indeed, axiom 1 is a consequence of the definition of the cohomology classes ω_k while axiom 3 is a consequence of Cartan formula. Axiom 2 is a consequence of the fact that the Steenrood cohomological operations are natural transformations (i.e. functorial). Finally, if we consider the canonical vector line $\lambda_1^1 = (E(\lambda_1^1), \pi, S^1)$ we obtain

$$\Phi(\omega_1) = Sq_1^1(\tau) = \tau \cup \tau \in H^2(\mathbf{D}, \partial\mathbf{D}; \mathbb{Z}_2) \simeq H^1(S^1, \mathbb{Z}_2) = \mathbb{Z}_2.$$

Thus we conclude that the class ω_1 is the generator of $H^1(S^1, \mathbb{Z}_2)$. Hence $\omega_1 = \omega_1(\lambda_1^1)$.

11.4 The Third Construction of Stiefel-Whitney Classes

In this section, we will describe elements of obstruction theory first introduced by Steenrod [6] (See also Whithead [7]). Using this theory, we shall construct the Stiefel-Whitney characteristic classes as obstruction classes to the existence of a family of k-fields.

In this section, it is assumed some knowledge on the theory of cellular complexes and some notions of cellular cohomology with local coefficients. The reader can find the elements of these two theories in [6] and [7]. Also, I will work with the notion of fibre bundles which generalizes the notion of vector bundles defined in 1.2.

11.4.1 The Machinery of the Obstruction Theory

Let B be a CW-complex and let $\zeta = (E, \pi, B)$ be a fibre bundle whose F is an arcwise connected topological space. We suppose that on the subcomplex $L \subset B$ there exists a continuous cross-section,

$$\begin{array}{rccc} s: & L & \rightarrow & E \\ & \cap & & \downarrow \pi \\ id: & B & \rightarrow & B \end{array}$$

Problem : Extend continuously the map $s : L \rightarrow E$ to a cross-section defined over the base space B.

To solve this problem we shall work by induction on the dimension of the cells of B which do not belong to the subcomplex L. Precisely, as we shall explain below, we solve the problem over the n-skeleton $L \cup X_n \subset L \cup X_{n+1} \subset B$.

Step 1. For each vertex $x_i \in X_0$ which dooes not belong to L we associate one point $y_i \in F_{x_i} = \pi^{-1}(x_i)$. Note that if we put $s_0(x_i) = y_i$ we extend s over $L \cup X_0$.

Step 2. Now, we consider one 1-cell $\sigma^1 = (D^1, \psi)$ which is not in L but we suppose that this boundary is contained in $L \cup X_0$. As we know that the fibrations verify the property of homotopy lifting, this allows us to extend $s : L \rightarrow E$ over the subcomplex $L \cup X_1$.

Step 3. In this step we suppose that the cross-section $s : L \rightarrow E$ is continuously extended it on $L \cup X_n$. We denote this section by $s_n : L \cup X_n \rightarrow E$ and we intend to extend over the subcomplex $L \cup X_{n+1}$.

Let $\sigma^{n+1} = (D^{n+1}, \psi)$ be a $(n+1)$-cell which does not belong to the subcomplex L. Its center will be denoted by $x_\sigma = \psi(O)$. Since the disk D^{n+1} is contractible the induced fibre bundle $\psi^*(E)$ is trivial :

$$\begin{array}{ccccc} D^{n+1} \times F_{x_\sigma} & \simeq & \psi^*(E) & \overset{\overline{\psi}}{\to} & p^{-1}(\sigma^{n+1}) & \subset E \\ & & pr_1 \downarrow & & \downarrow \pi & \downarrow \\ & & D^{n+1} & \overset{\psi}{\to} & \sigma^{n+1} & \subset B \end{array}$$

Note that since π admits a continuous cross-section s_n over $L \cup X_n$, the restriction $s_{n|} : \partial\sigma^{n+1} \subset L \cup X_{n+1} \to E$ can be lifted in $\psi^*(E)$. We denote it by $\psi : S^n \to \psi^*(E)$. On the other hand, since the subset $\psi^*(E)$ is homotopic to the fibre F_{x_σ} this gives us a homotopy class $[\widetilde{\psi}] \in [S^n, F_{x_\sigma}]$. We denote this class by $c^{n+1}(\sigma^{n+1})$.

Proposition 11.4.1 *The section $s_n : L \cup X_n \to E$ can be extended continuously over the subcomplex $L \cup X_{n+1}$ if and only if for each $(n+1)$-cell of σ^{n+1} the homotopy class $c^{n+1}(\sigma^{n+1})$ is trivial.*

Proof. Observe that a continuous function $f : S^n \to X$ can be extended over the disk D^{n+1} if and only if it is null-homotopic, apply this fact for extending the map $\widetilde{\psi} : S^n \to \psi^*(E)$ over the subcomplex $L \cup X_n \cup_\psi \sigma^{n+1}$.

Hypothesis : In the rest of this section, we assume that the fibre type F of the fibration ζ is *simple*, that is, the action of the fundamental group $\pi_1(B)$ on the homotopy group $\pi_n(F)$ is trivial for every $n \geq 1$.

Under this assumption we see that the homotopy group $\pi_n(F)$ is independent of the base point base of F for every n. This allows us to identify the set of homotopy classes $[S^n, F]$ with the group $\pi_n(F)$. With the system of groups $\{\pi_n(F_x), x \in B\}$ we can define a local coefficient $\pi_n(\mathcal{F})$ in the base space B (See [6] and [7]). For example, we note that topological groups, Lie groups and homogeneous spaces, are simple [6] and [7].

Definition 11.4.2 *Let $\zeta : (E, p, B)$ be a fibration containing a simple fibre type. Let $L \subset B$ be a subcomplex over which there exists a cross-section $s : L \to B$. The map $c_s : C_{n+1}(B, L) \to \pi_n(\mathcal{F})$ which sends an $(n+1)$-cell σ^{n+1} over the homotopy class $c^{n+1}(\sigma^{n+1}) \in \pi_n(F_{x_\sigma})$ is called the obstruction cochain to extend the cross-section s over the subcomplex $L \cup X_{n+1} \subset B$.*

The following theorem is a fundamental result in obstruction theory. Its proof is very long and difficult, it is developed in detail in the books [6] and [7].

Theorem 11.4.3 *With the above notations we have the following propositions:*

1. *The obstruction cochain c_s is a cocycle in the differential complex of cellular cochains with local coefficients $(C^*(B, L; \pi_n(\mathcal{F})), \delta_*)$.*

2. *If s and $s' : L \cup X_n \to E$ are two homotopic cross-sections relative to L, then $c_s = c_{s'}$.*

3. *Let $f : (K, K') \to (B, L)$ be a cellular map. Then, the obstruction cocyle to lift f via the fibration $p : E \to B$ is $c_f = f^*(c_s) \in C^{n+1}(K, K'; \pi_n(\mathcal{F}))$. In other words, the obstruction cocycle is natural.*

4. *The cross-section $s_n : L \cup X_n \to E$ can be extended over $L \cup X_{n+1}$ if and only if the cohomology class $[c_s] = 0 \in H^{n+1}(B, L; \pi_n(\mathcal{F}))$.*

Consequently, we conclude that the cohomology class $[c_s] \in H^{n+1}(B, L; \pi_n(\mathcal{F}))$ is a homotopy invariant, only depending on the fibration $\zeta = (E, p, B)$ and on the subcomplex $L \subset B$. In the literature of algebraic topology, according to the result of the fourth proposition of this theorem, the cohomology class $[c_s]$ is called *the primary obstruction to the extension of* $s : L \to p^{-1}(E)$. Note that in the case where the subcomplex $L \subset B$ is empty we prefer to call the cohomology class $[c_s] \in H^{n+1}(B, \pi_n(\mathcal{F}))$ *characteristic class of the fibration* ζ and we denote it by c_ζ.

11.4.2 Euler Characteristic Class of a Sphere Bundle

Let M^n be a Riemannian manifold with a Riemaniann metric $<,>$. We denote by $\zeta = (T_1M, p, M)$ the tangent sphere bundle, that is, for every $x \in M$ we have

$$p^{-1}(\{x\}) = \{X_x \in T_xM / < X_x, X_x >= 1\} \simeq S^{n-1}.$$

Since the homotopy group $\pi_k(S^n)$ all vanish for every $1 \leq k < n-1$ and since $\pi_{n-1}(S^{n-1}) = \mathbb{Z}$, the primary obstruction to construct a cross-section for the fibration ζ is a cohomology class $e(\zeta) \in H^n(M, \mathbb{Z})$. Now by the result of the fourth proposition of Theorem 4.3, we conclude that $e(\zeta)$ vanishes if and only if the manifold M admits a vector field without zero. Consequently, this characteristic cohomology class $e(\zeta)$ coincides with the Euler class we had defined in 2.3.

11.4.3 Characteristic Classes of k-frames Bundle

In this paragraph, we shall use the elements of obstruction theory we had discussed in the previous section to construct the Stiefel-Whitney characteristic classes.

Let M^n be a differentiable manifold. Recall that the manifold M admits a k-frame if there exists k vector fields $X_1, \cdots, X_k$ on M nowhere dependent. Observe that if we assume that there are k-fields $X_1, \cdots, X_k$ nowhere dependent on M then for each point $x \in M$ we may define a k-frame $X_1(x) \wedge X_2(x) \wedge \cdots \wedge X_k(x)$ in the tangent space $T_xM \simeq \mathbb{R}^n$. Let us consider the k-frame bundle $\mathfrak{R}_k(M^n) = (E(n,k), \pi_k, M^n)$.

Recall that the fiber type of the $\mathfrak{R}_k(M^n)$ is by definition the Stiefel manifold of all k-frames denoted by $\mathbb{S}(n,k)$ which is canonically homeomorphic to the homogeneous space $\dfrac{O(n)}{O(n-k)}$. Thus if we consider the homotopy sequence of the main bundle $O(n-k) \hookrightarrow$

$O(n) \to \dfrac{O(n)}{O(n-k)}$ we verify easily that $\pi_i(\mathbb{S}(n,k)) = 0$ for every $0 \leq i < n-k$. Therefore, by using obstruction theory, we conclude :

1. For every differentiable manifold M^n there is at least a k-frame over the $(n-k)$-skeleton $X_{n-k} \subset M^n$.
2. The characteristic class of the k-frames bundle $\mathfrak{R}_k(M^n)$ is a cohomology class,

$$c(\mathfrak{R}_k(M^n)) \in H^{n-k+1}(M, \pi_{n-k}(\mathfrak{S}(n,k)))$$

with local coefficients $\pi_{n-k}(\mathfrak{S}(n,k))$ associated to the k-frames bundle $\mathfrak{R}_k(M^n)$.

Theorem 11.4.4 (N. Steenrod [6]) *The reduction modulo 2 of the obstruction classes*

$$c(\mathfrak{R}_k(M^n)) \in H^{n-k+1}(M, \pi_{n-k}(\mathfrak{S}(n,k))), \qquad k = 1, 2, \cdots, n$$

are equals to the Stiefel-Whitney characteristic classes $w_k(TM^n) \in H^k(M^n, \mathbb{Z}_2)$.

In [4], the reader will find a proof of this theorem in which the authors use the fact that Stiefel-Whitney characteristic classes of the universal vector bundle $\gamma_n = (E(\gamma_n), \pi_n, G_n)$ generate the cohomology ring $H^*(G_n, \mathbb{Z}_2)$.

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Differential Geometry and Topology, Discrete and Computational Geometry
M. Boucetta and J.-M. Morvan (Eds.)
IOS Press, 2005
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Chapter 12

On Curvatures

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The goal of this Chapter is to introduce the notion of *curvature measures* on a manifold with or without singularities. In particular, we define curvature measures on polyhedra. Our approach begins with the classical *Steiner formula* in the convex case. Then, we introduce the space of currents, which appears to be the good framework for this theory.

12.1 Background

We shall extensively use the background of Riemannian geometry introduced by M. Boucetta in the previous Chapter.

12.1.1 Riemannian metric

Let (M^n, g) be a n dimensional smooth Riemanian manifold endowed with its canonical Levi-Civita connexion ∇. We denote by TM its tangent bundle and by $\mathcal{T}M$ the space of smooth tangent vector fields. Its curvature tensor R satisfies the following equalities:

1. $R(X,Y)Z + R(Y,X)Z = 0$;
2. $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$;
3. $g(R(X,Y)Z,W) + g(R(X,Y)W,Z) = 0$;
4. $g(R(X,Y)Z,W) = g(R(Z,W)X,Y)$.

The two first equalities derive only on the definition R. The two last equalities depend on the metric. Equality (2) is called the first Bianchy identity. The curvature tensor R

satisfies another identity, called the second Bianchi identity, which involves its covariant derivative:

$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0.$$

For each point m of M^n and each 2-plane π of $T_m M^n$ one can define the sectional curvature $K(\pi)$ by

$$K(\pi) = g(R(X,Y)Y,X),$$

where X, Y denote orthogonal non zero vectors of $T_m M^n$. This definition does not depend on X, Y, but only on π.

The Ricci curvature S and the scalar curvature r are derived from R by the following equalities:

$$S(X,Y) = \sum_1^n g(R(e_i,X)Y,e_i);$$

$$r = \frac{1}{n(n-1)} \sum_i S(e_i,e_i);$$

where (e_i) is a local orthonormal frame. The previous formalism can be written by using the so called moving frame theory. Basically, one writes the equation in a local orthonormal frame, introducing new connexion forms and curvature forms. Let $(e_1, ..., e_n)$ be a local orthonormal frame of M^n. We denote by $\omega^1, ..., \omega^n$ the dual frame, (in the duality defined by the metric). The connexion forms ω_i^j are 1-forms defined by

$$\nabla_X e_j = \sum_{i=1}^n \omega_j^i(X) e_i.$$

Moreover, we put

$$R(e_j,e_k)e_l = \sum R_{ljk}^i e_i,$$

and we introduce the 2-form

$$\Omega_j^i = \frac{1}{2} \sum R_{ljk}^i \omega^k \wedge \omega^l.$$

The Cartan structural equations are:

$$d\omega_i = -\sum \omega_j^i;$$

$$d\omega_j^i = -\sum \omega_k^i \wedge \omega_j^k + \Omega_j^i.$$

12.2 Riemannian Submanifolds of Euclidean Spaces

Let M^n be a n-dimensional manifold, and

$$x : M^n \to \mathbb{E}^N,$$

be an immersion of M^n into $\mathbb{E}^N$, endowed with its standard scalar product $<,>$. Denote by $g = x^*(<,>)$ the induced metric on M^n. Then x becomes an isometric immersion of (M^n, g) into $\mathbb{E}^N$. In the following we identify systematically a tangent vector X to M^n and its image x_*X. Using this identification, the orthogonal complement of TM^n in $T\mathbb{E}^N$ is the normal bundle of M^n in $\mathbb{E}^N$. We denote it by $T^\perp M^n$. Consequently, one can consider three bundles over M^n: the tangent bundle to M^n, $TM^n \to M^n$, the restriction to M^n of the tangent bundle of $\mathbb{E}^N$ $T_{M^n}\mathbb{E}^N \to M^n$, and the normal bundle $T^\perp M^n \to M^n$. These three bundles are related by the following equation:

$$T_{|M^n}\mathbb{E}^N = TM^n \oplus T^\perp M^n.$$

Let ∇ (*resp.* $\tilde{\nabla}$) denote the Levi-Civita connexion on (M^n, g) (*resp.* $(\mathbb{E}^N, <,>)$). For every vector fields X, Y tangent to M^n, $\tilde{\nabla}_X Y$ can be considered as a section of $T_{M^n}\mathbb{E}^N$. As shown in the previous Chapter, it appears that its component tangent to M^n is exactly $\nabla_X Y$, so that the two connexions ∇ and $\tilde{\nabla}$ are related by *Gauss formula*:

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{12.1}$$

where $h(X, Y) \in T^\perp M^n$ denotes the normal component of $\tilde{\nabla}_X Y$. It is well known that

$$h : TM^n \times TM^n \to T^\perp M^n,$$

is a symmetric tensor. It is called the *second fundamental form* of M^n with respect to the immersion x.

Let ξ be a normal vector field and X a tangent vector field on M^n. One can decompose $\tilde{\nabla}_X \xi$ into its tangent component $-A_\xi X$ and its normal component $\nabla^\perp_X \xi$. We get *Weingarten formula*:

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi, \forall X \in TM^n, \forall \xi \in T^\perp M^n. \tag{12.2}$$

The tensor A is the *Weingarten tensor*, and is related to h by the equation:

$$< A_\xi X, Y > = < h(X, Y), \xi >, \forall X, Y \in TM^n. \tag{12.3}$$

On the other hand, $\tilde{\nabla}^\perp$ is a linear connexion on the normal bundle.

The geometric meaning of h is well known: it describes the *shape* of the immersion. For instance, M^n is totally geodesic in $\mathbb{E}^N$, that is $h = 0$, if the geodesics of M^n are sent

onto geodesics, (straigth lines), of $\mathbb{E}^N$. So in this case, $x(M^n)$ is nothing but a portion of an affine n-subspace of $\mathbb{E}^N$.

The *Gauss equation* relates the curvature tensor R of M^n to the second fundamental form of the submanifold M^n. It can be stated as follows: $\forall X, Y, Z, W \in TM^n$,

$$< R(X,Y)Z, W >=< h(X,W), h(Y,Z) > - < h(X,Z), h(Y,W) > . \tag{12.4}$$

The *Codazzi* equation is obtained by writing that the normal component of the curvature tensor of $\mathbb{E}^N$ is null:

$$(\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z) = 0, \forall X, Y, Z \in TM^n, \tag{12.5}$$

where

$$(\overline{\nabla}_X h)(Y,Z) = \nabla_X^{\perp} h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

Finally, the *Ricci* equation relates the normal curvature tensor to the Weingarten tensor: one more time, it is obtained by writing that the normal component of the curvature tensor of $\mathbb{E}^N$ is null:

$$R^{\perp}(X,Y,\xi,\eta) =< [A_\xi, A_\eta](X), Y >, \tag{12.6}$$

where

$$[A_\xi, A_\eta] = A_\xi A_\eta - A_\eta A_\xi.$$

If M^n is a hypersurface of $\mathbb{E}^{n+1}$, there are many simplifications in the previous theory, as we have seen in the previous Chapter, (M. Boucetta). Since there is only one normal direction, one can choose an orientation and define the second fundamental form as a tensor

$$h : TM \times TM \to \mathcal{C}^\infty(M).$$

At each point m, h_m is a bilinear symmetric form, which can be diagonalized in an orthonormal frame. Its eigenvalues are called the principal curvatures, and the eigenvectors are called the principal directions. The trace of h_m is the mean curvature of M^n at m and its determinant is the Gauss curvature of M^n at m. More generally, one have the following:

Definition 91 *Let M^n be an oriented hypersurface of $\mathbb{E}^{n+1}$. The k^{-th}-elementary symmetric function of the principal curvatures of M^n*

$$\Xi_k(\xi) = \{\lambda_{i_1}^\xi, ..., \lambda_{i_k}^\xi\}$$

is called the k^{-th}-mean curvature of M^n.

When the point m varies on M^n, one gets principal vector fields which can be integrated; one obtains curves, called lines of curvatures.

On can extend the notion of k^{-th}-mean curvatures of a hypersurface on a submanifold of any codimension, (see [10] for instance).

At each point m of M^n and for any normal direction ξ at m, the eigenvalues $(\lambda_1^\xi, ..., \lambda_n^\xi)$ of A_ξ are called *the principal curvatures* of M^n at m in the direction ξ.

Definition 92 *Let M^n be a n-dimensional Riemannian manifold (with or without boundary), isometrically embedded in $\mathbb{E}^N$. Let ξ be a (unit) normal vector field. The k^{-th}-elementary symmetric function of the principal curvatures of M^n in the direction ξ*

$$\Xi_k(\xi) = \{\lambda^{\xi}_{i_1}, ..., \lambda^{\xi}_{i_k}\}$$

is called the k^{-th}-mean curvature of M^n in the direction ξ.

Remark that

$$\det\,(I + tA_\xi) = \sum_{k=0}^{k=N-1} \Xi_k(\xi)t^k, \tag{12.7}$$

with $\Xi_k(\xi) = \sum_{I_k} \Delta_{I_k}(\xi)$, where $\Delta_{I_k}(\xi)$ is the sum of all k-minors of the matrix A_ξ, (I_k denotes the class of subsets of $\{1, ..., (N-1)\}$ with k elements).

