# Optimization in Finance

## Maximum Entropy Distribution of an Asset Inferrred from Option Prices

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# Numerical implementation steps

1. Option prices snap from Google Finance API
2. Data cleaning:
   - Calculate mid from bid / ask
   - Keep only relevant option prices (put-call parity/liquidity)
3. Apply the Cross Entropy Minimization algorithm
4. Compute option prices from asset distribution
5. Convert option prices into Black-Scholes Implied Volatility

# GOOG.OQ Option quotes

**Google Inc** (NASDAQ:GOOG)

chg | %

GOOG 548.34 -6.83

View options by expiration Sep 18, 2015

| Calls | | | | | | | Puts | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Price | Change | Bid | Ask | Volume | Open Int | Strike | Price | Change | Bid | Ask | Volume | Open Int |
| 68.70 | -0.30 | 67.40 | 69.70 | 1 | 9 | 495.00 | 12.40 | +3.95 | 12.40 | 13.30 | 1 | 26 |
| 65.10 | -5.40 | 63.50 | 66.00 | 3 | 40 | 500.00 | 13.90 | +2.90 | 13.70 | 14.60 | 29 | 200 |
| 60.29 | 0.00 | 59.90 | 62.50 | - | 26 | 505.00 | 16.50 | 0.00 | 15.00 | 16.00 | - | 31 |
| 73.50 | 0.00 | 56.30 | 58.80 | - | 58 | 510.00 | 16.00 | +3.20 | 16.40 | 17.40 | 3 | 28 |
| 60.78 | 0.00 | 53.10 | 54.60 | - | 10 | 515.00 | 13.00 | 0.00 | 17.90 | 19.00 | - | 32 |
| 58.22 | 0.00 | 49.80 | 51.20 | - | 23 | 520.00 | 19.80 | +1.80 | 19.50 | 20.70 | 30 | 276 |
| 53.40 | 0.00 | 46.60 | 48.00 | - | 36 | 525.00 | 16.90 | 0.00 | 21.30 | 22.50 | - | 68 |
| 49.00 | 0.00 | 43.60 | 44.90 | - | 35 | 530.00 | 23.60 | +5.10 | 23.20 | 24.40 | 1 | 58 |
| 55.70 | 0.00 | 40.70 | 41.80 | - | 14 | 535.00 | 23.10 | 0.00 | 25.20 | 26.50 | - | 33 |
| 40.30 | -2.70 | 37.80 | 38.90 | 1 | 50 | 540.00 | 28.00 | +2.00 | 27.30 | 28.60 | 4 | 111 |
| 36.80 | -3.90 | 35.10 | 36.20 | 6 | 37 | 545.00 | 24.50 | 0.00 | 29.50 | 31.00 | - | 61 |
| 33.20 | -6.50 | 32.60 | 33.40 | 5 | 110 | 550.00 | 31.00 | +1.50 | 32.00 | 33.40 | 8 | 186 |
| 31.54 | -2.16 | 30.10 | 31.00 | 1 | 65 | 555.00 | 34.40 | +1.31 | 34.60 | 35.90 | 2 | 47 |
| 29.50 | -2.41 | 27.90 | 28.60 | 16 | 127 | 560.00 | 37.00 | +2.50 | 37.20 | 38.60 | 4 | 108 |
| 27.00 | -3.19 | 25.60 | 26.40 | 1 | 331 | 565.00 | 39.70 | +9.66 | 40.00 | 41.50 | 4 | 34 |
| 24.50 | -2.90 | 23.50 | 24.30 | 19 | 149 | 570.00 | 42.59 | 0.00 | 42.90 | 44.30 | - | 43 |
| 22.42 | -3.18 | 21.60 | 22.40 | 3 | 217 | 575.00 | 35.70 | 0.00 | 45.90 | 47.30 | - | 86 |
| 20.30 | -2.40 | 19.70 | 20.60 | 1 | 76 | 580.00 | 47.90 | 0.00 | 49.10 | 50.50 | - | 44 |
| 18.82 | -2.88 | 18.00 | 18.80 | 3 | 50 | 585.00 | 50.09 | 0.00 | 52.30 | 53.80 | - | 16 |

# Put-Call parity

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[ln\left(\frac{S}{K}\right) + (r - \frac{\sigma^2}{2})(T-t)\right]$$

