R.W. Sharpe

Differential Geometry

Cartan's Generalization of Klein's Erlangen Program

Foreword by S.S. Chern

With 104 Illustrations

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Foreword

I am honored by Professor Sharpe's request to write a forward to his beautiful book.

In his preface he asks the innocent question, "Why is differential geometry the study of a connection on a principal bundle?" The answer is of course very simple; because Euclidean geometry studies a connection on a principal bundle, and all geometries are in a sense generalizations of Euclidean geometry.

In fact, let E^n be the Euclidean space of n dimensions. We call an orthonormal frame x, $e_1, \ldots, e_n$ ($n+1$ vectors), where x is the position vector and e_i have the scalar products

$$(e_i, e_j) = \delta_{ij}, \qquad 1 \leq i, j \leq n.$$

Then the space of all orthonormal frames is a principal fiber bundle with group $O(n)$ and base space E^n, the projection being defined by mapping x, $e_1, \ldots, e_n$ to x. The equations

$$de_i = \sum_{1 \leq j \leq n} \omega_{ij} e_j, \qquad 1 \leq i \leq n,$$

define the Maurer–Cartan forms ω_{ij}, with

$$\omega_{ij} + \omega_{ji} = 0, \qquad 1 \leq i, j \leq n.$$

They satisfy the Maurer–Cartan equations

$$d\omega_{ij} = \sum_{1 \leq k \leq n} \omega_{ik} \wedge \omega_{kj}, \qquad 1 \leq i, j \leq n.$$

This is Euclidean geometry by moving frames. The ω_{ij} define the parallelism or connection. The Maurer–Cartan equations say that the connection is flat. This formulation has a great generalization.

As in all disciplines, the development of differential geometry is tortuous. The basic notion is that of a manifold. This is a space whose coordinates are defined up to some transformation and have no intrinsic meaning. The notion is original, bold, and powerful. Naturally, it took some time for the concept to be absorbed and the technology to be developed. For example, the great mathematician Jacques Hadamard "felt insuperable difficulty ... in mastering more than a rather elementary and superficial knowledge of the theory of Lie groups," a notion based on that of a manifold [1]. Also, it took Einstein seven years to pass from his special relativity in 1908 to his general relativity in 1915. He explained the long delay in the following words: "Why were another seven years required for the construction of the general theory of relativity? The main reason lies in the fact that it is not so easy to free oneself from the idea that coordinates must have an immediate metrical meaning." [2]

On the technology side the breakthrough was achieved by the tensor analysis of Ricci calculus. The central theme was Riemannian geometry, which Riemann formulated in 1854. Its fundamental problem is the "form problem": To decide when two Riemannian metrics differ by a change in coordinates. This problem was solved by E. Christoffel and R. Lipschitz in 1870. Christoffel's solution introduces a covariant differentiation, which could be given an elegant geometrical setting through the parallelism of Levi-Civita. Tensor analysis is extremely effective and has dominated differential geometry for a century.

Another technical tool, which has not quite received the recognition it deserves, is the exterior differential calculus of Elie Cartan. This was introduced by Cartan in 1922, following the work of Frobenius and Darboux. All the exterior differential forms on a manifold form a ring. It depends only on the differentiable structure of the manifold and not on any additional structure such as a Riemannian metric or an affine connection. Topologically it leads to the de Rham theory. Less known is its effectiveness in treating local problems.

A fundamental question is the equivalence problem for G-structures: Given, on an n-dimensional manifold with coordinates u^i, a set of linear differential forms ω^i, a similar set ω^{*j} with coordinates u^{*j}, and a subgroup $G \subset Gl(n, \mathbf{R})$, determine the conditions under which there exist functions

$$u^{*j} = u^{*j}(u^1, \ldots, u^n), \qquad 1 \le i, j \le n,$$

such that after substitution the ω^{*j} differ from the ω^j by a transformation of G. The form problem in Riemannian geometry is the case $G = O(n)$.

The solution of the form problem by Cartan's method of equivalence leads automatically to the tensor analysis. Thus, the method of equivalence is more general. In the case $G = O(n)$, this leads to the Levi-Civita parallelism and the Riemannian geometry. In this way Euclidean geometry generalizes to Riemannian geometry. For a general G, the solution of the equivalence problem is not always easy (cf. the Preface), although it is proved that it can always be achieved in a finite number of steps. Philosophically nice problems have nice answers.

Klein geometry can be developed through the Maurer–Cartan equations. The generalization of the above discussion, from $O(n)$ to G, gives Cartan's generalized spaces, essentially a connection in a principal bundle.

A fundamental problem is the relation of the local geometry with the global properties of the spaces in question. Such a result is the so-called Chern–Weil theorem that the characteristic classes can be represented by differential forms constructed explicitly from the curvature. The simplest result is the Gauss–Bonnet formula.

I wish to take this occasion to mention some recent developments on Finsler geometry [3]. This is the geometry of a very simple integral and was discussed in problem 23 of Hilbert's Paris address in 1900. By a proper interpretation of the analytical results, Finsler geometry now assumes a very simple form showing it to be a family of geometries quite analogous to the Riemannian case.

Differential geometry offers an open vista of manifolds with structures, finite or infinite dimensional. There are also simple and difficult low-dimensional problems, of the garden variety. If one switches between the two, life is indeed very enjoyable.

It is a great mystery that the infinitesimal calculus is a source of such depth and beauty.

References

[1] J. Hadamard, *Psychology of Invention in the Mathematical Field*, Princeton University Press, Princeton, 1945, p. 115.

[2] A. Einstein, Autobiographical Notes, in *Albert Einstein: Philosopher Scientists*, p. 67, 2nd ed., 1949, in vol. 7 of the series The Library of Living Philosophers, Evanston, IL, edited by P.A. Schilpp.

[3] D. Bao and S.S. Chern, On a notable connection in Finsler geometry, *Houston J. Math.* **19** (1993), 135–180.

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Preface

This book is a study of an aspect of Elie Cartan's contribution to the question "What is geometry?"

In the last century two great generalizations of Euclidean geometry appeared. The first was the discovery of the non-Euclidean geometries. These were organized into a coherent whole by Felix Klein, who recognized them as various examples of coset spaces G/H of Lie groups. In this book we refer to these latter as Klein geometries. The second generalization was Georg Riemann's discovery of what we now call Riemannian geometry. These two theories seemed largely incompatible with one other.[1]

In the early 1920s Elie Cartan, one of the pioneers of the theory of Lie groups, found that it was possible to obtain a common generalization of these theories, which he called *espaces généralizés* and we call Cartan geometries (see diagram).

$$\begin{tikzcd}[column sep=large, row sep=large]
\begin{array}{c}\textit{Euclidean}\\ \textit{Geometry}\end{array} \arrow[r, "\textit{generalization}"] \arrow[d, "\begin{array}{c}\textit{generali-}\\ \textit{zation}\end{array}"] & \begin{array}{c}\textit{Klein}\\ \textit{Geometries}\end{array} \arrow[d, "\begin{array}{c}\textit{generali-}\\ \textit{zation}\end{array}"'] \\
\begin{array}{c}\textit{Riemannian}\\ \textit{Geometry}\end{array} \arrow[r, "\textit{generalization}"] & \begin{array}{c}\textit{Cartan}\\ \textit{Geometries}\end{array}
\end{tikzcd}$$

[1] The only relationship was the "accident" that some of the non-Euclidean geometries could be regarded as special cases of Riemannian geometries.

Looking at this diagram vertically, we can say that just as a Riemannian geometry may be regarded, locally, as modeled on Euclidean space but made "lumpy" by the introduction of a curvature, so a Cartan geometry may be regarded, locally, as modeled on one of the Klein geometries but made "lumpy" by the introduction of curvature appropriate to the model in question. Looking at the same diagram horizontally, a Cartan geometry may be regarded as a non-Euclidean analog of Riemannian geometry.

Cartan actually gave the first example of a Cartan geometry more than a decade earlier, in the remarkable *tour de force* [E. Cartan, 1910]. In that paper he considered the case of a two-dimensional distribution on a five-dimensional manifold. He showed that such a distribution determined, and was determined by, a Cartan geometry modeled on the homogeneous space G_2/H, where H is a certain nine-dimensional subgroup of the fourteen-dimensional exceptional Lie group G_2. This process of associating a Cartan geometry to a raw geometric entity (the distribution) is an example of "solving the equivalence problem" for the entity in question. Although the solution of an equivalence problem is not always a Cartan geometry, in many important cases it is. When it is, the invariants of the geometry (curvature, etc.) are a priori invariants of the raw geometric entity. We recommend [R.B. Gardner, 1989] for an account of the method of equivalence.

