Exercise 1:

1. Let (B^1, B^2) be two independent Brownian motions.

a) (2 points) Find for $\rho \in [-1,1]$ constants a and b such that

$$W^1 := B^1 \text{ and } \quad W^2 := aB^1 + bB^2$$

satisfy $[W^1]_t = [W^2]_t = t$ and $[W^1, W^2]_t = \rho t$. Compute $\operatorname{corr}(W_t^1, W_t^2)$.

b) (2 points) Consider adapted and left-continuous bounded processes x_s, y_s, z_s and v_s. Define

$$X_t := \int_0^t x_s dW_s^1, \quad Y_t := \int_0^t y_s dW_s^2,$$
$$Z_t := \int_0^t z_s dX_s, \quad V_t := \int_0^t v_s dY_s.$$

Find $[Z, V]_t, t > 0$.

a)

$$\begin{aligned}
\rho t &= \left[W^1, W^2\right] \\
&= \left[B^1, aB^1 + bB^2\right] \\
&\text{, since the quadratic covariation is bilinear in it's arguments} \\
&= a\left[B^1, B^1\right] + b \underbrace{\left[B^1, B^2\right]}_{=0, \text{since} B^1 \text{and} B^2 \text{independent}} \\
&= a\left[B^1\right] \\
&= at \\
\Rightarrow a &= \rho
\end{aligned}$$

$$\begin{aligned}
t &= \left[W^2\right] \\
&= \left[W^2, W^2\right] \\
&= \left[aB^1 + bB^2, aB^1 + bB^2\right] \\
&= \left[aB^1 + bB^2, aB^1\right] + \left[aB^1 + bB^2, bB^2\right] \\
&= \left[aB^1, aB^1\right] + \underbrace{\left[bB^2, aB^1\right]}_{=0} + \underbrace{\left[aB^1, bB^2\right]}_{=0} + \left[bB^2, bB^2\right] \\
&= a^2 \underbrace{\left[B^1, B^1\right]}_{[B^1]=t} + b^2 \underbrace{\left[B^2, B^2\right]}_{[B^2]=t} \\
\Rightarrow b^2 &= 1 - a^2
\end{aligned}$$

$$
\begin{aligned}
\operatorname{corr}\left(W_t^1, W_t^2\right) &= \frac{\operatorname{Cov}\left(W_t^1, W_t^2\right)}{\sigma_{W_t^1}\sigma_{W_t^2}} \\
&= \frac{\operatorname{Cov}\left(B_t^1, aB_t^1 + bB_t^2\right)}{\sigma_{W_t^1}\sigma_{W_t^2}} \\
&= \frac{a\operatorname{Cov}\left(B_t^1, B_t^1\right) + b\operatorname{Cov}\left(B_t^1, B_t^2\right)}{\sigma_{B_t^1}\sigma_{aB_t^1+bB_t^2}}\textbf{, since } \sigma_{aB_t^1+bB_t^2} = \sqrt{\left(a^2+b^2\right)t} \\
&= \frac{a\operatorname{Cov}\left(B_t^1, B_t^1\right) + b\operatorname{Cov}\left(B_t^1, B_t^2\right)}{\sqrt{t}\sqrt{\left(a^2+b^2\right)t}} \\
&= \frac{at}{t\sqrt{a^2+b^2}} \\
&= \frac{a}{\sqrt{a^2+b^2}} = \rho
\end{aligned}
$$

b)

$$
\begin{aligned}
[Z, V]_t &= \left[\int_0^{\cdot} z_s dX_s, \int_0^{\cdot} v dY_s\right]_t \\
&= \int_0^t z_s v_s d\,[X, Y]_s \\
&= \int_0^t z_s v_s d\left[\int x_u dW_u^1, \int y_u dW_u^2\right]_s \\
&= \int_0^t z_s v_s d\left(\int_0^s x_u y_u d\,\underbrace{\left[W^1, W^2\right]}_{\left[B^1, aB^1+bB^2\right]}\right) \\
&= \int_0^t z_s v_s d\left(\int_0^s x_u y_u d\left(a\underbrace{\left[B^1, B^1\right]}_{=u} + b\underbrace{\left[B^1, B^2\right]}_{=0}\right)\right) \\
&= \int_0^t z_s v_s d\left(\int_0^s x_u y_u d a u\right) \\
&= a\int_0^t z_s v_s x_s y_s ds
\end{aligned}
$$

