Exercise 1:

1. Spot- and forward rates. (4 points) Consider some arbitrage-free term-structure model where bonds $p(t,T)$, $t \leq T$ of arbitrary maturities are traded. Denote by Q^T the measure corresponding to the numeraire $p(t,T)$ (this measure is known as T-forward measure). Show that for $S > T$ the forward price of the S-bond $p(t,S)/p(t,T)$, $0 \leq t \leq T$ is a Q^T martingale. Use this to show that the instantaneous forward rate satisfies the relation

$$f(t,T) = E^{Q^T}(r_T \mid \mathcal{F}_t);$$

in particular, $f(\cdot,T)$ is a Q^T-martingale.

Exercise 2:

2. Moment generating function in the Heston model. The Heston stochastic volatility model for the logarithmic stock price $Y_t = \ln S_t$ and the instantaneous variance V_t has dynamics

$$\begin{aligned} dY_t &= (r - \frac{1}{2}V_t)dt + \sqrt{V_t}dW_{t,1} \\ dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_{t,2} \end{aligned}$$

for two Brownian motions with $\langle W_1, W_2\rangle_t = \rho t$. Show that the conditional moment generating function

$$\psi(u_1,u_2) = E\left(\exp(-u_1Y_T - u_2V_T) \mid Y_t = y, V_t = v\right)$$

is of the form $\exp\left(a(t,T) + b_1(t,T)y + b_2(t,T)v\right)$ and derive an ODE-system for a, b_1 and b_2. Hint: use similar arguments as in the analysis of the affine short-rate models.