To be a little more precise, a Cartan geometry on M consists of a pair (P, ω), where P is a principal bundle $H \to P \to M$ and ω, the Cartan connection, is a differential form on P. The bundle generalizes the bundle $H \to G \to G/H$ associated to the Klein setting, and the form ω generalizes the Maurer–Cartan form ω_G on the Lie group G. In fact, the *curvature* of the Cartan geometry, defined as $d\omega + \frac{1}{2}[\omega, \omega]$, is the complete local obstruction to P being a Lie group.

One reason for the power of Cartan's method comes from the fact that these new geometries maintain the same intimate relation with Lie groups that one sees in the case of homogeneous spaces. This means, for example, that constructions in the theory of homogeneous spaces often generalize in a simple manner to the general "curved" case of Cartan geometries. It also means that the differential forms that appear are always related to components of the Maurer–Cartan form of the Lie group, a context in which their significance remains clear.

In the particular case of a Riemannian manifold M, Cartan's point of view offered a new and profound vantage point that is largely responsible for the modern insistence on "doing differential geometry on the bundle P of orthonormal frames over M."

The history of the study of Cartan geometries is somewhat troubled. First is the difficulty Cartan faced in trying to express notions for which there was no truly suitable language.[2] Next is the widely noted difficulty in reading

[2] This difficulty was resolved with the introduction of the notion of a principal bundle and of vector-valued forms on such a bundle.

Cartan.[3] In his paper [C. Ehresmann, 1950] Charles Ehresmann gave for the first time a rigorous global definition of a Cartan connection as a special case of a more general notion now called an Ehresmann connection (or more simply, a connection). For various reasons[4] the Ehresmann definition was taken as the definitive one, and Cartan's original notion went into a more or less total eclipse for a long time. The beautiful geometrical origin and insight connected with Cartan's view were, for many, simply lost. In short, although the Ehresmann definition gives us a good notion, it hides the real story about why it is so good. In this connection, the following quotation is interesting [S.S. Chern, 1979]:

> The physicist C.N. Yang wrote [C.N. Yang, 1977]: "That non-abelian gauge fields are conceptually identical to ideas in the beautiful theory of fibre bundles, developed by mathematicians without reference to the physical world, was a great marvel to me." In 1975 he mentioned to me: "This is both thrilling and puzzling, since you mathematicians dreamed up these concepts out of nowhere."

Far from arising "out of nowhere," the simple and compelling geometric origin of a connection on a principal bundle is that it is a generalization of the Maurer–Cartan form. Moreover, a study of the Cartan connection itself can illuminate and unify many aspects of differential geometry.

Novelties

Aside from the fact that one cannot find a fully developed, modern exposition of Cartan connections elsewhere, what is new or different in this book?

New Treatment

This book is written at a level that can be understood by a first- or second-year graduate student. In particular, we include the relevant theory of manifolds, distributions and Lie groups. For us, a manifold is, by definition, a

[3] To paraphrase Robert Bryant, "You read the introduction to a paper of Cartan and you understand nothing. Then you read the rest of the paper and still you understand nothing. Then you go back and read the introduction again and there begins to be the faint glimmer of something very interesting."

[4] At that stage it was easier to read Ehresmann than Cartan. There was also the attraction of a more general and global notion.

locally Euclidean, paracompact Hausdorff space. This is the same as a locally Euclidean Hausdorff space each of whose components has a countable basis.[5] In particular, Lie groups are defined to be manifolds in this sense. The result of Yamabe and Kuranishi ([H. Yamabe, 1950]) that a connected subgroup of a Lie group is a Lie subgroup implies that *any* subgroup of a Lie group is a Lie group in the present sense. The discussion of submanifolds given in Chapter 1 is broad enough to include these subgroups as submanifolds.

In our coverage of bundle theory, we emphasize the abstract principal bundles rather than bundles of frames.[6] Of course, these two views are really equivalent. In the case of the "first-order" geometries, the equivalence is quite simple. However, in the case of "higher-order" geometries, the choice of the higher-order frames usually seems to be decided on a rather ad hoc basis and can be complicated. Here the bundle approach gives a real advantage, and the right choice of frames becomes clear (if needed) once the bundle is understood. Another important advantage of working with the bundles themselves is that they give a common language, facilitating comparison between geometries and emphasizing the relation to the model space. In this sense, comparing Cartan geometries is like comparing Klein geometries.

Chapter 3 contains a complete and economical development of the Lie group—Lie algebra correspondence based on the *fundamental theorem of non-abelian calculus*. One of the novelties here is the characterization of a Lie group as a manifold equipped with a Lie algebra-valued form on it satisfying certain properties. This characterization prepares the reader for the generalization to Cartan geometries in Chapter 5.

Finally, in Appendix B we explain how one manifold may roll without slipping or twisting on another in Euclidean space. We also show how this notion yields a differential system that contains both the Levi–Civita connection and the Ehresmann connection on the normal bundle for a submanifold of Euclidean space.

New Results

Let us move on to some results we believe are new. In Chapter 4 we introduce the fundamental property of Klein geometries characterizing the kernel of such a geometry. This result is used in Chapter 5 in an essential way to show the equivalence of the base and bundle definitions of Cartan geometries in the effective case. In Chapter 5 we introduce and classify Cartan space forms. These geometries generalize the classical Riemannian

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[5]The usual definition requires a manifold to have a countable basis (cf., e.g., [Boothby, W. 1986, p. 6]).

[6]In much the same way, one might emphasize an abstract Lie group rather than a matrix group realizing it.

space forms.[7] One important ingredient of this classification is the property (apparently new) of a Cartan geometry called "geometric orientability." Another is the notion of "model mutation." Finally, in Chapter 7 we give a classification of the submanifolds of a Möbius geometry. This classification is more general than that of [A. Fialkow, 1944] in that ours allows the presence of umbilic points.

Prerequisites and Conventions

This book assumes very few prerequisites. The reader needs to be familiar with some basic ideas of group theory, including the notion of a group acting on a set. Results from the calculus of several variables, point set topology, and the theory of covering spaces are used in various places, and the long, exact sequence of homotopy theory is used once (at the end of Chapter 5). Aside from this, most of the material is developed *ab initio*. However, the reader is invited to shoulder some of the burden of the work in that essential use is made of a few of the exercises. These exercises are denoted by an asterisk to the right of the exercise number.

The numbering follows a single sequence throughout the book, with all items (definitions, theorems, figures, etc.) in a single stream. Thus 4.3.2 refers to Chapter 4, Section 3, item 2. For references to items occurring in the same chapter, we omit the chapter number, so that in Chapter 4, 4.3.2 becomes 3.2.

We use the following dictionary of symbols to denote the ends of various items:

symbol	*end of*
❀	definition
❑	exercise
■	proof
◆	example

Although it will often be convenient for us to write column vectors as row vectors, the reader should remember that all vectors are in fact column vectors.

[7] In fact, this notion is general enough to immediately allow a description of general symmetric spaces.

Limitations

The reader will find no mention here of some basic topics in differential geometry, such as Stokes' theorem, characteristic classes, and complex geometries. Also, our approach to Lie theory is "elementary" in that we do not discuss or use the classification theory of Lie groups, with its attendant study of roots, weights, and representations.

Originally, we had wished to include more than the three examples of Cartan geometries studied here; but in the end, the pressures of time, space, and energy limited this impulse. The three geometries we do study are not developed in complete analogy to each other. For example, the discussion of immersed curves in a Möbius geometry in terms of the normal forms given in Chapter 7 does of course have a Riemannian analog, but that is not studied in this book. And one may study subgeometries of projective geometries just as one studies subgeometries of Riemannian and Möbius geometries, but we do not do so here. We have also resisted the impulse to make a "dictionary" translating among the various versions of Cartan's view, Ehresmann's view,[8] and the view expressed in [L.P. Eisenhart, 1964]. In the end, however, for those who are interested in it, it should be abundantly clear how Cartan's view does illuminate the others.

Some Personal Remarks

An author often writes a book in order to sort out his or her own understanding of the subject. This is the circumstance in the present case. When I was an undergraduate, differential geometry appeared to me to be a study of curvatures of curves and surfaces in $\mathbf{R}^3$. As a graduate student I learned that it is the study of a connection on a principal bundle. I wondered what had become of the curves and surfaces, and I studied topology instead.

The reawakening of my interest in this subject began in 1987 when Tom Willmore very kindly wrote me a note thanking me for a preprint and mentioning his great interest in what is known as the Willmore conjecture (cf. 7.6). This led me once again to look at principal bundles and connections. In particular, I wondered whether there was an intrinsically defined Ehresmann connection on a surface in S^3 that was invariant under the group of Möbius transformations of S^3. It turns out there is no such connection. However, after calculating normal forms for surfaces in the Möbius sphere S^3 (cf. [G. Cairns, R. Sharpe, and L. Webb, 1994]), it became clear to me that there must be some other kind of invariantly defined structure inherited on the surface from its embedding in S^3. (In Chapter 7 it is shown

[8]See, however, the discussion in Appendix A dealing with the relationship between Cartan and Ehresmann connections.

that a Cartan connection is defined in this situation, and, in fact, Cartan also knew this [E. Cartan, 1923].)