Moreover, if we replace ξ by $-\xi$, Ξ_k is changed into $-\Xi_k$ when k is odd, and is unchanged when k is even.

Definition 93 *Let M^n be a Riemannian manifold (with or without boundary), isometrically embedded in $\mathbb{E}^N$.*

1. *At each point of $M^n \backslash \partial M^n$, the j-th Lipschitz-Killing curvature R_j is the n-differential form locally defined as follows:*
   - *if j is odd, R_j is null;*
   - *if j is even, R_j is given by*

$$R_j = (-1)^{j/2}((n-j)!2^j\pi^{j/2}(j/2)!)^{-1} \sum_{\sigma\in\mathcal{S}^n} (-1)^{\sigma_{i_1\cdots i_n}} \Omega^{i_1}_{i_2} \wedge ... \wedge \Omega^{i_{(j-1)}}_{i_j} \wedge \omega^{i_{(j+1)}} \wedge ... \wedge \omega^{i_n}; \tag{12.8}$$

2. *Similarly, the j^{-th} Lipschitz-Killing curvature of ∂M^n in M^n is the $(n-1)$-differential form defined locally defined as follows:*

$$H_j = \sum_k Q_{kj}, \tag{12.9}$$

where

$$Q_{kj} = c_{kj} \sum_{\sigma\in\mathcal{S}^{n-1}} (-1)^{\sigma_{i_1\cdots i_{(n-1)}}} \Omega^{i_1}_{i_2} \wedge ... \wedge \Omega^{i_{(2k-1)}}_{i_{2k}} \wedge \omega^{i_{(2k+1)}}_n \wedge ... \wedge \omega^{i_{(j-1)}}_n \wedge \omega^{i_j} \wedge ... \wedge \omega^{i_{n-1}}, \tag{12.10}$$

with

$$c_{kj} = (-1)^{k+1}((n-j)!2^j\pi^{(j-1)/2}k!(\frac{j-1}{2} - k)!)^{-1},$$

if j is odd, and

$$c_{kj} = (-1)^k((n-j)!2^{k+j/2}\pi^{j/2}k!(j-2k-1)!!)^{-1},$$

if j is even, where $(j-2k-1)!! = 1.3.5....(j-2k-1)$.

(The 1-forms $(\omega^1, ..., \omega^n)$ are the dual forms of a local orthonormal frame $(e_1, ..., e_n)$ such that e_n is normal to ∂M^n; the 1-forms ω_i^j are the corresponding connexion forms, the 2-forms Ω_j^i are the curvature forms M^n. The sum is over all the permutations σ of n elements.)

In particular, R_0 is the volume form of M^n and R_n is null if n is odd and proportional to $\sum_\sigma \Omega_{i_2}^{i_1} \wedge ... \wedge \Omega_{i_n}^{i_{n-1}}$ if n is even. The form R_n is involved in Gauss-Bonnet Theorem as we shall see later.

The link between the notion of k^{-th} mean curvature and k^{-th}-*Lipschitz-Killing curvature* is given by the following proposition, (in which we identify, for every $m \in M^n \setminus \partial M^n$, $ST_m^\perp M^n$ and S^{N-n-1}, -*resp.* $ST_m^\perp \partial M^n$ and $\frac{1}{2}S^{N-n-1}$):

Proposition 34 *Let M^n be a compact oriented n-dimensional Riemannian manifold (with or without boundary), isometrically embedded in $\mathbb{E}^N$.*

1. *Let m be a (fixed) point of $(M^n \setminus \partial M^n)$. Then, $\Xi_k(\xi)_m$ is a function defined on $ST_m^\perp M^n$, satisfying the following equality:*

   - *if k is odd,*
   $$\int_{S^{N-n-1}} \Xi_k(\xi) d\xi_m = 0, \tag{12.11}$$
   - *if k is even,*
   $$(\int_{\frac{1}{2}S^{N-n-1}} \Xi_k(\xi) d\xi_m) dv_M = R_k(m), \tag{12.12}$$
   this equality being considered as an equality between two n-forms at each point m of $(M^n \setminus \partial M^n)$.

2. *Let m be a (fixed) point of ∂M^n. Then, $\Xi_k(\xi)_m$ is a function defined on $ST_m^\perp \partial M^n = \frac{1}{2}S^{N-n}$, satisfying the following equality:*
$$(\int_{S^{N-n}} \Xi_k(\xi) d\xi_m) dv_{\partial M^n} = H_k(m), \tag{12.13}$$
this equality being considered as an equality between two $(n-1)$-forms at each point of ∂M^n.

The proof of this proposition uses extensively Gauss and Codazzi equations. It can be found in [8]

12.3 Steiner Formula for Convex Subsets

The idea here is to define invariant which look like the previous Lipschitz-Killing curvatures, on objects which have low differentiability. We shall restrict our frame to convex subsets of $\mathbb{E}^N$. Such subsets have a boundary which is not C^2 in general. Concretely we shall associate to each convex subsets $\mathcal{K}$ of $\mathbb{E}^N$ a sequence of $N+1$ real numbers $(\Phi_0(\mathcal{K}), ..., \Phi_k(\mathcal{K}), ..., \Phi_N(\mathcal{K}))$. This sequence can be obtained by computing the volume of the parallel convex subsets of $\mathcal{K}$ at distance ϵ of it. This volume is polynomial in ϵ. Its coefficients $(\Phi_k(\mathcal{K}), 0 \leq k \leq N)$ depend on the geometry of $\mathcal{K}$. If the boundary of $\mathcal{K}$ is smooth enough, these coefficients can be expressed as integral of the kth mean curvatures of the boundary, considered as a submanifold of the ambiant space.

12.3.1 *Steiner* formula

Definition 94 *Let $\mathcal{K}$ be a convex subset of $\mathbb{E}^N$.*

- *The parallel body $\mathcal{K}_\epsilon$ of $\mathcal{K}$ is defined by:*

$$\mathcal{K}_\epsilon = \{m \in \mathbb{E}^N,\ d(m, \mathcal{K}) \leq \epsilon\}.$$

- *Its boundary $\partial\mathcal{K}_\epsilon$ is called the parallel hypersurface of $\mathcal{K}$ at distance ϵ, ($d(m, \mathcal{K})$ denotes the usual Euclidean distance from m to $\mathcal{K}$).*

The *Steiner* formula gives the behaviour of the volume of the parallel body $\mathcal{K}_\epsilon$ of a convex subset $\mathcal{K}$ of $\mathbb{E}^N$; it will be proven that it is a polynomial in ϵ, whose coefficients depend only on $\mathcal{K}$ and the dimension N of the ambiant space.

Theorem 102 *Let $\mathcal{K}$ be a convex subset of $\mathbb{E}^N$. Then,*

$$Vol_N(\mathcal{K}_\epsilon) = \sum_{k=0}^{N} \Phi_k(\mathcal{K})\epsilon^k, \forall\epsilon \geq 0. \tag{12.14}$$

Moreover,

$$\Phi_0(\mathcal{K}) = Vol_N(\mathcal{K}). \tag{12.15}$$

The coefficients $\Phi_k(\mathcal{K})$ are called the *Quermassintegrale* of Minkowski. The following proof uses *Cauchy formula*, [26].

Theorem 103 - Cauchy formula - *Let $\mathcal{K}$ be a convex body of $\mathbb{E}^N$. Then,*

$$Vol_{N-1}(\partial\mathcal{K}) = \frac{1}{(N-1)s_{N-1}} \int_{\xi \in S^{N-1}} Vol_{N-1}(\mathcal{K}_{P^{\xi^\perp}}) d\xi, \tag{12.16}$$

where $\mathcal{K}_{P^{\xi^\perp}}$ denotes the orthogonal projection of $\mathcal{K}$ onto the hyperplane $\xi^\perp$.

Of course, using the two-fold covering of S^{N-1} onto Grassmannian $G(N, N-1)$ of (unoriented) $N-1$-subspaces of $\mathbb{E}^N$, one can integrate over $G(N, N-1)$. Since the volume of $G(N, N-1)$ is one half the volume of S^{N-1}, one obtains

$$\mathrm{Vol}_N(\partial\mathcal{K}) = \frac{2}{(N-1)s_{N-1}} \int_{G(N,N-1)} \mathrm{Vol}_{N-1}(\mathcal{K}_{P^{N-1}}) dP^{N-1}, \tag{12.17}$$

where $\mathcal{K}_{P^{N-1}}$ denotes the projection of $\mathcal{K}$ on the hyperplane P^{N-1}, and $G(N, N-1)$ is the Grassmann manifold of (unoriented) $(N-1)$-hyperplanes of $\mathbb{E}^N$, endowed with its standard left invariant Haar measure.

The proof of Cauchy formula can be found in [26].

Proof of Theorem 102. We proceed by induction on the dimension N of the ambiant space.

- for $N = 1$, it is trivial, since in this case, $\mathcal{K}$ is a point or segment, and

$$\mathrm{Vol}_1(\mathcal{K}_\epsilon) = \mathrm{Vol}_1(\mathcal{K}) + 2\epsilon.$$

- Suppose that the formula is true up to $N-1$. Let ρ be a real number such that $0 \leq \rho \leq \epsilon$. Consider the projection $\mathcal{K}_{\rho_{P^{N-1}}}$ of $\mathcal{K}_\rho$ on any hyperplane P^{N-1} of $\mathbb{E}^N$. $\mathcal{K}_{\rho_{P^{N-1}}}$ is still convex and one can apply the induction assumption:

$$\mathrm{Vol}_{N-1}(\mathcal{K}_{\rho P^{N-1}}) = \sum_{k=0}^{N-1} \Phi_k(\mathcal{K}_{P^{N-1}})\rho^k, \forall \rho \geq 0. \tag{12.18}$$

- Using an integration by slice, one has:

$$\mathrm{Vol}_N(\mathcal{K}_\epsilon) = \mathrm{Vol}_N(\mathcal{K}) + \int_0^\epsilon \mathrm{Vol}_{N-1}(\partial\mathcal{K}_\rho) d\rho.$$

- Now, we evaluate $\mathrm{Vol}_N(\partial\mathcal{K}_\rho)$ by *Cauchy formula*, (103) :

$$\mathrm{Vol}_N(\mathcal{K}_\epsilon) = \mathrm{Vol}_N(\mathcal{K}) + \int_0^\epsilon c(N) \int_{G(N,N-1)} \mathrm{Vol}_{N-1}(\mathcal{K}_{\rho_{P^{N-1}}}) dP^{N-1} d\rho.$$

- We replace $\mathrm{Vol}_{N-1}(\mathcal{K}_{\rho_{P^{N-1}}})$ by its value in (12.18):

$$\mathrm{Vol}_N(\mathcal{K}_\epsilon) = \mathrm{Vol}_N(\mathcal{K}) + \int_0^\epsilon c(N) \int_{G(N,N-1)} (\sum_{k=0}^{N-1} \Phi_k(\mathcal{K}_{P^{N-1}})\rho^k) dP^{N-1} d\rho,$$

  and integrating in ρ, we find

$$\mathrm{Vol}_N(\mathcal{K}_\epsilon) = \sum_{k=0}^{N} \Phi_k(\mathcal{K})\epsilon^k, \tag{12.19}$$

  with suitable Φ_k.

12.3.2 Examples: Segments, Discs and Balls

Steiner formula can be checked in particular cases by a direct computation: As we have seen in the proof of Theorem 102, the simplest (trivial) example is given with $N = 1$. In this case, $\mathcal{K}$ is an interval I and we have:

$$l(I_\epsilon) = l(I) + 2\epsilon;$$

(l denotes the length). When $N = 2$ or 3, we have the following, (A denotes the area):

1. If $N = 2$, and
   - p is a point in $\mathbb{E}^2$, then
   $$A(\{p\}_\epsilon) = \pi\epsilon^2;$$
   - I is a segment in $\mathbb{E}^2$, then
   $$A(I_\epsilon) = 2l(I)\epsilon + \pi\epsilon^2;$$
   - if D is a disc in $\mathbb{E}^2$, then
   $$A(D_\epsilon) = A(D) + l(\partial D)\epsilon + \pi\epsilon^2;$$
   - more generally, if $\mathcal{K}$ is a any convex body in $\mathbb{E}^2$, then
   $$A(\mathcal{K}_\epsilon) = A(\mathcal{K}) + l(\partial\mathcal{K})\epsilon + \pi\epsilon^2.$$
2. If $N = 3$, and
   - p is a point in $\mathbb{E}^3$, then
   $$\mathrm{Vol}_3(\{p\}_\epsilon) = \frac{4}{3}\pi\epsilon^3;$$
   - I is a segment in $\mathbb{E}^3$, then
   $$\mathrm{Vol}_3(I_\epsilon) = \pi l(I)\epsilon^2 + \frac{4}{3}\pi\epsilon^3;$$
   - if B is a ball in $\mathbb{E}^3$, then
   $$\mathrm{Vol}_3(B_\epsilon) = \mathrm{Vol}_3(B) + 4\pi R^2\epsilon + 4\pi R\epsilon^2 + \frac{4}{3}\pi\epsilon^3,$$
   (where R denotes the radius of the ball B);

12.3.3 The Case of a Convex Body in $\mathbb{E}^N$ Whose Boundary Is a Polyhedron

In this paragraph, we shall deal with the case where the boundary $\partial\mathcal{K}_\epsilon$ of the convex body $\mathcal{K}$ is a polyhedron. We shall compute the coefficients of *Steiner polynomial.*

Theorem 104 *et $\mathcal{K}$ be a convex body of $\mathbb{E}^N$ whose boundary $\partial\mathcal{K}$ is a polyhedron. Then,*

$$Vol_N(\mathcal{K}_\epsilon) = \sum_{k=0}^{k=N} \Phi_k(\mathcal{K})\epsilon^k, \tag{12.20}$$

with

$$\Phi_0(\mathcal{K}) = Vol_N(\mathcal{K}), \Phi_1(\mathcal{K}) = Vol_{N-1}(\partial\mathcal{K}), \Phi_N(\mathcal{K}) = Vol_N(S^N),$$

and

$$\Phi_k(\mathcal{K}) = \sum_{F^{N-k}\subset\partial\mathcal{K}} Vol_{N-k}(F^{N-k})\angle(C^\perp(F^{N-k})), \forall k, 1 < k < N,$$

where F^{N-k} denotes a generic $(N-k)$-face of $\partial\mathcal{K}$, $\angle(C^\perp(F^{N-k}))$ denotes the k-volume of the basis $L(F^{N-k})$ of the normal cone of F^{N-k} taken at any point of F^{N-k}.

When $N = 3$, the formula becomes the following, (we denote by $l(a)$ the length of the edge a, and by $\angle(a)$ the diedral angle of the edge a):

$$\mathrm{Vol}_3(\mathcal{K}_\epsilon) = \mathrm{Vol}_3(\mathcal{K}) + A(\partial\mathcal{K})\epsilon + (\sum_a \angle(a)l(a))\epsilon^2 + \frac{4}{3}\pi\epsilon^3. \tag{12.21}$$

Proof of Theorem .
We shall decompose $\mathcal{K}_\epsilon$ into a union of subsets "lying above the k-faces", $(0 \leq k \leq N)$, and evaluate the volume of each portion. Remark that

$$\partial\mathcal{K} = \cup_{0\leq k\leq N-1}\mathcal{F}^k,$$

where $\mathcal{F}^k$ is the union of the k-faces of $\partial\mathcal{K}$. Let F^k be any k-face of $\partial\mathcal{K}$. We define

$$B_k(F^k, \epsilon) = \cup_{\xi\in L_m, m\in F^k}[m, m + \epsilon\xi],$$

where L_m denotes the basis of the normal cone $C^\perp(F^k)$ at any point m of F^k . Remark that

$$\mathrm{Vol}_N(B_k(F^k, \epsilon)) = \mathrm{Vol}_N(L(F^k))\epsilon^k.$$

On the other hand,

$$\mathcal{K}_\epsilon = \mathcal{K} \cup_{0\leq k\leq N-1} \cup_{F^k\in\mathcal{F}^k} B_k(F^k, \epsilon).$$

Since the intersection of two portions of the previous union is empty or has a null N-measure, we deduce that

$$\mathrm{Vol}_N(\mathcal{K}_\epsilon) = \mathrm{Vol}_N(\mathcal{K}) + \sum_{0\le k\le N-1} \mathrm{Vol}_N(B_k(F^k,\epsilon)),$$

from which we deduce (12.20). Clearly, $\Phi_0(\mathcal{K}) = \mathrm{Vol}_N(\mathcal{K})$, and $\Phi_1(\mathcal{K}) = \mathrm{Vol}_{N-1}(\partial\mathcal{K})$, since for any $(N-1)$-face F^{N-1} and any $m \in F^{N-1}$, L_m is reduced to a point. Finally, the union of the basis $L(\sigma^0)$ over all 0-simplices is a $(N-1)$-sphere S^{N-1}, from which we deduce that $\Phi_N(\mathcal{K}) = \mathrm{Vol}_N(S^N)$.

12.3.4 The Case of a Convex Body With Smooth Boundary

Now we deal with the case where $\mathcal{K}$ is a convex body with smooth boundary $\partial\mathcal{K}$. Using classical differential geometry, we shall prove that the coefficients $\Phi_k(\mathcal{K})$ are proportional to the $(k-1)$-mean curvatures $\mathcal{M}_{k-1}(\partial\mathcal{K})$ of $\partial\mathcal{K}$, defined by

$$\mathcal{M}_r(\partial\mathcal{K}) = \frac{1}{C_{N-1}^r}\int_{\partial\mathcal{K}} \Xi_k dv_{\partial\mathcal{K}},$$

Ξ_k denoting the rth elementary symmetric function of the principal curvatures $\lambda_1, ..., \lambda_{N-1}$ of ∂K. *Steiner formula* can be stated as follows:

Theorem 105 *Let $\mathcal{K}$ be a compact convex body in $\mathbb{E}^N$ whose boundary $\partial\mathcal{K}$ is a hypersurface of class C^2. Then,*

$$Vol_N(\mathcal{K}_\epsilon) = \sum_{k=0}^{k=N} \Phi_k(\mathcal{K})\epsilon^k, \tag{12.22}$$

with

$$\Phi_0(\mathcal{K}) = Vol_N(\mathcal{K}), \tag{12.23}$$

and, for each $k \ge 1$,

$$\Phi_k(\mathcal{K}) = \frac{C_N^k}{N}\mathcal{M}_{k-1}(\partial\mathcal{K}). \tag{12.24}$$

So, the *Steiner formula* can be stated as follows:

$$\mathrm{Vol}_N(\mathcal{K}_\epsilon) = \mathrm{Vol}_N(\mathcal{K}) + \sum_{k=1}^{N} \frac{C_N^k}{N}\mathcal{M}_{k-1}(\partial\mathcal{K})\epsilon^k, \forall\epsilon \ge 0. \tag{12.25}$$

In particular, if $\mathcal{K}$ is a compact convex body in $\mathbb{E}^3$ with smooth boundary, then

$$\mathrm{Vol}_3(\mathcal{K}_\epsilon) = \mathrm{Vol}_3(\mathcal{K}) + A(\partial\mathcal{K})\epsilon + (\int_{\partial\mathcal{K}} H da)\epsilon^2 + \frac{1}{3}(\int_{\partial\mathcal{K}} G da)\epsilon^3, \tag{12.26}$$

where H, (resp. G), denotes the mean curvature, (resp. Gauss curvature) of $\partial\mathcal{K}$.

Remark that the *Gauss-Bonnet* theorem implies that the last term is equal to $\frac{4}{3}\pi\epsilon^3$, since $\mathcal{K}$ is convex. Consequently, we can state:

$$\mathrm{Vol}_3(\mathcal{K}_\epsilon) = \mathrm{Vol}_3(\mathcal{K}) + A(\partial\mathcal{K})\epsilon + (\int_{\partial\mathcal{K}} H da)\epsilon^2 + \frac{4}{3}\pi\epsilon^3. \tag{12.27}$$

Proof of Theorem 105. Let

$$x : \partial\mathcal{K} \to \mathbb{E}^N$$

be an isometric immersion of $\partial\mathcal{K}$ into $\mathbb{E}^N$. If ξ denotes a generic unit normal vector field, then

$$\phi(t) = x + t\xi$$

generates $\partial\mathcal{K}_t$ in $\mathbb{E}^N$.

