Degradation of the signal-to-noise ratio of Gaussian signals in Gaussian noise transmitted over a digital channel

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This paper presents a mathematical analysis of the signal-to-noise power ratio degradation incurred when Gaussian signals in Gaussian noise are level-quantized, time-sampled, transmitted over a noisy digital channel, and then reconstructed by a low-pass filter. Application of this analysis occurs when very weak acoustic or seismic signals in strong white noise are collected and transmitted over a digital channel to an analysis location remote from the collection point, and where detectability of the information-bearing signals at the analysis location depends on the original signal-to-noise power ratio in small frequency bands. Therefore, it is important to understand the degradation of the original signal-to-noise ratio occurring during transmission. Curves are presented which show bounds on the degradation as a function of the quantizer step size and the channel bit error probability. Also considered is a scheme for transmitting a low-rate digital sequence by periodically interrupting the level-and-time-quantized analog signal. Three schemes are proposed for doing this, and one scheme is mathematically analyzed. It is shown that this scheme increases the degradation of the reconstructed analog signal by about 0.15 dB over that of the uninterrupted case.

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INTRODUCTION

Consider the PCM system shown in Fig. 1. The signal $x(t)$ is quantized into one of a discrete set of output levels $\{\eta_1, \eta_2, \ldots, \eta_N\}$. The output of the quantizer is then transmitted over a discrete memoryless channel and filtered to obtain an approximation of the original input signal $x(t)$. The nonlinear quantization operation and the effects of the noisy channel affect the quality of the reconstructed analog signal $z_0(t)$.

The random process $x(t)$ is assumed to be the sum of two statistically independent zero mean Gaussian processes $s(t)$ and $n(t)$, where $s(t)$ is the useful information signal and $n(t)$ is a noise signal. The quantizer is assumed to be uniform with the input–output characteristics shown in Fig. 2. For a uniform quantizer we have the relations

$$x_k - x_{k-1} = Q, \quad 1 \leq k \leq N-1$$

$$\eta_k - \eta_{k-1} = Q, \quad k = 1, 2, \ldots, N.$$

The quantizer output $y_1(t)$ will equal η_k whenever

$$x_{k-1} \leq x(t) < x_k.$$

In order to determine the signal-to-noise ratio degradation in a particular frequency band resulting from quantization and channel errors it is necessary to calculate the power spectrum of the filter output $z_0(t)$.

I. CORRELATION FUNCTION AND SPECTRAL DENSITY OF OUTPUT

The correlation function and spectral density of a quantized Gaussian process have been calculated by Velichkin,[1] and Bruce.[2] Their results have been recently extended to the case in which a noisy channel is present by Kozlenko,[3] Kozlenko *et al.*,[4] and Chan and Donaldson.[5] This paper uses methods, ideas, and notations from all of the above.

As previously stated, the process $x(t)$ is assumed to be the sum of a desired signal $s(t)$ and an independent background noise $n(t)$,

$$x(t) = s(t) + n(t).$$

Since $s(t)$ and $n(t)$ are Gaussian it follows that $x(t)$ is Gaussian with variance

$$\sigma^2 = \sigma_s^2 + \sigma_n^2$$

where σ_s^2 and σ_n^2 are the variances of $s(t)$ and $n(t)$, respectively.

Define the normalized autocorrelation function $\rho(\tau)$ by

$$\rho(\tau) = \frac{E[x(t)x(t+\tau)]}{\sigma^2} = \frac{R_x(\tau)}{\sigma^2}.$$

Using this notation, the joint density function of $x(t)$ and $x(t+\tau)$ is given by

$$p_x(\alpha, \beta; \tau) = \frac{1}{2\pi\sigma^2[1-\rho^2(\tau)]^{1/2}} \times \exp\left(-\frac{[\alpha^2+\beta^2-2\alpha\beta\rho(\tau)]}{2\sigma^2(1-\rho^2(\tau))}\right)$$

or equivalently by[6]

$$p_x(\alpha, \beta; \tau) = \frac{1}{2\pi\sigma^2} e^{-\alpha^2/2\sigma^2} e^{-\beta^2/2\sigma^2} \times \sum_{n=0}^{\infty} \frac{\rho^n(\tau)}{n!} H_n\left(\frac{\alpha}{\sigma}\right) H_n\left(\frac{\beta}{\sigma}\right) \quad (1a)$$

where the $H_n(\cdot)$ are nth-order Hermite polynomials defined by

$$H_n(r) = (-1)^n e^{r^2/2} \frac{d^n(e^{-r^2/2})}{dr^n}. \quad (1b)$$

Sampling of the quantized process is achieved by

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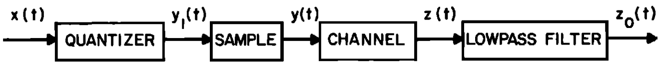

FIG. 1. PCM communciation system.

means of a periodic sequence of rectangular pulses $H_0(t)$ as shown in Fig. 3.

The time-sampled and level-quantized process $y(t)$ can be written as

$$y(t)=\sum_{i=-\infty}^{\infty} y_1(iT+t_0)H_0(t-t_0-iT)\,, \tag{2}$$

where $y_1(iT+t_0)$ is the quantizer output at time $t=iT+t_0$.

The channel output $z(t)$ is similarly written as

$$z(t)=\sum_{i=-\infty}^{\infty} z(iT+t_0)H_0(t-t_0-iT)\,. \tag{3}$$

The time autocorrelation function $\tilde{R}_z(\tau)$ of $z(t)$ is given by

$$\tilde{R}_z(\tau)=\lim_{A\to\infty}\frac{1}{2A}\int_{-A}^{A} z(t)z(t+\tau)\,dt \tag{4}$$

and for a wide-sense stationary process the ensemble autocorrelation function denoted by $R_z(\tau)$ is given by[7]

$$R_z(\tau)=E[\tilde{R}_z(\tau)]\,.$$

In order to evaluate the time correlation function of Eq. (4) define $z_N(t)$ by

$$z_N(t)=\sum_{i=-N}^{N} z(iT)H_0(t-iT)\,, \tag{5}$$

where without loss of generality the quantity t_0 has been set to zero. Using Eqs. (4) and (5) the expression for $\tilde{R}_{z_N}(\tau)$ is

$$\tilde{R}_{z_N}(\tau)=\frac{1}{2(NT+2\tau_0)}\int_{-NT-2\tau_0}^{NT+2\tau_0} z_N(t)z_N(t+\tau)\,dt \tag{6a}$$

$$=\frac{1}{2(NT+2\tau_0)}\sum_{i=-N}^{N} z(iT)z(iT+\tau)\times\int_{-NT-2\tau_0}^{NT+2\tau_0} H_0(t-iT)H_0(t-iT+\tau)\,dt \tag{6b}$$