During this time it began to seem strange to me that Ehresmann connections play such a prominent role in modern differential geometry. In some cases, such as the Levi–Civita connection, the connection is determined by the geometry. In many cases, however, one makes use of an arbitrary connection that one proves to exist by a general technique. This is the appropriate point of view for the construction of the characteristic classes of Chern and Pontryagin. There one may use any connection, since the aim is to obtain topological invariants for which the particular choice of connection does not matter. But these considerations seem to be at their base topological rather than differential geometric. My innocent question, left over from my undergraduate days, was "Why is differential geometry the study of a connection on a principal bundle?" And I began, rather impertinently, to ask this question at every opportunity, usually picking on some unsuspecting differential geometer who did not know me very well.

During one of these sessions, Min Oo remarked that Elie Cartan had considered connections with values in a Lie algebra larger than that of the fiber.[9] Later I read, and translated,[10] Cartan's book [E. Cartan, 1935]. I browsed through Cartan's collected works and through those of his successors and interpreters. It became clear to me that Cartan had a subtle and really wonderful idea, which gives a fully satisfying explanation for the modern, and approximately true, notion that differential geometry is the study of an Ehresmann connection on a principal bundle. There seems to be no treatment of these things in the standard texts on differential geometry. In the few books where the Cartan connections are mentioned at all (e.g., [J. Dieudonné 1974], [W.A. Poor, 1981], and [M. Spivak, 1979]), they make only a brief appearance, perhaps in the exercises or toward the end of the book, and one is left with the impression that the notion is only a quaint curiosity left over from bygone days. Six years ago I began to scribble some notes about these things and to talk about them; after a number of months had passed, I realized I was writing a book on the subject.

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I would like to thank everyone who has had an influence on this book. In addition to those mentioned above, I am grateful to Bernard Kamte, Joe Repka, Qunfeng Yang, and my wife, Mary, for their comments on portions of the manuscript. I would also like to acknowledge my gratitude to

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[9]See [E. Ruh, 1993] for a brief recent overview of Cartan connections and some of their applications.

[10]A copy of my translation, which is only a rough draft, can be found in the Mathematics Library at the University of Toronto.

the National Science and Engineering Research Council of Canada for its support of this project through grant #OGP0004621.

Finally, I would like to thank Velamir Jurdjevic for his encouragement over the years. It was Vel who suggested that, although it is perhaps impossible to catch all the errors before a book reaches print, the principal demand is that a book be interesting. As for the first part of his remark, the responsibility for any remaining errors lies with me. I will leave it to the reader to judge whether or not this principal demand has been met.

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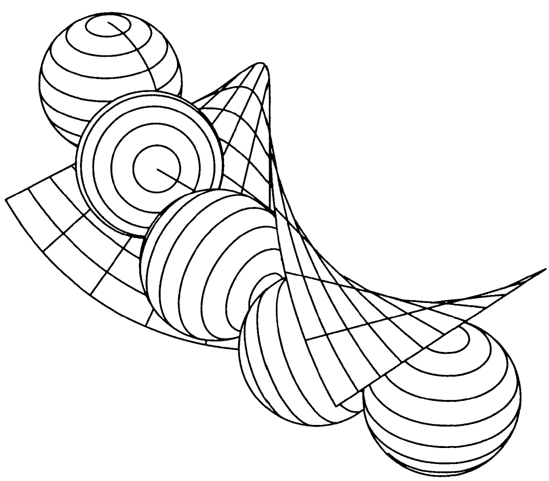
October 16, 1995

Contents

Appendix B

Rolling Without Slipping or Twisting

In this appendix we study the most classical of all nonholonomic differential systems, the system describing how one manifold may roll without slipping or twisting on another in Euclidean space.



A sphere rolling along the central line of a helicoid without slipping or twisting.

Such a system plays a role, for instance, in robotics, where one wishes to use robotic fingers to manipulate some object. Our interest, however, is to show the relationship of this concrete physical notion to the idea of Levi–Civita connection of a Riemannian manifold and the canonical Ehresmann connection on its normal bundle in Euclidean space.[1]

In §1 we give a definition of the notion of rolling without slipping or twisting in an "elementary" form which appeals directly to the intuition in Euclidean space. In §2 we reformulate this definition in terms of a differential sysem (i.e., a distribution) on a state space. In this formulation we easily obtain the existence and uniqueness of rolling without slipping or twisting. In particular, this yields isometries between the ambient tangent spaces at each point of the rolling. In §3 we show how this notion gives rise to both the Levi–Civita connection and the normal connection for a submanifold of Euclidean space. The section ends by relating the notion of rolling a manifold M^n on $\mathbf{R}^n$ to the notion of development described in Chapter 5. In particular, we show that a curve on M rolls without slipping on a straight line in $\mathbf{R}^n$ if and only if the curve is a geodesic. We also show that a vector field along a curve in M is parallel if and only if its image in $\mathbf{R}^n$ is parallel. This result (Proposition 3.5) makes the notion of the Levi–Civita connection and the normal connection for a submanifold of Euclidean space quite concrete. Finally, in §4, we study what happens when we roll one manifold on a second while the second is itself rolling on a third.

§1. Rolling Maps

In order to study rolling maps we need to have a way to represent elements of $\mathrm{Euc}_N(\mathbf{R})$, the group of isometries of Euclidean N space. If $g \in \mathrm{Euc}_N(\mathbf{R})$, then we may write

$$g\colon \mathbf{R}^n \to \mathbf{R}^n, \quad \nu \mapsto A\nu + p$$

where $A \in O_n(\mathbf{R})$ and $p \in \mathbf{R}^N$. It is easy to verify that the map

$$\mathrm{Euc}_N(\mathbf{R}) \to \left\{ \begin{pmatrix} 1 & 0 \\ p & A \end{pmatrix} \in Gl_{N+1}(\mathbf{R}) \mid A \in O_n(\mathbf{R}), p \in \mathbf{R}^n \right\}$$

is an isomorphism.

A rolling map will be a certain kind of one-parameter family $g\colon I \to \mathrm{Euc}_N(\mathbf{R})$. We write $g(t)v = A(t)v + p(t)$. For fixed $t \in I$, we have $g(t)\colon \mathbf{R}^N \to \mathbf{R}^N$ with derivative $g(t)_* = A(t)\colon \mathbf{R}^N \to \mathbf{R}^N$; here "$*$" refers to the partial derivative with respect to the space coordinates. We write the

[1]Both of these connections are described in Chapter 6. See also [C. Dodson and T. Poston, 1991], pp. 207ff., for a treatment of the tangential part of this result. We also refer the reader to [R.L. Bryant and L. Hsu, 1993], p. 456, for a more advanced study of aspects of this kind of differential system.

partial derivative of $g(t)$ with respect to the parameter t as $\dot{g}(t)\colon \mathbf{R}^n \to \mathbf{R}^n$. In particular, the composite mapping $\dot{g}(t)g(t)^{-1}\colon \mathbf{R}^N \to \mathbf{R}^N$ makes sense.

The following definition formalizes the intuitive notion of one manifold rolling on another in Euclidean space without slipping or twisting. The subsequent interpretation is meant to explain the sense in which it does so.

Definition 1.1. Let M_0^n, $M_1^n \subset \mathbf{R}^N$ (where $N = n + r$) be submanifolds. Then a map $g\colon I \to \mathrm{Euc}_N(\mathbf{R})$ satisfying the following properties for each $t \in I$ is called a *rolling of M_1 on M_0 without slipping or twisting.* More briefly, we shall call g a *rolling map.*

The "rolling" condition:

(1) There is a piecewise smooth "rolling curve on M_1" $\sigma_1\colon I \to M_1$ such that

(a) $g(t)\sigma_1(t) \in M_0$, and

(b) $T_{g(t)\sigma_1(t)}(g(t)M_1) = T_{g(t)\sigma_1(t)}(M_0)$ for all t.

The curve $\sigma_0\colon I \to M_0$ defined by $\sigma_0(t) = g(t)\sigma_1(t)$ is called the *development*[2] *of σ_1 on M_0.*

The "no-slip" condition:

(2) $\dot{g}(t)g(t)^{-1}\sigma_0(t) = 0$ for all t.

The tangential part of the "no-twist" condition:

(3) $(\dot{g}(t)g(t)^{-1})_* T_{\sigma_0(t)}(M_0) \subset T_{\sigma_0(t)}(M_0)^\perp$ for all t.