Let $(e_i, 1 \leq i \leq N-1)$ be a local frame on $\partial\mathcal{K}$. Without loss of generality, we can assume that they are principal vectors of $\mathcal{K}$ in the direction ξ. For every $i, 1 \leq i \leq N-1$, we have

$$A_\xi e_i = \lambda_i^\xi e_i,$$

where λ_i^ξ denotes the principal curvatures of $\mathcal{K}$ (with respect to ξ). Then

$$d\phi_t(e_i) = e_i + A_{t\xi} e_i = e_i + t\lambda_i^\xi e_i.$$

Consequently, the volume form $dv_{\partial\mathcal{K}_t}$ of $\partial\mathcal{K}_t$ satisfies

$$dv_{\partial\mathcal{K}_t} = (\omega^1 + t\lambda_1^\xi\omega^1) \wedge ... \wedge (\omega^{N-1} + t\lambda_{N-1}^\xi\omega^{N-1}) =$$

$$\sum_k \Xi_k(\xi) t^k \omega^1 \wedge ... \wedge \omega^{N-1}.$$

Then,

$$\mathrm{Vol}_N(\mathcal{K}_\epsilon) = \mathrm{Vol}_N(\mathcal{K}) + \sum_k \int_{t=0}^{t=\epsilon} \Xi_k(\xi) dv_{\partial\mathcal{K}_t} t^k dt,$$

that is

$$\mathrm{Vol}_N(\mathcal{K}_\epsilon) = \mathrm{Vol}_N(\mathcal{K}) + \sum_{k=1}^{N} \frac{C_N^k}{N} \mathcal{M}_{k-1}(\partial\mathcal{K})\mathcal{M}_{k-1}(\partial\mathcal{K})\epsilon^k,$$

which implies (12.25).

12.3.5 Continuity of the Φ_k

It is interesting to notice that the Φ_k satisfies continuity properties. It is well known that Φ_0, (that is, the N-volume), and Φ_1, (that is, the $N-1$-volume of the boundary), are continuous on the set $\mathcal{C}$ of convex bodies of $\mathbb{E}^N$ endowed with the Hausdorff topology. This can be precised and generalized in the following way:

Theorem 106 *Let $\mathcal{K}$ and $\mathcal{K}'$ to convex bodies of $\mathbb{E}^N$ whose Hausdorff distance is smaller than ρ.*

- *Then for each $k, 0 \leq k \leq N$, there exists a constant $C_k(\mathcal{K}, \mathcal{K}', N)$ (depending only on $\mathcal{K}$, $\mathcal{K}'$ and the dimension N of the ambiant space), such that*

$$|\Phi_k(\mathcal{K}) - \Phi_k(\mathcal{K}')| \leq C_k(\mathcal{K}, \mathcal{K}', N)\rho.$$

- *In particular, if $\mathcal{K}_n, n \geq 0$, be a sequence of (compact) convex bodies of $\mathbb{E}^N$ whose Hausdorff limit is the (compact) convex body $\mathcal{K}$, then,*

$$\forall k \geq 0, \lim_{n\to\infty} \Phi_k(\mathcal{K}_n) = \Phi_k(\mathcal{K}). \tag{12.28}$$

Here, it must be noticed that *the assumption of convexity* is crucial to obtain continuity results for every Φ_k. We shall see at the end of this paper that the problem is completely different in the non convex case.

Remark: In $\mathbb{E}^3$, the only interesting information given by Theorem 106 is the behaviour of the mean curvature of closed surfaces: Recall that if $\mathcal{K}$ is a convex body of $\mathbb{E}^3$ with smooth boundary, *Steiner formula* (12.26) gives

$$\mathrm{Vol}_3(\mathcal{K}_\epsilon) = \mathrm{Vol}_3(\mathcal{K}) + A(\partial\mathcal{K})\epsilon + (\int_{\partial\mathcal{K}} H da)\epsilon^2 + \frac{4}{3}\pi\epsilon^3; \tag{12.29}$$

and if $\mathcal{K}$ is a convex body bounded by a polyhedron, the *Steiner formula* (12.22) gives, ($l(a)$ denoting the length of the edge a, and $\angle(a)$ the diedral angle of the edge a):

$$\mathrm{Vol}_3(\mathcal{K}_\epsilon) = \mathrm{Vol}_3(\mathcal{K}) + A(\partial\mathcal{K})\epsilon + (\sum_a \angle(a).l(a))\epsilon^2 + \frac{4}{3}\pi\epsilon^3. \tag{12.30}$$

Consequently, (with obvious notations), we have the following

Corollary 21 *Let P_n be a sequence of convex polyhedra in $\mathbb{E}^3$ whose Hausdorff limit is a (closed convex) surface S. Then the mean curvature H of S satisfies:*

$$\int_S H da = \lim_{n\to\infty} \sum_{a_n} \angle(a_n).l(a_n). \tag{12.31}$$

12.4 Continuity of the Area

The previous paragraph dealt with convex objects. Before dealing with more general ones, we shall see that, even the area has strange behaviour with respect to continuity, if we avoid the convexity condition. In the following, we shall work in $\mathbb{E}^3$, but it is obvious to generalize the results in any dimension. If S is a smooth (compact orientable) surface S of the Euclidean space $\mathbb{E}^3$, and a subset M is *"close to S"* a basic problem is to know when

does the area $\mathcal{A}(M)$ of M approximate the area $\mathcal{A}(S)$ of S. (Basically, we must mention here that we make the crucial assumption that the area of M can be defined). Practically, we shall assume that M is *smooth almost everywhere.*

First of all, remark that one can begin to study the same problem for curves in the plane, dealing with their length. Consider the following example: Consider a curve C in $\mathbb{E}^3$ and a sequence of polygons P_n whose vertices lie in C, and whose initial and final points are the initial and final points of C. If the length of the edges of the polygons tends to zero, then *by definition of the length*, the length of C is the limit of the length of P_n. The situation is completely different for surfaces: the well known *"Lampion de Schwarz"* shows that the area of a sequence of triangulations inscribed in a fixed cylinder can tend to infinity. In order to describe succintly this phenomenon, let us build a 2-parameter family of *generalized Lampions de Schwarz.* Let C be a cylinder of finite height H and radius R in $\mathbb{E}^3$. Let $P(n, N)$ denote the triangulation inscribed in C defined as follows: consider N circles on the cylinder C obtained by intersecting C by 2-planes orthogonal to the axis of C. Inscribe on each circle a regular n-gon such that the n-gon on the slice k is deduced from the n-gon of the slice $k-1$ by a rotation of angle $\frac{2\pi}{n}$. Then join each vertex v of the slice $k-1$ to the two vertices of the slice k which are on each side of v. We obtain a triangulation whose vertices $v_{i,j}$ are defined as follows:

$$\begin{array}{ll} \forall i \in \{0,..n-1\}, & v_{i,j} = (R\ \cos(i\alpha), R\ \sin(i\alpha), jh) \text{ if } j \text{ is even,} \\ \forall j \in \{0,..N\}, & v_{i,j} = (R\ \cos(i\alpha + \frac{\alpha}{2}), R\ \sin(i\alpha + \frac{\alpha}{2}), jh) \text{ if } j \text{ is odd,} \end{array}$$

and whose faces are:

$$\begin{array}{l} v_{i,j}\ v_{i+1,j}\ v_{i,j+1}, \\ v_{i,j}\ v_{i-1,j+1}\ v_{i,j+1}, \end{array}$$

where $\alpha = \frac{\pi}{n}$ and $h = \frac{H}{N}$.

Then when n tends to infinity, the area $\mathcal{A}(P(n, n^3))$ of $P(n, n^3)$ tends to infinity. The obvious consequence is that, in general, there is no continuity theorem for the area.

12.4.1 The General Framework

In the following, we shall give a framework in which a continuity theory holds. Details can be found in the Thesis of B. Thibert, [28]. New tools will be introduced. Let S be a smooth compact surface of $\mathbb{E}^3$. We know that there exists an open set U_S of $\mathbb{E}^3$ containing S where the orthogonal projection ξ onto S is well defined. Moreover ξ is continuous in U_S. Let us define on U_S new geometric quantities:

Definition 95 *Let S be a smooth surface of $\mathbb{E}^3$.*

- *The function $\omega_S : U_S \to \mathbb{R}$ defined by*

$$\omega_S(m) = ||\xi(m) - m||\rho_{\xi(m)}$$

(where $\rho_{\xi(m)}$ denotes the maximum of the two principal curvatures of S at $\xi(m)$) is called the relative curvature function of S on U_S.

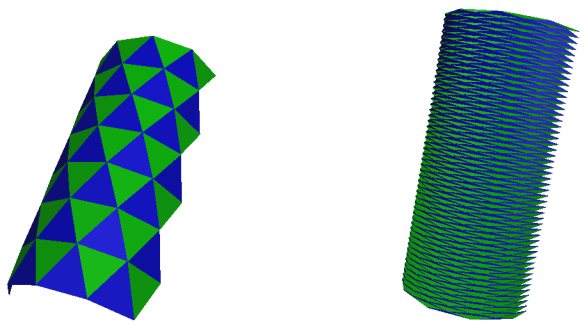

Figure 12.1: Examples of *half lampions of Schwarz* - from Boris THIBERT thesis

- *If M is any subset lying in $U_r(S)$ the real number*

$$\omega_S(M) = \sup_{m \in M} \omega_S(m),$$

is called the relative curvature of M with respect to S.

Since M lies in $U_r(S)$, one has $\omega_S(M) \leq 1$. In fact, for every $m \in M$, the point $m - 2\overrightarrow{m\xi(m)}$ -which is the orthogonal symmetric of m with respect to the tangent plane of S at $\xi(m)$- belongs to $U_r(S)$).

Finally, we give the following

Definition 96 *A subset M of $\mathbb{E}^3$ is* **closely near** *S if it lies in $U_r(S)$, where r is the reach of S and if the restriction of ξ to M is one-to-one.*

If M is a surface closely near S and differentiable almost everywhere, one can define at almost every point m of M the angle $\alpha_m \in \left[0, \frac{\pi}{2}\right]$ between the tangent planes T_mM and $T_{\xi(m)}S$. We put

$$\alpha_{\max} = \max_{m \in M} \alpha_m,$$

$$\alpha_{\min} = \min_{m \in M} \alpha_m.$$

12.4.2 The Approximation of the Area

In this paragraph, we shall show that if M is closely near S and has enough regularity to have a tangent plane almost everywhere, then the area of S is bounded from above and from below explicit geometric quantities as curvatures, Hausdorff distance, angles, depending of the geometry of the problem. In the following, M is a surface which admits a tangent plane almost everywhere. The details can be found in [24]

Theorem 107 *Let S be a (compact orientable) C^2 surface in $\mathbb{E}^3$. Let M be a surface closely near S. Then the area satisfies:*

$$\mathcal{A}(S) = \int_M \frac{cos\ \alpha_m}{1 + \delta_m \epsilon_m H_{\xi(m)} + \delta_m^2 G_{\xi(m)}} da_M(m),$$

where $\delta_m = \|\xi(m) - m\|$ and $\epsilon_m \in \{-1, +1\}$ is the scalar product between $\nu_{\xi(m)}$ and $\frac{\xi(m)-m}{||\xi(m)-m||}$.

Corollary 22 *Let M be a surface closely near S.*

1. *Then,*
$$\frac{\cos \alpha_{\max}}{(1+\omega_S(M))^2}\mathcal{A}(M) \leq \mathcal{A}(S) \leq \frac{\cos \alpha_{\min}}{(1-\omega_S(M))^2}\mathcal{A}(M). \tag{12.32}$$
2. *In particular*
$$|\mathcal{A}(S) - \mathcal{A}(M)| \leq K_S \ \left(\alpha_{max}^2 + \omega_S(M)\right), \tag{12.33}$$
and
$$|\mathcal{A}(S) - \mathcal{A}(M)| \leq K(S) \ \left(\alpha_{max}^2 + \delta_{Hauss}(S, M)\right), \tag{12.34}$$

We deduce from the previous corollary the following

Corollary 23 *Let S be a (compact orientable) C^2 surface in $\mathbb{E}^3$. Let M_n be a sequence surfaces closely near S. If*

1. *the Hausdorff limit of M_n is S, when n goes to infinity,*
2. *the angle between tangent planes $T_m M_n$ and $T_{\xi(m)}S$ tends to 0 almost everywhere, when n goes to infinity,*

then,

$$\lim_{n\to\infty} \mathcal{A}(M_n) = \mathcal{A}(S).$$

The rest of this section is devoted to the proof of Theorem 107.

Proof of Theorem 107. The following proposition gives the main differential properties of the function ξ for surfaces in $\mathbb{E}^3$, when it is defined.

Proposition 35 *Let S be a smooth surface of $\mathbb{E}^3$ without boundary, U_S an open subset of $\mathbb{E}^3$ where the map $\xi : U_S \to S$ is well defined. Then,*

1. *the map ξ is C^1 in U_S and satisfies for every $m \in U_S$:*
$$D\xi(m)(Z_m) = 0, \forall Z_m \text{ orthogonal to } T_{\xi(m)}S,$$
$$D\xi(m)(X_m) = (Id + \delta_m \epsilon_m A_{\xi(m)})^{-1}(X_m), \forall X_m \text{ parallel to } T_{\xi(m)}S,$$
where $\epsilon_m =< \nu_{\xi(m)}, \frac{\xi(m)-m}{||\xi(m)-m||} >\in \{-1, +1\}$ and $A_{\xi(m)} = -D\nu(\xi(m))$ is the Weingarten endomorphism of S at the point $\xi(m)$.

2. *In particular, the matrix of* $D\xi(m) : \mathbb{E}^3 \to T_{\xi(m)}S$ *(in local orthonormal frames* $(e^1_{\xi(m)}, e^2_{\xi(m)}, \nu^S_{\xi(m)})$ *and* $(e^1_{\xi(m)}, e^2_{\xi(m)})$*) is given by:*

$$\begin{pmatrix} \frac{1}{1+\delta_m\epsilon_m\lambda^1_{\xi(m)}} & 0 & 0 \\ 0 & \frac{1}{1+\delta_m\epsilon_m\lambda^2_{\xi(m)}} & 0 \end{pmatrix}.$$

where $e^1_{\xi(m)}$ *and* $e^2_{\xi(m)}$ *are unitary principal vectors and* $\nu^S_{\xi(m)}$ *is the oriented normal of* S *at* $\xi(m)$.

Now consider the surface M lying in U_S. The function ξ is differentiable for almost every point of M. We shall apply the general area-coarea formula [22]:

$$\mathcal{A}(S_i) = \int_S da_{S_i}(s) = \int_M |J_2 D\xi(m)| \; da_M(m),$$

where $J_2 D\xi(m)$ is the 2-dimensional Jacobian of ξ at m. At a point m of M admitting a tangent plane, $J_2 D\xi(m)$ equals the determinant of the linear map

$$D\xi_{|M} : T_m M \to T_{\xi(m)}S.$$

We have:

$$\mathcal{A}(S) = \int_M |D\xi_{|M}(m)| \; da_M(m).$$

Therefore

$$\mathcal{A}(S) = \int_M |D\xi_{|M}(m))| \; da_M(m).$$

The following lemma evaluates the Jacobian of $\xi_{|M}$. Its proof is obvious and left to the reader.

Lemma 44 *Let* S *be a smooth surface of* $\mathbb{E}^3$ *without boundary,* U_S *an open subset of* $\mathbb{E}^3$ *where the map* $\xi : U_S \to S$ *is well defined. Let* $M \subset U_S$ *be a smooth surface. Then the Jacobian of the differential of* $\xi_{|M}$ *is given by:*

$$|D\xi_{|M}(m)| = \frac{cos\alpha_m}{(1+\delta_m\epsilon_m\lambda^1_{\xi(m)})(1+\delta_m\epsilon_m\lambda^2_{\xi(m)})},$$

where $\epsilon_m =< \nu_{\xi(m)}, \frac{\xi(m)-m}{||\xi(m)-m||} > \in \{-1, +1\}$, *and* $\delta_m = \|\xi(m) - m\|$.

Remark that the condition of the theorem on the angle between corresponding tangent planes implies that $\xi_{|M}$ is a diffeomorphism between M and S.

End of the Proof of Theorem 107. Since M is closely inscribed in S, $\xi_{|M}$ is one-to-one between M and S. Therefore:

$$\mathcal{A}(S) = \int_M |D\xi_{|M}(m)| \; da_M(m) = \int_M \frac{cos\ \alpha_m}{(1+\delta_m\epsilon_m\lambda^1_{\xi(m)})(1+\delta_m\epsilon_m\lambda^2_{\xi(m)})} \; da_M(m).$$

12.5 Invariant Differential Forms and Normal Cycle

We have seen that even for the simplest geometric invariant for surfaces, that is, the area, it is not trivial to understand the continuity phenomenon. One can imagine that the difficulty is higher if one wants to get a general theory of continuity of all curvatures defined on a submanifold. The first problem is the definition itself of such invariants on non smooth, non convex objects. Our goal here is to define geometric invariants on "any" compact subsets, which generalize the Lipschitz-Killing curvatures. Then, we shall present a continuity theorem. The theory takes place in a very general context: the one of differential forms and currents defined on the tangent bundle of the Euclidean space.

12.5.1 Invariant Differential Forms

Let $\Lambda^n(\mathbb{E}^N)$ be the space of n-differential forms with compact support of $\mathbb{E}^N$, endowed with the *topology of uniform convergence of differential of all orders on every compact*: A sequence of differential forms

$$\omega_q = \sum_{(i_1 \ldots i_p)} f_q^{(i_1 \ldots i_p)}(x) dx_{i_1} \wedge \ldots \wedge dx_{i_p},$$

converges to 0 if and only if

$$\forall (i_1 \ldots i_p), \forall n \in \mathbb{N}, \lim_{q \to \infty} d^{(n)} f_q^{(i_1 \ldots i_p)} = 0,$$

uniformally on every compact subset of $\mathbb{E}^N$.