Exercise 2:

2. Kunita Watanabe decomposition, hedging with correlated assets.

a) (2 points) Consider two martingales $N, M \in \mathcal{M}^{2,c}$ and denote by $N_t = N_0 + \int_0^t H_s dM_s + L_t$, $t \leq T$, the Kunita Watanabe decomposition of N wrt M. Show that the integrand H satisfies the relation

$$[M,N]_t = \int_0^t H_s d[M]_s, \quad t \leq T.$$

Suppose that - as in the case of martingales driven by Brownian motion - $d[M,N]_t = \alpha_t^{[M,N]}dt$ and $d[M]_t = \alpha_t^{[M]}dt$ with $\alpha_t^{[M]} > 0$. Conclude that $H_t = \alpha_t^{[M,N]}/\alpha_t^{[M]}$.

b) (2 points) Consider a model with two correlated assets with dynamics $dS_t^i = \sigma_i S_t^i dB_t^i$ for two standard Brownian motions B^1, B^2 with $[B^1, B^2]_t = \rho t$ for some $\rho \in [-1,1]$. Compute the Kunita Watanabe decomposition of S^2 wrt S^1. (This is one possible approach for hedging the non-tradable asset S^2 with the tradable asset S^1.)

a)

$$\begin{aligned}
[M,N]_t &= \left[M, N_0 + \int_0^t H_s dM_s + L_t\right]_t \\
&= [M, N_0] + \left[M, \int_0^t H_s dM_s\right] + [M, L_t] \\
&= \left[\int_0^t 1 dM_s, \int_0^t H_s dM_s\right] \\
&= \int_0^t 1 H_s d\left[M_s, M_s\right] \\
&= \int_0^t H_s d\left[M_s\right]
\end{aligned}$$

and

$$\begin{aligned}
[M,N]_t &= \int_0^t H_s d\left[M\right]_s \\
\Rightarrow d\left[M,N\right]_t &= H_s d\left[M\right]_s \\
\Rightarrow \alpha_t^{[M,N]}dt &= H_s \alpha_t^{[M]}dt \\
\Rightarrow H_s &= \frac{\alpha_t^{[M,N]}}{\alpha_t^{[M]}}
\end{aligned}$$

b)

We have to calculate the Kunita Watanabe decomposition of S^2 wrt. S^1. We need to find H and L to find the characterization:

$$S^2 = S_0^2 + \int_0^t H_s d[S^1]_s + L_t \tag{1}$$

From a) we get:

$$H_t = \frac{[S^1, S^2]_t}{[S^1]_t} = \frac{[\int_0^\cdot \sigma_1 S_s^1 dB_s^1, \int_0^\cdot \sigma_2 S_s^2 dB_s^2]_t}{[\int_0^\cdot \sigma_1 S_s^1 dB_s^1]_t} = \tag{2}$$

$$= \frac{\int_0^t \sigma_1\sigma_2 S_s^1 S_s^2 d[B^1, B^2]_s}{\int_0^t (\sigma_1 S_s^1)^2 d[B^1]_s} = \frac{\int_0^t \sigma_1\sigma_2 S_s^1 S_s^2 \rho ds}{\int_0^t (\sigma_1 S_s^1)^2 ds} = \frac{\rho\sigma_2 S_t^2}{\sigma_1 S_t^1} \tag{3}$$

