Optimization in Finance

Maximum Entropy Distribution of an Asset Inferrred from Option Prices

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Numerical implementation steps

1. Option prices snap from Google Finance API
2. Data cleaning:
   - Calculate mid from bid / ask
   - Keep only relevant option prices (put-call parity/liquidity)
3. Apply the Cross Entropy Minimization algorithm
4. Compute option prices from asset distribution
5. Convert option prices into Black-Scholes Implied Volatility

GOOG.OQ Option quotes

Google Inc (NASDAQ:GOOG)

chg | %

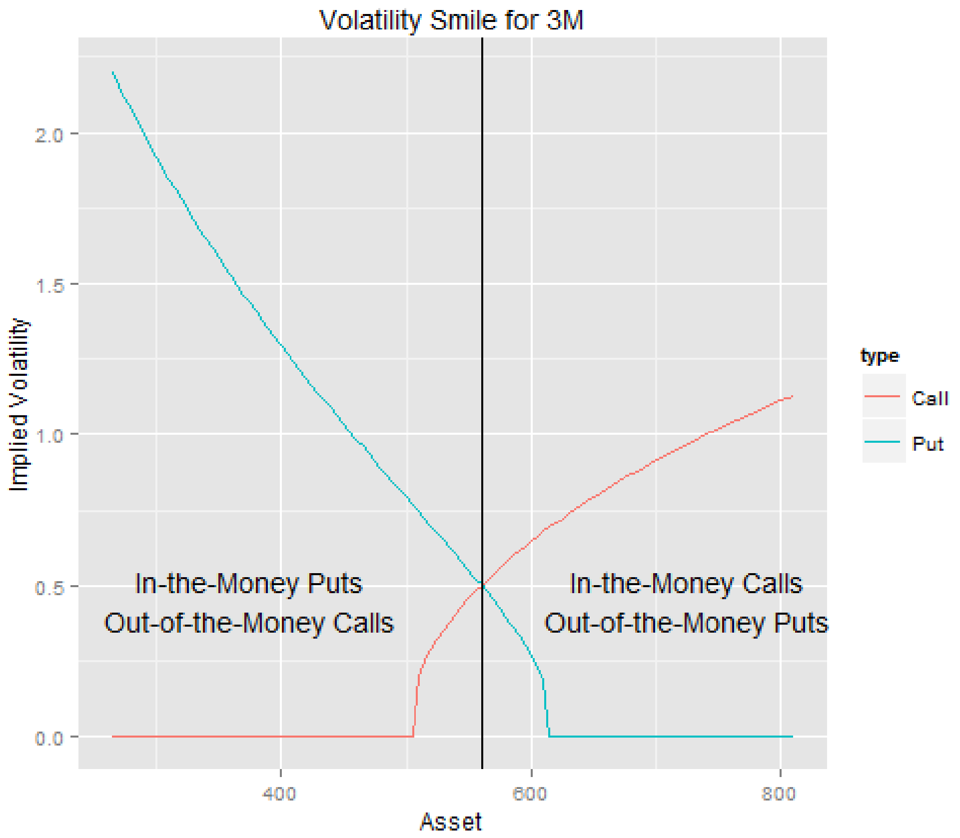
GOOG 555.17 -3.61

View options by expiration Mar 27, 2015 ▼

Calls							Puts					
Price	Change	Bid	Ask	Volume	Open Int	Strike	Price	Change	Bid	Ask	Volume	Open Int
36.90	0.00	19.20	21.50	-	15	535.00	0.05	0.00	0.05	0.10	127	219
18.97	-1.93	16.60	18.80	2	17	537.50	0.11	+0.03	0.05	0.10	37	267
14.20	-13.34	14.50	16.30	10	35	540.00	0.12	-0.13	0.05	0.15	463	306
13.60	-3.45	12.20	13.90	2	73	542.50	0.16	+0.08	0.15	0.25	335	139
8.80	-6.50	9.80	11.60	13	108	545.00	0.30	-0.20	0.20	0.40	379	645
9.40	-6.40	7.60	9.30	5	72	547.50	0.51	-0.09	0.45	0.60	255	193
6.10	-6.00	5.50	6.90	220	161	550.00	1.00	0.00	0.85	1.00	687	385
5.00	-6.20	4.00	4.70	1649	339	552.50	1.69	+0.23	1.50	1.90	607	211
2.85	-3.65	2.60	3.10	975	142	555.00	2.65	+0.65	2.40	2.95	2406	573
1.70	-3.30	1.55	1.80	924	204	557.50	4.03	+1.02	3.30	4.30	417	877
0.90	-2.70	0.80	1.10	2165	872	560.00	5.70	+1.50	5.40	6.10	451	983
0.50	-2.00	0.50	0.60	872	1241	562.50	7.30	+2.01	7.40	8.30	211	809
0.35	-1.15	0.20	0.30	627	2160	565.00	11.25	+4.32	9.00	10.50	77	1106
0.10	-0.89	0.05	0.15	375	960	567.50	12.00	+3.00	11.40	12.90	31	2231
0.05	-0.55	0.05	0.10	716	1812	570.00	14.47	+3.47	13.80	15.40	72	1104
0.05	-0.35	0.05	0.10	471	1631	572.50	18.80	+5.10	16.30	17.80	8	984
0.02	-0.18	0.05	0.05	160	930	575.00	20.10	+6.80	18.80	20.50	12	110
0.05	-0.22	0.05	0.05	27	203	577.50	11.80	0.00	21.30	23.10	-	19

