Nesterov Accelerated Shuffling Gradient Method for Convex Optimization

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Problem Statement

We consider the following **finite-sum minimization:**

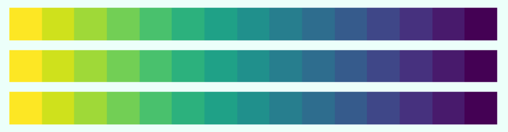
$$\min_{w \in \mathbb{R}^d} \left\{ F(w) := \frac{1}{n} \sum_{i=1}^{n} f(w; i) \right\}, \tag{1}$$

where $f(\cdot; i) : \mathbb{R}^d \to \mathbb{R}$ is **a Lipschitz smooth function** for $i \in [n] := \{1, \ldots, n\}$, and F is **convex**. Assume that we have access to the first order oracle of $f(\cdot; i)$. Below are some common sampling schemes:

Regular (Standard) Scheme: Uniformly at random: at each iteration i_t of epoch t, sample an index uniformly at random from $[n]$.

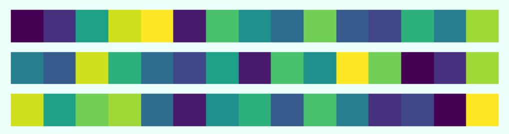
Shuffling Schemes:

Incremental Gradient: use a fixed permutation $\{1, \ldots, n\}$ for all epochs.



Shuffle Once: random shuffle one permutation and use it for all epochs.

Random Reshuffling: random shuffle a new permutation at every epoch.



Nesterov Accelerated Shuffling Gradient

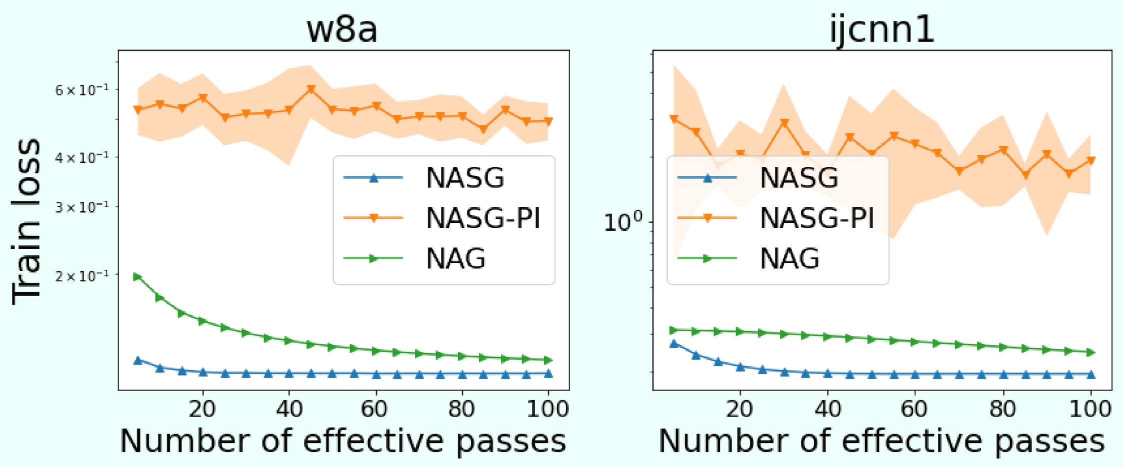
Algorithm 1: Nesterov Accelerated Shuffling Gradient (NASG) Method

1. **Initialization:** Choose an initial point $\tilde{x}_0, \tilde{y}_0 \in \mathbb{R}^d$.
2. **for** $t = 1, 2, \cdots, T$ **do**
3. Set $y_0^{(t)} := \tilde{y}_{t-1}$;
4. Generate any permutation $\pi^{(t)}$ of $[n]$ (either deterministic or random);
5. **for** $i = 1, \cdots, n$ **do**
6. Update $y_i^{(t)} := y_{i-1}^{(t)} - \eta_i^{(t)} \nabla f(y_{i-1}^{(t)}; \pi^{(t)}(i))$;
7. **end for**
8. Set $\tilde{x}_t := y_n^{(t)}$;
9. Update $\tilde{y}_t := \tilde{x}_t + \gamma_t(\tilde{x}_t - \tilde{x}_{t-1})$;
10. **end for**

