

## Exercise 1:

1. (6 points) Consider some asset value process  $V_t$  that solves the SDE  $dV_t = \mu V_t dt + \sigma V_t dW_t$  for constants  $\mu \in \mathbb{R}$ ,  $\sigma > 0$  and a Brownian motion  $W$  on some  $(\Omega, \mathcal{F}, Q)$ . Suppose that we want to price a dividend stream of the form  $D_T = \int_0^T d(V_s) ds$  for a bounded function  $d(v)$ , that is we want to compute the conditional expectation

$$E_{(t, V_t)}^Q \left( \int_t^T e^{-r(s-t)} d(V_s) ds \right) \text{ for some } r \geq 0.$$

Suppose that the bounded function  $F: [0, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}$  satisfies the PDE

$$F_t(t, v) + \mu v F_v(t, v) + \frac{1}{2} \sigma^2 v^2 F_{vv}(t, v) + d(t, v) = r F(t, v)$$

with terminal condition  $F(T, v) \equiv 0$ . Show in analogy to the proof of Feynman Kac that

$$F(t, V_t) = E_{(t, V_t)}^Q \left( \int_t^T e^{-r(s-t)} d(V_s) ds \right).$$

Since the Feynman-Kac Formula works in both directions, we choose to show that from the conditional expectation one can derive the PDE.

$$\begin{aligned} F(t, V_t) &= E_{(t, V_t)}^Q \left( \int_t^T e^{-r(s-t)} d(V_s) ds \right) \\ \text{adding to both sides } E_{(t, V_t)}^Q \left( \int_0^t e^{-r(s-0)} d(V_s) ds \right) \\ F(t, V_t) + E_{(t, V_t)}^Q \left( \int_0^t e^{-r(s-0)} d(V_s) ds \right) &= E_{(t, V_t)}^Q \left( \underbrace{\int_0^T e^{-r(s-0)} d(V_s) ds}_Z \right) \end{aligned}$$

since  $Z$  does not depend on  $t$  we can conclude from the properties of conditional expectation that it is a martingale hence the left hand side is also a martingale. We therefore can apply Ito's Lemma to the left hand side and end up with the PDE given the condition that  $F(T, v) = 0$ .

## Exercise 2:

**2. MLE for the OU-process.** (6 points) Suppose that  $X$  follows an Ornstein-Uhlenbeck process with dynamics  $dX_t = \kappa(\theta - X_t)dt + dW_t$  for a Brownian motion  $W$  and a speed of mean reversion  $\kappa > 0$ . Use the Girsanov theorem to derive the maximum likelihood estimator for the unknown parameter  $\theta \in \mathbb{R}$  (the mean-reversion level), given an observed trajectory  $\hat{X}_t$ ,  $0 \leq t \leq T$ .

The OU-process satisfies the following SDE:

$$dX_t = \kappa(\theta - X_t)dt + dW_t \tag{1}$$

The distribution of  $X_t$  is  $P^\mu$  and depends on  $\mu$ . To find the MLE of  $\theta$  we have to maximize the likelihood of our process  $L(X; \mu)$ . Suppose there is measure such that  $P^\mu \sim \tilde{P}$  and

$$\frac{dP^\mu}{\tilde{P}} = L(X; \mu) \tag{2}$$

We choose  $\tilde{P}$  equal to the Wiener measure and we get:

$$\frac{dP^\mu}{\tilde{P}} = \exp(\kappa(\theta - X_t)X_t - \frac{1}{2}\kappa^2(\theta - X_t)^2t) \quad (3)$$

To maximize the exponential is to maximize its argument. We derive w.r.t.  $\theta$  and set equal to zero:

$$\kappa\hat{X}_t - \kappa^2(\theta - \hat{X}_t)t = 0 \quad (4)$$

$$\frac{\hat{X}_t}{\kappa} = \theta - \hat{X}_t \quad (5)$$

$$\hat{\theta}_{MLE} = \frac{\hat{X}_t}{\kappa} + \hat{X}_t = \hat{X}_t(1 + \frac{1}{\kappa}) \quad (6)$$

### Exercise 3:

**3. Positive local martingales.** (2 points) Consider a strictly positive local martingale  $Z$  with continuous trajectories. Show that  $Z$  can be written in the form  $Z_t = \exp(M_t - \frac{1}{2}[M]_t)$  for a suitable local martingale  $M$ .

Consider  $M$  as an Ito integral  $M = \int_0^t \theta_s dW_s$ . Then we have

$$\begin{aligned} [M] &= [M, M] \\ &= \left[ \int_0^t \theta_s dW_s, \int_0^t \theta_s dW_s \right] \\ &= \int_0^t \theta_s^2 d[W, W]_s \\ &= \int_0^t \theta_s^2 ds \end{aligned}$$

Therefore

$$\exp(M - \frac{1}{2}[M]) = \exp\left(\int_0^t \theta_s dW_s - \frac{1}{2} \int_0^t \theta_s^2 ds\right)$$

Now we can apply Girsanov Theorem and know  $\exp(M - \frac{1}{2}[M])$  is a local martingale.