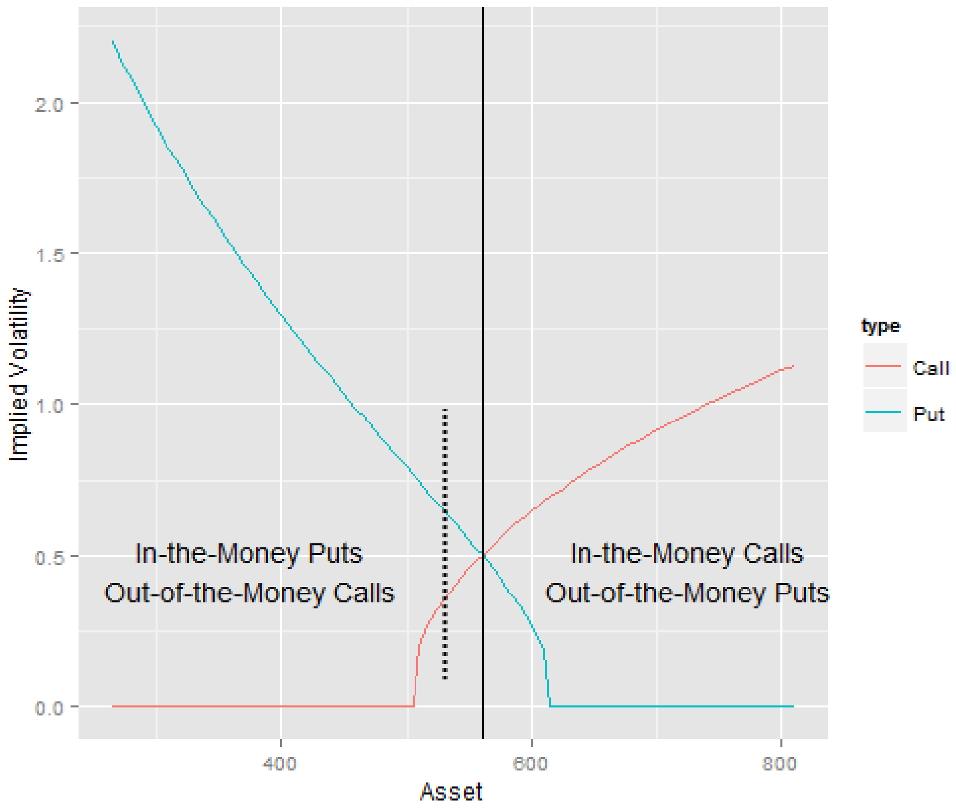
$$d_2 = \frac{1}{\sigma\sqrt{T-t}}\left[ln\left(\frac{S}{K}\right) + (r - \frac{\sigma^2}{2})(T-t)\right]$$

