

**Exercise 1:**

1. Let  $(B^1, B^2)$  be two independent Brownian motions.

a) (2 points) Find for  $\rho \in [-1, 1]$  constants  $a$  and  $b$  such that

$$W^1 := B^1 \text{ and } W^2 := aB^1 + bB^2$$

satisfy  $[W^1]_t = [W^2]_t = t$  and  $[W^1, W^2]_t = \rho t$ . Compute  $\text{corr}(W_t^1, W_t^2)$ .

b) (2 points) Consider adapted and left-continuous bounded processes  $x_s, y_s, z_s$  and  $v_s$ . Define

$$X_t := \int_0^t x_s dW_s^1, \quad Y_t := \int_0^t y_s dW_s^2,$$

$$Z_t := \int_0^t z_s dX_s, \quad V_t := \int_0^t v_s dY_s.$$

Find  $[Z, V]_t, t > 0$ .

a)

$$\begin{aligned} \rho t &= [W^1, W^2] \\ &= [B^1, aB^1 + bB^2] \\ &, \text{ since the quadratic covariation is bilinear in it's arguments} \\ &= a[B^1, B^1] + b \underbrace{[B^1, B^2]}_{=0, \text{ since } B^1 \text{ and } B^2 \text{ independent}} \\ &= a[B^1] \\ &= at \\ \Rightarrow a &= \rho \end{aligned}$$

$$\begin{aligned} t &= [W^2] \\ &= [W^2, W^2] \\ &= [aB^1 + bB^2, aB^1 + bB^2] \\ &= [aB^1 + bB^2, aB^1] + [aB^1 + bB^2, bB^2] \\ &= [aB^1, aB^1] + \underbrace{[bB^2, aB^1]}_{=0} + \underbrace{[aB^1, bB^2]}_{=0} + [bB^2, bB^2] \\ &= a^2 \underbrace{[B^1, B^1]}_{[B^1]=t} + b^2 \underbrace{[B^2, B^2]}_{[B^2]=t} \\ \Rightarrow b^2 &= 1 - a^2 \end{aligned}$$

$$\begin{aligned}
\text{corr}(W_t^1, W_t^2) &= \frac{\text{Cov}(W_t^1, W_t^2)}{\sigma_{W_t^1} \sigma_{W_t^2}} \\
&= \frac{\text{Cov}(B_t^1, aB_t^1 + bB_t^2)}{\sigma_{W_t^1} \sigma_{W_t^2}} \\
&= \frac{a\text{Cov}(B_t^1, B_t^1) + b\text{Cov}(B_t^1, B_t^2)}{\sigma_{B_t^1} \sigma_{aB_t^1 + bB_t^2}} \\
&= \frac{a\text{Cov}(B_t^1, B_t^1) + b\text{Cov}(B_t^1, B_t^2)}{ab\sigma_{B_t^1}^2 \sigma_{B_t^2}} \\
&= \frac{at}{abt\sqrt{t}} = \frac{1}{b\sqrt{t}}
\end{aligned}$$

b)

$$\begin{aligned}
[Z, V]_t &= \left[ \int_0^\cdot z_s dX_s, \int_0^\cdot v dY_s \right]_t \\
&= \int_0^t z_s v_s d[X, Y]_s \\
&= \int_0^t z_s v_s d \left[ \int_0^\cdot x_s dW_s^1, \int_0^\cdot y_s dW_s^2 \right]_s \\
&= \int_0^t z_s v_s d \left( \int_0^t x_s y_s d \underbrace{[W^1, W^2]}_{[B^1, aB^1 + bB^2]} \right) \\
&= \int_0^t z_s v_s d \left( \int_0^t x_s y_s d \left( \underbrace{a[B^1, B^1]}_{=t} + \underbrace{b[B^1, B^2]}_{=0} \right) \right) \\
&= \int_0^t z_s v_s d \left( \int_0^t x_s y_s da \right)
\end{aligned}$$

## Exercise 2:

### 2. Kunita Watanabe decomposition, hedging with correlated assets.

- a) (2 points) Consider two martingales  $N, M \in \mathcal{M}^{2,c}$  and denote by  $N_t = N_0 + \int_0^t H_s dM_s + L_t$ ,  $t \leq T$ , the Kunita Watanabe decomposition of  $N$  wrt  $M$ . Show that the integrand  $H$  satisfies the relation

$$[M, N]_t = \int_0^t H_s d[M]_s, \quad t \leq T.$$

Suppose that - as in the case of martingales driven by Brownian motion -  $d[M, N]_t = \alpha_t^{[M, N]} dt$  and  $d[M]_t = \alpha_t^{[M]} dt$  with  $\alpha_t^{[M]} > 0$ . Conclude that  $H_t = \alpha_t^{[M, N]} / \alpha_t^{[M]}$ .

- b) (2 points) Consider a model with two correlated assets with dynamics  $dS_t^i = \sigma_i S_t^i dB_t^i$  for two standard Brownian motions  $B^1, B^2$  with  $[B^1, B^2]_t = \rho t$  for some  $\rho \in [-1, 1]$ . Compute the Kunita Watanabe decomposition of  $S^2$  wrt  $S^1$ . (This is one possible approach for hedging the non-tradable asset  $S^2$  with the tradable asset  $S^1$ .)

**Exercise 3:**

**3. Quadratic variation of compensated Poisson process** . Consider a Poisson process  $N_t$  with parameter  $\lambda$  and recall that  $M_t = N_t - \lambda t$  is a square integrable martingale. Use the characterization of quadratic variation to show that  $[M_t] = N_t$ .

**Exercise 4:**

**4. Generator of the Heston model** (3 points) Consider a two-dimensional process  $X, v$  with

$$\begin{aligned}dX_t &= \mu X_t dt + \sigma_1 \sqrt{v_t} dB_{t,1} \\ dv_t &= \kappa(\theta - v_t)dt + \sigma_2 \sqrt{v_t} dB_{t,2}\end{aligned}$$

for constants  $\mu, \sigma_1, \sigma_2, \kappa, \theta > 0$  and two Brownian motions  $B_1, B_2$  with  $[B_1, B_2]_t = \rho t$  for some  $-1 \leq \rho \leq 1$ . Compute the generator of the process  $(X, v)$ .