Exercise 3:

3. Absence of arbitrage (6 points) Consider asset price dynamics of the form

$$dS_{t,i} = \mu_i S_{t,i} dt + \sigma_i S_{t,i} dW_t, \quad i = 1, 2$$

(the two price processes are driven by the same Brownian motion). Assume that the numeraire is of the form $S_{t,0} = e^{rt}$ for a constant $r \geq 0$.

a) Show that the model is arbitrage-free if $\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}$.

b) Assume that $r = 0$ and that $\frac{\mu_1}{\sigma_1} > \frac{\mu_2}{\sigma_2}$. Show that in that case there is a selffinancing strategy ϕ with value $dV_t^\phi = c_t dt$ for some $c_t > 0$ (and hence an arbitrage opportunity). (The restriction to the case $r = 0$ serves to keep the example simple).

We have 2 stocks who follow the following SDE:

$$dS_t^i = \mu S_t^i dt + \sigma S_t^i dW_t \tag{1}$$

a) To show that the market is arbitrage free we have to show that the market admits an equivalent martingale measure. We will show this by finding one.

The discounted value process is a Q-martingale if and only if $S_t^1 = S_0^1 + \int_0^t r S_s^1 ds + M_t^Q$ with M_t^Q being a Q-local martingale.
We define $\tilde{W}_t = W_t + \int_0^t \frac{\mu - r}{\sigma} ds$ and use Girsanov to get Q under which $\tilde{W}$ is again a brownian motion.

$$\frac{dQ}{dP} = exp(-\frac{\mu_i - r}{\sigma_i} W_T - \frac{1}{2}(\frac{\mu_i - r}{\sigma_i})^2 T) \tag{2}$$

As we can see the unique martingale measure only depends on $\frac{\mu_i - r}{\sigma_i}$. As this expression is equal for our two assets they are both martingales under Q. From the existence of a martingale measure it follows that there can be no arbitrage.

b) In the given example the market value of risk is higher for the second asset. Intuitively both assets move together due to brownian motion that drives both processes. But asset 1 has a higher drift. This means for our strategy we buy asset 1 and sell asset 2. We assume $S_{0,1} = S_{0,2}$ and buy $\frac{1}{\sigma_1}$ of asset 1 and sell $\frac{1}{\sigma_2}$ units of asset 2. Then we get:

$$dV_t^\phi = \frac{1}{\sigma_1} dS_{t,1} - \frac{1}{\sigma_2} dS_{t,2} \tag{3}$$

$$dV_t^\phi = \frac{\mu_1}{\sigma_1} dt - \frac{\mu_2}{\sigma_2} dt + dW_t - dW_t = c_t dt \tag{4}$$