The normal part of the "no-twist" condition:

(4) $(\dot{g}(t)g(t)^{-1})_* T_{\sigma_0(t)}(M_0)^\perp \subset T_{\sigma_0(t)}(M_0)$ for all t. ❀

Interpretation

Condition (1) says that M_1 moves under the action of $g(t)$ so as to be tangent to M_0 at the point $\sigma_0(t) = g(t)\sigma_1(t)$ at time t. Condition (2) says that the infinitesimal isometry $\dot{g}(t)g(t)^{-1}$ fixes $\sigma_0(t)$, namely, that it is an infinitesimal rotation about the point $\sigma_0(t)$. This is the "no-slip" condition. Next we consider conditions (3) and (4). Each tangent vector $w \in T_q(M_1)$ (respectively, normal vector $w \in T_q(M_1)^\perp$), as it is carried along by the motion $g(t)$, determines a one-parameter family of vectors $w_t = g(t)_* w \in T_{g(t)q}(g(t)M_1)$ (respectively, $T_{g(t)q}(g(t)M_1)^\perp$) with velocity $\dot{w}_t$. Now suppose $v \in T_{g(t_0)\sigma_1(t_0)}(g(t_0)M_1)$ $(= T_{\sigma_0(t_0)}(M_0))$. Setting $w = g(t_0)_*^{-1}v$, we have $v = w_{t_0} = g(t_0)_* w$ and

[2]See §5 for the relationship of this to the definition of development given in §4 of Chapter 5.

$$\dot{w}_t\big|_{t=t_0} = \dot{g}(t_0)_* w = \dot{g}(t_0)_* g(t_0)_*^{-1} v = (\dot{g}(t_0) g(t_0)^{-1})_* v.$$

Thus, condition (3) says that the motion of a vector $v \in T_{g(t_0)\sigma_1(t_0)}(g(t_0)M_1)$ $(= T_{\sigma_0(t_0)}(M_0))$ that is "stuck to $g(t)M_1$" has no component of velocity in the tangential direction. This is the tangential part of the "no-twist" condition. Condition (4) says that a vector $v \in T_{\sigma(t)}(g(t)M_1)^\perp$, as it is carried along by $g(t)$, has no component of velocity in the normal direction. This is the normal part of the "no twist" condition.

§2. The Existence and Uniqueness of Rolling Maps

The interpretation given in §1 makes it clear that the conditions of Definition 1.1 should be required of a rolling map. What is not clear is the existence and uniqueness, for each curve $\sigma(t)$, of a rolling map with $\sigma(t)$ as its rolling curve.

The most convenient[3] way to deal with the existence and uniqueness question is to reformulate the equations of Definition 1.1 as a differential system, that is, as an n-dimensional distribution on a certain configuration space Σ. Roughly speaking, the configuration space Σ is the space of all positions of M_1 in which it is tangent to M_0. More precisely, we define the configuration space to be

$$\Sigma = \{(p, A, q) \in M_0 \times O_N(\mathbf{R}) \times M_1 \mid AT_q(M_1) = T_p(M_0)\}$$

(where we are identifying $T_p(M_0)$ and $T_q(M_1)$ with subspaces of $\mathbf{R}^N$ in the usual way).

Why should Σ be regarded as the configuration space? First note that there is a canonical smooth map

$$\begin{aligned} \phi\colon \Sigma &\longrightarrow \mathrm{Euc}_n(\mathbf{R}), \\ (p,A,q) &\longmapsto g \end{aligned}$$

where $g = \phi(p, A, q)$ is defined by

$$\begin{aligned} g\colon \mathbf{R}^n &\longrightarrow \mathbf{R}^n. \\ x &\longmapsto A(x-q)+p \end{aligned}$$

Since $gq = p$ and $g_* = A$, it follows that $g_* T_q(M_1) = T_p(M_0)$. Thus we see that a point in Σ determines a point $p \in M_0$ and an isometry $g \in \mathrm{Euc}_N(\mathbf{R})$ such that gM_1 is tangent to M_0 at $p = gq$. This explains the sense in which Σ may be called the configuration space of all positions of M_1 in which it is tangent to M_0.

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[3]This is instructive as well as convenient, as it introduces the reader to a beautiful application of differential systems.

The first order of business is to determine the tangent bundle of Σ. It is not difficult to verify that the canonical projection $\Sigma \to M_0 \times M_1$ is a fiber bundle with fibers $O_n(\mathbf{R}) \times O_r(\mathbf{R})$. It follows that Σ is a manifold of dimension $2n + \frac{1}{2}n(n-1) + \frac{1}{2}r(r-1)$. We shall also need to use the second fundamental form as it is described in Exercise **1**.2.24. In particular, we shall interpret the second fundamental form as having values in the normal bundle of the manifold on which it is defined.

Proposition 2.1. *Let B_0 and B_1 denote the second fundamental forms for M_0 and M_1, respectively. Let $V \subset T_p(M_0) \times T_A(O_N(\mathbf{R})) \times T_p(M_0)$ be given by*

$$V = \{(\dot{p}, \dot{A}, \dot{q}) \mid \dot{A}A^{-1}u = AB_1(\dot{q}, A\dot{u}) - B_0(\dot{p}, u) \bmod T_p(M_0), \\ \forall u \in T_p(M_0)\}.$$

Then $T_{(p,A,q)}(\Sigma) = V$.

Proof. We know that $(\dot{p}, \dot{A}, \dot{q}) \in T_{(p,A,q)}(\Sigma)$ if and only if it is the derivative at (p, A, q) of a curve $(p(t), A(t), q(t))$ on Σ. Let $u_1(t)$ be a vector field tangent to M_1 defined along $q(t)$. Then, by the definition of Σ, the vector field $u_0(t) = A(t)u_1(t)$ is tangent to M_0 along $p(t)$. Differentiating produces

$$\dot{u}_0(t) = \dot{A}(t)u_1(t) + A(t)\dot{u}_1(t).$$

By Exercise **1**.2.24, we have

$$\dot{u}_0(t) = -B_0(\dot{p}, u_0) \bmod T_p(M_0) \quad \text{and} \quad \dot{u}_1(t) = -B_1(\dot{q}, u_1) \bmod T_q(M_1).$$

Thus, at (p, A, q) we have

$$-B_0(\dot{p}, u_0) = \dot{A}u_1 - AB_1(\dot{q}, u_1) \bmod T_p(M_0), \quad \text{or}$$

$$-B_0(\dot{p}, u) = \dot{A}A^{-1}u - AB_1(\dot{q}, A^{-1}u) \bmod T_p(M_0) \ \text{ for all } u \in T_p(M). \tag{2.2}$$

Thus, $T_{(p,A,q)}(\Sigma) \subset V$.

On the other hand, we claim that $\dim V \le 2n + \frac{1}{2}n(n-1) + \frac{1}{2}r(r-1)$, from which the proposition follows.

To see this, we note that $\dot{A}A^{-1}$ is skew symmetric (since $AA^t = I \Rightarrow \dot{A}A^t + A\dot{A}^t = 0 \Rightarrow \dot{A}A^{-1} + (\dot{A}A^{-1})^t = 0$). If we express the skew-symmetric matrix $\dot{A}A^{-1}$ with respect to an orthonormal basis of $\mathbf{R}^N$ that is adapted to $T_q(M_1)$ (i.e., the first n basis elements span $T_q(M_1)$, which implies that the last r basis elements span $T_q(M_1)^{\perp}$), then Eq. (2.2) determines the $(2,1)$ block of $\dot{A}A^{-1}$ in terms of $\dot{p}$ and $\dot{q}$, so we have (in the given basis)

$$\dot{A}A^{-1} = \left\{ \begin{matrix} \begin{pmatrix} \text{unknown } n \times n \\ \text{skew-symmetric matrix} \end{pmatrix} & \begin{pmatrix} \text{known} \\ \text{by symmetry} \end{pmatrix} \\ (\text{known}) & \begin{pmatrix} \text{unknown } r \times r \\ \text{skew-symmetric matrix} \end{pmatrix} \end{matrix} \right\}.$$

Thus, we see that specifying the variables $\dot{p} \in T_p(M_0)$ and $\dot{q} \in T_q(M_1)$, each with at most n degrees of freedom, forces $\dot{A}A^{-1}$ to lie in a subspace of dimension $\frac{1}{2}n(n-1) + \frac{1}{2}r(r-1)$. It follows that we have the inequality

$$\dim V \leq 2n + \frac{1}{2}n(n-1) + \frac{1}{2}r(r-1). \qquad \blacksquare$$

A smooth curve $c\colon I \to \Sigma$ determines a smooth curve $\phi \circ c\colon I \to \mathrm{Euc}_N(\mathbf{R})$. The next order of business is to determine the conditions on c that will ensure that $\phi \circ c$ is a rolling map.