Call option: $C(S,t) = N(d_1)S - N(d_2)Ke^{-r(T-t)}$

Put Option: $P(S,t) = -N(-d_1)S + N(-d_2)Ke^{-r(T-t)}$

Put-Call parity: $C(t) + Ke^{-r(T-t)} = P(t) + S(t)$

# Volatility smile



# Cross Entropy Minimization algorithm

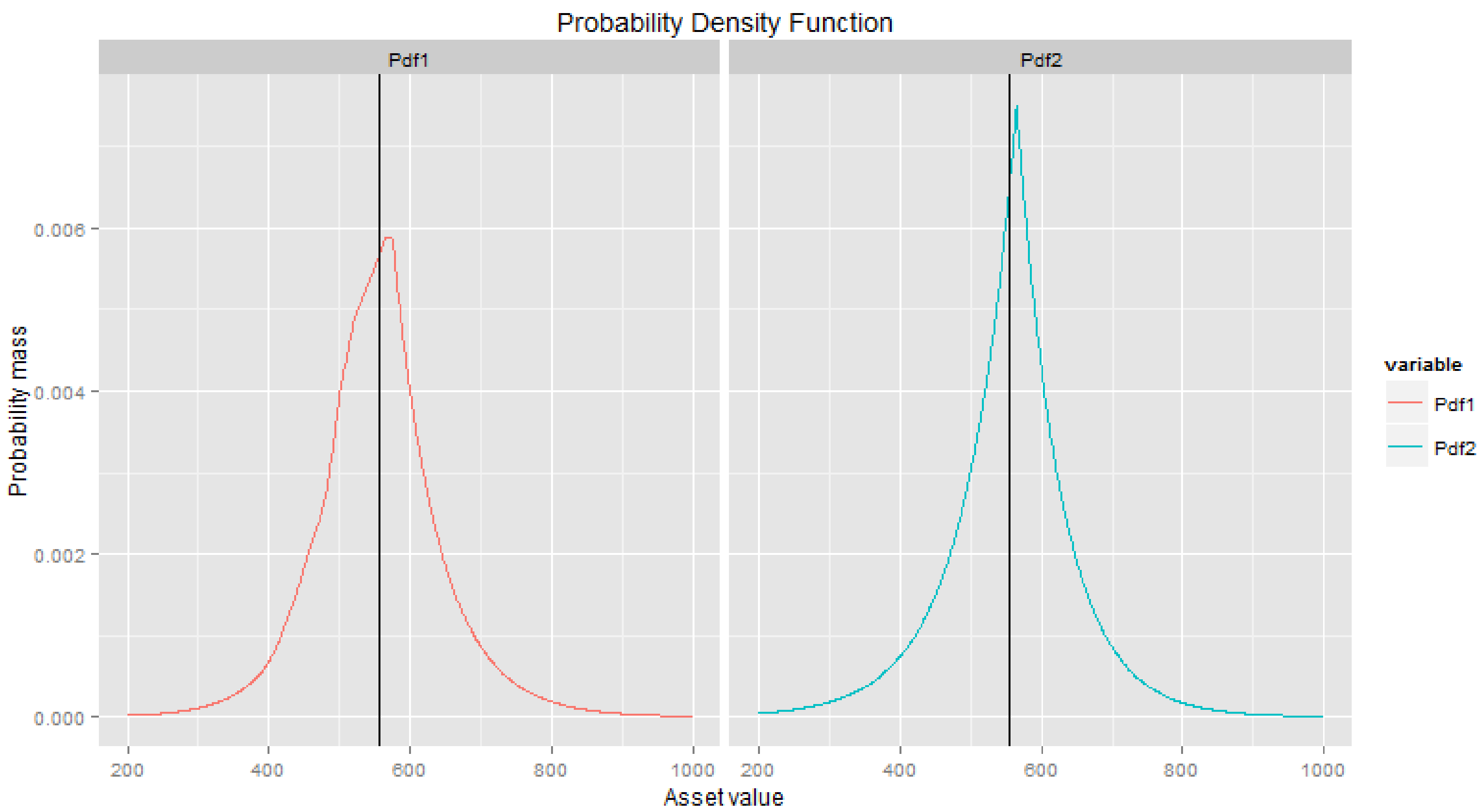
Minimize $S(p,q) = \int_{-\infty}^{\infty} p(x) \log\left[\frac{p(x)}{q(x)}\right] dx$

Subject to:

- $\int_0^{\infty} p(x)dx = 1$
- $\forall i = 1,2,..m \; \mathbb{E}[c_i(X)] = \int_0^{\infty} p(x) \; c_i(x)dx = ci$

1. A uniform distribution is used as a prior (non-informed prior)
2. Integrability constraint
3. Option prices used as constraints should be linearly independant
4. Forward is also used as a constraint (mean)
5. Choice of asset step for the discretisation / boundaries
6. Lagrangian function is minimized using an optimization routine (Limited-memory Broyden-Fletcher-Goldfarb-Shanno)