$$=\frac{1}{2(NT+2\tau_0)}\sum_{i=-N}^{N} z(iT)z(iT+\tau)f(\tau)\,, \tag{6c}$$

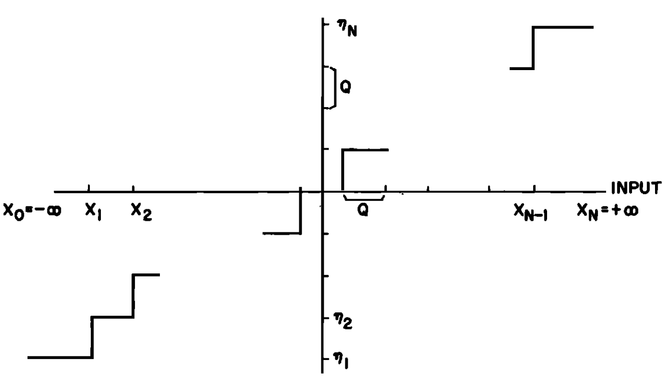

FIG. 2. Uniform quantizer characteristics.

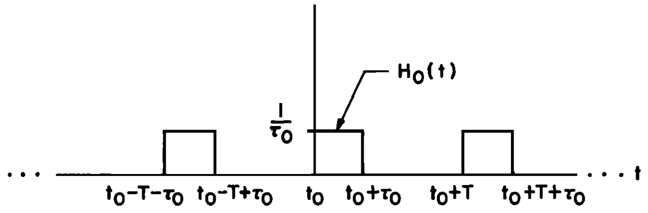

FIG. 3. Sampling waveform.

where $f(\tau)$ is the time correlation function of the sampling pulse $H_0(t)$ and is given by

$$f(\tau)=\int_{-\tau_0}^{\tau_0} H_0(t)H_0(t+\tau)\,dt \tag{7}$$

and is shown in Fig. 4. If the process $z(t)$ is at least wide-sense stationary it is known that[7]

$$R_z(\tau)=E[\tilde{R}_z(\tau)]=\lim_{A\to\infty}\frac{1}{2A}\int_{-A}^{A} E[z(t)z(t+\tau)]\,;$$

therefore, in terms of $\tilde{R}_{z_N}(\tau)$ we have

$$R_z(\tau)=E[\tilde{R}_z(\tau)]=\lim_{N\to\infty} E[\tilde{R}_{z_N}(\tau)]\,. \tag{8}$$

Assuming that $z(t)$ is at least wide-sense stationary and performing the operations indicated in Eq. (8) yields

$$R_z(\tau)=\lim_{N\to\infty}\frac{f(\tau)}{2(NT+2\tau_0)}\sum_{i=-N}^{N} E[z(iT)z(iT+\tau)]$$

$$=f(\tau)\lim_{N\to\infty}\frac{2N+1}{2(NT+2\tau_0)}E[z(T)z(T+\tau)]$$

$$=[f(\tau)/T]E[z(t)z(t+\tau)]\,, \tag{9a}$$

where t is some arbitrary time.

In order to visualize $R_z(\tau)$ it is convenient to consider some special values of τ:

(1) $-\tau_0\leqslant\tau\leqslant\tau_0$

$$R_z(\tau)=\frac{f(\tau)}{T}E[z(t)z(t+\tau)]=\frac{f(\tau)}{T}E[z^2]\,.$$

(2) $\lambda T-\tau_0\leqslant\tau\leqslant\lambda T+\tau_0$ for integer λ

$$R_z(\tau)=[f(\tau)/T]E[z(t)z(t+\lambda T)]\,.$$

Thus, in general we have

$$R_z(\tau)=\frac{f(\tau)}{T}E[z(t)z(t+\lambda T)],\quad \tau\in\bigcup_{\lambda=-\infty}^{\infty}[\lambda T-\tau_0,\lambda T+\tau_0]$$

$$R_z(\tau)=0,\quad \tau \text{ otherwise}\,. \tag{9b}$$

The form of $R_z(\tau)$ is illustrated in Fig. 5.

In order to completely determine $R_z(\tau)$, the expected values indicated in Eq. (9b) must be determined. There are two cases corresponding to $\lambda=0$ or $\lambda\neq 0$ which are

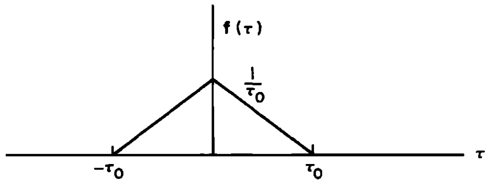

FIG. 4. Correlation function of sampling pulse.

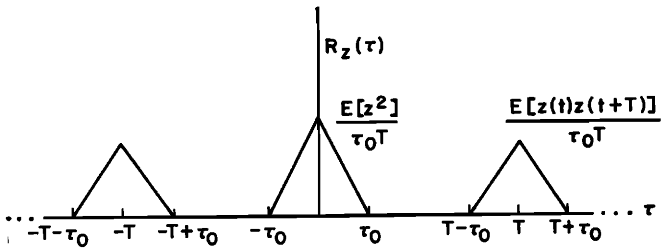

FIG. 5. Illustration of the form of $R_Z(\tau)$.

best treated separately.

Case I: $\lambda = 0$

$$E[z^2] = \sum_{j=1}^{N} \eta_j^2 \operatorname{Prob}[z(t) = \eta_j] .$$

Define channel transition probabilities P_{ij} by

$$P_{ij} = \operatorname{Prob}[z(t) = \eta_j | y(t) = \eta_i] . \tag{10}$$

Then $\operatorname{Prob}[z(t) = \eta_j]$ is given by

$$\begin{aligned}\operatorname{Prob}[z(t) = \eta_j] &= \sum_{i=1}^{N} \operatorname{Prob}[z(t) = \eta_j | y(t) = \eta_i] \\ &\quad \times \operatorname{Prob}[y(t) = \eta_i] \\ &= \sum_{i=1}^{N} P_{ij} \operatorname{Prob}[y(t) = \eta_i] ,\end{aligned} \tag{11}$$

where

$$\operatorname{Prob}[y(t) = \eta_i] = \frac{1}{(2\pi\sigma^2)^{1/2}} \int_{x_{i-1}}^{x_i} e^{-x^2/2\sigma^2} dx .$$

Thus,

$$E[z^2] = \sum_{j=1}^{N} \sum_{i=1}^{N} \eta_j^2 P_{ij} \operatorname{Prob}[y(t) = \eta_i] .$$

Case II: $\lambda \neq 0$, λ integer

$$E[z(t)z(t+\lambda T)] = \sum_{j=1}^{N} \sum_{k=1}^{N} \eta_j \eta_k \operatorname{Prob}[z(t) = \eta_j, z(t+\lambda T) = \eta_k] .$$