Lemma 2.3. *$\phi \circ c$ is a rolling map if and only if c is tangent to the n-dimensional distribution $\mathcal{R}$ on Σ given by*

(i) $\dot{p} = A\dot{q}$,

(ii) $\dot{A}A^{-1}u = AB_1(A^{-1}\dot{p}, A^{-1}u) - B_0(\dot{p}, u)$ *for all* $u \in T_p(M_0)$,

(iii) $\dot{A}A^{-1}\nu = -AB_1^t(A^{-1}\dot{p}, A^{-1}\nu) + B_0^t(\dot{p}, \nu)$ *for all* $\nu \in T_p(M_0)^\perp$. *(See Exercise* 1.2.24(iii) *for the definition of* B^t.)

Moreover, the derivative of the canonical map $\Sigma \to M$ induces the isomorphism shown in the following diagram.

$$\begin{tikzcd}[column sep=tiny] T_{(p,A,q)}(\Sigma) \arrow[dr] & \supset & \mathcal{R}_{(p,A,q)} \arrow[dl, "\approx"] \\ & T_p(M) & \end{tikzcd}$$

Proof. Let $c(t)$ be any curve on Σ. We write $c = (p, A, q)$ and $\dot{c} = (\dot{p}, \dot{A}, \dot{q})$ and we try to express conditions (2), (3), and (4) of Definition 1.1 in terms of $\dot{c}$. (Condition (1) is, of course, built into the very definiton of Σ.) We have

$$gv = A(v-q) + p \quad \text{so} \quad \dot{g}v = \dot{A}(v-q) + \dot{p} - A\dot{q}$$

and

$$g^{-1}v = A^{-1}(v-p) + q,$$

so that

$$\dot{g}g^{-1}v = \dot{A}(A^{-1}(v-p)) + \dot{p} - A\dot{q}$$

and

$$(\dot{g}g^{-1})_* v = \dot{A}A^{-1}v.$$

Since $\dot{g}g^{-1}\sigma = \dot{g}g^{-1}p = \dot{p} - A\dot{q}$, it follows that condition 1.1(2), namely, $\dot{g}g^{-1}\sigma = 0$, is equivalent to $\dot{p} = A\dot{q}$.

Now $(\dot{g}g^{-1})_* = \dot{A}A^{-1}$ implies that condition 1.1(3), namely,

$$(\dot{g}g^{-1})_* T_{\sigma_0(t)}(M_0) \subset T_{\sigma_0(t)}(M_0)^\perp$$

(and, respectively, condition 1.1(4), i.e., $(\dot{g}g^{-1})_* T_p(M_0)^\perp \subset T_p(M_0)$), is equivalent to $\dot{A}A^{-1}T_p(M_0) \subset T_p(M_0)^\perp$ (respectively, $\dot{A}A^{-1}T_p(M_0)^\perp \subset T_p(M_0)$), which by Proposition 2.1, is equivalent to the equation $\dot{A}A^{-1}u = AB_1(A^{-1}\dot{p}, A^{-1}u) - B_0(\dot{p}, u)$ for all $u \in T_p(M_0)$ (respectively, to the equation $\dot{A}A^{-1}v = -AB_1^t(A^{-1}\dot{p}, A^{-1}v) + B_0^t(\dot{p}, v)$ for all $v \in T_p(M_0)^\perp$). Of course, by Exercise **1**.2.24, this latter equation is always true mod $T_p(M_0)$ (respectively, mod $T_p(M_0)^\perp$), and the right-hand side always takes values in the normal bundle (respectively, the tangent bundle) of M_0. But here the left side also lies in $T_p(M_0)^\perp$ (respectively, mod $T_p(M_0)$), so we get equality.

Thus, $\phi \circ c$ is a rolling map if and only if c is tangent to the distribution $\mathcal{R}$.

To calculate the dimension of $\mathcal{R}$, we again fix, as in the proof of Proposition 2.1, an orthonormal basis of $\mathbf{R}^{n+r}$ that is adapted to $T_q(M_1)$ and consider the matrix $\dot{A}A^{-1}$ which is skew symmetric in this basis. By Definition 1.1(3) and 1.1(4), $\dot{A}A^{-1}$ must, in this basis, have the form

$$\begin{pmatrix} 0 & -S^t \\ S & 0 \end{pmatrix}.$$

Thus $\dot{A}A^{-1}$ is completely determined by its value on $T_p(M_0)$, which is known by equation 2.3(ii) at the point (p, A, q) once either $\dot{p}$ or $\dot{q}$ is known. By equation 2.3(i), $\dot{q}$ determines $\dot{p}$, so it follows that dim $\mathcal{R} \leq n$. On the other hand, for any choice of $\dot{q} \in T_q(M_1)$ equations (i), (ii), and (iii) can be solved for $\dot{p}$ and $\dot{A}$ so that dim $\mathcal{R} = n$. Finally, this same argument shows that the canonical projection $\pi: \Sigma \to M$ inducing the map $\pi_{*(p,A,q)}: T_{(p,A,q)}(\Sigma) \to T_p(M)$ has the property that $\pi_{*(p,A,q)} \mid \mathcal{R}$ is an isomorphism. ∎

Now we are in a position to obtain the existence and uniqueness of rolling maps.

Proposition 2.4. *Let $M_0^n, M_1^n \subset \mathbf{R}^{n+r}$ be submanifolds and let $(p_0, A_0, q_0) \in \Sigma$. Assume we are given a piecewise smooth curve $\sigma_1: (I, 0) \to (M_1, q_0)$ (respectively, $\sigma_0: (I, 0) \to (M_0, p_0)$). Then there is a unique rolling map $g: (I, 0) \to (Euc_n(\mathbf{R}), id)$ with rolling curve σ_1 (respectively, development σ_0).*

Proof. We may as well assume that the curve is smooth since we can patch the smooth pieces together to obtain the general case. We will deal only with the case of $\sigma_1: (I, 0) \to (M_1, q_0)$, as the other case is similar in the configuration space setting we are using. Pulling back the bundle

$$M_0 \times O_n(\mathbf{R}) \times O_r(\mathbf{R}) \to \Sigma \to M_1$$

over I along σ_1 yields a bundle

$$M_0 \times SO_n(\mathbf{R}) \times SO_r(\mathbf{R}) \to \Sigma_{\sigma_1} \to I$$

on which the restriction $\mathcal{R}_{\sigma_1}$ of the distribution $\mathcal{R}$ has dimension 1. Since a one-dimensional distribution is always integrable, there exists a unique curve in Σ_{σ_1} through $(p_0, A_0, q_0) \in \Sigma_{\sigma_1}$ which is tangent to $\mathcal{R}_{\sigma_1}$. The image of this curve in Σ is tangent to $\mathcal{R}$ and covers σ_1. By Lemma 2.3, this proves the existence and uniqueness of the rolling map given in Definition 1.1.

§3. Relation to Levi–Civita and Normal Connections

In this section we relate our discussion of "rolling without slipping or twisting" to the Levi–Civita connection and to the Ehresmann connection on the normal bundle. Some of this material necessarily covers ground similar to that of §**6**.5.

We consider the case of the standard n-plane $\mathbf{R}^n$ ($= M_1$ in the notation above) $\subset \mathbf{R}^N$ rolling on a manifold M^n ($= M_0$ in the notation above) $\subset \mathbf{R}^N$. Then

$$\Sigma = \{(p, A, q) \in M \times O_N(\mathbf{R}) \times \mathbf{R}^n \mid A\mathbf{R}^n = T_p(M)\}$$

is the configuration space for this pair of manifolds, and we again have the distribution $\mathcal{R}$ on it described in Lemma 2.3. We are going to study the following diagram of fiber bundles.

$$\begin{tikzcd} & & P_{\tan} \\ \Sigma \arrow[r] & P_\tau \arrow[ur] \arrow[dr] & \\ & & P_{\mathrm{nor}} \end{tikzcd}$$

The three spaces $P_{\tan}$, P_{nor} and P_τ are the bundles over M that we have already considered in §**6**.5. We are going to describe the images of $\mathcal{R}$ under these maps and show in particular that its images in $T(P_{\tan})$ and $T(P_{\mathrm{nor}})$ are (the kernels of) the Levi–Civita and the normal connections. Since, by Exercise A.1.3, these kernels determine the connections, we see that for a manifold in Euclidean space the notion of rolling without slipping or twisting includes the notions of the Levi–Civita connection on the tangent bundle and the Ehresmann connection on the normal bundle.