To find L_t we just need to insert H_t into the characterization and we get:

$$\begin{aligned}
S_t^2 &= S_0^2 + \int_0^t \frac{\rho\sigma_2 S_t^2}{\sigma_1 S_t^1} ds + L_t \\
L_t &= S_t^2 - S_0^2 - \int_0^t \frac{\rho\sigma_2 S_s^2}{\sigma_1 S_s^1} ds \\
&= \int_0^t S_s^2 ds - \int_0^t \frac{\rho\sigma_2 S_s^2}{\sigma_1 S_s^1} ds \\
&= \int_0^t S_s^2 (1 - \frac{\rho\sigma_2}{\sigma_1 S_s^1}) ds
\end{aligned}$$

Exercise 3:

3. Quadratic variation of compensated Poisson process . Consider a Poisson process N_t with parameter λ and recall that $M_t = N_t - \lambda_t$ is a square integrable martingale. Use the characterization of quadratic variation to show that $[M_t] = N_t$.

We start with:

$$[M_t] = [N_t - \lambda t] = [N_t] - 2[N_t, \lambda t] + [\lambda t] = [N_t] \tag{4}$$

The increments of λt go asymptotically to 0 as the size of the increments gets smaller. Therefore their covariation with N_t goes to 0 as well.
We divide t into sub-intervals: $0 = t_0 < t_1 < t_2 < ... < t_n = t$

We know that for sufficiently small increments ΔN is either 1 when there is a jump and 0 when there is no jump and therefore:

$$[N_t] = lim_{n\to\infty} \sum_{i=1}^{n} (N_{t_i} - N_{t_{i-1}})^2 = N_t \tag{5}$$

Exercise 4:

4. Generator of the Heston model (3 points) Consider a two-dimensional process X, v with

$$\begin{aligned}
dX_t &= \mu X_t dt + \sigma_1 \sqrt{v_t} dB_{t,1} \\
dv_t &= \kappa(\theta - v_t) dt + \sigma_2 \sqrt{v_t} dB_{t,2}
\end{aligned}$$

for constants $\mu, \sigma_1, \sigma_2, \kappa, \theta > 0$ and two Brownian motions B_1, B_2 with $[B_1, B_2]_t = \rho t$ for some $-1 \leq \rho \leq 1$. Compute the generator of the process (X, v).

$$Af(t,x) = \sum_{i=1}^{n} \mu_i(t,x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^{n} c_{ij}(t,x) \frac{\partial^2 f}{\partial x_i \partial x_j}(t,x)$$

In our case the generator of (X, v) is given by:

$$Af(t,X,v) = \mu_X(t,X,v)^t \Delta f + \frac{1}{2}C_{XX}f_{XX} + C_{Xv}f_{Xv}(t,X,v) + \frac{1}{2}C_{vv}f_{vv}$$

μ is given by the deterministic part of the stochastic process:

$$\mu(t,X,v) = (\mu X_s dt, \kappa(\theta - v_s)dt)^t$$

We get the variance term by:

$$\begin{aligned}
\sigma(t,X,v) &= (\sigma_1\sqrt{v_s}dB_1, \sigma_2\sqrt{v_s}dB_2)^t \\
C(t,X,v) &= \sigma(t,X,v)\sigma(t,X,v)^t \\
C(t,X,v)_{XX} &= \sigma_1^2 v_s d[B_1,B_1] = \sigma_1^2 v_s dt \\
C(t,X,v)_{vv} &= \sigma_2^2 v_s d[B_2,B_2] = \sigma_2^2 v_s dt
\end{aligned}$$

We know:

$$\begin{aligned}
[B_1,B_2]_t &= \rho \\
C(t,X,v)_{Xv} = C(t,X,v)_{vX} &= \sigma_1\sigma_2 v_s d[B_1,B_2] = \sigma_1\sigma_2 v_s \rho dt
\end{aligned}$$

We use this to rewrite the very first expression as:

$$Af(t,X,v) = \mu X_t f_X + \kappa(\theta - v_t)f_v + \frac{1}{2}\sigma_1^2 v_t f_{XX} + \sigma_1\sigma_2 v_t \rho f_{Xv} + \frac{1}{2}\sigma_2^2 v_t f_{vv} \tag{6}$$