Consider the *Poincaré's group* G of rigid motions of $\mathbb{E}^N$. It is well known that G is the semi-direct product $\mathbb{E}^N \propto SO(N)$. The action of G on $\mathbb{E}^N$ can be extended in a natural way to $\mathbb{E}^N \times S^{N-1}$: We put

$$\forall \phi = (t, a) \in G, \phi(x, y) = ((t, a).x, ay).$$

Consider now the space $\Lambda^{N-1}(\mathbb{E}^N \times S^{N-1})$ of $(N-1)$-forms on $\mathbb{E}^N \times S^{N-1}$. G acts on this space by:

$$\phi^* \omega(e_1, e_2, \ldots, e_{N-1}) = \omega(d\phi(e_1), d\phi(e_2), \ldots, d\phi(e_{N-1})). \tag{12.35}$$

We say that a $(N-1)$-form ω on $\mathbb{E}^N \times S^{N-1}$ is *invariant by rigid motion* if

$$\phi^* \omega = \omega, \forall \phi \in G.$$

In this paragraph, we shall give an explicit expression of the $(N-1)$-invariant forms. Let (m, ξ) be a point of $\mathbb{E}^N \times S^{N-1}$. Consider an orthonormal frame of $\mathbb{E}^N$, whose $(n+1)^{-th}$ vector is ξ, $(n+1 \leq N)$:

$$(e_1, e_2, \ldots, e_{N-1}, e_N = \xi).$$

(The choice consisting to put ξ at the last rank is totally arbitrary and will appear to be convenient latter).
Then, an orthonormal frame of $T_{(m,\xi)}\mathbb{E}^N \times S^{N-1}$ is given by the vectors

$$\varepsilon_1 = (e_1, 0), \varepsilon_2 = (e_2, 0), \ldots, \varepsilon_N = (e_N = \xi, 0),$$

$$\widetilde{\varepsilon}_1 = (0, e_1), \widetilde{\varepsilon}_2 = (0, e_2), \ldots, \widetilde{\varepsilon}_{N-1} = (0, e_{N-1}).$$

Identifying the Euclidean space $\mathbb{E}^N$ with its dual, (and consequently $(N-1)$-vectors with $(N-1)$-forms, we can consider the following $(N-1)$-form

$$(\varepsilon_1 + t\widetilde{\varepsilon}_1) \wedge \ldots \wedge (\varepsilon_{N-1} + t\widetilde{\varepsilon}_{N-1})_{(m,\xi)}.$$

This defines a differential $(N-1)$-form on STM. This expression can be considered as a polynomial in the variable t; (remark that the coefficient $\mathcal{W}_k$ of t^k is a differential form which does not depend on the orthonormal frame $(e_1, \ldots, e_N = \xi)$). Trivially one has:

$$\mathcal{W}_k = \sum_{\pi} (-1)^{\pi} \varepsilon_{\pi(1)} \wedge \ldots \wedge \varepsilon_{\pi(k)} \wedge \widetilde{\varepsilon}_{\pi(k+1)} \wedge \ldots \wedge \widetilde{\varepsilon}_{\pi(n-1)}.$$

Classically, for $1 \leq i \leq n-1$, the $(N-1)$-form $\mathcal{W}_k)$ on $\mathcal{T}M$ is called the kth-*Lipschitz-Killing curvature form.*

Theorem 108 *The space of $(N-1)$-invariant forms on $\mathbb{E}^N \times S^{N-1}$ has dimension N if N is even, and $N+1$ if N is odd. It is spanned by the forms $(\mathcal{W}_k)_{k=0\ldots(N-1)}$ and, if N is odd, by $\Omega^{\frac{N-1}{2}}$ (where Ω denotes the standard symplectic form of $\mathbb{E}^N \times \mathbb{E}^N$) restricted to $\mathbb{E}^N \times S^{N-1}$).*

In particular, suppose that $N = 2$. Then the space of 1-invariant forms of $\mathbb{E}^2 \times S^1$ has dimension 2. At a point $(p, \xi = e_2)$, it is spent by the two 1-forms

$$\mathcal{W}_0 = \varepsilon_1,$$

$$\mathcal{W}_1 = \widetilde{\varepsilon}_1.$$

Suppose now that $N = 3$. Then the space of 2-invariant forms has dimension 4. At a point $(p, \xi = e_3)$, it is spent by the four 2-forms:

$$\mathcal{W}_0 = \varepsilon_1 \wedge \varepsilon_2,$$

$$\mathcal{W}_1 = \varepsilon_1 \wedge \widetilde{\varepsilon}_2 + \widetilde{\varepsilon}_1 \wedge \varepsilon_2,$$

$$\mathcal{W}_2 = \widetilde{\varepsilon}_1 \wedge \widetilde{\varepsilon}_2,$$

$$\mathcal{W}_3 = \Omega_{|\mathbb{E}^N \times S^{N-1}} = \varepsilon_1 \wedge \widetilde{\varepsilon}_1 + \varepsilon_2 \wedge \widetilde{\varepsilon}_2.$$

12.5.2 The Notion of Normal Cycle

Background on currents

Definition 97 *The space $\Lambda_n(\mathbb{E}^N)$ of n-currents of $\mathbb{E}^N$ is the topological dual of the space $\Lambda^n(\mathbb{E}^N)$ of n-differential forms with compact support defined on $\mathbb{E}^N$.*

Remark that any smooth oriented submanifold S, or even any smooth submanifold with "nice" singularities can be considered as a current: If the dimension of S is n, we put

$$< S, \omega >= \int_S \omega dv_S,$$

for all n-form ω on $\mathbb{E}^N$. More generally, any oriented rectifiable subset can be considered as a current. If A is rectifiable and q is an integer, we define the current qA by

$$< qA, \omega >= q < A, \omega >,$$

for all n-form ω on $\mathbb{E}^N$. Conversely, we give the following

Definition 98 *A n-current A is rectifiable if A is a finite sum of currents associated to oriented n-rectifiable subsets, eventually with integer multiplicities.*

We also need the notion of the boundary of a current. If the current is a submanifold, its boundary has the intuitive meaning. More generally, we give the following

Definition 99 *To every n-current T is associated a $(n-1)$-current ∂T called the boundary of T, defined by*

$$< \partial T, \omega >=< T, d\omega >, \forall \omega.$$

Moreover

1. *If ∂T is null T is a cycle.*
2. *If T and ∂T are rectifiable, T is an integral current.*

The Normal Cycle of Standard Objects

To each object A of $\mathbb{E}^N$ on which we want to define curvatures, we would like to associate a current $N(A)$, called the normal cycle associated to A. It must be remarked that $N(A)$ will not be a current on $\mathbb{E}^N$, but on its unit sphere bundle, that is $\mathbb{E}^N \times S^{N-1}$. Since the invariant differential forms we delt with, are defined on this space, it will be possible to integrate them on normal cycles.

Up to the present, nobody knows how to define a normal cycle associated to any "very general compact subset". Here, we shall restrict ourselves to simple cases, (which is enough for our applications). More precisely, we shall define the *normal cycle* of

- a smooth submanifold of $\mathbb{E}^N$;
- a simplex and a polyhedron of $\mathbb{E}^N$.

The *"normal cycle"* is a (closed) $(N-1)$-integral current of $\mathbb{E}^N \times S^{N-1}$ associated to objets of $\mathbb{E}^N$:

Definition 100 *Let M^n be a submanifold of $\mathbb{E}^N$. The normal cycle of M^n is its unit normal bundle, (denoted by $N(M^n)$).*

Remarks:

1. Here, the normal cycle is a smooth oriented closed submanifold of $\mathbb{E}^N \times S^{N-1}$ of dimension $(N-1)$, endowed with its canonical orientation.
2. Let $p = (N-n)$ be the codimension of M^n. Let $(e_1, e_2, \ldots, e_n)$ be an orthonormal frame of $T_m M^n$ and $(\xi, \xi_2, \xi_3, \ldots, \xi_p)$ be an orthonormal frame of $T_m^{\perp} M^n$. Then,
$$(e_1, D_{e_1}\xi), (e_2, D_{e_2}\xi), \ldots, (e_n, D_{e_n}\xi), (0, \xi_2), (0, \xi_3), \ldots, (0, \xi_p), \tag{12.36}$$
is a frame of $TN(M)$ at (m, ξ), (D denotes the Levi-Civita connexion on $\mathbb{E}^N$).
3. For small ϵ, there is an obvious canonical one-to-one correspondance between ∂M_ϵ and $N(M^n)$: with obvious notations, to each point $(m+\epsilon\xi)$, one associates the point (m, ξ). The map
$$\phi : (m + \epsilon\xi_m) \to (m, \xi)$$
induces an orientation on $N(M^n)$.

Normal Cycle of a Polyhedron

We begin with the definition of the normal cycle of a simplex:

Definition 101 *Let σ^k be an elementary k-simplex of $\mathbb{E}^N$. We call* **normal cycle** *of σ^k the (closed) current $N(\sigma^k)$ of $\mathbb{E}^N \times S^{N-1}$ whose support $Spt(N(\sigma^k))$ is the (smooth) hypersurface defined by*
$$Spt(N(\sigma^k)) = \{(q, \xi) \in \mathbb{E}^N \times S^{N-1} : q \in \sigma^k, \text{ and } \xi \in C_q\}, \tag{12.37}$$
and whose orientation is given by the outward normal.

To define the normal cycle for any polyhedron, we use the fact that it is the union of simplexes, and use the relation
$$N(A \cup B) = N(A) + N(B) - N(A \cap B). \tag{12.38}$$

We let to the reader the proof that the normal cycle obtained by this process is independent of the decomposition.

12.5.3 Definition of Curvatures Via the Normal Cycle

We give here a general framework to compute the kth-*generalized curvature* of a subset of $\mathbb{E}^N$ admitting a normal cycle. That is why we give the following general

Definition 102 *If C is any $(N-1)$-current on $\mathbb{E}^N \times S^{N-1}$, the k^{-th}-curvature $\phi_k(C)$ of C is the real number $\phi_k(C)$ given by the formula:*

$$\phi_k(C) =< C, \mathcal{W}_k > .$$

This definition can be adapted to normal cycles: a complete theory must assign curvatures to any (compact) subsets of $\mathbb{E}^N$ in the following way: if a subset A of $\mathbb{E}^N$ admits a normal cycle $N(A)$, the k^{-th}-curvature $\phi_k(A)$, of A is the real number $\phi_k(A)$ given by the formula:

$$\phi_k(A) =< N(A), \mathcal{W}_k > .$$

However, we must be sure that for a given A, $N(A)$ exists and is unique. This is true for a large class of compact subsets. In particular cases, (smooth submanifolds, subsets with positive reach, polyhedra, subanalytic sets), a simple construction is possible and we can evaluate explicitly the curvatures ϕ_k in each case. In all these cases, these ϕ_k coincide (up to a constant) with the coefficients Φ_k which appear from the beginning in Steiner polynomial.

The Case of Submanifolds

Let M^n be a n-dimensional submanifold of $\mathbb{E}^N$. Let m be a point of M^n, and ξ a unit vector of $T_m^{\perp}M^n$. $(e_1, e_2, \ldots, e_n)$ be an orthonormal frame of T_mM^n and $(\xi_1, \xi_2, \xi_3, \ldots, \xi_{N-n-1}, \xi_{N-n} = \xi)$ be an orthonormal frame of $T_m^{\perp}M^n$. We can complete the previous frame to get an orthonormal frame of $T_{(m,\xi)}\mathbb{E}^N \times S^{N-1}$ adapted to M^n in the following way: The vectors

$$\varepsilon_1 = (e_1, 0), \varepsilon_2 = (e_2, 0), \ldots, \varepsilon_n = (e_n, 0),$$

$$\varepsilon_{n+1} = (\xi_1, 0), \varepsilon_{n+2} = (\xi_2, 0), \ldots, \varepsilon_N = (\xi_{N-n}, 0),$$

$$\widetilde{\varepsilon}_1 = (0, e_1), \widetilde{\varepsilon}_2 = (0, e_2), \ldots, \widetilde{\varepsilon}_n = (0, e_n),$$

$$\widetilde{\varepsilon}_{n+1} = (0, \xi_1), \ldots, \widetilde{\varepsilon}_{N-1} = (0, \xi_{N-n-1})$$

define an orthonormal frame of $T_{(m,\xi)}\mathbb{E}^N \times S^{N-1}$.

Using (12.36), we see immediately that the $(N-1)$-form

$$\varepsilon_{M^n} = (e_1, D_{e_1}\xi) \wedge (e_2, D_{e_2}\xi) \wedge \ldots \wedge (e_n, D_{e_n}\xi) \wedge (0, \xi_1), (0, \xi_2), \ldots, (0, \xi_{N-n-1}),$$

is a volume form of $ST^{\perp}M^n$.

Let $(h_{ij}^{\xi}) =< D_{e_i}\xi, e_j >$ be the second fundamental form in the direction ξ. We have the

Proposition 36 *At every point $(m_x i)$ of $ST^{\perp}M^n$,*

$$< \mathcal{W}_k, \varepsilon_{M^n} >_{(m,\xi)} = 0 \text{ if } k < N-n-1, \tag{12.39}$$

$$< \mathcal{W}_{(N-n-1)}, \varepsilon_{M^n}) >_{(m,\xi)} = 1, \tag{12.40}$$

$$< \mathcal{W}_{(N-n-1+k)_{(m,\xi)}}, \varepsilon_{M^n} >= \sum_{I \subset \{1 \ldots n\}} \Delta_I, \tag{12.41}$$

where the cardinal of I is $(n-k)$, and Δ_I is the k-minor determinant obtained by taking off lines and columns of index I to the matrix (h^{ξ}_{ij}). In other word,

$$< \mathcal{W}_{(N-n-1+k)_{(m,\xi)}}, \varepsilon_{M^n} >= \Xi_k(\xi). \tag{12.42}$$

In particular,

$$< \mathcal{W}_{(N-1),\varepsilon} >_{(m,\xi)} = \det h^{\xi}. \tag{12.43}$$

Consequently, integrating these forms on the unit normal bundle of the submanifold M^n, we obtain, *via* Gauss equation, the classical invariants $\Phi_k(M)$ introduced in the previous paragraph:

Theorem 109

$$< N(M^n), \mathcal{W}_{N-n-1+k} >= \frac{n!}{(N-k)!k!s_{k-1}} \Phi_{N-n+k}(M^n). \tag{12.44}$$

for all $k \geq 0$ and

$$< N(M^n), \mathcal{W}_k >= 0, \tag{12.45}$$

in the other cases.

Remarks:

1. In particular,
$$< N(M^n), \mathcal{W}_{N-n-1} >= \frac{s_{N-n-1}}{s_{N-1}} \mathrm{Vol}_n(M^n). \tag{12.46}$$

2. On the other hand, using Gauss-Bonnet theorem, we see that when n is even,
$$< N(M^n), \mathcal{W}_{N-1} >= \frac{b_N}{s_{n-1}} \chi(M^n). \tag{12.47}$$

3. If M^n has no boundary, we deduce that if k is odd,
$$< N(M^n), \mathcal{W}_{k)} >= 0, \tag{12.48}$$
(since for every (m,ξ), $\mathcal{W}_{(N-n+2l)_{(m,-\xi)}} = -\mathcal{W}_{(N-n+2l)_{(m,\xi)}}$).

As a consequence, we can compute the *Steiner polynomial* in terms of the forms $\mathcal{W}_k$ and the normal cycle:

Corollary 24 *Let M^n be a compact oriented n-dimensional submanifold of $\mathbb{E}^N$. Then, Steiner polynomial can be stated as follows:*

$$Vol_N(M^n_\epsilon) = \sum_{k=0}^{k=N} \Phi_k(M^n)\epsilon^k, \tag{12.49}$$

with

$$\Phi_k(M^n) = 0, \forall k,\ 0 \leq k \leq (N-n-1), \tag{12.50}$$

and

$$\Phi_k(M^n) =< N(M^n), \mathcal{W}_{k-N+n+1} >, \forall k,\ N-n \leq k \leq N. \tag{12.51}$$

The Case of Polyhedra

To evaluate the invariant forms on the normal cycle of a polyhedron, we begin to evaluate them on the elementary simplices, and then, we use the fundamental formula:

$$N(A \cup B) = N(A) + N(B) - N(A \cap B). \tag{12.52}$$

Consequently, we need only to evaluate the curvatures on the simplices:

Theorem 110 *Let σ^l be an l-simplex of $\mathbb{E}^N$. Then, for every k,*

$$\Phi_k(\sigma^l) = \sum_{\sigma^q \subset \sigma^l} Vol_q(\sigma^q).(\sigma^q, \sigma^l)^*. \tag{12.53}$$

Proof. We know that

$$\Phi_k(\sigma^l) =< N(\sigma^l), \mathcal{W}_k > .$$

Let F be an affine l-dimensional subspace of $\mathbb{E}^N$. Let $(e_1, e_2, \ldots, e_l)$ be an orthonormal frame of F, and ξ be an unit vector orthogonal to F. Complete $(e_1, e_2, \ldots, e_n, \xi)$ to obtain an orthonormal frame $(e_1, e_2, \ldots, e_n, \xi, \xi_2 \ldots, \xi_p)$ of $\mathbb{E}^N$, $(p = N - l)$.
With these notations, a frame of $TN(F)$ at (m, ξ) is

$$(\varepsilon) = (e_1, 0), (e_2, 0), \ldots, (e_l, 0), (0, \xi_2), \ldots, (0, \xi_p).$$

We have

$$\mathcal{W}_k(\varepsilon) = 0 \text{ if } k \neq N-n-1, \mathcal{W}_{N-n-1}(\varepsilon) = 1. \tag{12.54}$$

Now let P be an open set of $N(F)$ of the form $U \times U'$, where U is an open set of F and U' is an open set of $F^\perp \cap S^{N-1}$. We have:

$$\begin{aligned} &< \mathcal{W}_k, P >= 0 \text{ if } k \neq N-n-1, \\ &< \mathcal{W}_{(p-1)}, P >= \mathrm{Vol}(U).\mathrm{Vol}(U'). \end{aligned} \tag{12.55}$$

Remarks:

1. We still have:
$$< N(P), \mathcal{W}_m >= \Phi_0(P) = \mathrm{Vol}(P). \tag{12.56}$$

2. On the other hand, an easy computation gives:
$$< N(P), \mathcal{W}_{N-1} >= \Phi_N(M) = \frac{N!}{n!}\mathrm{Vol}_{N-1}(S^{N-1})\chi(P). \tag{12.57}$$

3. If P has no boundary, we have
$$< N(P), \mathcal{W}_{(N-n+2l)}, >= 0, \tag{12.58}$$
(since $W_{(N-n-1)+2l+1_{(m,-\xi)}} = -W_{(N-n-1)+2l+1_{(m,\xi)}}$).

4. Of course, if P is the boundary of a convex domain, then the results found here are coherent with theorem 12.3.3:
$$\Phi_k(P) = \sum_{\sigma^{(N-j)} \subset \bar{\sigma}^k} (-1)^{k-(N-j)}\mathrm{Vol}(\sigma^{N-j}).(\sigma^{N-j}, \sigma^k)^*. \tag{12.59}$$

12.5.4 On the Continuity of the Φ_k

If A_q is a sequence of subsets admitting normal cycles $N(A_q)$, such that $N(A_q)$ converges (as a current for the weak topology), to a current N of $\mathbb{E}^N \times S^{N-1}$, then by definition of the weak convergence,
$$\forall k \in (0, ..., N), \lim_{q\to\infty} < N(A_q), \mathcal{W}_k >=< N, \mathcal{W}_k > .$$
So,
$$\forall k \in (0, ..., N), \lim_{q\to\infty} \Phi_k(A_q) >=< N, \mathcal{W}_k > .$$
But nobody knows if $< N, \mathcal{W}_k >$ can represent the k^{-th}-curvature of some subset of E^N. However, we mention the following theorem of J. Fu, [19]. It claims that *"under suitable geometric conditions"*, one has a convergence theorem. We need to introduce the notion of *fatness* of a polyhedron.

Definition 103 *Let σ be a k-simplex whose vertices are $v_0, v_1, ..., v_k$, and P a polyhedron of $\mathbb{E}^n$.*

- *The size of σ is the real number $\eta(\sigma) = \max|v_j - v_i|$. Moreover, the fatness of σ is the real number*
$$\Theta(\sigma) = \min_{\mu}(\frac{Vol^j(\mu)}{\eta(\sigma)^j}),$$
where μ describes the j-dimensional faces of σ

- *The fatness of P is the supremum of $\Theta(\sigma)$, where σ describes all the k-simplices of all possible triangulations of P.*

We can now state the

Theorem 111 *Let M be a submanifold of $\mathbb{E}^n$. Let P_i be a sequence of polyhedra (with or without boundary), closely inscribed in M (with or without boundary). Suppose that P_i tends to M and ∂P_i tends to ∂M for the Hausdorff topology when i tends to infinity. If $\Theta(P_i) > c$ for all i, where c is a positive constant, then the curvature measures $\Phi_k(P_i)$ tend to the curvature measures $\Phi_k(M)$ for every k.*

12.6 Some Examples

In this section, we show some examples of applications of this theory and other geometrical approachs to graphism. Using the theory of normal cycle, D. Colhen Steiner has visualized "the principal directions" of Michelangelo's David. F. Cazals and M. Pouget used other technics to find the ridges, the crest linesand other geometric quantities of triangulations. We show here some of the images they obtained recently.

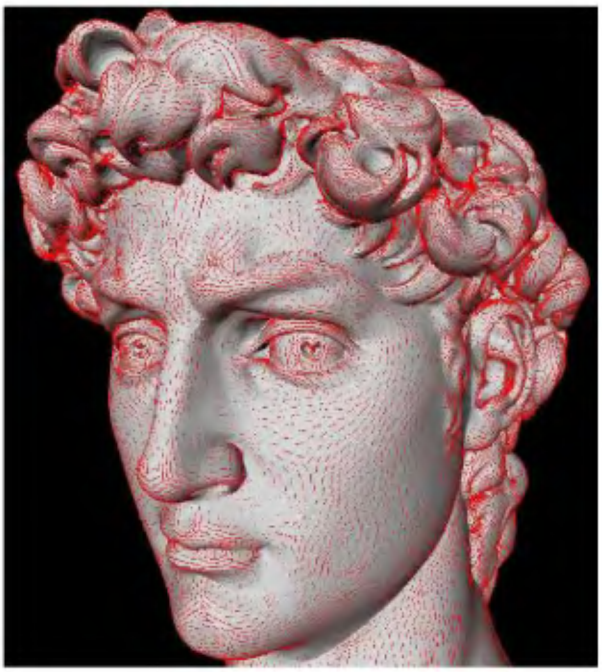

Figure 12.2: Michelangelo's David: principal directions associated with k_{max}

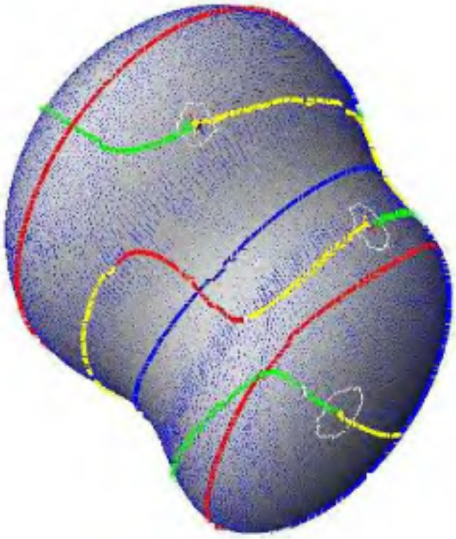

Figure 12.3: Ridges and umbilics on the implicit blending of two ellipsoids (40k points)

Figure 12.4: Mechanical part (37k pts): (a) All crest lines, (b) crests filtered with the strength and (c) crests filtered with the sharpness. Notice that any point on a flat or cylindrical part lies on two ridges, so that the noise observed on the top two Figs. is unavoidable. It is however easily filtered out with the sharpness on the bottom figure.