We define

$$\mathrm{Iso}(\mathbf{R}^k, \mathbf{R}^N) = \{\phi \in \mathrm{Hom}(\mathbf{R}^k, \mathbf{R}^N) \mid \langle \varphi(u), \varphi(v)\rangle = \langle u, v\rangle \forall u, v \in \mathbf{R}^k\}$$

and set

$$\begin{aligned} P_\tau &= \{(p,A) \in M \times O_N(\mathbf{R}) \mid A(\mathbf{R}^n) = T_p(M)\}, \\ P_{\text{tan}} &= \{(p,\varphi) \in M \times \operatorname{Iso}(\mathbf{R}^n, \mathbf{R}^N) \mid \varphi(\mathbf{R}^n) = T_p(M)\}, \\ P_{\text{nor}} &= \{(p,\varphi) \in M \times \operatorname{Iso}(\mathbf{R}^r, \mathbf{R}^N) \mid \varphi(\mathbf{R}^r) = T_p(M)^\perp\}. \end{aligned}$$

Lemma 3.1. *The map $P_\tau \to M$ (respectively, $P_{tan} \to M$, $P_{nor} \to M$) induced by canonical projection is a principal $O_n(\mathbf{R}) \times O_r(\mathbf{R})$ (respectively, $O_n(\mathbf{R})$, $O_r(\mathbf{R})$) bundle map.*

Proof. We deal with the map $P_\tau \to M$ only, the others being similar. Consider the right action

$$\underset{((p,A),a,b) \mapsto \left(p, A\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right)}{P_\tau \times O_n(\mathbf{R}) \times O_r(\mathbf{R}) \to P_\tau.}$$

This action is free and proper (note that $O_n(\mathbf{R}) \times O_r(\mathbf{R})$ is compact) and acts transitively on the fibers of the projection $P_\tau \to M$. It follows from Theorem **4**.2.4 that $P_\tau \to M$ is a principal bundle with group $O_n(\mathbf{R}) \times O_r(\mathbf{R})$. ■

Exercise 3.2. Verify that P_{tan} and P_{nor} are the tangent and normal orthonormal frame bundles associated to the inclusion $M \subset \mathbf{R}^N$ as in §**6**.5. Show also that P_τ corresponds to the bundle of the same name in §**6**.5. ◇

Lemma 3.3. *For $p \in M$ let $V \subset T_p(\mathbf{R}^N) \times T_A(O_N(\mathbf{R}))$ be defined by*

$(\dot p, \dot A) \in V \Leftrightarrow \dot p$ and $\dot A$ satisfy the following conditions

(i) $\dot A A^{-1} u = -B(\dot p, u) \bmod T_p(M)$ *for all* $u \in T_p(M)$.

(ii) $\dot A A^{-1} v = B^t(\dot p, v) \bmod T_p(M)^\perp$ *for all* $v \in T_p(M)^\perp$.

Then $T_{(p,A)}(P_\tau) \subset V$.

Proof. Let $(p, A) \in P_\tau$, and consider the vector space V. In an orthonormal basis of $T_p(\mathbf{R}^N)$ adapted to $T_p(M)$ (cf. the proof of Proposition 2.1), conditions (i) and (ii) imply that the matrix $\dot A A^{-1}$ has the block form

$$\dot A A^{-1} = \begin{pmatrix} \text{arbitrary} & \text{determined by } \dot p \\ \text{determined by } \dot p & \text{arbitrary} \end{pmatrix}.$$

In particular, the dimension of V is

$$\dim M + \dim O_n(\mathbf{R}) + \dim O_r(\mathbf{R}) = \dim P_\tau.$$

Thus, it suffices to verify the inclusion $T_{(p,A)}(P_\tau) \subset V$. Suppose that $(\dot p, \dot A) \in T_{(p,A)}(P_\tau)$. Let $(p(t), A(t))$ be a curve on P_τ with tangent $(\dot p, \dot A)$ at

(p, A). Let $u(t)$ be a tangent field (respectively, normal field) to M along the curve $p(t)$. Then $A^{-1}(t)u(t) \in \mathbf{R}^n \times 0$ (respectively, $0 \times \mathbf{R}^r$). Differentiating, we get

$$-A^{-1}(t)\dot{A}(t)A^{-1}(t)u(t) + A^{-1}(t)\dot{u}(t) \in \mathbf{R}^n \times 0 \text{ respectively, } 0 \times \mathbf{R}^r)$$

or

$$-\dot{A}(t)A^{-1}(t)u(t) + \dot{u}(t) \in T_p(M) \text{ (respectively, } T_p(M)^{\perp})).$$

But we know from Exercise 1.2.24(i) (respectively, (iii)) that

$$\dot{u}(t) = -B(\dot{p}(t), u(t)) \operatorname{mod} T_p(M)$$

respectively,

$$\dot{u}(t) = B^t(\dot{p}(t), u(t))\operatorname{mod} T_p(M)^{\perp}),$$

so

$$\dot{A}(t)A^{-1}(t)u(t) + B(\dot{p}(t), u(t)) \in T_p(M)$$

respectively,

$$\dot{A}(t)A^{-1}(t)u(t) - B^t(\dot{p}(t), u(t)) \in T_p(M)^{\perp}).$$

This verifies the inclusion $\subset$. ∎

Exercise 3.4. Show that

$$\begin{aligned} T_{(p,\varphi)}(P_{tan}) &= \{(\dot{p}, \dot{\varphi}) \in T_p(M) \times \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^N) \mid \dot{\varphi} = -B(\dot{p}, \varphi(-)) \\ &\qquad \operatorname{mod} T_p(M)\}, \\ T_{(p,\varphi)}(P_{\mathrm{nor}}) &= \{(\dot{p}, \dot{\varphi}) \in T_p(M) \times \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^N) \mid \dot{\varphi} = -B^t(\dot{p}, \varphi(-)) \\ &\qquad \operatorname{mod} T_p(M)^{\perp}\}. \end{aligned}$$ ∎

It is convenient now to regard the group of Euclidean isometries $\mathrm{Euc}_N(\mathbf{R})$ as the matrix group (with 1×1 and $N \times N$ blocks down the diagonal in the block form)

$$\mathrm{Euc}_N(\mathbf{R}) = \left\{ \begin{pmatrix} 1 & 0 \\ p & A \end{pmatrix} \in Gl_{N+1}(\mathbf{R}) \mid p \in \mathbf{R}^N, \ A \in O_N(\mathbf{R}) \right\}$$

with Lie algebra

$$\mathfrak{euc}_N(\mathbf{R}) = \left\{ \begin{pmatrix} 0 & 0 \\ \dot{p} & \dot{A} \end{pmatrix} \in M_{N+1}(\mathbf{R}) \mid \dot{p} \in \mathbf{R}^N, \ \dot{A} \in \mathfrak{o}_N(\mathbf{R}) \right\}.$$

In this picture, the isometry given by $gv = Av+p$ corresponds to the matrix $\begin{pmatrix} 1 & 0 \\ p & A \end{pmatrix}$, and the (left invariant) Maurer–Cartan form on $\mathrm{Euc}_N(\mathbf{R})$ may be expressed as

$$\omega_{\mathrm{Euc}_N(\mathbf{R})} = \begin{pmatrix} 1 & 0 \\ p & A \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \dot{p} & \dot{A} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -A^{-1}p & A^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \dot{p} & \dot{A} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ A^{-1}\dot{p} & A^{-1}\dot{A} \end{pmatrix}.$$

Let us express the Maurer–Cartan form on $\mathrm{Euc}_N(\mathbf{R})$ (with 1×1, $n \times n$, and $r \times r$ blocks down the diagonal in the block form) as

$$\omega_{\mathrm{Euc}_{n+r}(\mathbf{R})} = \begin{pmatrix} 0 & 0 & 0 \\ \star & \alpha & -\beta^t \\ \star & \beta & \gamma \end{pmatrix}.$$

Then we have

Proposition 3.5. *The inclusion $M \subset \mathbf{R}^{n+r}$ is covered by the right $O_n(\mathbf{R}) \times O_r(\mathbf{R})$ equivariant map*

$$\iota\colon P_\tau \subset \mathrm{Euc}_{n+r}(\mathbf{R}).$$
$$(p, A) \mapsto \begin{pmatrix} 1 & 0 \\ p & A \end{pmatrix}$$

Let us define $\mathcal{H} = \ker(\alpha \mid P_\tau) \cap ker(\delta \mid P_\tau)$. Then, for each $(p, A, q) \in \Sigma$, the derivative of the canonical map $\Sigma \to P_\tau$ induces the isomorphism in the following diagram.

$$\begin{array}{ccc} T_{(p,A,q)}(\Sigma) & \supset & \mathcal{R}_{(p,A,q)} \\ \downarrow & & \downarrow \approx \\ T_{(p,A)}(P_\tau) & \supset & \mathcal{H}_{(p,A)} \end{array}$$

Proof. Since

$$\begin{aligned} &(\iota^*\omega_{\mathrm{Euc}_N(\mathbf{R})})_{(p,A)}(\dot{p}, \dot{A}) \\ &= (\omega_{\mathrm{Euc}_N(\mathbf{R})})_{\begin{pmatrix} 1 & 0 \\ p & A \end{pmatrix}}(\iota_*(\dot{p}, \dot{A})) = \begin{pmatrix} 0 & 0 \\ A^{-1}\dot{p} & A^{-1}\dot{A} \end{pmatrix}, \end{aligned}$$

using the block form of the Maurer-Cartan form, we have

$$A^{-1}\dot{A} = \begin{pmatrix} \alpha(\dot{p}, \dot{A}) & -\beta^*(\dot{p}, \dot{A}) \\ \beta(\dot{p}, \dot{A}) & \gamma(\dot{p}, \dot{A}) \end{pmatrix}.$$