The joint probability in the above equation may be written as

$$\begin{aligned}&\operatorname{Prob}[z(t) = \eta_j, z(t+\lambda T) = \eta_k] \\ &\quad = \sum_{l=1}^{N} \sum_{m=1}^{N} \operatorname{Prob}[z(t) = \eta_j, z(t+\lambda T) \\ &\quad = \eta_k | y(t) = \eta_l, y(t+\lambda T) = \eta_m] \\ &\qquad \times \operatorname{Prob}[y(t) = \eta_l, y(t+\lambda T) = \eta_m] .\end{aligned}$$

Furthermore, the discrete memoryless channel is characterized by[8]

$$\begin{aligned}&\operatorname{Prob}[z(t) = \eta_j, z(t+\lambda T) = \eta_k | y(t) = \eta_l, y(t+\lambda T) = \eta_m] \\ &\quad = \operatorname{Prob}[z(t) = \eta_j | y(t) = \eta_l] \\ &\qquad \times \operatorname{Prob}[z(t+\lambda T) = \eta_k | y(t+\lambda T) = \eta_m] \\ &\quad = P_{lj} P_{mk} .\end{aligned}$$

Thus,

$$\begin{aligned}&\operatorname{Prob}[z(t) = \eta_j, z(t+\lambda T) = \eta_k] \\ &\quad = \sum_{l=1}^{N} \sum_{m=1}^{N} P_{lj} P_{mk} \operatorname{Prob}[y(t) = \eta_l, y(t+\lambda T) = \eta_m] .\end{aligned}$$

The joint probability involving the y variables can be calculated as follows:

$$\begin{aligned}\operatorname{Prob}[y(t) = \eta_l, y(t+\lambda T) = \eta_m] &= \operatorname{Prob}[x_{l-1} \leq x(t) < x_l, x_{m-1} \leq x(t+\lambda T) < x_m] = \int_{x_{l-1}}^{x_l} \int_{x_{m-1}}^{x_m} p_x(\alpha, \beta; \lambda T) \, d\alpha \, d\beta \\ &= \int_{x_{l-1}}^{x_l} \int_{x_{m-1}}^{x_m} \left[\frac{1}{2\pi\sigma^2} e^{-\alpha^2/2\sigma^2} e^{-\beta^2/2\sigma^2} + \frac{1}{2\pi\sigma^2} e^{-\alpha^2/2\sigma^2} e^{-\beta^2/2\sigma^2} \sum_{n=1}^{\infty} \frac{\rho^n(\lambda T)}{n!} H_n\left(\frac{\alpha}{\sigma}\right) H_n\left(\frac{\beta}{\sigma}\right) \right] d\alpha \, d\beta .\end{aligned}$$

The above series may be integrated term by term (Cramer[9]) so that

$$\begin{aligned}\operatorname{Prob}[y(t) = \eta_l, y(t+T) = \eta_m] &= \operatorname{Prob}[x_{l-1} \leq x(t) < x_l] \operatorname{Prob}[x_{m-1} \leq x(t+\lambda T) < x_m] \\ &\quad + \sum_{n=1}^{\infty} \frac{\rho^n(\lambda T)}{n!} \int_{x_{l-1}}^{x_l} \frac{e^{-\alpha^2/2\sigma^2} H_n(\alpha/\sigma) d\alpha}{(2\pi\sigma^2)^{1/2}} \int_{x_{m-1}}^{x_m} \frac{e^{-\beta^2/2\sigma^2}}{(2\pi\sigma^2)^{1/2}} H_n\left(\frac{\beta}{\sigma}\right) d\beta \\ &= \operatorname{Prob}[x_{l-1} \leq x(t) < x_l] \operatorname{Prob}[x_{m-1} \leq x(t+\lambda T) < x_m] + \sum_{n=1}^{\infty} \frac{\rho^n(\ T)}{n!} \Gamma_n(l) \Gamma_n(m) ,\end{aligned}$$

where $\Gamma_n(\cdot)$ is defined by[5]

$$\Gamma_n(m) = \frac{(-1)^n}{(2\pi)^{1/2}} \left(\frac{d^{n-1}(e^{-r^2/2})}{dr^{n-1}} \bigg|_{r=x_m/\sigma} - \frac{d^{n-1}(e^{-r^2/2})}{dr^{n-1}} \bigg|_{r=x_{m-1}/\sigma} \right) .$$

Thus, we have

$$\operatorname{Prob}[z(t) = \eta_j, z(t+\lambda T) = \eta_m] = \sum_{l=1}^{N} \sum_{m=1}^{N} P_{lj} P_{mk} \left(\operatorname{Prob}[x_{l-1} \leq x(t) < x_l] \operatorname{Prob}[x_{m-1} \leq x(t+\lambda T) < x_m] + \sum_{n=1}^{\infty} \frac{\rho^n(\lambda T)}{n!} \Gamma_n(l) \Gamma_n(m) \right) .$$

Using the above expression and some straightforward algebra the expression for $E[z(t)z(t+\lambda T)]$ becomes

$$\begin{aligned}E[z(t)z(t+\lambda T)] &= \left(\sum_{j=1}^{N} \sum_{l=1}^{N} \eta_j P_{lj} \operatorname{Prob}[x_{l-1} \leq x(t) < x_l] \right) \left(\sum_{k=1}^{N} \sum_{m=1}^{N} P_{mk} \eta_k \operatorname{Prob}[x_{m-1} \leq x(t+\lambda T) < x_m] \right) \\ &\quad + \sum_{n=1}^{\infty} \frac{\rho^n(\lambda T)}{n!} \left(\sum_{j=1}^{N} \sum_{l=1}^{N} \eta_j P_{lj} \Gamma_n(l) \right) \left(\sum_{k=1}^{N} \sum_{m=1}^{N} \eta_k P_{mj} \Gamma_n(m) \right) = (E[z])^2 + \sum_{n=1}^{\infty} \rho^n(\lambda T) b_n^2 ,\end{aligned} \tag{12}$$

where the b_n^2 terms are defined by

$$b_n^2=\frac{1}{n!}\left(\sum_{j=1}^{N}\sum_{i=1}^{N}\eta_j P_{ij}\Gamma_n(l)\right)^2 . \tag{13}$$

Assuming a uniform quantizer and symmetric channel $(P_{ij}=P_{ji})$ we have $E[z]=0$.