Comparison with deterministic NAG:

- Inner loop of deterministic NAG
  1. **for** $i = 1, \cdots, n$ **do**
  2. Update $y_i^{(t)} := y_{i-1}^{(t)} - \eta_i^{(t)} \nabla f(y_0^{(t)}; \pi^{(t)}(i))$; ← fixed point
  3. **end for**
- Inner loop of stochastic NASG
  1. **for** $i = 1, \cdots, n$ **do**
  2. Update $y_i^{(t)} := y_{i-1}^{(t)} - \eta_i^{(t)} \nabla f(y_{i-1}^{(t)}; \pi^{(t)}(i))$; ← moving continuously
  3. **end for**

Our binary classification experiments for `w8a` and `ijcnn1` datasets show our motivation. NASG-PI is the stochastic version that applies Nesterov momentum per iteration, while our method is per epoch.



Assumptions

Problem (1) satisfies:

(a) (Bounded below and convexity for F) We assume the existence of a minimizer for F, and F is convex.

(b) (L-smoothness) $f(\cdot; i)$ is L-smooth for all $i \in [n]$: , i.e., there exists $L > 0$:

$$\forall w, w' \in \mathrm{dom}(F) \; \|\nabla f(w; i) - \nabla f(w'; i)\| \le L\|w - w'\|. \tag{2}$$

We let x_* be any minimizer of F and consider the variance of F at x_*:

$$\sigma_*^2 := \frac{1}{n} \sum_{i=1}^{n} \|\nabla f(x_*; i)\|^2 \in [0, +\infty). \tag{3}$$

In addition, we assume either (c1) or (c2):

(c1) (Individual convexity) $f(\cdot; i)$ is convex for all $i \in [n]$.

(c2) (Generalized bounded variance) There exist two finite constants $\Theta, \sigma \ge 0$:

$$\forall w \in \mathrm{dom}(F) : \frac{1}{n} \sum_{i=1}^{n} \|\nabla f(w; i) - \nabla F(w)\|^2 \le \Theta \|\nabla F(w)\|^2 + \sigma^2. \tag{4}$$

Main results

Theorem 1 - Unified Schemes (Informal)

We assume Assumption (a) and (b) with either (c1) or (c2) is satisfied. Let $\Delta := \|\tilde{x}_0 - x_*\|^2$ with the initial point $\tilde{x}_0$ and the minimizer x_*. With an appropriate choice of the learning rate, $F(\tilde{x}_T) - F(x_*)$ is upper bounded by

either $\mathcal{O}\left(\frac{\sigma_*^2/L + L\Delta}{T}\right)$, for individual convexity (c1)

or $\mathcal{O}\left(\frac{\sigma^2/(\Theta L) + L\Theta^{1/3}\Delta}{T}\right)$, for generalized bounded variance (c2)

The convergence rate of NASG is better than the current state-of-the-art rate in term of T for convex problems with general shuffling-type strategies [1, 3].

Theorem 2 - Randomized Schemes (Informal)

Suppose that Assumption (a), (b) and (c1) hold. Let $\Delta := \|\tilde{x}_0 - x_*\|^2$ with the initial point $\tilde{x}_0$ and the minimizer x_*. With an appropriate choice of the learning rate and randomized shuffling schemes, we have

$$\mathbb{E}[F(\tilde{x}_T) - F(x_*)] \le \mathcal{O}\left(\frac{\sigma_*^2/L}{nT} + \frac{L\Delta}{T}\right)$$

This rate has a factor of n improved, and is better than the corresponding rate for randomized schemes in the literature for convex problems [1, 3]. In the table below, we show **the complexity to reach an ϵ-accurate solution** x that satisfies $F(x) - F(x_*) \le \epsilon$ (or $\mathbb{E}[F(x) - F(x_*)] \le \epsilon$ in random case).