Volatility smile - Put-Call parity

$$C(t) + K \cdot B(t,T) = P(t) + S(t)$$



Cross Entropy Minimization algorithm

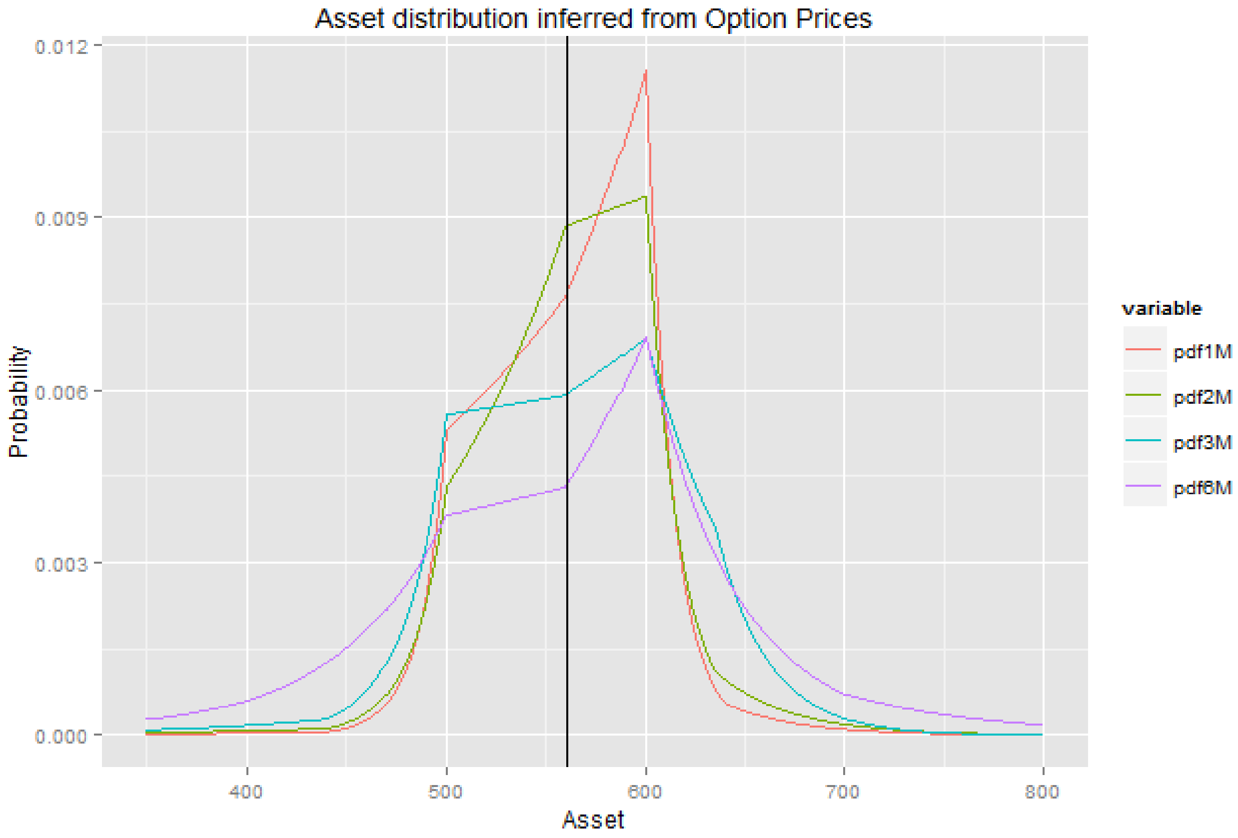
Minimize $S(p,q) = \int_{-\infty}^{\infty} p(x) \log\left[\frac{p(x)}{q(x)}\right] dx$

