4. Random processes and Markov chains.

a) Fix a parameter $\lambda \in (0, 1)$ and let $X_0, X_1, X_2, \ldots$ be a sequence of independent random variables, whose distribution satisfies $P(X_j = -1) = P(X_j = 1) = 1/2$. Consider the following random sequence

$$Y_n = \sum_{i=0}^{n} X_i \lambda^i, \qquad n = 0,1,2,3,\ldots$$

i) Show $\{Y_n\}$ is a martingale. [3]

ii) Derive the characteristic function of Y_n. [4]

iii) Show that for any set E

$$P(Y_{n+1} \in E) = \frac{1}{2} P(Y_n \in T_1^{-1}(E)) + \frac{1}{2} P(Y_n \in T_2^{-1}(E))$$

where $T_1(x) = \lambda x + 1$ and $T_2(x) = \lambda x - 1$. [4]

iv) What is the limiting distribution of Y_n as $n \to \infty$ if $\lambda = 1/2$? [4]

b) Consider the random walk with left barrier as in Lecture 9 with infinite state space $E = \{0,1,2,\ldots\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1 & & & & 0 \\ q & 0 & p & & & \\ & q & 0 & p & & \\ & & \cdot & \cdot & \cdot & \\ 0 & & \cdot & \cdot & \cdot & \end{pmatrix}$$

where $0 < p < 1$, $q = 1 - p$. Write a computer program to simulate the random walk and show the realizations of $X(t)$ as a function of t, for

i) $p = 1/3$; [3]

ii) $p = 1/2$; [3]

iii) $p = 2/3$. [3]

iv) Discuss your findings. [3]

[Only show the simulation results as in Lecture 9 (not the program). Obviously, such a question cannot be tested in this way in the exam!]