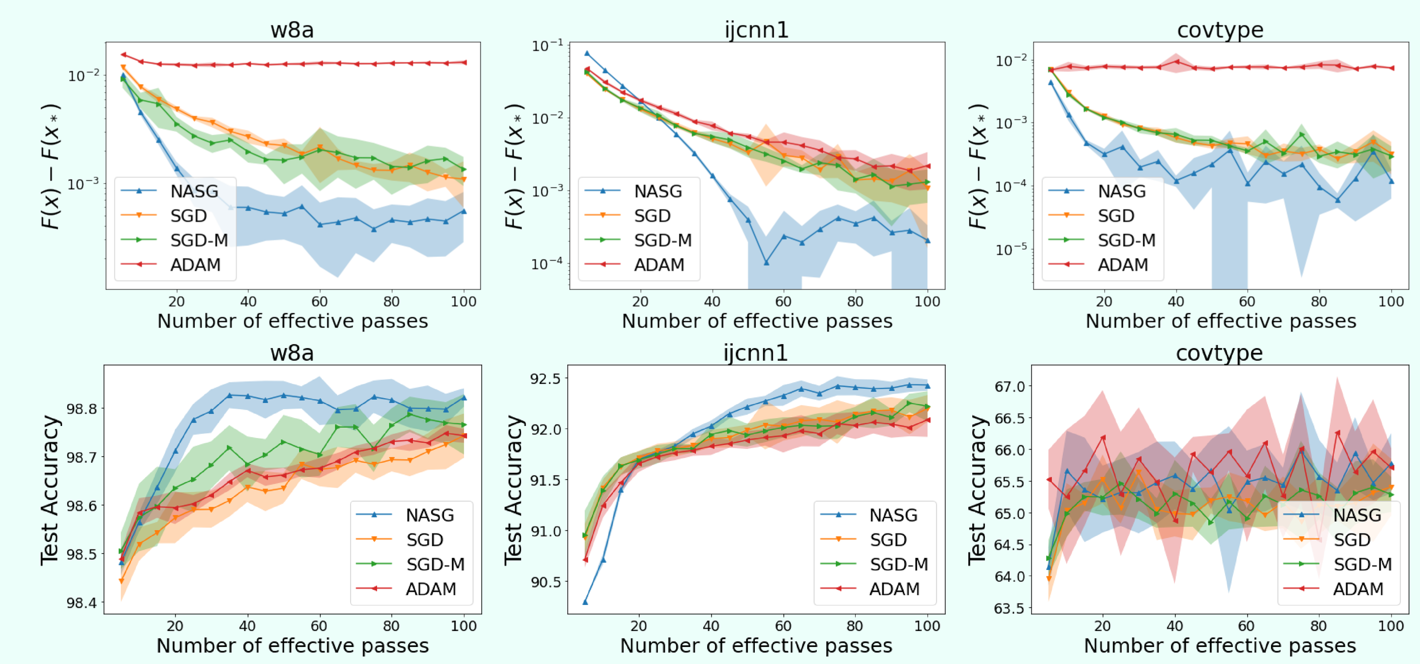
Algorithms	Complexity	References
Standard SGD(1)	$\mathcal{O}\left(\frac{\Delta_0^2 + G^2}{\epsilon^2}\right)$ (1)	[2, 4]
SGD - Unified Schemes	$\mathcal{O}\left(\frac{nL\Delta}{\epsilon} + \frac{n\sqrt{L}\sigma_*\Delta}{\epsilon^{3/2}}\right)$	[1, 3]
SGD - Randomized Schemes	$\mathcal{O}\left(\frac{nL\Delta}{\epsilon} + \frac{\sqrt{nL}\sigma_*\Delta}{\epsilon^{3/2}}\right)$	[1, 3]
NASG - Unified Schemes	$\mathcal{O}\left(\frac{nL\Delta}{\epsilon} + \frac{n\sigma_*^2}{L\epsilon}\right)$	Theorem 1
NASG - Randomized Schemes	$\mathcal{O}\left(\frac{nL\Delta}{\epsilon} + \frac{\sigma_*^2}{L\epsilon}\right)$	Theorem 2

(1) Standard results for SGD often use bounded domain that $\|x - x_*\|^2 \le \Delta_0$ for each iterate x and/or bounded gradient that $\mathbb{E}[\|\nabla f(x; i)\|] \le G^2$.