The proof of Lemma 3.1 shows that $P_\tau \subset \mathrm{Euc}_{n+r}(\mathbf{R})$ is a right $O_n(\mathbf{R}) \times O_r(\mathbf{R})$ equivariant map. Now the derivative of the map $\Sigma \to P_\tau$ is given by $(\dot{p}, \dot{A}, \dot{q}) \mapsto (\dot{p}, \dot{A})$, and so, according to the description of $\mathcal{R}$ in Lemma 2.3, (and in the present case, $B_0 = B$ and $B_1 = 0$) the image of $\mathcal{R}_{(p,A,q)}$ under this map is $W_{(p,A)} \subset T_p(M) \times T_A(O_n(\mathbf{R}))$ where

$$(\dot{p}, \dot{A}) \in W_{(p,A)} \Leftrightarrow \text{the following two conditions hold:}$$

(i) $\dot{A}A^{-1}u = -B(\dot{p}, u)$, for all $u \in T_p(M)$

(ii) $\dot{A}A^{-1}\nu = -B^t(\dot{p}, \nu)$, for all $\nu \in T_p(M)^\perp$.

Thus,

$$\begin{aligned}
(\dot{p}, \dot{A}) \in W_{(p,A)} &\Rightarrow \begin{cases} \dot{A}A^{-1}T_p(M) \subset T_p(M)^\perp, \text{ and} \\ \dot{A}A^{-1}T_p(M)^\perp \subset T_p(M) \end{cases} \\
&\Rightarrow \begin{cases} A^{-1}\dot{A}(\mathbf{R}^n \times 0) \subset 0 \times \mathbf{R}^r, \text{ and} \\ A^{-1}\dot{A}(0 \times \mathbf{R}^r) \subset \mathbf{R}^n \times 0 \end{cases} \\
&\Rightarrow \begin{cases} \alpha(\dot{p}, \dot{A}) = 0, \text{ and} \\ \delta(\dot{p}, \dot{A}) = 0 \end{cases} \\
&\Rightarrow (\dot{p}, \dot{A}) \in \mathcal{H}_{(p,A)}.
\end{aligned}$$

Thus $W_{(p,A)} \subset \mathcal{H}_{(p,A)}$. On the other hand, by Lemma 2.3, the composite $\mathcal{R}_{(p,A,q)} \to T_{(p,A)}(P_\tau) \to T_p(M)$ is an isomorphism, so that $\mathcal{R}_{(p,A,q)} \to T_{(p,A)}(P_\tau)$ is injective. Since P_τ is a principal $O_n(\mathbf{R}) \times O_r(\mathbf{R})$ bundle over M and $\alpha \oplus \gamma$ restricts to the fiber of P_τ to yield the Maurer–Cartan form on $O_n(\mathbf{R}) \times O_r(\mathbf{R})$, it follows that dim $\mathcal{H} = n =$ dim $\mathcal{R}$, and so the image of $\mathcal{R}_{(p,A,q)}$ is $\mathcal{H}_{(p,A)}$. ■

Corollary 3.6. *For each $(p, A) \in P_\tau$, the canonical projections $P_\tau \to P_{tan}$ and $P_\tau \to P_{nor}$ induces the isomorphisms in the following diagram.*

$$\begin{array}{ccc}
T_{(p,A)}(P_\tau) & \supset & \mathcal{H}_{(p,A)} \\
\downarrow & & \downarrow \approx \\
T_{(p,A|\mathbf{R}^n)}(P_{tan}) & \supset & ker\ (\alpha)_{(p,A|\mathbf{R}^n)}
\end{array}
\qquad
\begin{array}{ccc}
T_{(p,A)}(P_\tau) & \supset & \mathcal{H}_{(p,A)} \\
\downarrow & & \downarrow \approx \\
T_{(p,A|\mathbf{R}^r)}(P_{nor}) & \supset & ker\ (\gamma)_{(p,A|\mathbf{R}^r)}
\end{array}$$

Proof. The proofs are similar, so we consider only the projection $P_\tau \to P_{\tan}$. Now the form α on P_τ is basic for this projection (Proposition **6**.5.12) so the distribution ker α makes sense as a distribution on $P_{\tan}$. Since $\mathcal{H} = \ker(\alpha \mid P_\tau) \cap \ker(\delta \mid P_\tau)$, the canonical projection maps $\mathcal{H}$ to ker α. Finally, since the derivative of the composite map $P_\tau \to P_{\tan} \to M$ is injective on $\mathcal{H}$, it follows that $\mathcal{H} \to$ ker α is injective, and since the dimensions are the same, it is an isomorphism. ■

Proposition 3.5 establishes the link between "rolling without slipping or twisting" and the usual connection forms on a submanifold of $\mathbf{R}^N$ since we know from Proposition **6**.5.12 that $\ker(\alpha \mid P_\tau) \cap \ker(\delta \mid P_\tau)$ projects to the Levi–Civita connection on $P_{\tan}$ and to the canonical normal connection P_{nor}.

Proposition 3.7. *Let $t \mapsto (\sigma_0(t), A(t), \sigma_1(t)) \in \Sigma$ be an integral curve for the distribution $\mathcal{R}$.*

(i) *$t \mapsto \sigma_1(t) \in \mathbf{R}^n$ is the development (in the sense of Definition* 5.4.15) *of $t \mapsto \sigma_0(t) \in M$.*

(ii) *If $v(t)$ is a vector field along $t \mapsto \sigma_0(t) \in M$ which is tangent (respectively, normal) to M, then*

$$D_{\sigma_0(t)}v(t) = A(t)(A(t)^{-1}v(t)),$$

where $D_{\sigma_0(t)}v(t)$ is the Levi–Civita (respectively, the canonical normal) covariant derivative.

(iii) *A vector field $v(t)$ along $t \mapsto \sigma_0(t) \in M$ which is tangent (respectively, normal) to M is parallel if and only if $A(t)^{-1}v(t)$ is constant.*

Proof. (i) Exercise 5.4.14 tells us how to develop a curve $t \mapsto \sigma_0(t) \in M$ to a curve on the model space of a reductive geometry. It says we must first take the horizontal lift $\hat{\sigma}\colon (I, a) \to (P_{\tan}, p)$, which in the present case is the projection of $t \mapsto (\sigma_0(t), A(t), \sigma_1(t)) \in \Sigma$, to $P_{\tan}$. Then we must solve the equation $\hat{\sigma}^*\omega_{\tan} = d\tilde{\sigma}$ (i.e., $\omega_{\tan}(\dot{\hat{\sigma}}) = \dot{\tilde{\sigma}}$) for the development $\tilde{\sigma}\colon (I, a) \to (\mathbf{R}^n, 0)$. But this equation is, in the present circumstance, the equation $A(t)^{-1}\dot{\sigma}_0(t) = \dot{\tilde{\sigma}}(t)$. But since, by the definition of $\mathcal{R}$, $A(t)^{-1}\dot{\sigma}_0(t) = \dot{\sigma}_1(t)$, the uniqueness of development implies $\tilde{\sigma}(t) = \sigma_1(t)$ for all t.

(ii) Case (1): $v(t)$ is a tangent vector field. According to Definition A.4.1, the covariant derivative D_XY, for $X \in T_x(M)$ and Y a tangent vector field on M, is calculated as follows: express Y as a function $f_Y\colon P \to \mathfrak{g}/\mathfrak{h}$; lift X to a horizontal vector $\tilde{X}_p = T_p(P)$; compute $\tilde{X}_p(f_Y) \in \mathfrak{g}/\mathfrak{h}$; and then set $D_XY = \varphi_p^{-1}(\tilde{X}_p(f_Y)) \in T_x(M)$.

First consider the tangent vector field $v(t)$ on M. It determines the function $t \mapsto \varphi_{(\sigma(t),A(t))}(v(t)) \in \mathfrak{g}/\mathfrak{h}$.

Next, since the curve $t \mapsto \sigma_0(t)$ has the lift $t \mapsto (\sigma_0(t), A(t), \sigma_1(t)) \in \Sigma$, which is an integral curve for the distribution $\mathcal{R}$, it follows by Corollary 3.6 that the curve $t \mapsto (\sigma_0(t), A(t) \mid \mathbf{R}^n \times 0)$ is a horizontal lift to $P_{\tan}$, that is, $(\dot{\sigma}_0(t), \dot{A}(t) \mid \mathbf{R}^n \times 0))$ is a horizontal vector covering $\dot{\sigma}_0(t)$.