Using Eq. (12) the expression for $R_z(\tau)$ given by Eq. (9b) may be rewritten as

$$R_z(\tau)=\frac{f(\tau)}{T}E[z^2]+\left(\sum_{\lambda=1}^{\infty}\frac{f(\tau+\lambda T)}{T}+\frac{f(\tau-\lambda T)}{T}\right)\times\left(\sum_{n=1}^{\infty}b_n^2\rho^n(\lambda T)\right)\, ; \tag{14}$$

we note that $E[z^2]$ can be written as [Eq. (12) with $\lambda=0$]

$$E[z^2]=\sum_{n=1}^{\infty}b_n^2 .$$

Thus, Eq. (14) becomes

$$R_z(\tau)=\frac{1}{T}\sum_{\lambda=-\infty}^{\infty}f(\tau+\lambda T)\sum_{n=1}^{\infty}b_n^2\rho^n(\lambda T) . \tag{15a}$$

Due to the nature of $R_z(\tau)$ (see Fig. 5) the order of summation may be interchanged in Eq. (15a) to yield

$$R_z(\tau)=\frac{1}{T}\sum_{n=1}^{\infty}b_n^2\sum_{\lambda=-\infty}^{\infty}\rho^n(\lambda T)f(\tau+\lambda T) . \tag{15b}$$

The power spectral density $S_z(\omega)$ is given by

$$S_z(\omega)=\int_{-\infty}^{\infty}R_z(\tau)e^{-j\omega\tau}\,d\tau =\frac{1}{T^2}\sum_{n=1}^{\infty}b_n^2F(\omega)\sum_{\lambda=-\infty}^{\infty}S_n\left(\omega-\frac{2\pi\lambda}{T}\right), \tag{16}$$

where

$$S_n(\omega)=\int_{-\infty}^{\infty}\rho^n(\tau)e^{-j\omega\tau}\,d\tau ,$$

$$F(\omega)=\int_{-\infty}^{\infty}f(\tau)e^{-j\omega\tau}\,d\tau ,$$

and we have used the identity (Rowe[10])

$$\int_{-\infty}^{\infty}\sum_{\lambda=-\infty}^{\infty}\rho^n(\lambda T)f(\tau-\lambda T)e^{-j\omega\tau}\,d\tau=\frac{F(\omega)}{T}\sum_{\lambda=-\infty}^{\infty}S_n\left(\omega-\frac{2\pi\lambda}{T}\right).$$

The spectral component associated with the input signal $x(t)$ is $S_1(\omega)$; all other components are spectral distortions of the input due to the combined effects of quantization and channel errors.

Two observations concerning Eq. (16) are in order at this point. First, the expression for $S_z(\omega)$ is completely general in the sense that $F(\omega)$ need not be the transform of the correlation of a rectangular pulse. In fact Eq. (16) is true when any arbitrary function, time limited as is $H_0(t)$ of Fig. 3, is used as a sampling pulse. Second, when rectangular pulses are used two cases are of practical interest. They correspond to natural sampling when $\tau_0\to 0$, and sample and hold when $\tau_0=T$. The remainder of this paper will assume the former case, i.e., $\tau_0\to 0$. This assumption is made for ease of analysis only. Using the assumption we have

$$F(\omega)=1\quad \omega\in(-\infty,\infty)\, ,$$

so that

$$S_z(\omega)=\frac{1}{T^2}\sum_{n=1}^{\infty}b_n^2\sum_{\lambda=-\infty}^{\infty}S_n\left(\omega-\frac{2\pi\lambda}{T}\right) =\frac{1}{T^2}b_1^2\sum_{\lambda=-\infty}^{\infty}S_1\left(\omega-\frac{2\pi\lambda}{T}\right)+\frac{1}{T^2}\sum_{n=2}^{\infty}b_n^2\sum_{\lambda=-\infty}^{\infty}S_n\left(\omega-\frac{2\pi\lambda}{T}\right). \tag{17}$$

II. FILTER OUTPUT SPECTRAL DENSITY AND OUTPUT SIGNAL-TO-NOISE RATIO RESULTS

To proceed further we will make the following assumptions:

(1) The reconstruction filter is an ideal low-pass filter.

(2) The spectral density function $S_1(\omega)$ is bandlimited to $[-2\pi B, 2\pi B]$, i.e.,

$$S_1(\omega)=0,\quad \omega\notin[-2\pi B, 2\pi B] .$$

(3) The total power of the input process satisfies

$$\sigma^2=\sigma_s^2+\sigma_n^2\simeq\sigma_n^2 .$$

This corresponds to the case of weak signals in strong noise.

(4) The sampling time T satisfies the inequality

$$T\leq 1/2B .$$

Using these assumptions the spectral density of the output $z_0(t)$, denoted by $S_{z_0}(\omega)$ is given by

$$S_{z_0}(\omega)=\frac{1}{T^2}b_1^2S_1(\omega)+\left[\frac{1}{T^2}\sum_{n=2}^{\infty}b_n^2\sum_{\lambda=-\infty}^{\infty}S_n\left(\omega-\frac{2\pi\lambda}{T}\right)\right]|H(\omega)|^2, \tag{18}$$

where $|H(\omega)|^2$ is the low-pass filter transfer function shown in Fig. 6.

The spectral density of the noise contained in the output is

$$S_{z_0}(\omega)-\frac{1}{T^2}b_1^2S_1(\omega)=\left[\frac{1}{T^2}\sum_{n=2}^{\infty}b_n^2\sum_{\lambda=-\infty}^{\infty}S_n\left(\omega-\frac{2\pi\lambda}{T}\right)\right]|H(\omega)|^2 .$$

The above expression is an exact expression for the noise, but is very difficult to use to obtain numerical results. In order to obtain some useful results $S_1(\omega)$ will be assumed to be a bandlimited white spectrum given by

$$S_1(\omega)=\begin{cases}1/2B, & \omega\in[-2\pi B, 2\pi B]\\ 0, & \text{otherwise.}\end{cases}$$

Using this assumption the normalized autocorrelation function is

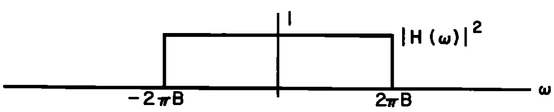

FIG. 6. Low-pass filter transfer function.

$\rho(\lambda T) = \sin 2\pi B\lambda T / 2\pi B\lambda T$.