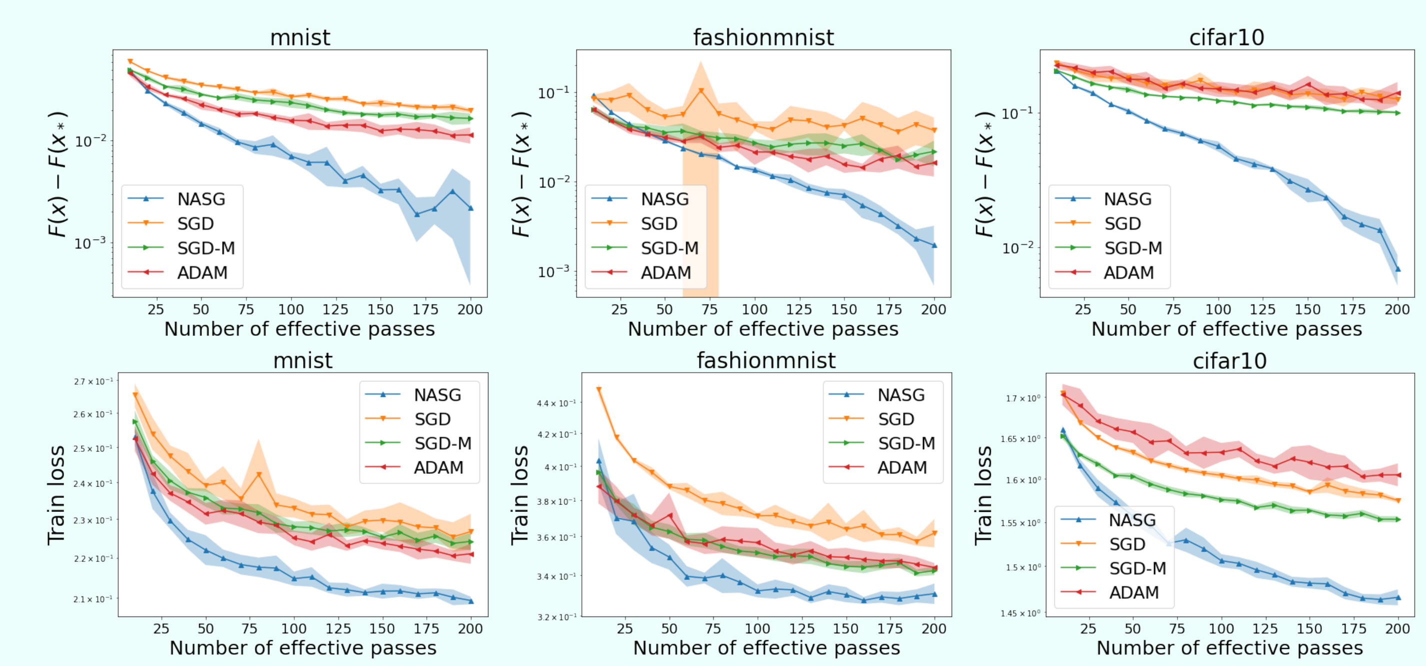
Experiments

We test **NASG method with SGD algorithm, SGD with momentum and ADAM**. Our tests have shown encouraging results for NASG.

(Convex Binary Classification). For the first experiment, we choose a binary classification problem. Below, we show comparisons of loss residual $F(x) - F(x_*)$ (top) and test accuracy (bottom) produced by first-order methods for `w8a`, `ijcnn1` and `covtype` datasets, respectively.



(Convex and Non-convex Image Classification). We test four methods for the second problem: training a neural network to classify images. Our figure below compares the loss residual $F(x) - F(x_*)$ (convex setting, top) and train loss $F(x)$ (non-convex setting, bottom) produced by first-order methods for `MNIST`, `Fashion-MNIST` and `CIFAR-10`, respectively.



Key References

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