Exercise 1:

1. Let (B^1, B^2) be two independent Brownian motions.

a) (2 points) Find for $\rho \in [-1,1]$ constants a and b such that

$$W^1 := B^1 \text{ and } \quad W^2 := aB^1 + bB^2$$

satisfy $[W^1]_t = [W^2]_t = t$ and $[W^1, W^2]_t = \rho t$. Compute $\operatorname{corr}(W_t^1, W_t^2)$.

b) (2 points) Consider adapted and left-continuous bounded processes x_s, y_s, z_s and v_s. Define

$$X_t := \int_0^t x_s dW_s^1, \quad Y_t := \int_0^t y_s dW_s^2,$$
$$Z_t := \int_0^t z_s dX_s, \quad V_t := \int_0^t v_s dY_s.$$

Find $[Z, V]_t, t > 0$.

a)

$$\begin{aligned}
\rho t &= \left[W^1, W^2\right] \\
&= \left[B^1, aB^1 + bB^2\right] \\
&\text{, since the quadratic covariation is bilinear in it's arguments} \\
&= a\left[B^1, B^1\right] + b \underbrace{\left[B^1, B^2\right]}_{=0,\text{since} B^1 \text{and} B^2 \text{independent}} \\
&= a\left[B^1\right] \\
&= at \\
\Rightarrow a &= \rho
\end{aligned}$$

$$\begin{aligned}
t &= \left[W^2\right] \\
&= \left[W^2, W^2\right] \\
&= \left[aB^1 + bB^2, aB^1 + bB^2\right] \\
&= \left[aB^1 + bB^2, aB^1\right] + \left[aB^1 + bB^2, bB^2\right] \\
&= \left[aB^1, aB^1\right] + \underbrace{\left[bB^2, aB^1\right]}_{=0} + \underbrace{\left[aB^1, bB^2\right]}_{=0} + \left[bB^2, bB^2\right] \\
&= a^2 \underbrace{\left[B^1, B^1\right]}_{[B^1]=t} + b^2 \underbrace{\left[B^2, B^2\right]}_{[B^2]=t} \\
\Rightarrow b^2 &= 1 - a^2
\end{aligned}$$

$$
\begin{aligned}
\operatorname{corr}\left(W_t^1, W_t^2\right) &= \frac{\operatorname{Cov}\left(W_t^1, W_t^2\right)}{\sigma_{W_t^1}\sigma_{W_t^2}} \\
&= \frac{\operatorname{Cov}\left(B_t^1, aB_t^1 + bB_t^2\right)}{\sigma_{W_t^1}\sigma_{W_t^2}} \\
&= \frac{a\operatorname{Cov}\left(B_t^1, B_t^1\right) + b\operatorname{Cov}\left(B_t^1, B_t^2\right)}{\sigma_{B_t^1}\sigma_{aB_t^1+bB_t^2}} \\
&= \frac{a\operatorname{Cov}\left(B_t^1, B_t^1\right) + b\operatorname{Cov}\left(B_t^1, B_t^2\right)}{ab\sigma^2_{B_t^1}\sigma_{B_t^2}} \\
&= \frac{a}{abt\sqrt{t}} = b^{-1}t^{-\frac{3}{2}}
\end{aligned}
$$

Exercise 2:

2. Kunita Watanabe decomposition, hedging with correlated assets.

a) (2 points) Consider two martingales $N, M \in \mathcal{M}^{2,c}$ and denote by $N_t = N_0 + \int_0^t H_s dM_s + L_t$, $t \leq T$, the Kunita Watanabe decomposition of N wrt M. Show that the integrand H satisfies the relation

$$[M,N]_t = \int_0^t H_s d[M]_s, \quad t \leq T.$$

Suppose that - as in the case of martingales driven by Brownian motion - $d[M,N]_t = \alpha_t^{[M,N]}dt$ and $d[M]_t = \alpha_t^{[M]}dt$ with $\alpha_t^{[M]} > 0$. Conclude that $H_t = \alpha_t^{[M,N]}/\alpha_t^{[M]}$.

b) (2 points) Consider a model with two correlated assets with dynamics $dS_t^i = \sigma_i S_t^i dB_t^i$ for two standard Brownian motions B^1, B^2 with $[B^1, B^2]_t = \rho t$ for some $\rho \in [-1, 1]$. Compute the Kunita Watanabe decomposition of S^2 wrt S^1. (This is one possible approach for hedging the non-tradable asset S^2 with the tradable asset S^1.)

Exercise 3:

3. Quadratic variation of compensated Poisson process . Consider a Poisson process N_t with parameter λ and recall that $M_t = N_t - \lambda_t$ is a square integrable martingale. Use the characterization of quadratic variation to show that $[M_t] = N_t$.

Exercise 4:

4. Generator of the Heston model (3 points) Consider a two-dimensional process X, v with

$$
\begin{aligned}
dX_t &= \mu X_t dt + \sigma_1\sqrt{v_t}dB_{t,1} \\
dv_t &= \kappa(\theta - v_t)dt + \sigma_2\sqrt{v_t}dB_{t,2}
\end{aligned}
$$

for constants $\mu, \sigma_1, \sigma_2, \kappa, \theta > 0$ and two Brownian motions B_1, B_2 with $[B_1, B_2]_t = \rho t$ for some $-1 \leq \rho \leq 1$. Compute the generator of the process (X, v).