The output noise spectral density in the band $[-2\pi B, 2\pi B]$ denoted by $N_1(\omega)$ is given by

$$N_1(\omega) = \frac{|H(\omega)|^2}{T^2}\sum_{n=2}^{\infty} b_n^2 \sum_{\lambda=-\infty}^{\infty} S_n\left(\omega - 2\pi\frac{\lambda}{T}\right)$$

$$= \frac{|H(\omega)|^2}{T^2}\sum_{n=2}^{\infty} b_n^2 T \sum_{\lambda=-\infty}^{\infty} \rho^n(\lambda T)e^{-j\omega\lambda T}$$

$$= \frac{|H(\omega)|^2}{T^2}\sum_{n=2}^{\infty} b_n^2 T\left[1 + 2\sum_{\lambda=1}^{\infty}\rho^n(\lambda T)\cos\omega\lambda T\right]. \quad (19)$$

We will denote by $N(\omega)$ the spectral density obtained from Eq. (19) when

$T = 1/2B$,

which is the minimum sampling time. Thus,

$$N(\omega) = \frac{1}{2BT^2}\sum_{n=2}^{\infty} b_n^2, \quad (20)$$

since

$\rho^n(\lambda T) = \rho(\lambda/2B) = 0$,

and the T^2 term is retained in Eq. (20) to remain consistent with the first term of Eq. (18).

A bound on the noise spectrum $N_1(\omega)$ can be established by noting that

$$N_1(\omega) \leq \frac{1}{T^2}\sum_{n=2}^{\infty} b_n^2 T\left[1 + 2\sum_{\lambda=1}^{\infty}\rho^2(\lambda T)\right],$$

but

$$T\left[1 + 2\sum_{\lambda=1}^{\infty}\rho^2(\lambda T)\right] = T\left[1 + \frac{2}{(2\pi BT)^2}\sum_{\lambda=1}^{\infty}\frac{\sin^2 2\pi B\lambda T}{\lambda^2}\right]$$

$$= T[1 + (1/2BT)(1 - 2BT)]$$

$$= 1/2B,$$

where the above summation is obtained from Ref. 11. Therefore, in general the noise spectral density satisfies the inequality

$$N_1(\omega) \leq N(\omega) = \frac{1}{2BT^2}\sum_{n=2}^{\infty} b_n^2, \quad \omega \in [-2\pi B, 2\pi B],$$

with equality holding whenever

$T = 1/2B$.

In terms of $N_1(\omega)$ the filter output spectral density is given by

$S_{z_0}(\omega) = (1/T^2)b_1^2 S_1(\omega) + N_1(\omega)$.

We also note for computational purposes that

$$\sum_{n=2}^{\infty} b_n^2 = E[z^2] - b_1^2. \quad (21)$$

The signal-to-noise ratio at the filter output will be evaluated for small frequency bands $\Delta\omega$ as illustrated in Fig. 7.

The following definitions will be used in the remainder of this report:

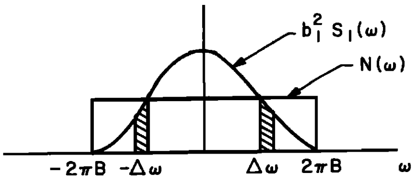

FIG. 7. Pertaining to the determination of signal-to-noise ratio.

$SNR_o(\Delta\omega)$ = Output signal-to-noise power ratio in the $\Delta\omega$ band.

$SNR_i(\Delta\omega)$ = Quantizer input signal to noise power ratio in the $\Delta\omega$ band.

$P_s(\Delta\omega)$ = Filter output signal power in $\Delta\omega$ band.

$P_n(\Delta\omega)$ = Filter output noise power in $\Delta\omega$ band associated with quantizer input noise.

$S_1(\omega)$ = Quantizer input noise spectral density. $S_1(\omega)$ is assumed to be white bandlimited noise with spectral density

$$S_1(\omega) = \frac{\sigma_n^2}{2B(\sigma_n^2 + \sigma_s^2)} \simeq \frac{1}{2B}, \quad \omega \in [-2\pi B, 2\pi B].$$

See assumption (3) at the beginning of Sec. II.

$N_1(\Delta\omega)$ = Filter output noise power in the $\Delta\omega$ band due to quantization and channel errors.

$\mathrm{Deg}(\Delta\omega) = 10\log SNR_o(\Delta\omega) - 10\log SNR_i(\Delta\omega)$

= Signal-to-noise power ratio degradation in dB in the $\Delta\omega$ band.

Using these definitions the following relations are obtained:

$$P_n(\Delta\omega) = \frac{b_1^2}{\pi T^2}\int_{\Delta\omega} S_n(\omega)\, d\omega = \frac{b_1^2}{\pi T^2}\frac{\Delta\omega}{2B},$$

$$P_s(\Delta\omega) = \frac{b_1^2}{\pi T^2}\int_{\Delta\omega} S_s(\omega)\, d\omega,$$

$$N_1(\Delta\omega) \leq \frac{1}{\pi}\int_{\Delta\omega} N(\omega)\, d\omega = \frac{\Delta\omega}{2\pi BT^2}\sum_{n=2}^{\infty} b_n^2,$$

$$SNR_i(\Delta\omega) = \frac{P_s(\Delta\omega)}{P_n(\Delta\omega)},$$

$$SNR_o(\Delta\omega) = \frac{P_s(\Delta\omega)}{P_n(\Delta\omega) + N(\Delta\omega)} = \frac{P_s(\Delta\omega)/P_n(\Delta\omega)}{1 + N_1(\Delta\omega)/P_n(\Delta\omega)}$$

$$= \frac{SNR_i(\Delta\omega)}{1 + N_1(\Delta\omega)/P_n(\Delta\omega)}.$$

Therefore,

$$\frac{SNR_o(\Delta\omega)}{SNR_i(\Delta\omega)} = \frac{1}{1 + N_1(\Delta\omega)/P_n(\Delta\omega)} \geq \left(1 + \frac{\sum_{n=2}^{\infty} b_n^2}{b_1^2}\right)^{-1}.$$

$$0 \geq \mathrm{Deg}(\Delta\omega) = 10\log SNR_o(\Delta\omega) - 10\log SNR_i(\Delta\omega)$$

$$= 10\log\left(1 + \sum_{n=2}^{\infty} b_n^2/b_1^2\right)^{-1}, \quad (22)$$

and Eq. (22) establishes a bound on the signal-to-noise power ratio degradation.