Subject to:

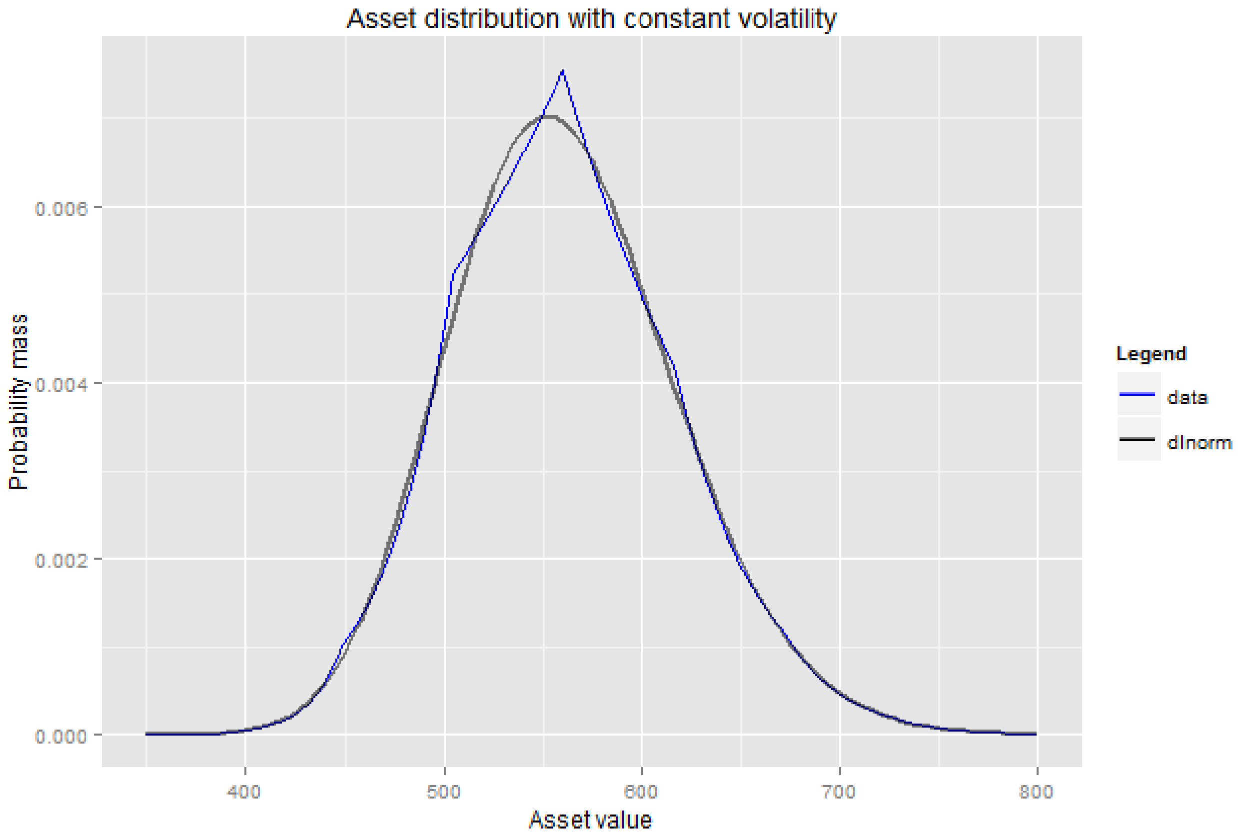
- $\int_0^\infty p(x)dx = 1$
- $\forall i = 1, 2, ..m\ \mathbb{E}[c_i(X)] = \int_0^\infty p(x)\ c_i(x)dx = ci$

1. A uniform distribution is used as a prior (non-informed prior)
2. Integrability constraint
3. The option prices used as constraints should be linearly independant
4. Forward is also used as a constraint
5. Choice of asset step for the discretisation / boundaries
6. The Lagrangian function is minimized using an optimization routine (Limited-memory Broyden-Fletcher-Goldfarb-Shanno)

GOOG.OQ Asset distribution (1m/2m/3m/6m)



