Exercise 1:

1. (6 points) Consider some asset value process V_t that solves the SDE $dV_t = \mu V_t dt + \sigma V_t dW_t$ for constants $\mu \in \mathbb{R}$, $\sigma > 0$ and a Brownian motion W on some $(\Omega, \mathcal{F}, Q)$. Suppose that we want to price a dividend stream of the form $D_T = \int_0^T d(V_s)ds$ for a bounded function $d(v)$, that is we want to compute the conditional expectation

$$E^Q_{(t,V_t)}\Big(\int_t^T e^{-r(s-t)} d(V_s)\,ds\Big) \text{ for some } r \geq 0.$$

Suppose that the bounded function $F\colon [0,T] \times \mathbb{R}^+ \to \mathbb{R}$ satisfies the PDE

$$F_t(t,v) + \mu v F_v(t,v) + \frac{1}{2}\sigma^2 v^2 F_{vv}(t,v) + d(t,v) = rF(t,v)$$

with terminal condition $F(T,v) \equiv 0$. Show in analogy to the proof of Feynman Kac that

$$F(t,V_t) = E^Q_{(t,V_t)}\Big(\int_t^T e^{-r(s-t)} d(V_s)\,ds\Big).$$

Since the Feynman-Kac Formula works in both directions, we choose to show that from the conditional expectation one can derive the PDE.

$$F\left(t, V_t\right) = E^Q_{(t,V_t)}\left(\int_t^T e^{-r(s-t)} d(V_s) ds\right)$$

$$\text{adding to both sides } E^Q_{(t,V_t)}\left(\int_0^t e^{-r(s-0)} d(V_s) ds\right)$$

$$F\left(t, V_t\right) + E^Q_{(t,V_t)}\left(\int_0^t e^{-r(s-0)} d(V_s) ds\right) = E^Q_{(t,V_t)}\left(\underbrace{\int_0^T e^{-r(s-0)} d(V_s) ds}_{Z}\right)$$

since Z does not debend on t we can conclude from the properties of conditional expectation that it is a martingale hance the left hand side is also a martingale. We therefore can apply Ito's Lemma to the left hand side and end up with the PDE given the condition that $F(T,v) = 0$.

$$F_t\left(t,v\right) + \mu v F_v\left(t,v\right) + \frac{1}{2}\sigma^2 v^2 F_{vv}\left(t,v\right) + d\left(t,v\right) - rF\left(t,v\right) = 0$$

Exercise 2:

2. MLE for the OU-process. (6 points) Suppose that X follows an Ornstein-Uhlenbeck process with dynamics $dX_t = \kappa(\theta - X_t)dt + dW_t$ for a Brownian motion W and a speed of mean reversion $\kappa > 0$. Use the Girsanov theorem to derive the maximum likelihood estimator for the unknown parameter $\theta \in \mathbb{R}$ (the mean-reversion level), given an observed trajectory $\hat{X}_t$, $0 \leq t \leq T$.

The OU-process satisfies the following SDE:

$$dX_t = \kappa(\theta - X_t)dt + dW_t \tag{1}$$

The distribution of X_t is P^μ and depends on μ. To find the MLE of θ we have to maximize the likelihood of our process $L(X;\mu)$. Suppose there is measure such that $P^\mu \sim \tilde{P}$ and

$$\frac{dP^\mu}{\tilde{P}} = L(X;\mu) \tag{2}$$

We choose $\tilde{P}$ equal to the Wiener measure and we get:

$$\frac{dP^\mu}{\tilde{P}} = exp(\kappa(\theta - X_t)X_t - \frac{1}{2}\kappa^2(\theta - X_t)^2 t) \tag{3}$$

To maximize the exponential is to maximize its argument. We derive w.r.t. θ and set equal to zero:

$$\kappa\hat{X}_t - \kappa^2(\theta - \hat{X}_t)t = 0$$
$$\frac{\hat{X}_t}{\kappa t} = \theta - \hat{X}_t$$
$$\hat{\theta}_{MLE} = \frac{\hat{X}_t}{\kappa t} + \hat{X}_t = \hat{X}_t(1 + \frac{1}{\kappa})$$

Exercise 3:

3. Positive local martingales. (2 points) Consider a strictly positive local martingale Z with continuous trajectories. Show that Z can be written in the form $Z_t = \exp(M_t - \frac{1}{2}[M]_t)$ for a suitable local martingale M.

$$Z_t = exp(M_t - \frac{1}{2}[M]_t)$$

For strictly positive processes ln and exp are bijective transformations. We use Ito's formula:

$$exp(ln(Z_t)) = exp(ln(Z_0) + \int_0^t \frac{1}{Z_s} dZ_s - \frac{1}{2}\int_0^t \frac{1}{(Z_s)^2} d[Z]_s)$$
$$Z_t = exp(ln(Z_0) + \int_0^t \frac{1}{Z_s} dZ_s - \frac{1}{2}[\int_0^\cdot \frac{1}{Z_s} dZ_s]_t)$$

Integrals with respect to a martingale are martingales. We can therefore conclude that:

$$M_t = \int_0^t \frac{1}{Z_s} dZ_s$$