In order to obtain numerical results it is necessary

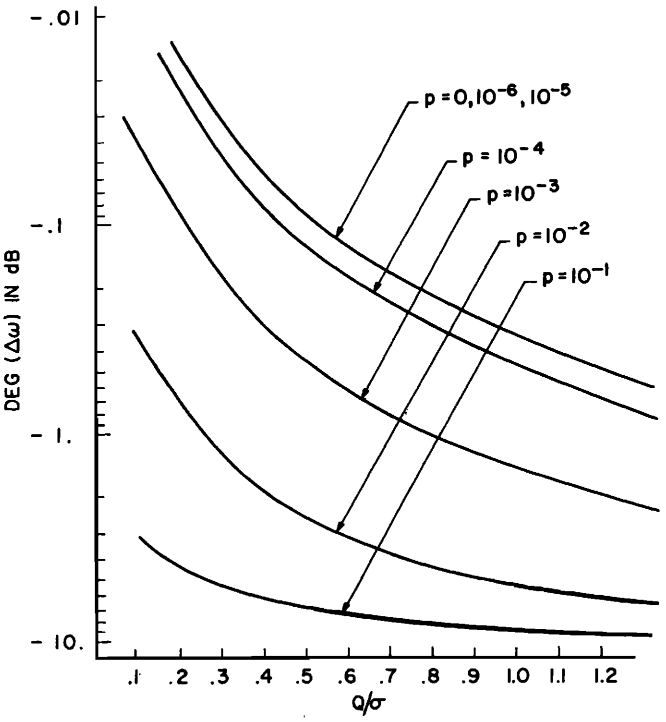

FIG. 8. Output signal-to-noise ratio degradation.

to evaluate the b_n^2 terms of Eq. (22). This in turn requires specification of the P_{ij} of Eq. (10). One specification, and the one used here, is given by

$$P_{ij}=p^{d_{ij}}(1-p)^{d-d_{ij}}, \tag{23}$$

where

$d=\log_2 N$, $N=2^k$ for some integer k

d_{ij} = Hamming distance between quantizer output levels i and j

p = bit error probability.

Equation (23) results when the quantizer output levels are coded in a natural binary code and transmitted over a discrete memoryless channel having bit error probability p (Ref. 5). Using the P_{ij} of Eq. (23) and a quantizer with $N=64$ levels, numerical values of the degradation bound were obtained and are presented in Fig. 8. In Fig. 8 the bound on the degradation Deg($\Delta\omega$) in dB is plotted on the vertical axis as a function of Q/σ with p as a running parameter. The quantity Q is the quantizer step size and σ is the rms power of the input to the quantizer. It should be noted that with an even number of steps the quantity $E[z]$ does not actually equal zero. However, for the case when $N=64$, and $Q/\sigma \leq 1.2$ the expected value of z is very small and may be assumed zero with negligible loss of accuracy in the results.

III. DEGRADATION DUE TO PERIODIC INTERRUPTION OF DATA SEQUENCE BY AN INDEPENDENT SEQUENCE

It has been proposed that a low rate digital sequence could be transmitted by periodically interrupting a higher rate sequence without a significant loss in signal-to-noise ratio of the analog signal reconstructed from the high rate signal. This kind of situation might occur, for example, if a teletype signal were transmitted by periodically interrupting a higher rate data sequence. Such a system is illustrated in Fig. 9. Every T_1 seconds ($T_1 > T$), the signal $y(t)$ is interrupted by an independent source $w(t)$. The value $y(kT_1)$ is replaced by the value $w(kT_1)$ at the interruption times kT_1, $k=0, \pm 1, \pm 2, \ldots$. The new signal thus formed is then transmitted over the discrete memoryless channel as before. The effect of the interrupting signal on the original signal is the only concern of this analysis. Therefore, since the quality of the received interrupting signal is not considered, the interrupting signal may be placed at the channel output as indicated by the dashed line shown in Fig. 9. Furthermore, the interrupting signal will be modeled as a discrete random variable, independent of the original data signal, the value of which is equally likely to be any of the allowable quantizer output values, and whose value at any one time is independent of its value at any other time. Mathematically this means

$$\begin{aligned}&\mathrm{Prob}[w(kT_1)=\eta_j]=1/N, && j=1,2,\ldots,N,\\ & && k=0,\pm1,\pm2,\ldots,\\ &E[w(kT_1)z(lT)]=0, && \text{all } k,l,\\ &E[w(k_1T_1)w(k_2T_1)]=0, && k_1\neq k_2.\end{aligned}$$

The $z(t)$ signal is interrupted every $T_1=KT$ seconds for some fixed integer K. The autocorrelation function of this interrupted signal is obtained in the same manner as in the Introduction. Define truncated signal $z_{Nk}(t)$ by [see Eq. (5)]

$$z_{NK}(t)=\sum_{i=-NK}^{NK} z(iT)H_0(t-iT), \tag{5a}$$

where it is understood that in Eq. (5a) the values $w(\cdot)$ are substituted for corresponding $z(\cdot)$ values at the appropriate times. The time autocorrelation function $R_{z_{NK}}(\tau)$ is evaluated in steps as follows:

(1) $-\tau_0 \leq \tau \leq \tau_0$

$$\tilde{R}_{z_{NK}}(\tau)=\frac{1}{2(NKT+2\tau_0)}\int_{-NK-2\tau_0}^{NK+2\tau_0} z_{NK}(t)z_{NK}(t+\tau)\,dt$$

$$=\frac{f(\tau)}{2(NKT+2\tau_0)}\sum_{i=-NK}^{NK} z(iT)z(iT+\tau)$$

and

$$E[\tilde{R}_{z_{NK}}(\tau)]=f(\tau)E[z^2]\frac{2NK+1-(2N+1)}{2(NKT+2\tau_0)}+f(\tau)E[w^2]\frac{2N+1}{2(NKT+2\tau_0)},$$

so that

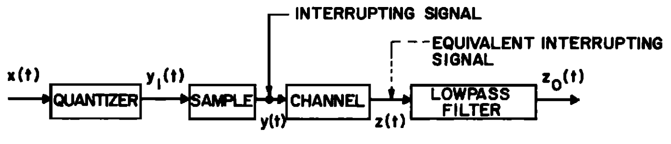

FIG. 9. Illustration of an interruption scheme.

$$R_z(\tau)=\lim_{N\to\infty}E[\tilde{R}_{z_{NK}}(\tau)]$$

$$=f(\tau)E[z^2]\frac{K-1}{KT}+f(\tau)E[w^2]\frac{1}{KT}.$$

(2) $mKT-\tau_0\leq\tau\leq mKT+\tau_0, m=\pm1,\pm2,\ldots$

$$\tilde{R}_{z_{NK}}(\tau)=\frac{f(\tau)}{2(NKT+2\tau_0)}\sum_{i=-NK}^{NK}z(iT)z(iT+\tau),$$

$$E[\tilde{R}_{z_{NK}}(\tau)]=f(\tau)\frac{2NK+1-(2N+1)}{2(NKT+2\tau_0)}E[z(t)z(t+\tau)].$$

Hence,

$$R_z(\tau)=\lim_{N\to\infty}E[\tilde{R}_{z_{NK}}(\tau)]=f(\tau)\frac{K-1}{KT}E[z(t)z(t+\tau)].$$

(3) Values of τ for which $f(\tau)\neq0$ and are not included in (1) or (2)

$$E[\tilde{R}_{z_{NK}}(\tau)]=f(\tau)E[z(t)z(t+\tau)]\frac{2NK+1-(4N+1)}{2(NKT+2\tau_0)}.$$