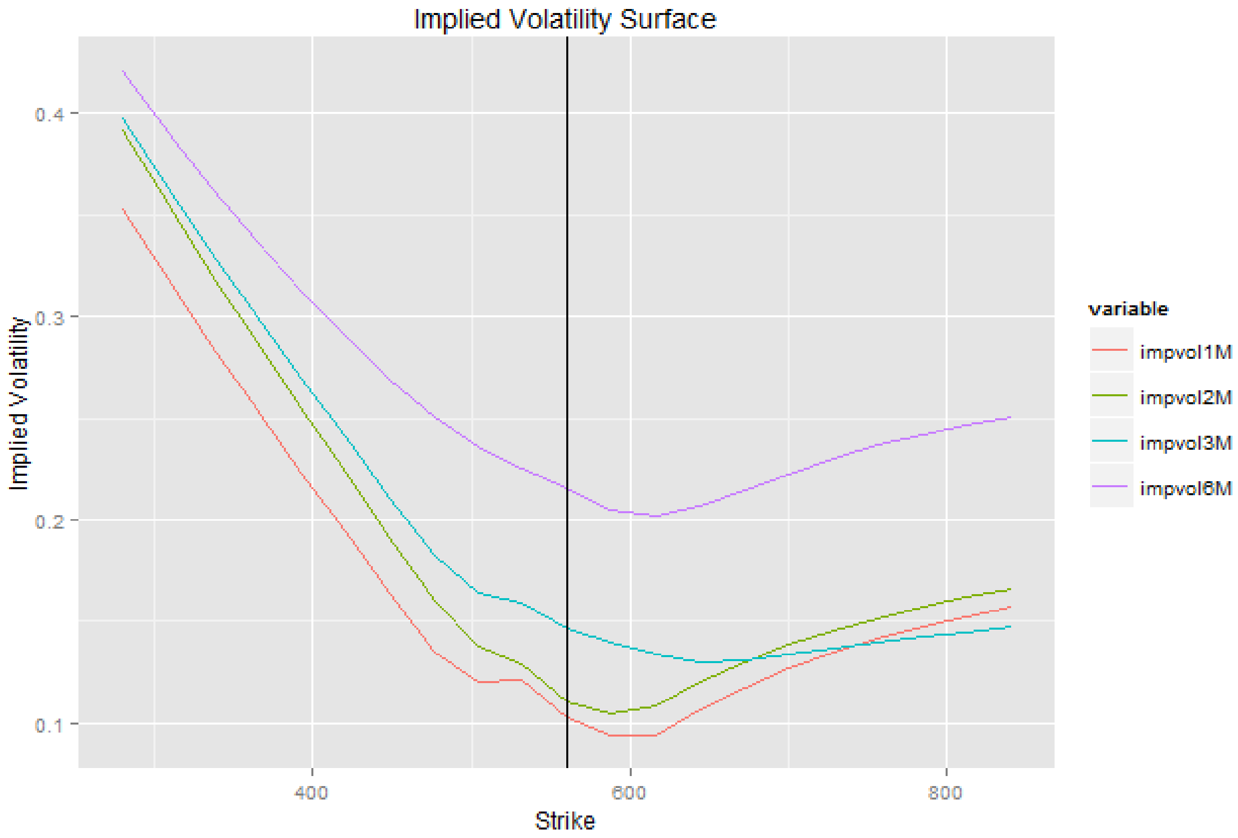
Asset distribution with constant volatility (25%)