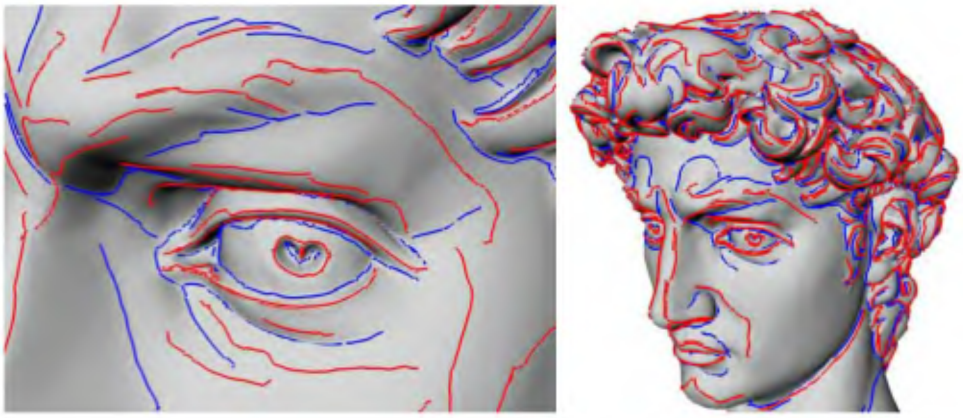

Figure 12.5: Filtered crest lines on a 380k pts model

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Differential Geometry and Topology, Discrete and Computational Geometry
M. Boucetta and J.-M. Morvan (Eds.)
IOS Press, 2005
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Chapter 13

Voronoi Diagrams, Triangulations and Surfaces

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13.1 Introduction

This chapter gives an introduction to the computational geometry of surfaces. The main motivation is to provide effective algorithms to construct PL approximations of surfaces. Instead of surveying the many approaches that have been proposed in the literature, we focus on provably correct methods and, among them, on methods based on a fundamental data structure that has been extensively studied in computational geometry, the Voronoi diagram of a finite set of points. Given n points (also called sites) of $\mathbb{R}^d$, the associated Voronoi diagram subdivides $\mathbb{R}^d$ into regions, each region consisting of the points of $\mathbb{R}^d$ that are closer to one of the sites.

In a first part (sections 13.2-13.3), we introduce Voronoi diagrams and their dual Delaunay triangulations. We establish a connexion between Voronoi diagrams and polytopes that allows to derive tight combinatorial bounds and efficient algorithms to construct Voronoi diagrams. This first part is a (very brief) introduction to computational geometry. The interested reader will find more material in the textbooks devoted to the subject [3, 4, 7, 9].

In a second part (sections 13.4-13.5), we show how Voronoi diagrams and Delaunay triangulations can be used to sample and approximate a surface. The main tool is the concept of Delaunay triangulation restricted to a surface. Under some sample condition, we will show that the restricted Delaunay triangulation of a surface S is a good approximation of S, both in a topological and a geometric sense. We will present and analyze an algorithm to compute such an approximation, and present some results.

Notation : We identify a point $x \in \mathbb{R}^d$ and the vector of its coordinates. We note $x \cdot y$

the dot product and $x^2 = x \cdot x = \|x\|^2$ the squared euclidean norm of x. $B(c, r)$ denotes the ball of center c and radius r, $\mathrm{aff}(f)$ the affine hull of f.

13.2 Voronoi diagrams and Delaunay triangulations

We first introduce Voronoi diagrams of points and establish a correspondence between those diagrams and polytopes in a one dimension higher space. This allows to derive tight bounds on their combinatorial complexity. The standard polarity allows to define a dual to a Voronoi diagram, called a Delaunay triangulation, which has many beautiful properties.

13.2.1 Voronoi Diagrams

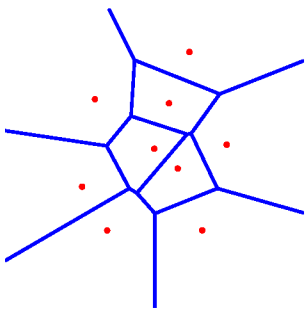
Definition

Let $E = \{p_1, \ldots, p_n\}$ be a set of points of $\mathbb{R}^d$. To each p_i, we associate its Voronoi region $V(p_i)$

$$V(p_i) = \{x \in \mathbb{R}^d : \|x - p_i\| \leq \|x - p_j\|, \forall j \leq n\}.$$

$V(p_i)$ is the intersection of the $n-1$ half-spaces bounded by the bisector planes of p_i and each of the other points of E. $V(p_i)$ is therefore a convex polytope, possibly unbounded.

Definition 104 *The Voronoi diagram of E, noted $\mathrm{Vor}(E)$, is the cell complex whose d-cells are the Voronoi regions.*



The space of spheres

Let σ be the sphere of equation

$$\sigma(x) = (x - c)^2 - r^2 = x^2 - 2c \cdot x + s = 0,$$

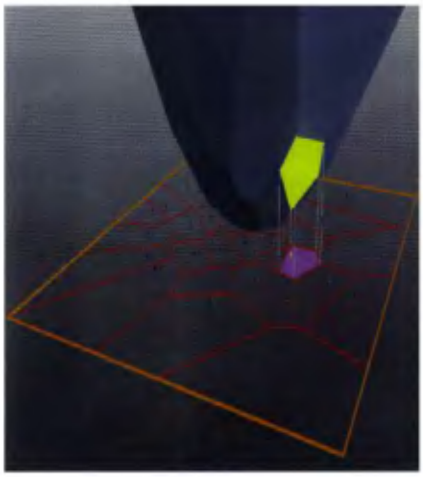
where c is the center of σ, r its radius and $s = \sigma(0) = c^2 - r^2$.

Let

$$\phi : \sigma \in \mathbb{R}^d \longrightarrow \phi(\sigma) = (c, s) \in \mathbb{R}^{d+1}$$

be the bijection that maps a sphere of $\mathbb{R}^d$ to a point of $\mathbb{R}^{d+1}$. We call $\mathbb{R}^{d+1}$ *the space of spheres*. For convenience, we call *vertical* axis the $(d+1)$-th coordinate axis and $(x, s) \in \mathbb{R}^{d+1}$ is said to be above (resp. below) (x, s') if $s > s'$ (resp. $s < s'$). We call *vertical projection* the orthogonal projection onto $x_{d+1} = 0$.

Observe that the image by ϕ of a point, considered as a sphere of radius 0 is a point of the paraboloid $\mathcal{Q}$ of equation $x_{d+1} = x^2$.



Let us consider a sphere passing through a given point p. We denote by x its center, r its radius and $s = x^2 - r^2$. We have

$$\sigma(p) = p^2 - 2x \cdot p + s = 0.$$

It follows that the image by ϕ of the spheres that pass through p is the *hyperplane* h_p of $\mathbb{R}^{d+1}$

$$x_{d+1} = 2p \cdot x - p^2.$$

The intersection of h_p and $\mathcal{Q}$ is reduced to the point $\phi(p) = (p, p^2)$: h_p is tangent to $\mathcal{Q}$ in p.

If x is closer to p_i than p_j

$$(x - p_i)^2 < (x - p_j)^2 \Longleftrightarrow 2\, p_i \cdot x - p_i^2 > 2\, p_j \cdot x - p_j^2.$$

In other words, the vertical line passing through x intersects h_{p_i} above h_{p_j}.

Pushing a little bit further this correspondence, we can establish a useful link with the Voronoi diagram of E. Let us consider the polytope $\mathcal{V}(E) = h_{p_1}^+ \cap \ldots \cap h_{p_n}^+$ of $\mathbb{R}^{d+1}$,

where $h^+_{p_i}$ denotes the half-space above h_{p_i}. Each point (x, x_{d+1}) of the facet f_i of $\mathcal{V}(E)$ contained in h_{p_i} is above all h_{p_j}. Hence, point x is closer to p_i than to any p_j, and the vertical projection of f_i is the Voronoi cell $V(p_i)$.

The faces of the Voronoi diagram of E are the vertical projections of the faces of $\mathcal{V}(E)$.

Combinatorics

It follows from the above correspondence that the combinatorial complexity of the Voronoi diagram of n points of $\mathbb{R}^d$ is the same as the combinatorial complexity of a polytope defined as the intersection of n half-spaces of $\mathbb{R}^{d+1}$, which is given by the celebrated upper bound theorem proved by McMullen and Shepard [10].

Theorem 112 (Upper Bound Theorem) *The total number of faces of a polytope defined as the intersection of n half-spaces of $\mathbb{R}^\delta$ is $\Theta\left(n^{\lfloor \frac{\delta}{2} \rfloor}\right)$.*

Proof. We present a simple proof of the asymptotic version of the upper bound theorem due to Seidel [15]. Let P be the polytope and p one of its vertices. We assume that all the vertices of P are incidents to exactly d edges (which is the generic situation). Otherwise, we can slightly perturb the hyperplanes bounding the half-spaces defining P so that the hyperplanes become in general position. During this process, the number of faces of P can only increase. Among the edges that are incident to p (there are d such edges), at least $\lceil \frac{d}{2} \rceil$ edges are in the half-space $x_d \geq x_d(p)$ or at least $\lceil \frac{d}{2} \rceil$ edges are in the half-space $x_d \leq x_d(p)$. If $k < d$ edges are incident to p, they belong to a same face of dimension k (k-face for short). Therefore, p is a vertex of extremal x_d-th coordinate for at least one face of dimension $\lceil \frac{d}{2} \rceil$. Since any face has at most one vertex of maximal x_d-th coordinate and one vertex of minimal x_d-th coordinate, the number of vertices of P is at most twice the number of $\lceil \frac{d}{2} \rceil$-faces of P.

Under the general position assumption, a k-face is the intersection of $d-k$ of the hyperplanes that define P. We conclude that the number of k-faces is $\binom{n}{d-k} = O(n^{d-k})$, which is $O(n^{\lfloor \frac{d}{2} \rfloor})$ if $k = \lceil \frac{d}{2} \rceil$. From the above discussion, we then conclude that the number of vertices of P is $O(n^{\lfloor \frac{d}{2} \rfloor})$.

Under the general position assumption, the number of faces (of any dimension) that are incident to a vertex is bounded by a constant that depends on d but not on n. Hence, the upper bound holds also for the number of faces of any dimension. □

An immediate consequence of the upper bound theorem and of the correspondence between Voronoi diagrams and polytopes is the following theorem.

Theorem 113 *The combinatorial complexity of the Voronoi diagram of a set of n points of $\mathbb{R}^d$ is $\Theta\left(n^{\lfloor \frac{d+1}{2} \rfloor}\right)$.*

In particular, the Voronoi diagram of n points of $\mathbb{R}^3$ may be quadratic.

Exercises 1

1. What do points above $\mathcal{Q}$ represent ?
2. What do the points of the boundary of $\mathcal{V}(E)$ represent ?
3. What does a line in the space of spheres represent ?
4. What is the vertical projection of the faces of $h^-_{p_1} \cap \ldots \cap h^-_{p_n}$?
5. Show that if we take points on two non coplanar lines of $\mathbb{R}^3$, say $n_1 + 1$ on one of the lines and $n_2 + 1$ on the other, their Voronoi diagram has $n_1 n_2$ vertices. See also exercices 2.

13.2.2 Delaunay Triangulation

We call *triangulation* of a finite set of points E of $\mathbb{R}^d$ a simplicial cell complex embedded in $\mathbb{R}^d$ whose set of vertices is E and whose domain (i.e. the union of its cells) is the convex hull $\mathrm{conv}(E)$ of E.

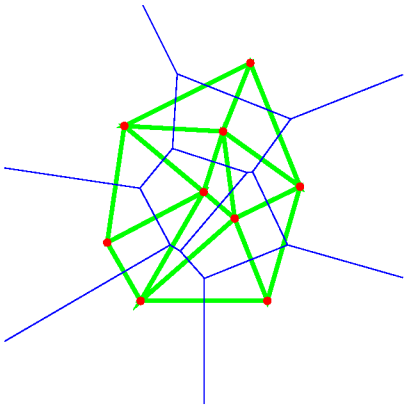
Duality

We introduce now a cell complex that is *dual* to the Voronoi diagram of a finite set of points E. Two cell complexes V and D are said to be dual if there exists an involutive correspondence between the faces of V and the faces of D that preserves the incidences and reverses the inclusions, i.e. for any two faces f and g of V, their dual faces f^* and g^* of D satisfy

$$\begin{aligned} f \cap g \neq \emptyset \quad &\Longrightarrow \quad f^* \cap g^* \neq \emptyset \\ f \subset g \quad &\Longrightarrow \quad g^* \subset f^* \end{aligned}$$

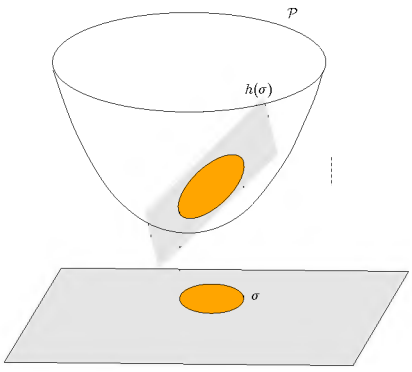
Let f be a face of the Voronoi diagram of E. All points in the interior of f have the same closest points in E. Let $E_f \subset E$ be the subset of those closest sites. The face dual to f is the convex hull of E_f. The *Delaunay triangulation of* E, noted $\mathrm{Del}(E)$, is the complex consisting of all the dual faces.

If E is in *general position*, no subset of $d+2$ points of E lie on a same sphere, $\mathrm{Del}(E)$ is a triangulation of E.



This duality is, in disguise, the well-known duality between polytopes in the space of spheres. In section 13.2.1, we have associated to the sphere σ of $\mathbb{R}^d$ of equation $\sigma(x) = x^2 - 2c \cdot x + s = 0$ the point $\phi(\sigma) = (c, s)$ of $\mathbb{R}^{d+1}$.

We associate to this point $\phi(\sigma)$ the so-called *polar hyperplane* h_σ of $\mathbb{R}^{d+1}$ of equation $x_{d+1} = 2c \cdot x - s$. Observe that if σ is reduced to a point c, h_σ is identical to the hyperplane h_c tangent to the paraboloid $\mathcal{Q}$ introduced in section 13.2.1. It should also be observed that the intersection of h_σ with $\mathcal{Q}$ projects onto the horizontal hyperplane $x_{d+1} = 0$ along σ.



We deduce the remarkable following property : $x \in \sigma$ iff $\phi(x) \in h_\sigma$ and σ encloses[1] x

[1] We say that a sphere σ of equation $\sigma(x) = 0$ encloses a point p if the open ball bounded by σ contains p, or equivalently if $\sigma(p) < 0$.

iff $\phi(x)$ is below h_σ. Indeed

$$\begin{aligned} \sigma(x) = 0 &\iff x^2 = 2c \cdot x - s \iff \phi(x) \in h_\sigma \\ \sigma(x) < 0 &\iff x^2 < 2c \cdot x - s \iff \phi(x) \in \operatorname{int} h_\sigma^- . \end{aligned}$$

The correspondence introduced above is an involution that preserves incidences and reverses inclusions. Hence, it is a duality. Indeed, if σ and σ' are two spheres, we have

$$\begin{aligned} \phi(\sigma) \in h_{\sigma'} &\iff s = 2c' \cdot c - s' \iff s' = 2c \cdot c' - s \iff \phi(\sigma') \in h_\sigma \\ \phi(\sigma) \in h_{\sigma'}^+ &\iff s < 2c' \cdot c - s' \iff s' = 2c \cdot c' - s \iff \phi(\sigma') \in h_\sigma^- . \end{aligned}$$

Consider now a set $E = \{p_1, \ldots, p_n\}$ of n points and let $\mathcal{V}(E)$ denote the polytope intersection of the n halfspaces above the n polar hyperplanes h_{p_i}. Let f be a face of $\mathcal{V}(E)$ and $h_{p_1}, \ldots, h_{p_{k+1}}$ be the $k+1$ hyperplanes that contain f. If σ denotes a sphere of $\mathbb{R}^d$, we deduce from the discussion above

$$\phi(\sigma) \in h_{p_1} \cap \ldots \cap h_{p_{k+1}} \iff \phi(p_1), \ldots, \phi(p_{k+1}) \in h_\sigma . \tag{13.1}$$

Moreover, if $\phi(\sigma) \in \mathcal{V}(E)$, $\phi(\sigma) \in h_{p_i}^+$ for any $p_i \in E$ et $\phi(\sigma) \in \operatorname{int} h_{p_i}^+$ for any $i > k+1$. Equivalently,

$$\forall i > k+1, \quad \phi(p_i) \in \operatorname{int} h_\sigma^+ . \tag{13.2}$$

We deduce from (13.1) and (13.2) that, if $\phi(\sigma) \in f$, h_σ is a hyperplane that supports the convex hull $\operatorname{conv}(\phi(E))$ of $\phi(E)$, and that $\operatorname{conv}(\{\phi(p_i)\}_{i=1,\ldots,k+1})$ is a face f^* of the lower convex hull $\mathcal{D}(E)$ of $\phi(\sigma_i)$[2]. Hence, duality associates to a face f of $\mathcal{V}(E)$ the face f^* of $\mathcal{D}(E)$. This correspondence is 1-1, preserves incidences and reverses inclusions.

We deduce a duality between the faces of $\operatorname{Vor}(E)$ and the faces of $\operatorname{Del}(E)$ which is the one we presented at the beginning of this section.

$$\begin{array}{ccc} \mathcal{V}(E) = h_{p_1}^+ \cap \ldots \cap h_{p_n}^+ & \longleftrightarrow & \mathcal{D}(E) = \operatorname{conv}^-(\{\phi(p_1), \ldots, \phi(p_n)\}) \\ \updownarrow & & \updownarrow \\ \text{Voronoi Diagram} & \longleftrightarrow & \text{Delaunay Triangulation} \end{array}$$

If the points $\phi(p_i)$ are affinely independent (which is equivalent to saying that no subset of $d+2$ points of E lie on a same sphere), all faces of $\mathcal{D}(E)$ are simplices.

Otherwise, it is always possible to triangulate the faces that are not simplices. We then obtain a Delaunay triangulation of E. Since there are several ways of triangulating the faces, the Delaunay triangulation of E is no longer unique.

It follows from the above correspondence that the combinatorial complexity of the Delaunay triangulation of n points is the same as the combinatorial complexity of the dual Voronoi diagram.

[2] The lower convex hull of a set of points P consists of the faces of the convex hull of P whose supporting hyperplanes are not above any point of P.

Fundamental property of the Delaunay triangulation

The following lemma can easily be proved either directly or as a consequence of the correspondence between $\mathrm{Del}(E)$ and $\mathcal{D}(E)$.

Lemma 45 *A triangulation T of a finite set of points E such that any d-simplex of T has a $(d-1)$-sphere circumscribing that does not enclose any point of E is a Delaunay triangulation of E. Any k-simplex with vertices in E that can be circumscribed by a $(d-1)$-sphere that does not enclose any point of E is a face of a Delaunay triangulation of E.*

Exercises 2

1. Show, using the fundamental property of Delaunay triangulations, that if we take points on two non coplanar lines of $\mathbb{R}^3$, say n_1+1 on one of the lines and n_2+1 on the other, their Delaunay triangulation has $n_1 n_2$ vertices. See also exercises 1.

