

Exercise 1:

1. Let (B^1, B^2) be two independent Brownian motions.

a) (2 points) Find for $\rho \in [-1, 1]$ constants a and b such that

$$W^1 := B^1 \text{ and } W^2 := aB^1 + bB^2$$

satisfy $[W^1]_t = [W^2]_t = t$ and $[W^1, W^2]_t = \rho t$. Compute $\text{corr}(W_t^1, W_t^2)$.

b) (2 points) Consider adapted and left-continuous bounded processes x_s, y_s, z_s and v_s . Define

$$X_t := \int_0^t x_s dW_s^1, \quad Y_t := \int_0^t y_s dW_s^2,$$

$$Z_t := \int_0^t z_s dX_s, \quad V_t := \int_0^t v_s dY_s.$$

Find $[Z, V]_t, t > 0$.

a)

$$\begin{aligned} \rho t &= [W^1, W^2] \\ &= [B^1, aB^1 + bB^2] \\ &, \text{ since the quadratic covariation is bilinear in it's arguments} \\ &= a[B^1, B^1] + b \underbrace{[B^1, B^2]}_{=0, \text{ since } B^1 \text{ and } B^2 \text{ independent}} \\ &= a[B^1] \\ &= at \\ \Rightarrow a &= \rho \end{aligned}$$

$$\begin{aligned} t &= [W^2] \\ &= [W^2, W^2] \\ &= [aB^1 + bB^2, aB^1 + bB^2] \\ &= [aB^1 + bB^2, aB^1] + [aB^1 + bB^2, bB^2] \\ &= [aB^1, aB^1] + \underbrace{[bB^2, aB^1]}_{=0} + \underbrace{[aB^1, bB^2]}_{=0} + [bB^2, bB^2] \\ &= a^2 \underbrace{[B^1, B^1]}_{[B^1]=t} + b^2 \underbrace{[B^2, B^2]}_{[B^2]=t} \\ \Rightarrow b^2 &= 1 - a^2 \end{aligned}$$

$$\begin{aligned}
\text{corr}(W_t^1, W_t^2) &= \frac{\text{Cov}(W_t^1, W_t^2)}{\sigma_{W_t^1} \sigma_{W_t^2}} \\
&= \frac{\text{Cov}(B_t^1, aB_t^1 + bB_t^2)}{\sigma_{W_t^1} \sigma_{W_t^2}} \\
&= \frac{a\text{Cov}(B_t^1, B_t^1) + b\text{Cov}(B_t^1, B_t^2)}{\sigma_{B_t^1} \sigma_{aB_t^1 + bB_t^2}}, \text{ since } \sigma_{aB_t^1 + bB_t^2} = \sqrt{(a^2 + b^2)t} \\
&= \frac{a\text{Cov}(B_t^1, B_t^1) + b\text{Cov}(B_t^1, B_t^2)}{\sqrt{t}\sqrt{(a^2 + b^2)t}} \\
&= \frac{at}{t\sqrt{a^2 + b^2}} \\
&= \frac{a}{\sqrt{a^2 + b^2}} = \rho
\end{aligned}$$

b)

$$\begin{aligned}
[Z, V]_t &= \left[\int_0^\cdot z_s dX_s, \int_0^\cdot v dY_s \right]_t \\
&= \int_0^t z_s v_s d[X, Y]_s \\
&= \int_0^t z_s v_s d \left[\int x_s dW_s^1, \int y_s dW_s^2 \right]_s \\
&= \int_0^t z_s v_s d \left(\int_0^t x_s y_s d \underbrace{[W^1, W^2]}_{[B^1, aB^1 + bB^2]} \right) \\
&= \int_0^t z_s v_s d \left(\int_0^t x_s y_s d \left(a \underbrace{[B^1, B^1]}_{=t} + b \underbrace{[B^1, B^2]}_{=0} \right) \right) \\
&= \int_0^t z_s v_s d \left(\int_0^t x_s y_s da t \right)
\end{aligned}$$

Exercise 2:

2. Kunita Watanabe decomposition, hedging with correlated assets.

- a) (2 points) Consider two martingales $N, M \in \mathcal{M}^{2,c}$ and denote by $N_t = N_0 + \int_0^t H_s dM_s + L_t$, $t \leq T$, the Kunita Watanabe decomposition of N wrt M . Show that the integrand H satisfies the relation

$$[M, N]_t = \int_0^t H_s d[M]_s, \quad t \leq T.$$

Suppose that - as in the case of martingales driven by Brownian motion - $d[M, N]_t = \alpha_t^{[M, N]} dt$ and $d[M]_t = \alpha_t^{[M]} dt$ with $\alpha_t^{[M]} > 0$. Conclude that $H_t = \alpha_t^{[M, N]} / \alpha_t^{[M]}$.