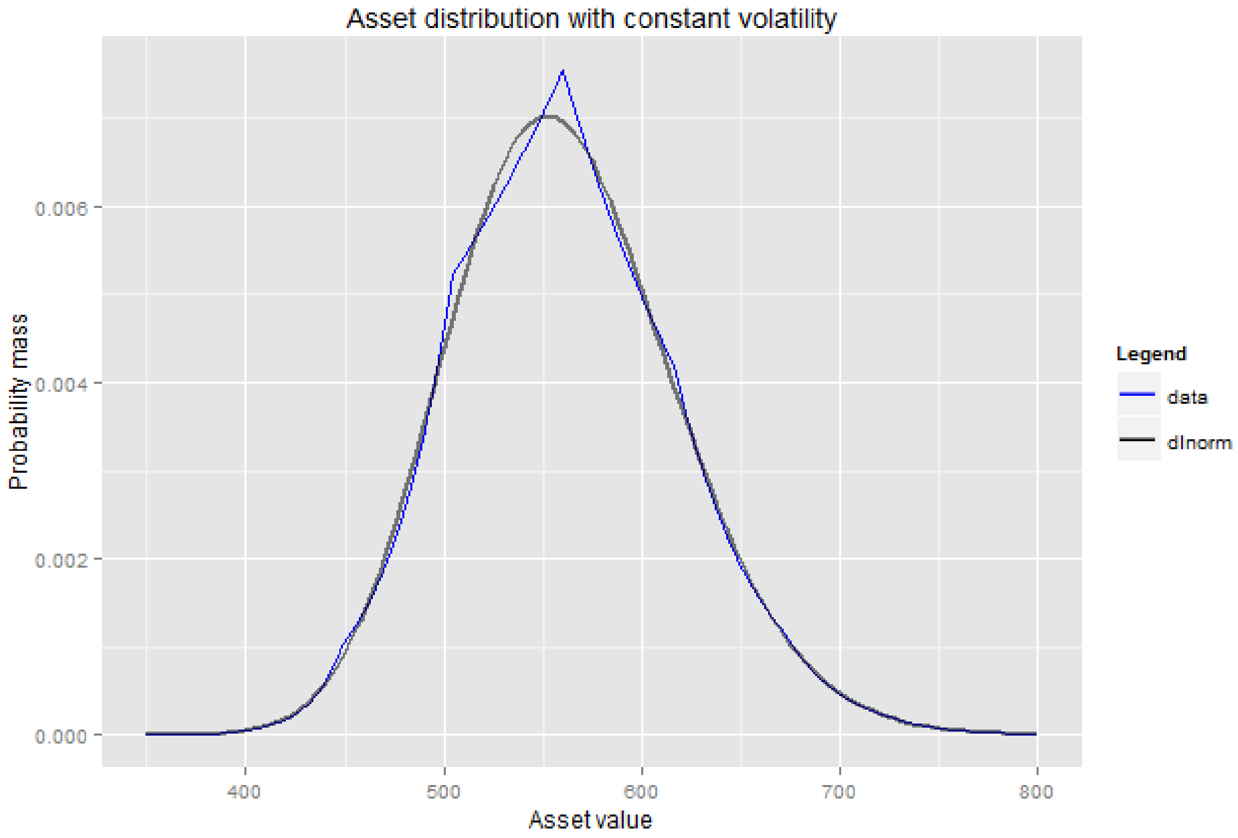
# GOOG.OQ Asset distribution




Pdf1: Asset distribution inferred from 9 option prices

Pdf2: Asset distribution inferred from 1 option price

# Asset distribution with constant volatility (25%)

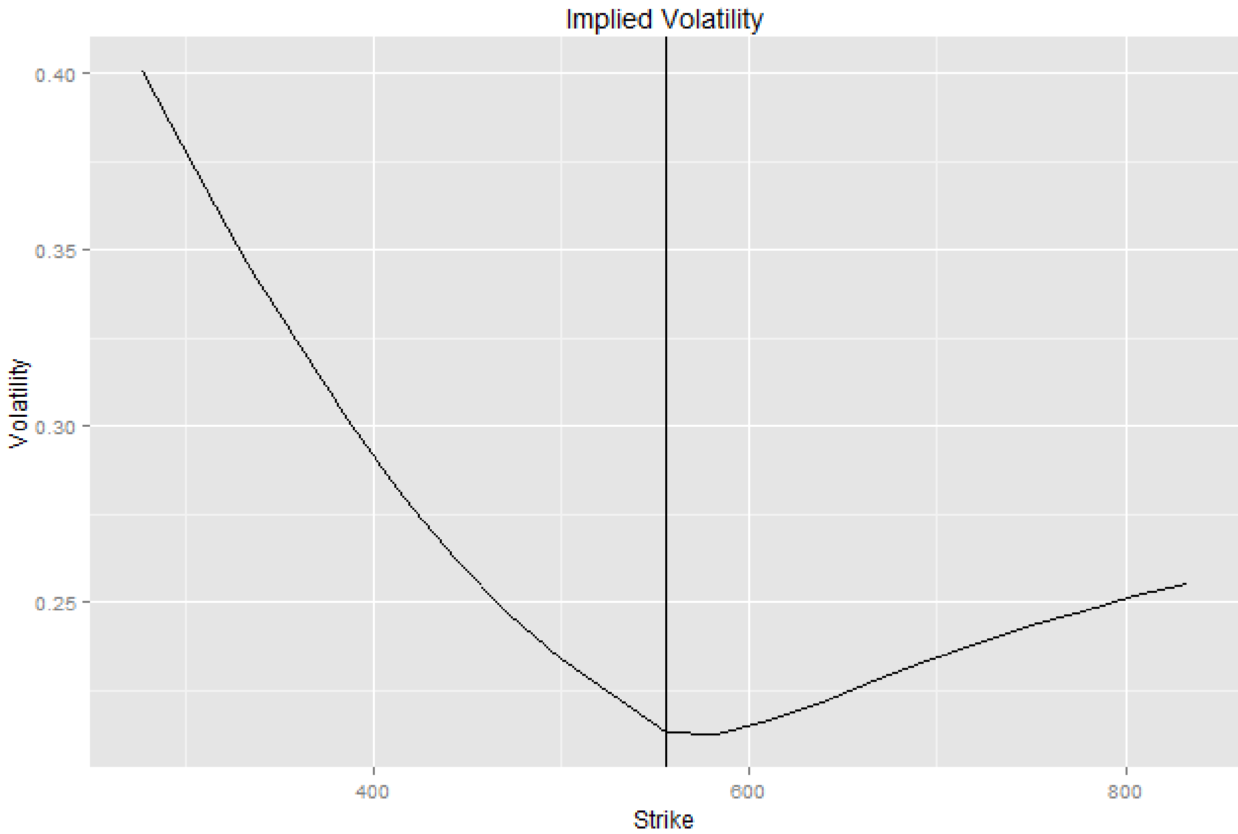



Asset distribution inferred from 6 option prices (constant volatility)

# GOOG.OQ Volatility smile

$c(K) = D(T) \int_0^\infty p(x)\ (x - K)^+ dx$

$p(K) = D(T) \int_0^\infty p(x)\ (K - x)^+ dx$



# Conclusion

1. We have observed that assets distribution inferred from options prices are not log-normal (i.e. Volatility is not constant)
2. Principle of Minimum Cross-Entropy can be used to estimate the distribution of an asset without assuming a particular distribution shape or family of distribution (non-parametric approach)
3. The algorithm is stable and fast
4. This method can be used to perform implied volatility interpolation/extrapolation from only few quotes