Putting this all together, we have

$$\begin{aligned}
\varphi_{(\sigma(t),A(t))}(D_{\sigma_0(t)}v(t)) &= (\varphi_{(\sigma(t),A(t))}(v(t)))\\
&= (A(t)^{-1}(v(t)))\\
&= \varphi_{(\sigma(t),A(t))}(A(t)(A(t)^{-1}(v(t)))).
\end{aligned}$$

Since $\varphi_{(\sigma(t),A(t))}$ is injective, this verifies case (1).

Case (2): $v(t)$ is a normal vector field. This time the covariant derivative is taken with respect to the Ehresmann connection on the principal $O_r(\mathbf{R})$ bundle $O_r(\mathbf{R}) \to Q \to M$ of normal frames on M.

According to Definition A.4.1, the covariant derivative D_XY, for $X \in T_x(M)$ and Y a normal vector field on M, may be calculated as follows: express Y as a function $f_Y\colon Q \to \mathbf{R}^r$ (i.e., write Y in terms of the basis given by the frame $q \in Q$); lift X to a horizontal vector $\tilde{X}_q \in T_q(Q)$; compute $\tilde{X}_q(f_Y) \in \mathbf{R}^r$. Then $D_X Y$ is the normal vector to M which, when expressed in the basis $q \in Q$, is $\tilde{X}_q(f_Y)$.

First consider the normal vector field $v(t)$ on M. It determines the function $t \mapsto A(t)^{-1}v(t) \in \mathbf{R}^r$ with derivative $t \mapsto (A(t)^{-1}v(t)) \in \mathbf{R}^r$.

Since the curve $t \mapsto \sigma_0(t)$ has the lift $t \mapsto (\sigma_0(t), A(t), \sigma_1(t)) \in \Sigma$, which is an integral curve for the distribution $\mathcal{R}$, it follows by Corollary 3.6 that the curve $t \mapsto (\sigma_0(t), A(t) \mid 0 \times \mathbf{R}^r)$ is a horizontal lift to P_{nor}, that is, for each t, $(\dot{\sigma}_0(t), \dot{A}(t) \mid 0 \times \mathbf{R}^r)$ is a horizontal vector covering $\dot{\sigma}_0(t)$.

Putting all this together, we have

$$D_{\sigma_0(t)} v(t) = A(t)(A(t)^{-1}(v(t))).$$

(iii) This is an immediate consequence of (ii). ∎

§4. Transitivity of Rolling Without Slipping or Twisting

In this section we will study the "composite" of two "rollings without slipping or twisting."

Theorem 4.1. *Let three submanifolds $M_0, M_1, M_2 \subset \mathbf{R}^N$ be given which are tangent to each other at some point $p \in M_0 \cap M_1 \cap M_2$, and let a path $\sigma_0: (I, 0) \to (M_0, p)$ be given. Suppose that*

(a) *M_1 rolls on M_0 along $\sigma_1: (I, 0) \to (M_1, p)$ without slipping or twisting via $g_1: (I, 0) \to (Euc_N(\mathbf{R}), I)$, with development $\sigma_0: (I, 0) \to (M_0, p)$,*

(b) *M_2 rolls on M_1 along σ_2 without slipping or twisting via $g_2: (I, 0) \to (Euc_N(\mathbf{R}), I)$, with development $\sigma_1: (I, 0) \to (M_1, p)$.*

Then it follows that

(c) *M_2 rolls on M_0 along σ_2 without slipping or twisting via $g_1 g_2: (I, 0) \to (Euc_N(\mathbf{R}), I)$, with development $\sigma_0: (I, 0) \to (M_0, p)$.*

Proof. It suffices to show that $g_1 g_2$ verifies properties (1), (2), (3), and (4) of Definition 1.1.

(1) By (a), $\sigma_0(t) = g_1(t)\sigma_1(t)$ and $T_{\sigma_0(t)}(g_1(t)M_1) = T_{\sigma_0(t)}(M_0)$.
By (b), $\sigma_1(t) = g_2(t)\sigma_2(t)$ and $T_{\sigma_1(t)}(g_2(t)M_2) = T_{\sigma_1(t)}(M_1)$.
Thus, $\sigma_1(t) = g_1(t)g_2(t)\sigma_3(t)$ and

$$\begin{aligned} T_{\sigma_1(t)}(g_1(t)g_2(t)M_3) &= g_1(t)_* T_{\sigma_2(t)}(g_2(t)M_3) \\ &= g_1(t)_* T_{\sigma_2(t)}(M_2) \\ &= T_{\sigma_1(t)}(g_1(t)M_2) \\ &= T_{\sigma_1(t)}(M_1). \end{aligned}$$

In the proof of the remaining parts, we shall need the (easily proved) formula

$$(g_1(t)g_2(t))^{\dot{}}(g_1(t)g_2(t))^{-1} = \dot{g}_1(t)g_1(t)^{-1} + g_1(t)\dot{g}_2(t)g_2(t)^{-1}g_1(t)^{-1}.$$

(2)

$$\begin{aligned}&(g_1(t)g_2(t))^{\dot{}}(g_1(t)g_2(t))^{-1}\sigma_1(t)\\ &\quad= \dot{g}_1(t)g_1(t)^{-1}\sigma_1(t) + g_1(t)\dot{g}_2(t)g_2(t)^{-1}g_1(t)^{-1}\sigma_1(t)\\ &\quad= 0 + g_1(t)\dot{g}_2(t)\dot{g}_2(t)^{-1}\sigma_2(t) = 0.\end{aligned}$$

(3)

$$\begin{aligned}&((g_1(t)g_2(t))^{\dot{}}(g_1(t)g_2(t))^{-1})_*T_{\sigma_1(t)}(M_1)\\ &\quad= (\dot{g}_1(t)g_1(t)^{-1})_*T_{\sigma_1(t)}(M_1) + (g_1(t)\dot{g}_2(t)g_2(t)^{-1}g_1(t)^{-1})_*T_{\sigma_1(t)}(M_1)\\ &\quad\subset T_{\sigma_1(t)}(M_1)^{\perp} + g_1(t)_*(\dot{g}_2(t)g_2(t)^{-1})_*g_1(t)_*^{-1}T_{\sigma_1(t)}(M_1)\\ &\quad\subset T_{\sigma_1(t)}(M_1)^{\perp} + g_1(t)_*(\dot{g}_2(t)g_2(t)^{-1})_*T_{\sigma_2(t)}(M_2)\\ &\quad\subset T_{\sigma_1(t)}(M_1)^{\perp} + g_1(t)_*T_{\sigma_2(t)}(M_2)^{\perp} \subset T_{\sigma_1(t)}(M_1)^{\perp}.\end{aligned}$$

(4) is similar to (3). ∎

Corollary 4.2. *Suppose that two manifolds $M_1, M_2 \subset \mathbf{R}^N$ are given which are tangent to each other at some point $p \in M_1 \cap M_2$, and let a path $\sigma_1: (I, 0) \to (M_1, p)$ be given. Suppose that*

(a) *M_2 rolls on M_1 along σ_1 without slipping or twisting via $g_1: (I, 0) \to (Euc_n(\mathbf{R}), I)$, with development $\sigma_2: (I, 0) \to (M_2, p)$.*

Then it follows that

(b) *M_1 rolls on M_2 along σ_2 without slipping or twisting via $g_2 = g_1^{-1}$: $(I, 0) \to (Euc_n(\mathbf{R}, I)$, with development $\sigma_1: (I, 0) \to (M_1, p)$.*

Proof. Apply the theorem to the case $M_3 = M_1$ to see that M_1 rolls on M_1 along σ_1 without slipping or twisting via $g_1g_2: (I, 0) \to (\mathrm{Euc}_n(\mathbf{R}), I)$, with development $\sigma_3: (I, 0) \to (M_2, p)$. But obviously $g_1g_2 = I$ and $\sigma_3 = \sigma_1$. ∎

Corollary 4.3. *The "rolling data" consisting of a fixed manifold M and a curve σ on it depend only on the curve in space and the tangent spaces to M along the curve.*

Proof. Construct a manifold M_1 out of σ and the tangent spaces along the curve by letting V_t be the subspace of $T_{\sigma(t)}(M)$ orthogonal to $\sigma'(t)$ and setting $M_1 = \cup V_t$. It is easy to see that M_1 is a manifold in some neighborhood of σ which is a smooth curve on M_1. Moreover, the rolling map of M_1 on M is the identity. Thus, by multiplication, if we wish to find the rolling map of $\tilde{M}$ along (M, σ), it is enough to roll it along (M_1, σ) or indeed any other manifold tangent to M along σ. ∎

Exercise 4.4. Verify that the picture given at the beginning of this appendix is correct. That is, show that a sphere rolling on the central line of a helicoid rolls along a line of longitude. [*Hint:* use the fact that the central line of a helicoid is a geodesic.] ❑