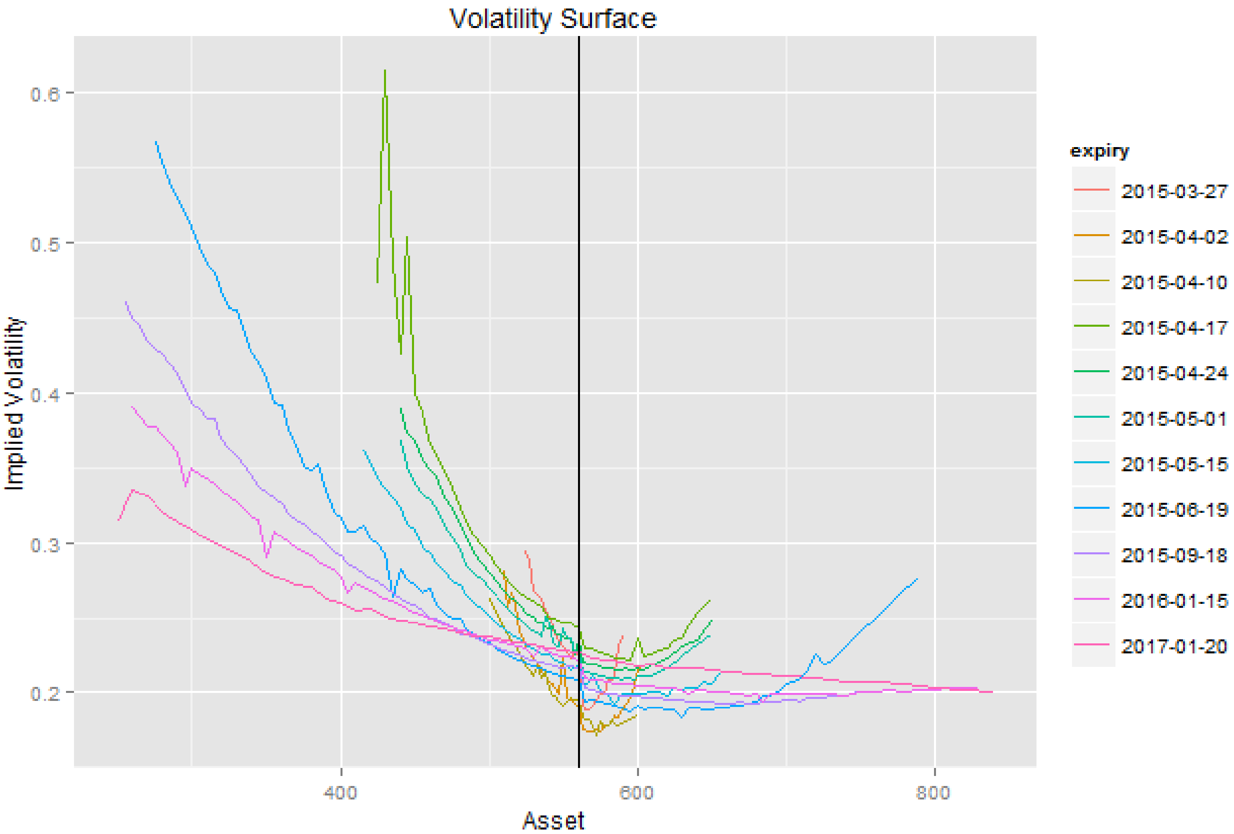
GOOG.OQ Volatility smile (from Asset distribution)

$$c(K) = D(T)\int_0^\infty p(x)\,(x-K)^+\,dx$$

$$p(K) = D(T)\int_0^\infty p(x)\,(K-x)^+\,dx$$



Volatility smile (from Quotes)



Conclusion

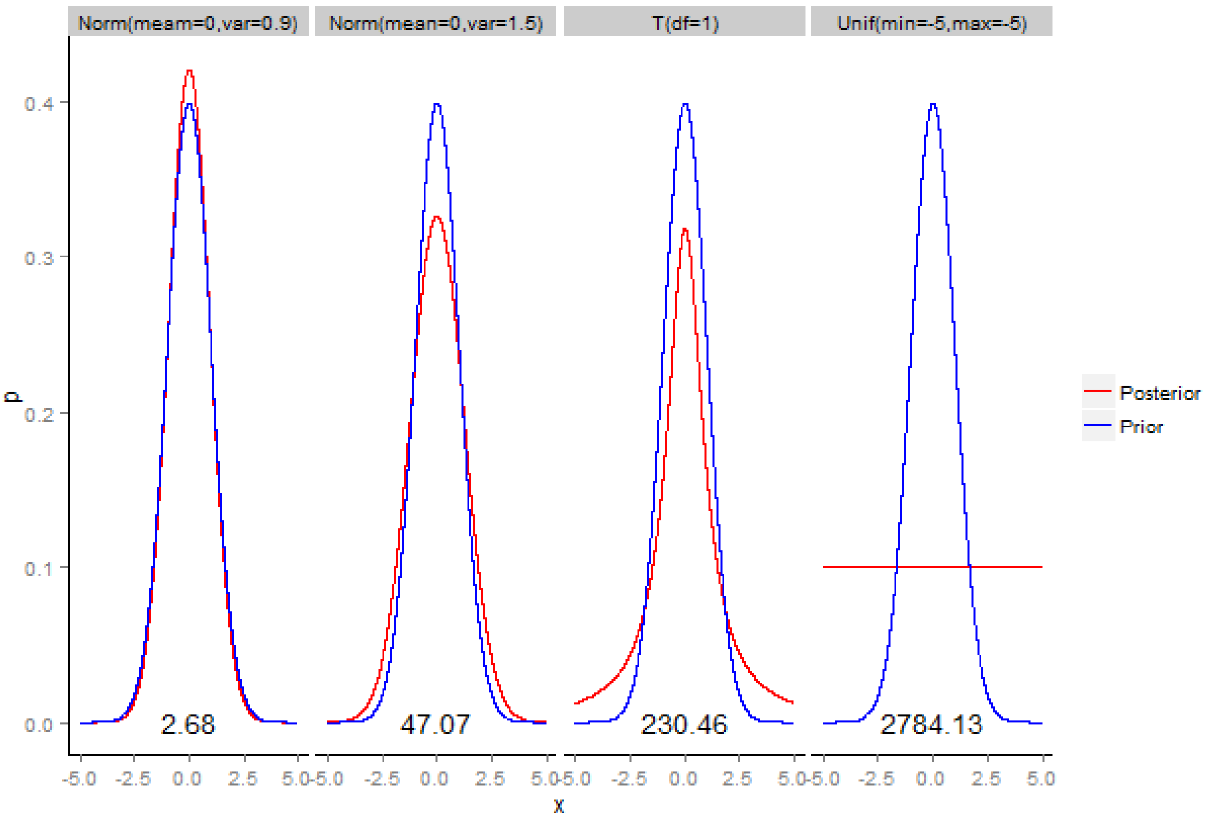
1. We have observed that assets distribution inferred from options prices are not log-normal (i.e. Volatility is not constant)
2. Principle of Minimum Cross-Entropy can be used to estimate the distribution of an asset without any assumption (non-parametric approach)
3. The algorithm is stable and fast
4. This method can be used to perform implied volatility interpolation/extrapolation from only few quotes