- b) (2 points) Consider a model with two correlated assets with dynamics $dS_t^i = \sigma_i S_t^i dB_t^i$ for two standard Brownian motions B^1, B^2 with $[B^1, B^2]_t = \rho t$ for some $\rho \in [-1, 1]$. Compute the Kunita Watanabe decomposition of S^2 wrt S^1 . (This is one possible approach for hedging the non-tradable asset S^2 with the tradable asset S^1 .)

a)

$$\begin{aligned} [M, N]_t &= \left[M, N_0 + \int_0^t H_s dM_s + L_t \right]_t \\ &= [M, N_0] + \left[M, \int_0^t H_s dM_s \right] + [M, L_t] \\ &= \left[\int_0^t 1 dM_s, \int_0^t H_s dM_s \right] \\ &= \int_0^t 1 H_s d[M_s, M_s] \\ &= \int_0^t H_s d[M]_s \end{aligned}$$

and

$$\begin{aligned} [M, N]_t &= \int_0^t H_s d[M]_s \\ \Rightarrow d[M, N]_t &= H_t d[M]_t \\ \Rightarrow \alpha_t^{[M, N]} dt &= H_t \alpha_t^{[M]} dt \\ \Rightarrow H_t &= \frac{\alpha_t^{[M, N]}}{\alpha_t^{[M]}} \end{aligned}$$

b)

We have to calculate the Kunita Watanabe decomposition of S^2 wrt S^1 . We need to find H and L to find the characterization:

$$S^2 = S_0^2 + \int_0^t H_s d[S^1]_s + L_t \quad (1)$$

From a) we get:

$$\begin{aligned}
 H_t &= \frac{[S^1, S^2]_t}{[S^1]_t} \\
 &= \frac{[\int_0^t \sigma_1 S_s^1 dB_s^1, \int_0^t \sigma_2 S_s^2 dB_s^2]_t}{[\int_0^t \sigma_1 S_s^1 dB_s^1]_t} \\
 &= \frac{\int_0^t \sigma_1 \sigma_2 S_s^1 S_s^2 d[B^1, B^2]_s}{\int_0^t (\sigma_1 S_s^1)^2 d[B^1]_s} \\
 &= \frac{\int_0^t \sigma_1 \sigma_2 S_s^1 S_s^2 d\rho_s}{\int_0^t (\sigma_1 S_s^1)^2 ds}
 \end{aligned}$$

Exercise 3:

3. Quadratic variation of compensated Poisson process . Consider a Poisson process N_t with parameter λ and recall that $M_t = N_t - \lambda t$ is a square integrable martingale. Use the characterization of quadratic variation to show that $[M_t] = N_t$.

We start with:

$$[M_t] = [N_t - \lambda t] = [N_t] - 2[N_t, \lambda t] + [\lambda t] = [N_t] \quad (2)$$

The increments of λt go asymptotically to 0 as the size of the increments gets smaller. Therefore their covariation with N_t goes to 0 as well.

We divide t into sub-intervals: $0 = t_0 < t_1 < t_2 < \dots < t_n = t$

We know that for sufficiently small increments ΔN is either 1 when there is a jump and 0 when there is no jump and therefore:

$$[N_t] = \lim_{n \rightarrow \infty} \sum_{i=1}^n (N_{t_i} - N_{t_{i-1}})^2 = N_t \quad (3)$$

Exercise 4:

4. Generator of the Heston model (3 points) Consider a two-dimensional process X, v with

$$\begin{aligned}
 dX_t &= \mu X_t dt + \sigma_1 \sqrt{v_t} dB_{t,1} \\
 dv_t &= \kappa(\theta - v_t) dt + \sigma_2 \sqrt{v_t} dB_{t,2}
 \end{aligned}$$

for constants $\mu, \sigma_1, \sigma_2, \kappa, \theta > 0$ and two Brownian motions B_1, B_2 with $[B_1, B_2]_t = \rho t$ for some $-1 \leq \rho \leq 1$. Compute the generator of the process (X, v) .