Exercise 1:

1. Stochastic volatility model. (4 points) Consider an asset price model with constant interest rate $r \geq 0$ and price dynamics of the form

$$dS_t = \mu_S S_t dt + \exp(V_t) S_t dW_{t,1}, \quad dV_t = \mu_V dt + \sigma_V dW_{t,2}$$

for independent Brownian motions W_1, W_2. Consider a change of measure with

$$\frac{dQ}{dP}_{|\mathcal{F}_T} = \exp\Big(\sum_{i=1}^{2} \int_0^T \theta_{s,i} dW_{s,i} - \frac{1}{2}\int_0^T \|\theta_s\|^2 \, ds\Big).$$

Give conditions on the process θ such that Q is an equivalent martingale measure. Is Q unique?

- θ must be adapted so θ_t has to be $\mathcal{F}_t$ measurable for all t
- $P\left(\frac{dQ}{dP} > 0\right) = 1$
- $1 = E\left(\frac{dQ}{dP}\right)$

For Q being a equivalent martingale measure the discounted asset price process $\tilde{S}_t = \frac{S_t}{S_t^0}$ with Since r is constant $S^0(t) = exp\left(\int_0^t r_s ds\right) = exp(rt)$ has to be a martingale so $\tilde{S}_0 = E_0^Q\left(\tilde{S}\right)$.

applying Ito's product formula to $\tilde{S}$ gives

$$d\tilde{S}_t = d\frac{S_t}{S_t^0} = \frac{1}{S_t^0} dS + Sd\frac{1}{S_t^0} \tag{1}$$

We see that $S^0 = e^{rt}$ solves the ODE $d\frac{1}{S_t^0} = -r\frac{1}{S_t^0}dt$ therefore we can substitud back into the equation (1) and get

$$\begin{aligned} d\tilde{S} &= e^{-rt} dS - Sre^{-rt}dt \qquad (2)\\ &= e^{-rt}\left(\mu_s S_t dt + e^{V_t} S_t dW_{t,1}\right) - Sre^{-rt}dt \qquad (3)\end{aligned}$$

applying Girsanov, 2-dimensions $W_{t,1} = \tilde{W}_{t,1} + \int_0^t \theta_{s,1} ds$ and that $\tilde{W}$ is a Brownian Motion under Q.

$$= e^{-rt}\left(\mu_s S_t dt + e^{V_t} S_t \left(d\tilde{W}_{t,1} + \theta_{d,1} dt\right)\right) - Sre^{-rt}dt \tag{4}$$

$$\begin{aligned} \tilde{S}_0 &= E_0^Q\left(\tilde{S}\right)\\ \tilde{S}_0 &= E_0^Q\left(\tilde{S}_0 + e^{-rt}\left(\int_0^t \mu_s S_u du + \int_0^t e^{V_u} S_u d\tilde{W}_{u,1} + \int_0^t \theta_{d,1} du\right) - \int_0^t Sre^{-ru} du\right)\end{aligned}$$

Since linearity of the expectation and $E\left(\int f dW\right) = 0$ we get

$$\begin{aligned} 0 &= -re^{-rt} S_t + e^{-rt}\mu_S S_t + e^{V_t - rt} S_t \theta_{t,1}\\ 0 &= -r + \mu_s + e^{V_t}\theta_{t,1}\\ \theta_{t,1} &= \frac{r - \mu_s}{e^{V_t}}\end{aligned}$$

The same reasoning also works for $\tilde{V}$ and $\theta_{t,2}$

$$d\tilde{V} = e^{-rt}dV_t - V_t r e^{-rt}dt$$
$$\dots$$
$$\theta_{t,2} = \frac{V_t r - \mu_V}{\sigma_V}$$

Exercise 2:

2. Modelling foreign exchange (6 points) Consider a model for the exchange rate between two countries f (foreign) and d (domestic), and denote by e_t the price in domestic currency of one unit of the foreign currency. Assume that the short rate in the two countries is constant and denote it by r^d respectively by r^f. $B_t^d = e^{r^d t}$ ($B_t^f = e^{r^f t}$) denote the price of the domestic (foreign) savings account. The exchange-rate dynamics under P is given by $de_t = \mu e_t dt + \sigma e_t dW_t$.

a) Denote by Q^d the domestic martingale measure, that is the martingale measure that corresponds to the numeraire S^d. Show that under Q^d the drift of e_t is equal to $r^d - r^f$. Hint: consider the drift of the domestic asset $e_t S_t^f$.

b) Compute the Q^d-dynamics for the inverse exchange rate $1/e_t$. Is Q^d also the appropriate martingale measure for a foreign investor who denominates in foreign currency and uses S_t^f as numeraire?

c) Use the change-of numeraire technique to identify the martingale measure Q^f that corresponds to the numeraire $e_t S_t^f$ (the foreign martingale measure) and compute the Q^f dynamics of $1/e_t$.

Exercise 3:

3. Absence of arbitrage (6 points) Consider asset price dynamics of the form

$$dS_{t,i} = \mu_i S_{t,i}dt + \sigma_i S_{t,i}dW_t, \quad i = 1, 2$$

(the two price processes are driven by the same Brownian motion). Assume that the numeraire is of the form $S_{t,0} = e^{rt}$ for a constant $r \geq 0$.

a) Show that the model is arbitrage-free if $\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}$.

b) Assume that $r = 0$ and that $\frac{\mu_1}{\sigma_1} > \frac{\mu_2}{\sigma_2}$. Show that in that case there is a selffinancing strategy ϕ with value $dV_t^\phi = c_t dt$ for some $c_t > 0$ (and hence an arbitrage opportunity). (The restriction to the case $r = 0$ serves to keep the example simple).

We have 2 stocks who follow the following SDE:

$$dS_t^i = \mu S_t^i dt + \sigma S_t^i dW_t \tag{5}$$

a) To show that the market is arbitrage free we have to show that the market admits an equivalent martingale measure. We will show this by finding one.

The discounted value process is a Q-martingale if and only if $S_t^1 = S_0^1 + \int_0^t rS_s^1 ds + M_t^Q$ with M_t^Q being a Q-local martingale.
We define $\tilde{W}_t = W_t + \int_0^t \frac{\mu - r}{\sigma} ds$ and use Girsanov to get Q under which $\tilde{W}$ is again a brownian motion.

$$\frac{dQ}{dP} = exp(-\frac{\mu_i - r}{\sigma_i} W_T - \frac{1}{2}(\frac{\mu_i - r}{\sigma_i})^2 T) \tag{6}$$

As we can see the unique martingale measure only depends on $\frac{\mu_i - r}{\sigma_i}$. As this expression is equal for our two assets they are both martingales under Q. From the existence of a martingale measure it follows that there can be no arbitrage.

b) In the given example the market value of risk is higher for the second asset. Intuitively both assets move together due to brownian motion that drives both processes. But asset 1 has a higher drift. This means for our strategy we buy asset 1 and sell asset 2. We assume $S_{0,1} = S_{0,2}$ and buy $\frac{1}{\sigma_1}$ of asset 1 and sell $\frac{1}{\sigma_2}$ units of asset 2. Then we get:

$$dV_t^\phi = \frac{1}{\sigma_1} dS_{t,1} - \frac{1}{\sigma_2} dS_{t,2} \tag{7}$$

$$dV_t^\phi = \frac{\mu_1}{\sigma_1} dt - \frac{\mu_2}{\sigma_2} dt + dW_t - dW_t = c_t dt \tag{8}$$