2. Let S be a sphere of $\mathbb{R}^d$ passing through $d+1$ points $p_0, \ldots, p_d$. Show that a point p_{d+1} of $\mathbb{R}^d$ lies on S, in the interior of the ball B_S bounded by S or outside B_S, depending whether the determinant of the $(d+2) \times (d+2)$ matrix

$$\texttt{in_sphere}(p_0, \ldots, p_{d+1}) = \begin{vmatrix} 1 & \ldots & 1 \\ p_0 & \ldots & p_{d+1} \\ \|p_0\|^2 & \ldots & \|p_{d+1}\|^2 \end{vmatrix}$$

   is 0, $<$ or $>$ 0. Show that this predicate is the only numerical operation that is required to check if a triangulation is a Delaunay triangulation.

3. Project vertically the faces of the upper hull $\mathrm{conv}^+(\{\phi(E)\})$. Show that we obtain a triangulation of the vertices of $\mathrm{conv}(E)$ such that each ball circumscribing a simplex contains all points of E. Define a dual and make a link with exercise 13.2.1 (question 4) .

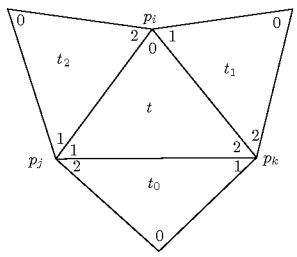
13.3 Construction of Voronoi diagrams and Delaunay triangulations

It follows from the previous sections that the construction of the Delaunay triangulation and, by duality, of the Voronoi diagram of a finite set of points E of $\mathbb{R}^d$ reduces to the construction of the convex hull of $\phi(E)$ in $\mathbb{R}^{d+1}$. The goal of this section is to present optimal algorithms that construct the convex hull of a set P of n points of $\mathbb{R}^\delta$. The result will apply to Delaunay triangulations by taking $P = \phi(E)$ and by replacing δ by $d+1$.

Some basic notions of complexity are recalled in the appendix.

13.3.1 Representation

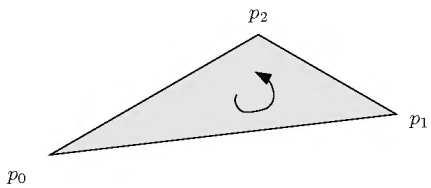
A vertex s of $\mathrm{conv}(P)$ is represented by a pointer to the corresponding point $\mathrm{point}(s)$. Each vertex points to one of its incident facets. Each facet of $\mathrm{conv}(P)$ has $\delta+1$ pointers to its $\delta+1$ vertices, $s_0, \ldots s_\delta$, and $\delta+1$ pointers to its $\delta+1$ incident facets, $f_0, \ldots, f_\delta$. By convention, f_i is the facet that has not s_i as a vertex.



13.3.2 Predicates

We call predicates the numerical tests that are used by the algorithm. The only predicate that is involved in the construction of the convex hull of points is the `orient` predicate. This predicate allows to determine the orientation of a δ-simplex of $\mathbb{R}^\delta$. Simplex $p_0 \ldots p_\delta$ is oriented positively, negatively or is degenerate (meaning that the dimension of the affine hull of the simplex is $< \delta$) whether the sign of the determinant of a $(\delta+1)\times(\delta+1)$ matrix is $+$, $-$ or 0. Namely, we have to evaluate

$$\texttt{orient}(p_0, \ldots, p_\delta) = \mathrm{signe}\begin{vmatrix} 1 & \ldots & 1 \\ p_0 & \ldots & p_\delta \end{vmatrix} = \mathrm{signe}|p_1 - p_0 \quad \ldots \quad p_\delta - p_0|$$



We say that the convex hull is *oriented* if all its facets are oriented negatively. A facet $f = (p_0, \ldots, p_\delta)$ is oriented positively if, for any $p \in P$, $\texttt{orient}(p_0, \ldots, p_\delta, p) < 0$. We require, in the sequel, that $\mathrm{conv}(P)$ is oriented.

One should verify that the `in_sphere` predicate that allows to determine whether a point $p_{\delta+1}$ is enclosed or not by the sphere circumscribing $\delta+1$ points $p_0, \ldots, p_\delta$ is nothing else thant the `orient` predicate in the space of spheres (see exercise 2).

Predicates play a crucial role in geometric algorithms because, depending on their evaluation, the program will branch differently. If the predicate is not evaluated correctly,

the correctness of the algorithm won't be ensured and the program may run forever or crash (a very common situation indeed). It is therefore of utmost importance to evaluate the predicates exactly. Observe however that this does not mean to resort to exact arithmetic. In the case of the `orient` predicate for example, we do not need to evaluate exactly the corresponding determinant but only its sign. Interval arithmetic and separation bounds can therefore be used to accelerate the evaluation. For a recent survey on robustness issues in computational geometry see [12].

13.3.3 An incremental algorithm

We start with the case of points in $\mathbb{R}^2$. The algorithm extends easily to higher dimensions.

1. We sort the points by lexicographic order. We therefore have for any two indices $i < j$

$$x(p_i) < x(p_j) \text{ or } \begin{cases} x(p_i) = x(p_j) \\ y(p_i) < y(p_j) \end{cases}$$

We represent $\mathrm{conv}(P)$ by two lists that share the same first element p_1 and the same last element p_n. The first one, L^+, represents the vertices of the *upper* hull, located above p_1p_2, the other one, L^-, represents the vertices of the *lower* hull, located below p_1p_2. We describe only the construction of L^+, the construction of L^- being entirely symetrical.

2. L^+ is initialized with p_1 and p_2.
3. **Loop :**

for $i = 3, \ldots, n$
while $\mathrm{tail}(L^+) \neq \mathrm{head}(L^+)$ **and** $\mathrm{orient}(p_i, \mathrm{tail}(L^+), \mathrm{pred}(\mathrm{tail}(L^+))) \leq 0$
    remove $\mathrm{tail}(L^+)$ from L^+ ;
insert p_i in $\mathrm{tail}(L^+)$.

The initial sorting can be done using $O(n \log n)$ comparisons. In step 3, a unique new element is inserted in L^+. The other elements of L^+ that are considered are, all except the last one, removed from the list and won't be inserted again. The total cost of step 3 is therefore $O(n)$. Observe that the cost of a single iteration can be $O(n)$; nevertheless the total *amortized* cost is also linear.

The overall complexity of the algorithm is therefore $O(n \log n)$. Note that the algorithm executes $O(n \log n)$ comparisons of coordinates and only $O(n)$ orientation tests.

13.3.4 Can we do better ?

It is known that sorting n elements requires at least $\log_2(n!) = \Omega(n \log n)$ comparisons (see the appendix). We deduce that computing the convex hull of n points requires at least $\Omega(n \log n)$ comparisons and orientation tests.

Indeed, take n numbers $x_1, \ldots, x_n$. We associate to each x_i the point $p_i = (x_i, x_i^2)$. The p_i lie on the parabola $y = x^2$. If we know the convex hull of the p_i, we can deduce in

linear time the list of the x_i sorted by increasing values. Observe that the orientation test reduces in this case to three comparisons. Indeed,

$$\begin{vmatrix} x_i - x_j & x_i - x_k \\ x_i^2 - x_j^2 & x_i^2 - x_k^2 \end{vmatrix} = (x_i - x_j)(x_j - x_k)(x_k - x_i).$$

As any sorting algorithm must execute $\Omega(n \log n)$ comparisons, we conclude that any convex hull algorithm using comparisons and orientation tests only must execute $\Omega(n \log n)$ such operations.

13.3.5 Randomisation helps

The algorithm of the section 13.3.3 can be extended to $\mathbb{R}^3$ but its complexity cannot remain $O(n \log n)$. In fact, no incremental algorithm can have an $O(n \log n)$ complexity in dimensions 3 and higher. Still, we can have an optimal algorithm provided we introduce some randomness in the algorithm[3].

The algorithm is incremental and introduces the points one by one. We first describe the case of point set P of $\mathbb{R}^2$. Let o be a point in the interior of $\mathrm{conv}(P)$, e.g. the barycenter of three non collinear points. We denote by P_i the subset of the i first points that have been inserted. The algorithm maintains two data structures. First, we maintain the current convex hull $\mathrm{conv}(P_i)$ at step i. Second, we maintain a graph called *conflict graph*. The conflict graph is a bipartite graph that links any point p_j that has not been inserted yet (i.e. $j > i$) to an edge e of the current convex hull $\mathrm{conv}(P_i)$ that is intersected by the line segment op_j. A point and an edge that are joined by an arc in the conflict graph are said to be in conflict.

Consider step $i+1$. An edge of $\mathrm{conv}(P_i)$ is said to be *red* if its supporting line intersects the line segment op_{i+1}. An edge that is not red is said to be *blue*. If $\mathrm{conv}(P_i)$ is oriented positively, an edge ab is red if and only if $\mathtt{orient}(a, b, p_{i+1}) < 0$. The union of the red edges is a connected portion of the boundary of $\mathrm{conv}(P_i)$. Let s and t be its endpoints. Updating $\mathrm{conv}(P_i)$ consists in replacing the red edges by the two new edges sp_{i+1} and $p_{i+1}t$.

Let e be an edge of $\mathrm{conv}(P_i)$ in conflict with p_{i+1}. As e is intersected by op_{i+1}, e is red and must be removed. To find the other red edges, we walk on the boundary of $\mathrm{conv}(P_i)$ (in both directions) until we meet a blue edge : we thus get s and t, and we can construct $\mathrm{conv}(P_{i+1})$ in time proportional to the number of red edges.

It remains to update the conflict graph. For each point of index $j > i+1$ whose conflict edge has just been removed, we look for the new edge that is intersected by the half-line $[op_j)$. Let $e' = a'b'$ be that edge. If $\mathtt{orient}(a'b'p_j) \geq 0$, p_j is not a vertex of $\mathrm{conv}(P)$ and is no longer considered. Otherwise, we add an arc in the conflict graph joining p_j and e'.

In higher dimensions, the algorithm is almost identical. The main difference comes from the fact that the boundary of the convex hull is a triangulation and not a polygonal curve. We identify the red facets by traversing the adjacency graph of the facets of $\mathrm{conv}(P_i)$. Let R

[3]The theory of randomized algorithms is well-developed and finds applications in many areas of computer science. See the book by Motwani and Raghavan for a broad perspective [13].

be the union of the red faces. To update $\mathrm{conv}(P_{i+1})$, we create new faces $f_k = \mathrm{conv}(p_{i+1}, g_k)$ for all faces g_k of the boundary of R, and remove the red faces.

To analyze this algorithm, we assume that the points are inserted in random order.

Let us bound the expected number of facets created at step i. In $\mathbb{R}^2$, this number is 2. Consider the situation in $\mathbb{R}^\delta$. Assume that P_i is given. A facet of $\mathrm{conv}(P_i)$ has been created at step i if and only if one of its δ vertices is p_i, which arises with probability $\frac{\delta}{i}$. Noting $\mathcal{E}$ the expectation and n_i the number of facets of $\mathrm{conv}(P_i)$ that are created at step i, we have, using the upper bound theorem

$$\mathcal{E}(n_i) = \sum_{f \in \mathrm{conv}(P_i)} \frac{\delta}{i} = \frac{\delta}{i}\, O\left(i^{\lfloor \frac{\delta}{2} \rfloor}\right) = O(n^{\lfloor \frac{\delta}{2} \rfloor - 1}).$$

In particular, when $\delta = 2$ or 3, $\mathcal{E}(n_i) = O(1)$. This bound does not depend on the choice of P_i and therefore remains valid when averaging on all subsets P_i of P of i elements. Therefore, the bound holds also for the expected number of facets created at step i. By summing over all i and using linearity of expectation, we obtain that the expected total number of created (and therefore also removed) facets is $O(n^{\lfloor \frac{\delta}{2} \rfloor})$.

We now bound the cost of updating the conflict graph using backwards analysis. Starting from the final convex hull, we remove points one-by-one in reverse order. Consider step i where point p_i is removed. The conflict associated to a point p_j, $j > i$, must be modified if and only if the facet f of the current convex hull $\mathrm{conv}(P_i)$ that is in conflict with p_j is removed at step i. This occurs if one of the δ vertices of f is precisely point p_i, which occurs with probability $\frac{\delta}{i}$. The expected number of conflicts that are modified at step i is therefore $\frac{\delta}{i}\, O\left((n-i)^{\lfloor \frac{\delta}{2} \rfloor}\right)$. By summing over i and using linearity of expectation, we obtain that the total cost of updating the conflict graph is $O(n \log n + n^{\lfloor \frac{\delta}{2} \rfloor}))$.

Theorem 114 *The convex hull of n points of $\mathbb{R}^\delta$ can be constructed in time $\Theta\left(n \log n + n^{\lfloor \frac{\delta}{2} \rfloor}\right)$.*

Corollary 25 *The Delaunay triangulation and the Voronoi diagram of n points of $\mathbb{R}^d$ can be constructed in time $\Theta\left(n \log n + n^{\lfloor \frac{d+1}{2} \rfloor}\right)$.*

Robust and efficient implementations exist for constructing convex hulls and Delaunay triangulations. We encourage the reader to experiment with the CGAL library [5].

Exercises 3

1. Show that any incremental algorithm that constructs the convex hull of n points of $\mathbb{R}^3$ takes $\Omega(n^2)$ time in the worst-case.

2. Show that we can remove a point in expected time $O(\log n)$.

13.4 Delaunay triangulation restricted to a surface

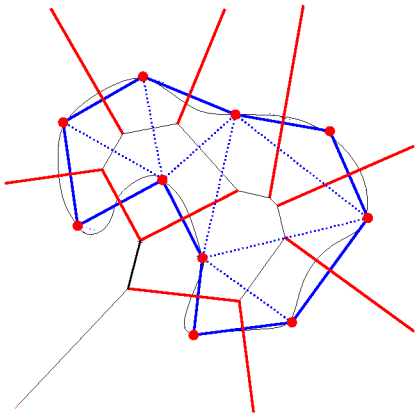
In this section, we introduce the concept of restricted Delaunay triangulation. This concept is related to the notion of *nerve* in algebraic topology [8]. Our main motivation is to approximate a surface S of $\mathbb{R}^3$ by the restriction to S of the Delaunay triangulation of a sample E of S, i.e. a finite set of points of S. Before we state our approximation results in sections 13.4.3, we need to define the sampling density in section 13.4.2.

13.4.1 Restricted Delaunay triangulation

Let S be a k-manifold of $\mathbb{R}^d$ and E a finite set of points of $\mathbb{R}^d$.

Definition 105 *If f is a face of the Voronoi diagram of E, we say that $f \cap S$ is the restriction of f to S. The subcomplex made of all non-empty restrictions of faces of* $\mathrm{Vor}(E)$ *to S is called the restriction of* $\mathrm{Vor}(E)$ *to S, and noted* $\mathrm{Vor}_{|S}(E)$.

Definition 106 *We call restriction of the Delaunay triangulation* $\mathrm{Del}(E)$ *to S the subcomplex of* $\mathrm{Del}(E)$*, noted* $\mathrm{Del}_S(E)$*, that consists of the k-faces of* $\mathrm{Del}(E)$ *whose dual Voronoi faces of codimension k intersect S.*



$\mathrm{Del}_S(E)$ is a simplicial 2-complex.

We call *surface Delaunay ball* a ball that is centered on S and that circumscribes a facet of $\mathrm{Del}_S(E)$. There may be several surface Delaunay balls associated to a given facet.

In the sequel, we denote by f a facet of $\mathrm{Del}_S(E)$ and by $B_f = B(c_f, r_f)$ be a surface Delaunay ball that circumscribes f.

As can be seen from the figure, $\mathrm{Del}_S(E)$ is a simple closed polygon that correctly approximates S. We will show, in the two next sections, that this is indeed the case under some sampling condition.

13.4.2 Sampling conditions

The medial axis $\mathcal{M}(S)$ of S will play a crucial role in the sequel. In some sense, $\mathcal{M}(S)$ is a generalization of the notion of Voronoi diagram to infinite point sets.

Definition 107 (Medial axis of a surface) *The* medial axis *of S is the closure of the set of points with two closest points on S.*

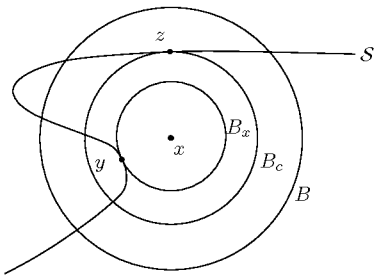
Observe that if S is a finite set of points of $\mathbb{R}^d$, $\mathcal{M}(S)$ is the subcomplex of $\mathrm{Vor}(E)$ obtained by removing from $\mathrm{Vor}(E)$ the cells of full dimension d.

A ball centered on the medial axis whose interior does not intersect S but whose boundary intersect S is called a *medial ball.*

Definition 108 (Local feature size) *The local feature size at a point $x \in S$, noted $\mathrm{lfs}(x)$, is the distance of x to the medial axis of S.*

It can easily be proved that lfs is 1-Lipschitz, i.e. $\forall x, y \in S$, $\mathrm{lfs}(x) \leq \mathrm{lfs}(y) + \|x - y\|$.

Lemma 46 *Let S be a manifold of $\mathbb{R}^d$ and B a ball centered at a point x of radius r that intersects S. If $B \cap S$ is not a $(d-1)$-topological ball, B contains a point of the medial axis of S.*



Proof. The result is trivial when x belongs to the medial axis $\mathcal{M}(S)$ of S. Therefore assume that $x \notin \mathcal{M}(S)$ and that $B \cap S$ is not a topological ball.

We denote by B_x the largest ball centered at x whose interior does not intersect S. Let r_x be its radius. B_x is tangent to S in a unique point y (since x does not belong to $\mathcal{M}(S)$). Hence $B_x \cap S = \{y\}$ is a topological ball. It follows that there exists a point z of S at distance $r_c > r_x$ from x such that the ball $B(x, r_c)$ is tangent to S in z. The medial ball tangent to S in z and centered on the half-line $[zx)$ is contained in B_c (since it is tangent to S in z and cannot contain y) and therefore in B. Its center belongs to $\mathcal{M}(S)$. □

Since, by definition of $\mathrm{lfs}(x)$, $B(x, r)$ cannot intersect the medial axis of S, Lemma 46 implies

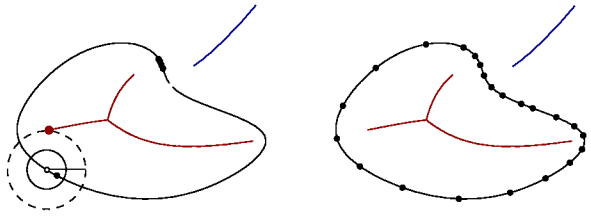
Lemma 47 *For any x of S, and any $r < \mathrm{lfs}(x)$, the intersection of S with the ball $B(x, r)$ centered at x of radius r is a topological $(d-1)$-ball.*

Definition 109 *Given a function $\phi : S \to \mathbb{R}$ and $\varepsilon < 1$, we say that a finite set of points E is a loose ε-sample for ϕ if*

1. $E \subset S$ and $\mathrm{Del}_S(E)$ has at least one vertex on each connected component of S,

2. for any facet f of $\mathrm{Del}_S(E)$ and any surface Delaunay ball $B_f = B(c_f, r_f)$ circumscribing f, we have $r_f \leq \varepsilon\phi(c_f)$. A loose 1-sample is simply called a loose sample.

Uniform samples are obtained with $\phi = 1$. A useful function that allows non uniform samples is $\phi = \mathrm{lfs}$ (See the figure).



13.4.3 Approximation results

The goal of this section is to prove that, under some sampling condition, $\mathrm{Del}_S(E)$ and S are isotopic and that the distance between a point and its image by the isotopy is a decreasing function of the sampling density.

We recall some definitions.

Definition 110 *Two subsets X and Y of $\mathbb{R}^d$ are said to be homeomorphic if there exists a continuous, bijective map $f : X \to Y$ with continuous inverse f^{-1}.*

Definition 111 *Two subsets X and Y of $\mathbb{R}^d$ are said to be isotopic if there exists a continuous map $f : X \times [0,1] \to \mathbb{R}^d$ such that $f(.,0)$ is the identity of X, $f(X,1) = Y$, and for each $t \in [0,1]$, $f(.,t)$ is a homeomorphism onto its image.*

Before considering the case of a surface of $\mathbb{R}^3$, we consider the simpler case of a curve in the next section.

