Exercise 1:

1. Stochastic volatility model. (4 points) Consider an asset price model with constant interest rate $r \geq 0$ and price dynamics of the form

$$dS_t = \mu_S S_t dt + \exp(V_t) S_t dW_{t,1}, \quad dV_t = \mu_V dt + \sigma_V dW_{t,2}$$

for independent Brownian motions W_1, W_2. Consider a change of measure with

$$\frac{dQ}{dP}_{|\mathcal{F}_T} = \exp\Big(\sum_{i=1}^{2} \int_0^T \theta_{s,i} dW_{s,i} - \frac{1}{2}\int_0^T \|\theta_s\|^2 \, ds\Big).$$

Give conditions on the process θ such that Q is an equivalent martingale measure. Is Q unique?

Exercise 2:

2. Modelling foreign exchange (6 points) Consider a model for the exchange rate between two countries f (foreign) and d (domestic), and denote by e_t the price in domestic currency of one unit of the foreign currency. Assume that the short rate in the two countries is constant and denote it by r^d respectively by r^f. $B_t^d = e^{r^d t}$ ($B_t^f = e^{r^f t}$) denote the price of the domestic (foreign) savings account. The exchange-rate dynamics under P is given by $de_t = \mu e_t dt + \sigma e_t dW_t$.

a) Denote by Q^d the domestic martingale measure, that is the martingale measure that corresponds to the numeraire S^d. Show that under Q^d the drift of e_t is equal to $r^d - r^f$. Hint: consider the drift of the domestic asset $e_t S_t^f$.

b) Compute the Q^d-dynamics for the inverse exchange rate $1/e_t$. Is Q^d also the appropriate martingale measure for a foreign investor who denominates in foreign currency and uses S_t^f as numeraire?

c) Use the change-of numeraire technique to identify the martingale measure Q^f that corresponds to the numeraire $e_t S_t^f$ (the foreign martingale measure) and compute the Q^f dynamics of $1/e_t$.

Exercise 3:

3. Absence of arbitrage (6 points) Consider asset price dynamics of the form

$$dS_{t,i} = \mu_i S_{t,i} dt + \sigma_i S_{t,i} dW_t, \quad i = 1, 2$$

(the two price processes are driven by the same Brownian motion). Assume that the numeraire is of the form $S_{t,0} = e^{rt}$ for a constant $r \geq 0$.

a) Show that the model is arbitrage-free if $\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}$.

b) Assume that $r = 0$ and that $\frac{\mu_1}{\sigma_1} > \frac{\mu_2}{\sigma_2}$. Show that in that case there is a selffinancing strategy ϕ with value $dV_t^\phi = c_t dt$ for some $c_t > 0$ (and hence an arbitrage opportunity). (The restriction to the case $r = 0$ serves to keep the example simple).