Exercise 1:

1. Spot- and forward rates. (4 points) Consider some arbitrage-free term-structure model where bonds $p(t,T)$, $t \leq T$ of arbitrary maturities are traded. Denote by Q^T the measure corresponding to the numeraire $p(t,T)$ (this measure is known as T-forward measure). Show that for $S > T$ the forward price of the S-bond $p(t,S)/p(t,T)$, $0 \leq t \leq T$ is a Q^T martingale. Use this to show that the instantaneous forward rate satisfies the relation

$$f(t,T) = E^{Q^T}(r_T \mid \mathcal{F}_t);$$

in particular, $f(\cdot,T)$ is a Q^T-martingale.

From the fundamental theorem of calculus we get following relationship between $f(t,\cdot)$ and $p(t,)$

$$p(t,T) = \exp\left(-\int_t^T f(t,u)du\right) \text{ see lecture notes (8.1)}$$

This implies

$$f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T}$$

since the instantaneous short-rate of interest is $r(t) = f(t,t)$
to show that $f(t,T) = \mathbb{E}^{Q^T}(f(T,T))$ it is sufficient to show

$$-\frac{\partial \ln p(t,T)}{\partial T} = \mathbb{E}^{Q^T}(r(T))$$

the price process of the zero coupon dicounted with the money market acount $B_t = \exp(\int_0^t r_s ds)$ under the bond price measure is a martingale

$$\frac{p(t,T)}{B_t} = \mathbb{E}^{Q^B}\left(\frac{p(T,T)}{B_T}\right)$$

since the price of a zero coupn bond at maturity is 1. We can rewrite this as follows

$$p(t,T) = \mathbb{E}^{Q^B}\left(\frac{B_t}{B_T}\right) = \mathbb{E}^{Q^B}\left(\exp\left(-\int_t^T r(s)ds\right)\right)$$

Taking the derivative with respect to T gives us

$$-\frac{\partial p(t,T)}{\partial T} = \mathbb{E}^{Q^B}\left(\exp\left(-\int_t^T r(s)ds\right)r(T)\right)$$

Now we just have to change measure from the money account meassure Q^B to the T-forward measure Q^T with the Radon Nikodyn derivative $\frac{dQ^B}{dQ^T} = \frac{p(t,T)}{\exp\left(-\int_t^T r(s)ds\right)}$

$$-\frac{\partial p(t,T)}{\partial T} = \mathbb{E}^{Q^T}\left(\exp\left(-\int_t^T r(s)ds\right)r(T)\frac{p(t,T)}{\exp\left(-\int_t^T r(s)ds\right)}\right) = p(t,T)\,\mathbb{E}^{Q^T}(r(T))$$

Since in general $\frac{d}{dx}\ln g(x) = \frac{g'(x)}{g(x)}$ we end up with the wanted result

$$-\frac{\partial \ln p(t,T)}{\partial T} = \mathbb{E}^{Q^T}(r(T))$$

Exercise 2:

2. Moment generating function in the Heston model. The Heston stochastic volatility model for the logarithmic stock price $Y_t = \ln S_t$ and the instantaneous variance V_t has dynamics

$$dY_t = (r - \frac{1}{2}V_t)dt + \sqrt{V_t}dW_{t,1}$$
$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_{t,2}$$

for two Brownian motions with $\langle W_1, W_2\rangle_t = \rho t$. Show that the conditional moment generating function

$$\psi(u_1, u_2) = E\left(\exp(-u_1 Y_T - u_2 V_T) \mid Y_t = y, V_t = v\right)$$

is of the form $\exp\left(a(t,T) + b_1(t,T)y + b_2(t,T)v\right)$ and derive an ODE-system for a, b_1 and b_2. Hint: use similar arguments as in the analysis of the affine short-rate models.