Hence,

$$R_z(\tau)=\lim_{N\to\infty}E[\tilde{R}_{z_{NK}}(\tau)]=f(\tau)\frac{K-2}{KT}E[z(t)z(t+\tau)].$$

Thus, the ensemble autocorrelation function $R_z(\tau)$ is given by

$$R_z(\tau)=\frac{K-1}{KT}f(\tau)\sum_{n=1}^{\infty}b_n^2+\frac{f(\tau)}{KT}E[w^2],\quad -\tau_0\leq\tau\leq\tau_0$$

$$=\frac{K-1}{KT}f(\tau-mKT)\sum_{n=1}^{\infty}b_n^2\rho^n(mKT),\quad mKT-\tau_0\leq\tau\leq mKT+\tau_0\ m=\pm1,\pm2,\ldots$$

$$=\frac{K-2}{KT}f(t-\lambda T)\sum_{n=1}^{\infty}b_n^2\rho^n(\lambda T),\quad \text{integer } \lambda \text{ and } \lambda\neq0,K,-K,2K,-2K,\ldots$$

Combining the above special cases, an expression for $R_z(\tau)$ valid for all values of τ is

$$R_z(\tau)=\frac{K-2}{KT}\sum_{\lambda=-\infty}^{\infty}f(\tau-\lambda T)\sum_{n=1}^{\infty}b_n^2\rho^n(\lambda T)$$
$$+\frac{1}{KT}\sum_{\lambda=-\infty}^{\infty}f(\tau-\lambda KT)\sum_{n=1}^{\infty}b_n^2\rho^n(\lambda KT)+\frac{1}{KT}f(\tau)E[w^2].\tag{24}$$

Using Eq. (24) the spectral density is determined to be

$$S_z(\omega)=\frac{K-2}{KT^2}F(\omega)\sum_{\lambda=-\infty}^{\infty}\sum_{n=1}^{\infty}b_n^2S_n\left(\omega-\frac{2\pi\lambda}{T}\right)$$
$$+\frac{1}{K^2T^2}F(\omega)\sum_{\lambda=-\infty}^{\infty}\sum_{n=1}^{\infty}b_n^2S_n\left(\omega-\frac{2\pi\lambda}{KT}\right)+\frac{1}{KT}F(\omega)E[w^2].\tag{25}$$

The first and second terms of Eq. (25) were obtained by application of the result given in Ref. 10. Certain components of the middle term of Eq. (25) may be combined with the first term to yield

$$S_z(\omega)=\left(\frac{K-1}{K}\right)^2\frac{F(\omega)}{T^2}\sum_{\lambda=-\infty}^{\infty}\sum_{n=1}^{\infty}b_n^2S_n\left(\omega-\frac{2\pi\lambda}{T}\right)$$
$$+\frac{F(\omega)}{K^2T^2}\sum_{\lambda=-\infty}^{\infty}\sum_{n=1}^{\infty}b_n^2S_n\left(\omega-\frac{2\pi\lambda}{KT}\right)+\frac{F(\omega)}{KT}E[w^2].\tag{25a}$$

$$\lambda\neq0,K,-K,2K,-2K,\ldots$$

Restricting attention as before to the case of impulses presented to the low-pass filter $[F(\omega)=1]$ the first term of Eq. (25a) is of the same form as Eq. (17), so that the filter output due to this term is given by

$$\left(\frac{K-1}{K}\right)^2\frac{b_1^2}{T^2}S_1(\omega)+\left(\frac{K-1}{K}\right)^2N_1(\omega),$$

where $N_1(\omega)$ is as defined previously. The middle term of Eq. (25a) is too complicated for computational purposes, but may be bounded as follows. Define the filter output spectral density $M_1(\omega)$ by

$$M_1(\omega)=\frac{|H(\omega)|^2}{KT^2}\sum_{n=1}^{\infty}b_n^2\sum_{\substack{\lambda=-\infty\\ \lambda\neq0,K,-K,\ldots}}^{\infty}S_n\left(\omega-\frac{2\pi\lambda}{KT}\right),$$

but

$$\sum_{\lambda=-\infty}^{\infty}S_n\left(\omega-\frac{2\pi\lambda}{KT}\right)=\sum_{\lambda=-\infty}^{\infty}S_n\left(\omega-\frac{2\pi\lambda}{T}\right),$$
$$+\sum_{\substack{\lambda=-\infty\\ \lambda\neq0,K,-K,\ldots}}^{\infty}S_n\left(\omega-\frac{2\pi\lambda}{KT}\right).$$

Assuming as before that $S_1(\omega)$ is white and band limited and considering the case for n greater than one we have

$$\sum_{\lambda=-\infty}^{\infty}S_n\left(\omega-\frac{2\pi\lambda}{KT}\right)=KT\left[1+2\sum_{\lambda=1}^{\infty}\rho^n(\lambda KT)\cos\omega\lambda KT\right]$$
$$\leq KT\left[1+2\sum_{\lambda=1}^{\infty}\rho^2(\lambda KT)\cos\omega\lambda KT\right]$$
$$\leq KT\left[1+2\sum_{\lambda=1}^{\infty}\rho^2(\lambda KT)\right]$$
$$\leq KT\left[1+2\sum_{\lambda=1}^{\infty}\rho^2(\lambda T)\right.$$
$$=K/2B,$$

where the above inequality holds with equality whenever

$T = 1/2B$.

Using this result it follows that

$$\sum_{\substack{\lambda=-\infty \\ \lambda\neq 0, K, -K, \ldots}}^{\infty} S_n\left(\omega - \frac{2\pi\lambda}{KT}\right) \leq \frac{K-1}{2B}, \quad n \geq 2,$$

with equality holding for the minimum sampling time. In order to bound $M_1(\omega)$ when n equals one, the sum over λ is broken into $(K-1)$ summands as follows:

$$\sum_{\substack{\lambda=-\infty \\ \lambda\neq 0, K, -K, \ldots}}^{\infty} = \sum_{\substack{\lambda=\pm(mK+1) \\ m=0,.,2,\ldots}}^{\infty} + \sum_{\substack{\lambda=\pm(mK+2) \\ m=0,1,2,\ldots}}^{\infty} + \cdots + \sum_{\substack{\lambda=\pm(mK+K-1) \\ m=0,1,2,\ldots}}^{\infty}.$$