# Thank You

# References

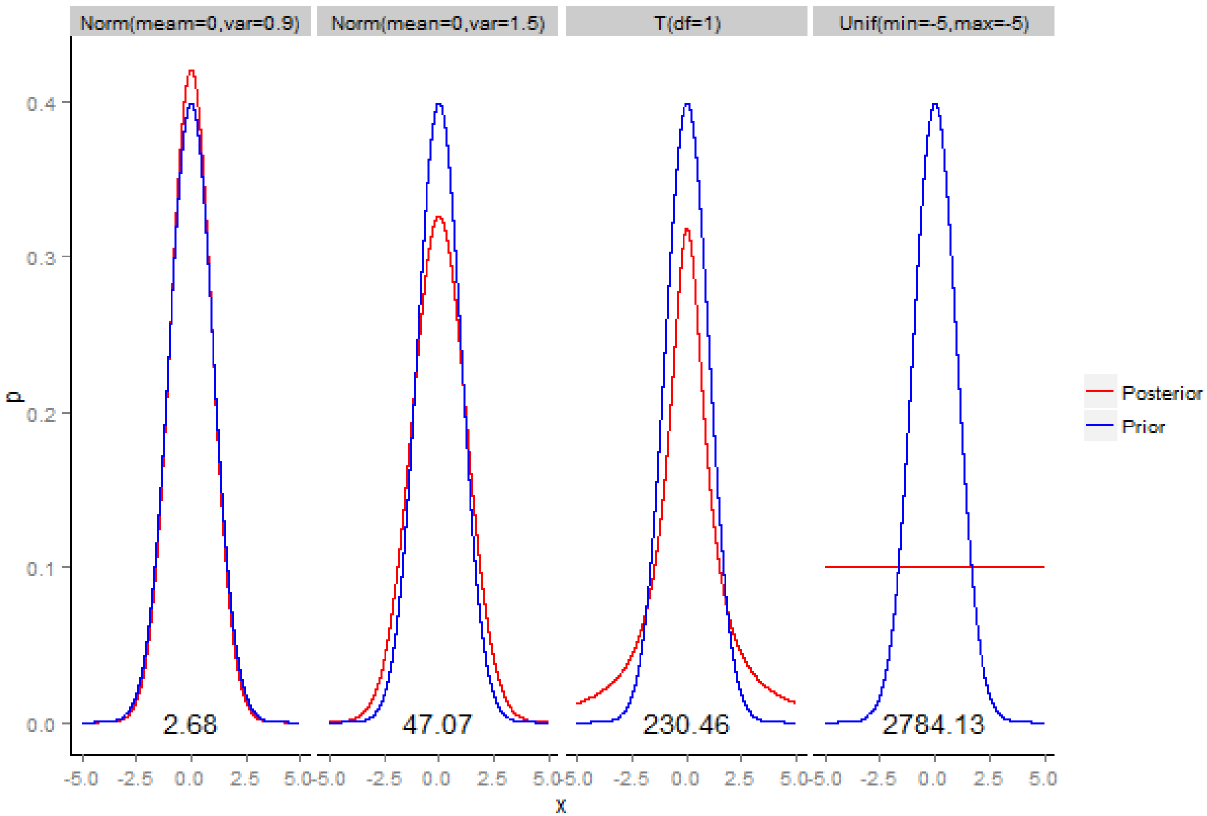

- [BuchenKelly96] The Maximum Entropy Distribution of an Asset Inferred from Option Prices (Peter W. Buchen, Michael Kelly)

# Appendix

# Kullback-Leibler Relative entropy

1. In Information Theory, Shannon defines the entropy as a measure of unpredictability of information content
2. the KL relative entropy (or KL divergence) is a non-suymetric mesure of the difference between two probability distribution P and Q.
3. $D_{KL}(P\|Q) = \int_{-\infty}^{\infty} p(x) \ln \frac{p(x)}{q(x)} \, dx$
4. Properties:
5. KL is equal to zero if P and Q are identical
6. KL relative entropy is always positive

# Calculation of Relative Entropy



# Formulating the Optimization problem

We have one prior probability density function $q(x)$

We have $m$ price constraints: $\forall i = 1, 2, .. m \; d_i = D(T)\mathbb{E}_{\mathbb{Q}}[c_i(X_T)]$where

- $D(T) = e^{-(r-q)T}$ represents the discount factor
- $c_i(X_t)$ denotes the $i$th option pay-off function at expiry depenent only on the asset value at expiry
- $d_i$ is the corresponding option price
- $r$ risk-free rate for $T$
- $q$ dividend yield

# Formulating the Optimization problem (2)

Minimize $S(p,q) = \int_{-\infty}^{\infty} p(x) \log\left[\frac{p(x)}{q(x)}\right] dx$

Subject to 2 constraints:

1. $\int_0^\infty p(x)dx = 1$
2. $\forall i = 1, 2, .. m\ \mathbb{E}[c_i(X)] = \int_0^\infty p(x)\ c_i(x)dx = ci$

This is a standard constrained optimization problem which solved by using the method of Lagrange which transforms a problem in n variable and m constraints into an unconstrainted optimization with n+m variables.

# Objective function

$$H(p) = -\int_0^\infty p(x)\log\left[\frac{p(x)}{q(x)}\right]dx + (1+\lambda_0)\int_0^\infty p(x)dx + \sum_{i=1}^m \lambda_i \int_0^\infty p(x)\; c_i(x)dx$$

From standard calculus, we know that the minimum $\lambda^* = (\lambda_0^*, \ldots, \lambda_M^*)$ is reached when:

- the gradient (vector of derivatives) $\delta H$ is equal to zero: $\delta H(\lambda^*) = \int_0^\infty \left[-\log\left[\frac{p(x)}{q(x)}\right] + \lambda_0 + \sum_{i=1}^m \lambda_i c_i(x)\right]\delta p(x)dx = 0$ (necessary condition)
- the hessian (matrix of second derivatives) is positive definite (sufficient condition)

# Objective function solution

This leads immediately to the following explicit representation of the MED:

$p(x) = \frac{q(x)}{\mu} \exp\left(\sum_{i=1}^{m} \lambda_i c_i(x)\right)$, $\mu = \int_0^\infty q(x) \exp\left(\sum_{i=1}^{m} \lambda_i c_i(x)\right) dx$

# Impact of constraints on MXED