A. Approximation of a curve

In this section, S denotes a simple curve of $\mathbb{R}^d$ and E a sample of S. We assume that E contains at least three points on each closed connected component of S (H1). If a component is an arc, we assume that the two end points of the arc belong to E (H2). S is then the union of *elementary arcs* joining two points of E that are consecutive along S. Note that two elementary arcs have at most one endpoint in common. In addition, we assume that no vertex of $\mathrm{Vor}(E)$ lie on S (H3), which can always be ensured by slightly perturbing E.

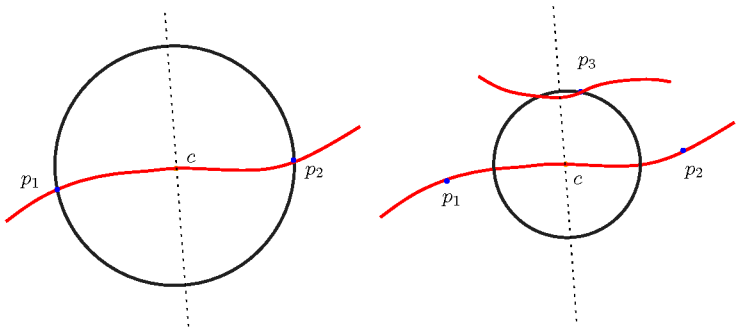
We say that a ball B is *regular* if $\mathrm{int}(B \cap S)$ is either empty or an arc.

Definition 112 *We call critical radius at point x, noted $r_c(x)$, the smallest r such that the ball $B(x,r)$ is regular.*

Observe that r_c is not necessarily Lipschitz. By lemma 47, $r_c(x) \geq \mathrm{lfs}(x)$.

Theorem 115 *If E be an ε-sample of S for r_c, then $\mathrm{Del}_S(E)$ is a polygonal curve isotopic to S and $d_H(\mathrm{Del}_S(E), S) \leq 2\varepsilon \sup_{x\in S} r_c(x)$.*

Proof. Let p_1 and p_2 be two points of E consecutive along S and let γ_{12} be the elementary arc of S that joins p_1 to p_2. We denote by c an intersection point of γ_{12} with the bisector hyperplane of p_1 and p_2, and σ the largest sphere centered at c that does not enclose any point of E. If σ passes by p_1 and p_2, p_1p_2 is an edge of $\mathrm{Del}(E)$ and therefore of $\mathrm{Del}_S(E)$. Otherwise, σ passes through a third point p_3 of E and the intersection of the ball bounded by σ with S contains at least two arcs. The radius of σ is therefore greater than $r_c(c)$, which contradicts the hypothesis.



Conversely, we show that any Delaunay edge p_ip_j whose Voronoi dual facet $p_ip_j^*$ intersects S joins two points of E that are consecutive along S. We denote by c an intersection point of $p_ip_j^*$ with S. The sphere σ that is centered at c and passes through p_i and p_j does not enclose any point of E : its radius is therefore $< r_c(c)$ since, by hypothesis, c is at distance $< r_c(c)$ from a point of E. By definition of r_c, the portion of S that is contained in the ball bounded by σ is an arc : p_i and p_j are therefore consecutive along S.

Two consecutive points defining a unique elementary arc (H1), we have a bijection between the elementary arcs of S and the edges of $\mathrm{Del}_S(E)$. Moreover, the surface Delaunay ball[4] of any arc γ of S intersects no other portion of S. Hence, there exists an isotopy that maps γ to the corresponding Delaunay edge. Since an elementary arc of S and the associated edge of $\mathrm{Del}_S(E)$ have the same endpoints, we obtain an isotopy between S and $\mathrm{Del}_S(E)$ by concatenating the elementary isotopies between the elementary arcs and their associated edges of $\mathrm{Del}_S(E)$.

The last claim comes from the observation made above that S is included in the union of the surface Delaunay balls. □

[4]Curve Delaunay ball should be more appropriate.

B. Approximation of a smooth surface

The goal of this section is to prove a result analogous to Theorem 115 for surfaces. S denotes now a surface of $\mathbb{R}^3$ that we assume to be of class C^2 and closed. E is a loose ε-sample for lfs of S.

The proof is elementary but involves several lemmas. We first prove some preliminary results. We then prove that $\mathrm{Del}_S(E)$ is a closed surface. Lastly, we exhibit an isotopy between S and $\mathrm{Del}_S(E)$ that does not move the points by more than $\varepsilon \sup_{x \in S} \mathrm{lfs}(x)$.

Preliminary lemmas

For $x \in S$, n_x denotes the unit vector normal to S, oriented according to the orientation of S. In this section, f denotes a facet of $\mathrm{Del}_S(E)$, B_f a surface Delaunay ball circumscribing f, c_f its center, r_f its radius, and n_f a unit vector perpendicular to f. Unless it is explicitly mentionned, we do not specify an orientation for n_f in this section. The module of the angle between two vectors n and n', measured in $[0, \pi]$, is noted (n, n'). The first three lemmas have been proved by Amenta and Bern in a seminal paper [1].

Lemma 48 (Normal variation lemma) *Let x and y be two points of S with $\|x - y\| \leq \varepsilon \min(\mathrm{lfs}(x), \mathrm{lfs}(y))$, $\varepsilon < \frac{1}{3}$. The angle (n_x, n_y) between the normals n_x and n_y is at most $\frac{\varepsilon}{1-3\varepsilon}$.*

Proof. We parametrize the line segment xy by length. Let $x(t)$ denote the point on xy with parameter value t and let $f(t)$ denote the nearest point to $x(t)$ on the surface S. Point $f(t)$ is unique, because otherwise $x(t)$ would be a point of the medial axis, contradicting $\|x - y\| \leq \varepsilon \mathrm{lfs}(x)$ with $\varepsilon < \frac{1}{3}$.

Let $n(t)$ denote the unit normal to S at $f(t)$, and let $\|n'(t)\|$ denote the magnitude of the derivative of $n(t)$ with respect to t. The change in normal between x and y is at most $\int_{xy} \|n'(t)\| \, dt$, which is at most $\|x - y\| \sup_t \|n'(t)\|$.

The surface S passes between the tangent balls of radius $\mathrm{lfs}(f(t))$ at $f(t)$, so the greater of the two principal curvatures at $f(t)$ is no more than the curvature of these tangent balls. The rate at which the normal changes with $f(t)$ is at most the greater principal curvature, and hence $\|n'(t)\|$ is at most the rate at which the normal turns (as a function of t) on one of these tangent balls. We have

$$dt \geq (\mathrm{lfs}(f(t)) - \|f(t) - x(t)\|) \sin d\theta.$$

Now $\sin d\theta$ approaches $d\theta$ as θ goes to zero, so

$$\|n'(t)\| = \frac{d\theta}{dt} \leq 1/(\mathrm{lfs}(f(t)) - \|f(t) - x(t)\|).$$

We have

$$\|f(t) - x(t)\| \leq \|x(t) - x\| \leq \varepsilon \, \mathrm{lfs}(x)$$

and

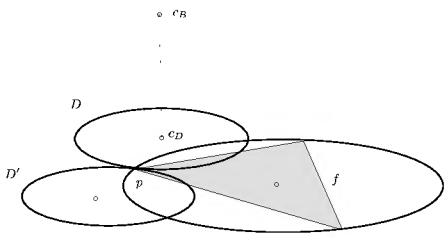
$$\|f(t) - x\| \leq \|f(t) - x(t)\| + \|x(t) - x\| \leq 2\varepsilon \operatorname{lfs}(x),$$

so, by the Lipschitz property of lfs, $\operatorname{lfs}(f(t)) \geq (1 - 2\varepsilon) \operatorname{lfs}(x)$. Altogether we obtain $\sup_t \|n'(t)\| \leq 1/((1 - 3\varepsilon) \operatorname{lfs}(x))$, which yields the lemma. □

Lemma 49 (Chord lemma) *For any two points x and y of S with $\|x - y\| \leq \varepsilon \operatorname{lfs}(x)$, $|\delta_{xy} \cdot n_x| \leq \frac{\varepsilon}{2}$, where $\delta_{xy} = \frac{x-y}{\|x-y\|}$.*

Proof. S and therefore y do not intersect the interior of the tangent balls of radius $\operatorname{lfs}(x)$ at x. The module of the angle between δ_{xy} and n_x is maximized when y lies on the boundary of one of the tangent balls. We then have $|\delta_{xy} \cdot n_x| = \frac{\|x-y\|}{2\operatorname{lfs}(x)} \leq \frac{\varepsilon}{2}$. □

Lemma 50 (Facet normal lemma) *Let $f = pqr$ be a facet of $\operatorname{Del}_S(E)$ with $r_f \leq \varepsilon \operatorname{lfs}(c_f)$, $\varepsilon < 1$, and assume that $\hat{p} \geq \frac{\pi}{3}$. Then $\sin(n_f, n_p) \leq \frac{\varepsilon\sqrt{3}}{1-\varepsilon}$.*



Proof. Let B and B' denote the two balls of radius $\operatorname{lfs}(p)$ tangent to S at p, $D = B \cap \operatorname{aff}(f)$ and $D' = B' \cap \operatorname{aff}(f)$. We call c_B and c_D the centers of B and D respectively. ρ_f denotes the circumradius of f, and ρ_D the radius of D and D'.

We first observe that $\rho_D \leq \rho_f\sqrt{3}$. Indeed, the interiors D and D' contain no vertex of f. ρ_D is then maximized with respect to ρ_f when f is equilateral with one vertex other than p, say q, on the boundary of D and r on the boundary of D'.

According to the hypothesis, $\rho_f \leq r_f \leq \varepsilon \operatorname{lfs}(c_f) \leq \frac{\varepsilon}{1-\varepsilon} \operatorname{lfs}(p)$.

We then have

$$\sin(n_f, n_p) = \sin(pc_Bc_D) \leq \frac{\|p - c_D\|}{\|p - c_B\|} = \frac{\rho_D}{\operatorname{lfs}(p)} \leq \frac{\varepsilon\sqrt{3}}{1 - \varepsilon}$$

□

Lemma 51 (Orientation lemma) *With the same notations as in the previous lemma, for $v = q, r$, $|n_f \cdot n_v| \geq \cos\left(\arcsin \frac{\varepsilon\sqrt{3}}{1-\varepsilon} + \frac{2\varepsilon}{1-7\varepsilon}\right)$. Moreover, if $\varepsilon < 0.1$, $n_f \cdot n_p$, $n_f \cdot n_q$ and $n_f \cdot n_r$ have the same sign.*

Proof. Let p be the vertex of f with largest inner angle and assume, without loss of generality, that $n_f \cdot n_p > 0$. By the facet normal lemma 50, we have $(n_f, n_p) \leq \arcsin \frac{\varepsilon\sqrt{3}}{1-\varepsilon}$. Let v be another vertex of f. $\|p - v\| \leq 2\varepsilon \mathrm{lfs}(c_f) \leq \frac{2\varepsilon}{1-\varepsilon} \min(\mathrm{lfs}(p), \mathrm{lfs}(v))$ since $\mathrm{lfs}(c_f) \leq \mathrm{lfs}(v) + \|v - c_f\| \leq \mathrm{lfs}(v) + \varepsilon \mathrm{lfs}(c_f)$. By the normal variation lemma 48, we then get $(n_p, n_v) \leq \frac{2\varepsilon}{1-7\varepsilon}$. From $(n_f, n_v) \leq (n_f, n_p) + (n_p, n_v)$, we deduce $n_f \cdot n_v \geq \cos\left(\arcsin \frac{\varepsilon\sqrt{3}}{1-\varepsilon} + \frac{2\varepsilon}{1-7\varepsilon}\right)$.

Hence, for $\varepsilon < 0.1$, $n_f \cdot n_v > 0$. □

Orientation convention *From now on, we assume that $\varepsilon < 0.1$ and orient the normals of the facets of $\mathrm{Del}_S(E)$ so that, for any facet f and any vertex v of f, $n_f \cdot n_v > 0$.*

Lemma 52 (Projection lemma) *Let f and f' be two facets of $\mathrm{Del}_S(E)$ with a common edge $e = uv$. If $r_f < \varepsilon\ \mathrm{lfs}(c_f)$ and $r_{f'} < \varepsilon\ \mathrm{lfs}(c_{f'})$, with $\varepsilon < 0.1$, the orthogonal projections of f and f' onto the tangent plane at v do not overlap, i.e. their interiors do not intersect.*

Proof. We assume, for a contradiction, that the projections of f and f' onto the tangent plane at v overlap. For convenience, we note $B = B_f$, $r = r_f$, $c = c_f$ and $B' = B_{f'}$, $r' = r_{f'}$, $c' = c_{f'}$. Since u and v are vertices of f and f', ∂B and $\partial B'$ have a non-empty intersection. Let P be the plane that contains $\partial B \cap \partial B'$, and therefore u and v. Let w be the third vertex of f, and w' be the third vertex of f'. P is perpendicular to cc'. On one side of P, B is included in the interior of B', whereas on the other side of P, B' is included in the interior of B. Since B and B' are Delaunay balls, B cannot contain w' and B' cannot contain w. Hence, w and w' cannot lie on the same side of P.

Recall that n_f has been oriented so that $n_f \cdot n_v \geq 0$. By the orientation lemma 50, the angle (n_f, n_v) between n_f and n_v is at most $\eta = \arcsin \frac{\varepsilon\sqrt{3}}{1-\varepsilon} + \frac{2\varepsilon}{1-7\varepsilon}$ and the same bound applies to the angle between $n_{f'}$ and n_v.

For any two points p and q, let $\delta_{pq} = \frac{q-p}{\|q-p\|}$ and consider a plane Π perpendicular to uv.

The two facets f and f' projects on Π on two line segments s and s' that have v as a common endpoint (the projection of u and v) and lie in the same halfplane bounded by the line supporting n_v. Since P contains u and v, P is perpendicular to Π and projects onto Π along a line l_P. P and therefore l_P are perpendicular to $\delta_{cc'}$. Since f and f' are not on the same side of P, s and s' cannot be on the same side of l_P. It follows that

$$(\delta_{cc'}, n_v) \leq (n_f, n_v) \leq \eta = \arcsin \frac{\varepsilon\sqrt{3}}{1-\varepsilon} + \frac{2\varepsilon}{1-7\varepsilon}. \tag{13.3}$$

Using the Lipschitz property of function lfs, we get

$$\|c - c'\| \leq \varepsilon(\mathrm{lfs}(c) + \mathrm{lfs}(c')) \leq \varepsilon(2\mathrm{lfs}(c) + \|c - c'\|) \leq \frac{2\varepsilon}{1-\varepsilon}\mathrm{lfs}(c). \tag{13.4}$$

By the chord lemma 49, we have $|\delta_{cc'} \cdot n_c| \leq \frac{\varepsilon}{1-\varepsilon}$. Moreover, since $\|c - v\| \leq \varepsilon \mathrm{lfs}(c) \leq \frac{\varepsilon}{1-\varepsilon}\mathrm{lfs}(v)$, the normal variation lemma 48 implies that $\phi = (n_c, n_v) \leq \frac{\varepsilon}{1-4\varepsilon}$. Noting $n_v = \cos\phi\ n_c + \sin\phi\ n_c^{\vdash}$, we obtain

$$|n_v \cdot \delta_{cc'}| \leq |\cos\phi\ \delta_{cc'} \cdot n_c| + |\sin\phi\ \delta_{cc'} \cdot n_c^{\vdash}| \leq \xi = \frac{\varepsilon}{1-\varepsilon} + \sin\frac{\varepsilon}{1-4\varepsilon} \tag{13.5}$$

We get a contradiction from equations 13.3 and 13.5 when $\varepsilon \leq 0.1$. □

Remark. Observe that the proof of the projection lemma still holds if B and B' are two surface Delaunay balls circumscribing the same facet $f = f'$. Hence, a facet of $\mathrm{Del}_S(E)$ cannot have two surface Delaunay balls $B_f = B(c_f, r_f)$ and $B'_f = B(c'_f, r'_f)$ with $r_f \leq \varepsilon\,\mathrm{lfs}(c_f)$ and $r'_f \leq \varepsilon\,\mathrm{lfs}(c'_f)$.

$\mathrm{Del}_S(E)$ is a closed (triangulated) surface

In this section, we prove that $\mathrm{Del}_S(E)$ is a closed (triangulated) surface of $\mathbb{R}^3$. Since $\mathrm{Del}_S(E)$ is a 2-complex, this is equivalent to proving that each edge of $\mathrm{Del}_S(E)$ is incident to exactly two facets, and that the boundary of the star of each vertex is a simple closed polygon. The *star* of a vertex v is the union of the facets of $\mathrm{Del}_S(E)$ that are incident to v.

From the projection lemma 52, we deduce the following result.

Lemma 53 *If E is a loose ε-sample for* lfs, $\varepsilon < 0.1$, *every edge of* $\mathrm{Del}_S(E)$ *is incident to exactly two facets of* $\mathrm{Del}_S(E)$.

Proof. Let e be an edge of $\mathrm{Del}_S(E)$. We denote by e^* the Voronoi face dual to e. Since S has no boundary, $S \cap \mathrm{aff}(e^*)$ is a union of simple closed curves, none of which intersects the boundary ∂e^* of e^* tangentially, by the facet normal lemma 50. Thus, by the Jordan curve theorem, each component of $S \cap \mathrm{aff}(e^*)$ intersects ∂e^* at an even number of points. It follows that S intersects ∂e^* at an even number of points. Moreover, by the remark after the projection lemma, each edge of ∂e^* cannot be intersected more than once by S. Thus, S intersects an even number of edges of ∂e^*, and e is incident to an even number of restricted Delaunay facets.

In addition, e cannot be incident to more than two facets of $\mathrm{Del}_S(E)$. Indeed, by the projection lemma 52, the projections onto $T(v)$ (where v is a vertex of e) of the restricted Delaunay facets incident to e pairwise do not overlap, thus they must lie on different sides of the line supporting the projection of e.

In conclusion, the number of facets of $\mathrm{Del}_S(E)$ that are incident to e is even, at least 1 and at most 2. The result follows. □

Consider a vertex v of $\mathrm{Del}_S(E)$ whose star boundary is not a simple closed polygon. By the discussion above, the star of v consists of several components we call *umbrellas*. Each umbrella is a triangulated topological disk. All umbrellas of v have v in common but two distinct umbrellas have distinct edges and facets. We can further prove

Lemma 54 *Under the same conditions as in the previous lemma, every vertex of* $\mathrm{Del}_S(E)$ *has exactly one umbrella.*

The next proposition follows from lemmas 53 and 54.

Proposition 37 *Let S be a smooth closed surface and E a loose ε-sample of S. If $\varepsilon < 0.1$, then* $\mathrm{Del}_S(E)$ *is a compact surface without boundary.*

$\mathrm{Del}_S(E)$ and S are isotopic

Let S be a surface of $\mathbb{R}^3$. For $\varepsilon < 1$, we define $\mathrm{tube}_\varepsilon(S)$ to be the tubular neighborhood of S

$$\mathrm{tube}_\varepsilon = \bigcup_{x \in S} B(x, \varepsilon \mathrm{lfs}(x)).$$

$\mathrm{tube}_\varepsilon$ does not intersect the medial axis of S and its boundary has two components S_ε^+ and S_ε^-. We call *fibre* at x the line segment normal to S at x with one endpoint on S_ε^+ and the other on S_ε^-.

The following result is a basic theorem in differential topology [11]. Let $\pi_S : \mathbb{R}^3 \to S$ map each point to its closest point of S. The restriction of π_S to $\mathrm{tube}_\varepsilon$ is a well-defined continuous function since $\mathrm{tube}_\varepsilon(S)$ does not intersect the medial axis of S.

Theorem 116 *Let S be a C^2, compact, oriented, closed surface and T be a compact closed surface (not necessarily smooth) such that :*

1. *$T \subset U = \mathrm{tube}_\varepsilon(S)$,*
2. *Any fibre intersects T in exactly one point.*

Then the restriction of π_S to T induces an isotopy that maps T to S. The isotopy does not move the points by more than $\varepsilon \sup_{x \in S} \mathrm{lfs}(x)$.