Appendix

Kullback-Leibler Relative entropy

1. In Information Theory, Shannon defines the entropy as a measure of unpredictability of information content
2. the KL relative entropy (or KL divergence) is a non-suymetric mesure of the difference between two probability distribution P and Q.
3. $D_{KL}(P \| Q) = \int_{-\infty}^{\infty} p(x) \ln \frac{p(x)}{q(x)} \, dx$
4. Properties:
5. KL is equal to zero if P and Q are identical
6. KL relative entropy is always positive

Calculation of Relative Entropy



Formulating the Optimization problem

We have one prior probability density function $q(x)$

We have m price constraints: $\forall i = 1, 2, ..m \; d_i = D(T)\mathbb{E}_{\mathbb{Q}}[c_i(X_T)]$where

- $D(T) = e^{-(r-q)T}$ represents the discount factor
- $c_i(X_t)$ denotes the ith option pay-off function at expiry depenent only on the asset value at expiry
- d_i is the corresponding option price
- r risk-free rate for T
- q dividend yield

Formulating the Optimization problem (2)

Minimize $S(p,q) = \int_{-\infty}^{\infty} p(x) \log\left[\frac{p(x)}{q(x)}\right] dx$

Subject to 2 constraints:

1. $\int_0^{\infty} p(x) dx = 1$
2. $\forall i = 1, 2, ..m \; \mathbb{E}[c_i(X)] = \int_0^{\infty} p(x) \; c_i(x) dx = ci$

This is a standard constrained optimization problem which solved by using the method of Lagrange which transforms a problem in n variable and m constraints into an unconstrainted optimization with n+m variables.

Objective function

$$H(p) = -\int_0^\infty p(x)\log\left[\frac{p(x)}{q(x)}\right]dx + (1+\lambda_0)\int_0^\infty p(x)dx + \sum_{i=1}^m \lambda_i \int_0^\infty p(x)\ c_i(x)dx$$

From standard calculus, we know that the minimum $\lambda^* = (\lambda_0^*, \ldots, \lambda_M^*)$ is reached when:

- the gradient (vector of derivatives) δH is equal to zero: $\delta H(\lambda^*) = \int_0^\infty \left[-\log\left[\frac{p(x)}{q(x)}\right] + \lambda_0 + \sum_{i=1}^m \lambda_i c_i(x)\right]\delta p(x)dx = 0$ (necessary condition)
- the hessian (matrix of second derivatives) is positive definite (sufficient condition)

Objective function solution

This leads immediately to the following explicit representation of the MED:

$p(x) = \frac{q(x)}{\mu} \exp\left(\sum_{i=1}^{m} \lambda_i c_i(x)\right)$, $\mu = \int_0^\infty q(x) \exp\left(\sum_{i=1}^{m} \lambda_i c_i(x)\right) dx$

Impact of constraints on MXED