

Exercise 1:

1. (6 points) Consider some asset value process V_t that solves the SDE $dV_t = \mu V_t dt + \sigma V_t dW_t$ for constants $\mu \in \mathbb{R}$, $\sigma > 0$ and a Brownian motion W on some (Ω, \mathcal{F}, Q) . Suppose that we want to price a dividend stream of the form $D_T = \int_0^T d(V_s) ds$ for a bounded function $d(v)$, that is we want to compute the conditional expectation

$$E_{(t, V_t)}^Q \left(\int_t^T e^{-r(s-t)} d(V_s) ds \right) \text{ for some } r \geq 0.$$

Suppose that the bounded function $F: [0, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfies the PDE

$$F_t(t, v) + \mu v F_v(t, v) + \frac{1}{2} \sigma^2 v^2 F_{vv}(t, v) + d(t, v) = r F(t, v)$$

with terminal condition $F(T, v) \equiv 0$. Show in analogy to the proof of Feynman Kac that

$$F(t, V_t) = E_{(t, V_t)}^Q \left(\int_t^T e^{-r(s-t)} d(V_s) ds \right).$$

Exercise 2:

2. **MLE for the OU-process.** (6 points) Suppose that X follows an Ornstein-Uhlenbeck process with dynamics $dX_t = \kappa(\theta - X_t)dt + dW_t$ for a Brownian motion W and a speed of mean reversion $\kappa > 0$. Use the Girsanov theorem to derive the maximum likelihood estimator for the unknown parameter $\theta \in \mathbb{R}$ (the mean-reversion level), given an observed trajectory \hat{X}_t , $0 \leq t \leq T$.

Exercise 3:

3. **Positive local martingales.** (2 points) Consider a strictly positive local martingale Z with continuous trajectories. Show that Z can be written in the form $Z_t = \exp(M_t - \frac{1}{2}[M]_t)$ for a suitable local martingale M .

$$Z_t = \exp(M_t - \frac{1}{2}[M]_t) \tag{1}$$

We use Ito's formula

$$M_t - \frac{1}{2}[M]_t = \ln(Z_0) + \int_0^t \frac{1}{Z_s} dZ_s - \frac{1}{2} \int_0^t \frac{1}{(Z_s)^2} d[Z]_s \tag{2}$$

$$M_t - \frac{1}{2}[M]_t = \ln(Z_0) + \ln(Z_t) - \ln(Z_0) - \frac{1}{2} \left[\int_0^t \frac{1}{Z_s} dZ_s \right]_t \tag{3}$$

$$M_t - \frac{1}{2}[M]_t = \ln(Z_t) - \frac{1}{2}[\ln(Z)]_t \tag{4}$$

From this we conclude that one solution is $M_t = \ln(Z_t)$.