This theorem will be used to prove the following result :

Theorem 117 *If E is a loose ε-sample, $\varepsilon < 0.1$, then the restriction of π_S to $\mathrm{Del}_S(E)$ induces an isotopy that maps $\mathrm{Del}_S(E)$ to S. The isotopy does not move the points by more than $\varepsilon \sup_{x \in S} \mathrm{lfs}(x)$.*

Before proving the theorem in its full generality, we prove that the restriction of π_S from $\mathrm{Del}_S(E)$ to $E \subset S$ is one-to-one.

Lemma 55 *Let E be a loose sample of S, $p \in E$, and let m_p be the center of a medial ball B_p tangent to S at p. The open segment pm_p can only intersect $\mathrm{Del}_S(E)$ at point p.*

Proof. Let $p \in E$, m_p the center of a medial ball B_p tangent to S at p. We prove that the open segment pm_p cannot intersect $\mathrm{Del}_S(E)$. Assume that pm_p intersects a facet f of $\mathrm{Del}_S(E)$. ∂B_p and ∂B_f necessarily intersect since the vertices of f do belong to the interior of B_p and p does not belong to the interior of B_f. Let H be the plane containing the intersection of the spheres bounding B_p and B_f, H^+ be the halfspace limited by H and containing f. Since p does not belong to the interior of B_f and is on the boundary of B_p, p must belong to H^- (possibly in H). On the other hand, m_p belongs to the open half space H^- since otherwise m_p would belong to B_f, which contradicts the assumption that $r_f < \mathrm{lfs}(c_f)$. Hence the open segment pm_p is contained in the open half space H^-, and therefore cannot intersect f which is contained in H^+ . □

We now proceed to the proof of theorem 117.

Proof. We prove that the hypotheses of theorem 116 are satisfied for S, $T = \mathrm{Del}_S(E)$ and $U = \mathrm{tube}_\varepsilon(S)$.

1. By proposition 37, T is a compact closed surface. We orient T as described in section 13.4.3.

2. Each facet of T is contained in the corresponding surface Delaunay ball. By definition of loose ε-samples, the radius of the surface Delaunay ball of any facet f of $\mathrm{Del}_S(E)$ is $\leq \varepsilon \mathrm{lfs}(c_f)$ where c_f is the center of the ball. Hence the first condition of theorem 116 is fulfilled.

3. Consider now the second condition. Let $S_1 \subset S$ be the subset of points of S that are images by π_S of exactly one point of T. $S_1 \neq \emptyset$ by lemma 55. Moreover, since a sufficiently small neighborhood of any vertex v of T projects 1-1 onto the tangent plane to S at v (lemma 54), S_1 contains an open neighborhood of each vertex of T.

If S_1 has no boundary, then, since E intersects all the connected components of S, $S_1 = S$ and the theorem is proved. Assume, for a contradiction, that $\partial S_1 \neq \emptyset$, and let $s \in \partial S_1$ and l be the fibre at s. l intersects T at a point t that necessarily belongs to an edge $e = uv$ of T. Let us assume that t is not an endpoint of e (otherwise move s slightly along ∂S_1). Since T is a closed surface, there are exactly two facets of T, say f and f', that are incident to e. We will now make use of the fact that, since $s \in \partial S_1$ and T is a closed surface, the orientation convention implies that $n_s \cdot n_f$ and $n_s \cdot n_{f'}$ have different signs.

With $\phi = (n_s, n_v)$ and $n_s = \cos\phi\, n_v + \sin\phi\, n_v^{\vdash}$,

$$0 > (n_f \cdot n_s)\,(n_{f'} \cdot n_s) \geq \cos^2\phi\,(n_f \cdot n_v)\,(n_{f'} \cdot n_v) - 2\sin\phi\,\cos\phi - \sin^2\phi$$

Hence $(n_f \cdot n_v)(n_{f'} \cdot n_v) \leq \tan\phi\,(2 + \tan\phi)$. We now bound ϕ. Let s' be the point of e closer to s and v the vertex of e closer to s'. By Pythagorus theorem, we have

$$\|s - v\|^2 \leq \|s - s'\|^2 + \|s' - v\|^2 \leq \|s - t\|^2 + r_f^2 \leq \varepsilon^2 \mathrm{lfs}^2(s) + \varepsilon^2 \mathrm{lfs}^2(c_f). \tag{13.6}$$

Using the Lipschitz property of lfs, we have

$$\begin{aligned} \mathrm{lfs}(c_f) &\leq \mathrm{lfs}(v) + \|c_f - v\| \leq \mathrm{lfs}(v) + \varepsilon\,\mathrm{lfs}(c_f) \leq \frac{\varepsilon}{1-\varepsilon}\,\mathrm{lfs}(v) \\ \mathrm{lfs}(s) &\leq \mathrm{lfs}(v) + \|s - v\| \end{aligned}$$

Inequality 13.6 then implies

$$(1-\varepsilon^2)\,\|s - v\|^2 - 2\varepsilon^2\,\mathrm{lfs}(v)\,\|s - v\| - \frac{\varepsilon^2(2 - 2\varepsilon + \varepsilon^2)}{(1-\varepsilon)^2}\,\mathrm{lfs}^2(v)$$

from which and $\varepsilon < 1$ we get

$$\|s - v\| \leq \frac{\varepsilon(\sqrt{2} + \varepsilon)}{1 - \varepsilon^2}\,\mathrm{lfs}(v).$$

The normal variation lemma 48 yields $\phi = (n_s, n_v) \leq \frac{\varepsilon(\sqrt{2}+\varepsilon)}{1-3\varepsilon\sqrt{2}+4\varepsilon^2}$. Together with the inequation above, we get a function of ε that bounds $(n_f \cdot n_v)\,(n_{f'} \cdot n_v)$ from above.

On the other hand, by the orientation lemma 51, we have

$$(n_f \cdot n_v)\,(n_{f'} \cdot n_v) \geq \cos^2\left(\arcsin(\frac{\varepsilon\sqrt{3}}{1-\varepsilon}) + \frac{2\varepsilon}{1-7\varepsilon}\right).$$

We obtain a contradiction for $\varepsilon < 0.1$.

It follows that $\partial S_1 = \emptyset$ and therefore that $S_1 = S$, which proves the theorem.

□

Exercises 4

1. Show that the Hausdorff distance between $\mathrm{Del}_S(E)$ and S is $O(\varepsilon^2)$.
2. (Research problem) Extend the approximation results to piecewise smooth surfaces.

13.5 Surface meshing

Let S be a surface of $\mathbb{R}^3$. If we know a loose ε-sample E for $\varepsilon < 0.1$ of S, then, according to section 13.4.3, $\mathrm{Del}_S(E)$ is a good approximation of S. In this section, we present an algorithm that can construct such a sample and the associated restricted Delaunay triangulation. We restrict the presentation to the case of surfaces that are compact, C^2 and closed. S being compact and of class C^2, for any $x \in S$, $\mathrm{lfs}(x) \geq \mathrm{lfs}_{\mathrm{inf}} > 0$.

13.5.1 Algorithm

The algorithm is greedy. It inserts points one by one and maintains the current set E, the Delaunay triangulation $\mathrm{Del}(E)$ and its restriction $\mathrm{Del}_S(E)$ to S, and a list L of bad facets of $\mathrm{Del}_S(E)$. Any point that is inserted is the center of the surface Delaunay ball of a bad facet of $\mathrm{Del}_S(E)$. The algorithm stops when there is no more bad facets (which eventually happens as we will see).

We define a *bad facet* as a facet f of $\mathrm{Del}_S(E)$ that has a circumscribing surface Delaunay ball $B_f = B(c_f, r_f)$ satisfying $r_f > \psi(c_f)$, where ψ is a function defined over S and that satisfies

$$\forall x \in S, \quad \psi(x) \geq \psi_{\mathrm{inf}} \geq 0.$$

The surface is only queried through an *oracle* that, given a line segment f^* (to be the edge of $\mathrm{Vor}(E)$ dual to a facet f of $\mathrm{Del}_S(E)$), determines whether f^* intersects S and, in the affirmative, returns an intersection point and the value of ψ at this point.

We initialize the construction with a (usually small) set of points $E_0 \subset S$. As we will see, three points per connected component of S are sufficient. The algorithm then executes the following loop

while L is not empty {
  take an element f of L;
  insert c_f in E and in $\mathrm{Del}(E)$;
  update $\mathrm{Del}_S(E)$, i.e.
    remove the facets that are no longer facets of $\mathrm{Del}(E)$;
    add the new facets of $\mathrm{Del}(E)$ whose dual Voronoi edge intersects S;
  update L, i.e.
    remove the elements of L that are no longer facets of $\mathrm{Del}_S(E)$;
    add the new facets of $\mathrm{Del}_S(E)$ that are bad; }

13.5.2 Termination

Proposition 38 *The algorithm terminates after a finite number of steps.*

Proof. We call *radius of insertion* of a point p the distance of p to the current set E, just before inserting p. We note it ρ_p.

We show that $\rho_p \geq \psi_{\inf}$ for all points $p \in E \setminus E_0$. Indeed, at each step, we insert the center of a surface Delaunay ball that circumscribes a bad facet f. If r_f denotes the radius of this ball and c_f its center, we have $\rho_{c_f} = r_f$. And, as f is a bad facet, $r_f > \psi(c_f) \geq \psi_{\inf} > 0$. The insertion radius of the points of $E \setminus E_0$ is therefore always greater than $\psi_{\inf}$. The balls of radius $\frac{\psi_{\inf}}{2}$ centered at the points of $E \setminus E_0$ thus have disjoint interiors. The surface being compact, we can only insert a finite number of points, which proves that the algorithm terminates. □

Upon termination, any facet f of $\mathrm{Del}_S(E)$ has a circumscribing surface Delaunay ball B_f of center c_f and radius r_f with $r_f < \psi(c_f)$. To be able to apply theorem 117, we need 1. to take $\psi \leq \varepsilon\,\mathrm{lfs}$, for a sufficiently small ε. 2. To ensure that $\mathrm{Del}_S(E)$ has at least one vertex on each connected component of S. This can be done by taking in E_0 three points per component of S that are sufficiently close (see exercise 5).

13.5.3 Optimality

We bound the number of points inserted by the algorithm when ψ is 1-Lipschitz, i.e. $\psi(x) \leq \psi(y) + \|x - y\|$ for any $x, y \in S$.

Proposition 39 *If $\psi(x)$ is 1-Lipschitz and at most $\frac{1}{2}\,\mathrm{lfs}(x)$, the number of points inserted by the algorithm after the initialization phase is $n = |E \setminus E_0| = O\left(\int_S \frac{dx}{\psi^2(x)}\right)$.*

Proof. Let $\tau(x) = \inf\{r : |B(x,r) \bigcap E| = 2\}$ and $B_p = B(p, \frac{\tau(p)}{2})$, $p \in E$. It is easy to see that τ is 1-Lipschitz.

$$\int_S \frac{dx}{\tau^2(x)} \geq \sum_{p \in E \setminus E_0} \int_{(B_p \cap S)} \frac{dx}{\tau^2(x)} \qquad \text{(the balls } B_p \text{ have disjoint interiors)}$$

$$\geq \frac{4}{9} \sum_{p \in E \setminus E_0} \frac{\mathrm{area}(B_p \cap S)}{\tau^2(p)} \qquad (\tau(x) \leq \tau(p) + \|p - x\| \leq \tfrac{3}{2}\,\tau(p))$$

$$\geq \frac{4}{9} \sum_{p \in E \setminus E_0} \frac{3\pi}{16} \;=\; \frac{\pi}{12}\, n$$

The last inequality comes from the fact that $\mathrm{area}(B_p \cap S) \geq \frac{3\pi\tau^2(p)}{16}$. We first observe that $\tau(p)$ is no more than twice the radius of the surface Delaunay balls incident to p. Hence

$$\tau(p) \leq 2r_f \leq 2\psi(c_f) \leq 2\psi(p) + \|p - c_f\| = 2\psi(p) + r_f \leq 4\psi(p).$$

Now consider the two balls B and B' tangent to S at p of radius $\frac{1}{2}\tau(p)$. Since $2\psi(p) \leq \mathrm{lfs}(p)$, $B_p \cap S$ is a topological disk by lemma 13.4.2. Moreover, this disk cannot intersect the interiors of the two balls B and B', and its boundary lies on ∂B_p. Hence the area of $B_p \cap S$ is larger than the area of the disk bounded by the circle $\partial B \cap \partial B_p$. Denoting by ρ its radius, we have $\rho = \frac{\tau(p)\sqrt{3}}{4}$. The claim follows.

Using the Lipschitz property of τ, $\forall x \in B_p$, we have $\frac{1}{2}\,\tau(p) \leq \tau(p) - \|x - p\| \leq \tau(x) \leq \tau(p) + \|x - p\| \leq \frac{3}{2}\,\tau(p)$. For the same reasons, we have $\frac{1}{2}\psi(p) \leq \psi(x) \leq \frac{3}{2}\psi(p)$. Let q be a point of E closest to p : $\tau(p) = \|p - q\|$.

According to the algorithm, $\|p - q\| > \psi(p)$ or $\|p - q\| > \psi(q)$ depending whether p has been inserted after or before q. In both cases, due to the Lipschitz property of ψ, we have $\tau(p) = \|p - q\| > \frac{1}{2}\,\psi(p)$.

Hence $\tau(x) \geq \frac{1}{2}\tau(p) \geq \frac{1}{4}\psi(p) \geq \frac{1}{6}\psi(x)$, from which we deduce

$$n \leq \frac{12}{\pi} \int_S \frac{36dx}{\psi(x)^2}.$$

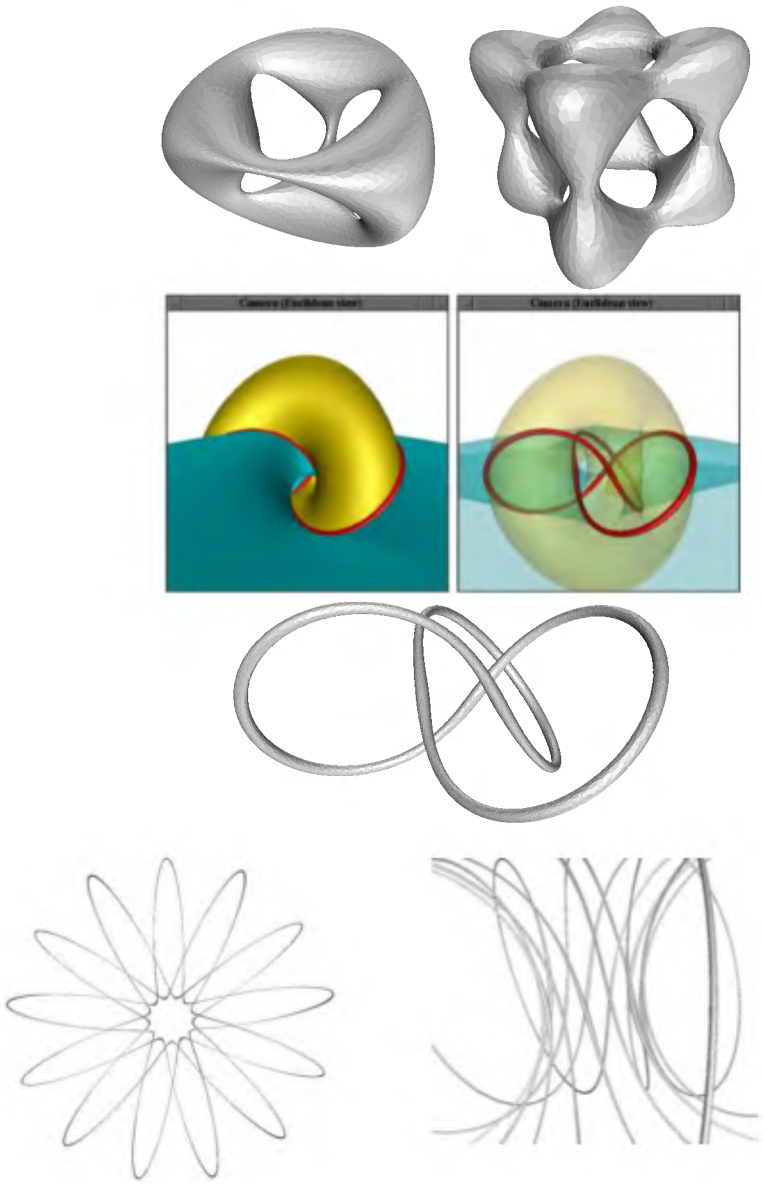
□

We sum up the results of this section in the following theorem.

Theorem 118 *Given a compact oriented C^2 surface S and a positive function on S, one can compute a loose ε-sample E of S for ψ. If ψ is 1-Lipschitz, the size of the sample is $O\left(\int_S \frac{dx}{\psi^2(x)}\right)$, which is optimal. If $\psi < 0.1\,\mathrm{lfs}$, $\mathrm{Del}_S(E)$ is isotopic to S and the isotopy does not move the points by more than $\sup_{x \in S} \psi(x)$.*

13.5.4 Experimental results

The algorithm has been implemented by S. Oudot. We present some experimental results on algebraic surfaces. The first two surfaces have genus respectively 3 and 5. We then show the standard trefoil knot and a more intricate knot in "sausage" format. A thorough discussion of the implementation of the algorithm and other experimental results can be found in [2].



Exercises 5

1. Adapt the algorithm to the case where S is a curve of $\mathbb{R}^d$.
2. Assume that ψ is a Lipschitz function. Show that a facet f circumscribed by Delaunay ball B_f of center c_f and radius r_f with $r_f \leq \frac{1}{3}\psi(c_f)$ will remain a restricted Delaunay facet throughout the course of the algorithm.
3. We modify the definition of a bad facet so as to eliminate facets with small angles. A facet f of $\mathrm{Del}_S(E)$ is now said to be bad if r_f is too large (as above) or if one of its angles is smaller than $\frac{\pi}{6}$. Show that the algorithm still terminates.
4. Show that the bound in proposition 39 is tight.
5. Since S is compact and lfs is continuous, there exists $x \in S$ such that $\mathrm{lfs}(x) = \inf_{y \in S} \mathrm{lfs}(y)$. Show that either $\mathrm{lfs}(x)$ is the minimal radius of curvature at x (in which case, one of the two medial spheres passing through x is osculating S), or the line normal to S at x intersects S at another point z and is also the line normal to S at z, and the sphere with diameter xz is a medial ball. Show then how to compute $\inf_{y \in S} \mathrm{lfs}(y)$.

Acknowledgements

The sections on surface approximation and meshing are derived from common work with Steve Oudot. The experimental results are due to Steve.

Appendix : notions of complexity

We recall some basic notions of complexity. A good introduction to algorithms and data structures are the books of Cormen, Leiserson and Rivest [6] and the book of Sedgewick [14]. The complexity of an algorithm is evaluated by bounding the number of elementary operations executed by the algorithm (time complexity) and the memory space required (space complexity). Any numerical value (either an input date or some intermediate value) is assumed to be stored in a memory unit. Elementary operations that have a unit cost are the arithmetic operations, comparaisons, as well as reading a number stored in a memory unit or writing a number in memory.

Complexity in time and space are evaluated asymptotically. We are only interested in orders of magnitude and we do not give the constants (that depend on the computer, the programming language, the compiler, etc). We use the following standard notations. Let f, g and h be three fonctions. The notations O (upper bound), Ω (lower bound) and Θ (tight bound) are defined by

$$\begin{aligned}
f(n) = O(g(n)) &\iff \exists\, n_0, \beta > 0, \forall n \geq n_0,\ f(n) \leq \beta\, g(n) \\
f(n) = \Omega(h(n)) &\iff \exists\, n_0, \alpha > 0, \forall n \geq n_0,\ f(n) \geq \alpha\, g(n) \\
f(n) = \Theta(g(n)) &\iff \exists\, n_0, \alpha, \beta > 0, \forall n \geq n_0,\ \alpha\, g(n) \leq f(n) \leq \beta\, g(n)
\end{aligned}$$

Exercises 6

1. Show that sorting a set E of n numbers requires to execute at least $\log_2(n!) = \Omega(n \log n)$ comparisons.

2. Propose an algorithm to sort E. Divide E into two sets E_1 and E_2 of (roughly) the same size, sort recursively E_1 and E_2, and merge the two sorted subsets. Show that merging the two sorted subsets can be done in time $O(n)$ and that the overall complexity of the sorting algorithm is $O(n \log n)$, which is optimal in view of the previous exercise.

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Differential Geometry and Topology, Discrete and Computational Geometry
M. Boucetta and J.-M. Morvan (Eds.)
IOS Press, 2005
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