Exercise 1:

1. Let (B^1, B^2) be two independent Brownian motions.

a) (2 points) Find for $\rho \in [-1,1]$ constants a and b such that

$$W^1 := B^1 \text{ and } \quad W^2 := aB^1 + bB^2$$

satisfy $[W^1]_t = [W^2]_t = t$ and $[W^1, W^2]_t = \rho t$. Compute $\operatorname{corr}(W^1_t, W^2_t)$.

b) (2 points) Consider adapted and left-continuous bounded processes x_s, y_s, z_s and v_s. Define

$$X_t := \int_0^t x_s dW^1_s, \quad Y_t := \int_0^t y_s dW^2_s,$$
$$Z_t := \int_0^t z_s dX_s, \quad V_t := \int_0^t v_s dY_s.$$

Find $[Z, V]_t, t > 0$.

a)

$$\begin{aligned}
\rho t &= \left[W^1, W^2\right] \\
&= \left[B^1, aB^1 + bB^2\right] \\
&\text{, since the quadratic covariation is bilinear in it's arguments} \\
&= a\left[B^1, B^1\right] + b \underbrace{\left[B^1, B^2\right]}_{=0, \text{since} B^1 \text{and} B^2 \text{independent}} \\
&= a\left[B^1\right] \\
&= at \\
\Rightarrow a &= \rho
\end{aligned}$$

$$\begin{aligned}
t &= \left[W^2\right] \\
&= \left[W^2, W^2\right] \\
&= \left[aB^1 + bB^2, aB^1 + bB^2\right] \\
&= \left[aB^1 + bB^2, aB^1\right] + \left[aB^1 + bB^2, bB^2\right] \\
&= \left[aB^1, aB^1\right] + \underbrace{\left[bB^2, aB^1\right]}_{=0} + \underbrace{\left[aB^1, bB^2\right]}_{=0} + \left[bB^2, bB^2\right] \\
&= a^2 \underbrace{\left[B^1, B^1\right]}_{[B^1]=t} + b^2 \underbrace{\left[B^2, B^2\right]}_{[B^2]=t} \\
\Rightarrow b^2 &= 1 - a^2
\end{aligned}$$

$$\begin{aligned}
\operatorname{corr}\left(W_t^1, W_t^2\right) &= \frac{\operatorname{Cov}\left(W_t^1, W_t^2\right)}{\sigma_{W_t^1}\sigma_{W_t^2}} \\
&= \frac{\operatorname{Cov}\left(B_t^1, aB_t^1 + bB_t^2\right)}{\sigma_{W_t^1}\sigma_{W_t^2}} \\
&= \frac{a\operatorname{Cov}\left(B_t^1, B_t^1\right) + b\operatorname{Cov}\left(B_t^1, B_t^2\right)}{\sigma_{B_t^1}\sigma_{aB_t^1+bB_t^2}}\textbf{, since } \sigma_{aB_t^1+bB_t^2} = \sqrt{\left(a^2+b^2\right)t} \\
&= \frac{a\operatorname{Cov}\left(B_t^1, B_t^1\right) + b\operatorname{Cov}\left(B_t^1, B_t^2\right)}{\sqrt{t}\sqrt{\left(a^2+b^2\right)t}} \\
&= \frac{at}{t\sqrt{a^2+b^2}} \\
&= \frac{a}{\sqrt{a^2+b^2}} = \rho
\end{aligned}$$