The reason for breaking the sum over λ into $K-1$ sums is that to each value of λ in each of the above sums there corresponds an interval on the ω axis such that the intervals are not overlapping in any sum. It is easy to demonstrate that the spectra in intervals corresponding to the λ's in a particular sum are nonoverlapping. For example, consider the first sum. To show that the spectra in this sum are nonoverlapping it is sufficient to show that the following inequality holds (see Fig. 10):

$$\frac{(mK+1)2\pi}{KT} + 2\pi B \leq \frac{[(m+1)K+1]2\pi}{KT} - 2\pi B.$$

This inequality reduces to

$$4\pi B \leq 2\pi/T$$

and is true by hypothesis since we have assumed $T < 1/2B$. Using the fact that the spectra in each sum are nonoverlapping, each of the sums satisfies the inequality

$$\sum_{\lambda} S_1\left(\omega - \frac{2\pi\lambda}{KT}\right) \leq \frac{1}{2B}.$$

There are $K-1$ such sums in the total sum. Therefore, it is clear that

$$\sum_{\substack{\lambda=-\infty \\ \lambda\neq 0, K-K, \ldots}}^{\infty} S_1\left(\omega - \frac{2\pi\lambda}{KT}\right) \leq \frac{K-1}{2B}.$$

Thus, the spectral density $M_1(\omega)$ is bounded by

$$M_1(\omega) \leq M(\omega) = \frac{K-1}{K^2(2BT^2)} \sum_{n=1}^{\infty} b_n^2, \quad \omega \in [-2\pi b, 2\pi B]$$
$$= 0, \qquad \text{otherwise}.$$

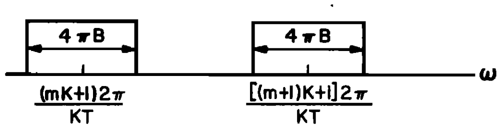

FIG. 10. Nonoverlapping intervals.

Using this expression for $M(\omega)$ the spectral density at the output of the low-pass filter is bounded by

$$S'_{z_o}(\omega) = \left(\frac{K-1}{K}\right)^2 \frac{b_1^2}{T^2} S_1(\omega) + \left(\frac{K-1}{K}\right)^2 \frac{1}{2BT^2} \sum_{n=2}^{\infty} b_n^2 + \frac{K-1}{K^2} \frac{1}{2BT^2} \sum_{n=1}^{\infty} b_n^2 + \frac{1}{KT} E[\omega^2],$$
$$\omega \in [-2\pi B, 2\pi B]. \tag{26}$$

Note that for $K=1$ all terms associated with the quantizer input reduce to zero as would be expected. The case $K=1$ is of no interest and will not be considered.

The output signal-to-noise power ratio is

$$SNR_o(\Delta\omega) = \frac{P_s(\Delta\omega)}{P_n(\Delta\omega) + N_1(\Delta\omega) + M_1(\Delta\omega) + (\Delta\omega/\pi KT)E[w^2]}$$
$$= SNR_i\left(1 + \frac{N_1(\Delta\omega)}{P_n(\Delta\omega)} + \frac{M_1(\Delta\omega)}{P_n(\Delta\omega)} \frac{(\cdot)}{P_n(\Delta\omega)}\right)^{-1}.$$

Therefore,

$$\frac{SNR_o(\Delta\omega)}{SNR_i(\Delta\omega)} \geq \left(1 + \frac{\sum_{n=2}^{\infty} b_n^2}{b_1^2} + \frac{1}{K-1} \cdot \frac{1}{b_1^2} \sum_{n=1}^{\infty} b_1^2 + (\cdot)\right)^{-1} \tag{27}$$

where $(\cdot)$ represents noise associated with $E[w^2]$, and the degradation is bounded by

$$0 \geq \mathrm{Deg}(\Delta\omega) \geq 10 \log\left(1 + \frac{\sum_{n=2}^{\infty} b_n^2}{b_1^2} + \frac{\sum_{n=1}^{\infty} b_n^2}{(K-1)b_1^2} + (\cdot)\right)^{-1}. \tag{28}$$

It is clear from Eq. (27) that if the interrupting signal is used in the reconstruction of $z_o(t)$ the signal-to-noise ratio decreases as a result of the noise power associated with $E[w^2]$. Thus, to minimize the effects of the interrupting signal it is best to remove this signal prior to analog reconstruction. There are three easily implemented schemes for doing this:

(1) Remove the interrupting signal and place a zero level in its place.

(2) Remove the interrupting signal and place the previous data signal level in its place.

(3) Remove the interrupting signal and place the average of the two adjacent data signal levels in its place.

It is believed that the resulting degradation would be least for case (3), then case (2), and most for case (1). It should be emphasized that case (1) is still preferable to using the interrupting signal in the reconstruction. This fact is obvious from Eqs. (27) and (28).

Numerical results have been obtained for case (1) in which a 2400 bps data stream composed of 6-bit quantized analog samples is interrupted by a 100 wpm (i.e., 75 bps) five-level teletype signal. The 7.5-bit teletype characters are buffered and stripped of the start and stop control bits (1 and 1.5 bits, respectively) leaving 5 data bits per character. At specified intervals a 6-bit (5 data bits plus space) teletype sample is

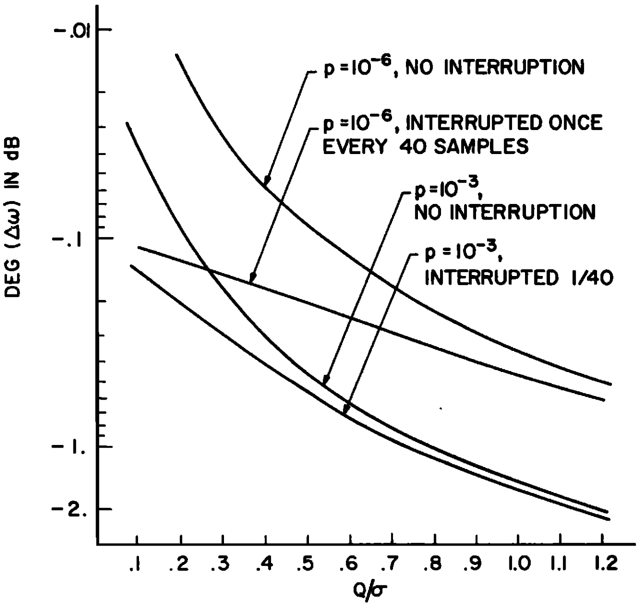

FIG. 11. Signal-to-noise ratio degradation.

clocked out of the buffer at 2400 bps and injected into the synchronous 2400 bps data stream where it replaces another 6-bit data signal sample from the data source (i.e., a time and level quantizer analog source). The effective teletype data rate is then 60 bps which requires that one out of every 40 samples of the 2400 bps data stream must be replaced by the teletype signal. Numerical results for this case, $K=40$, are presented in Figs. 11-13. The graphs in these figures are self-explanatory. The most important point to note from the figures is that interrupting the data signal every 40 samples increases the signal-to-noise ratio degradation of the reconstructed analog signal by about 0.15 dB

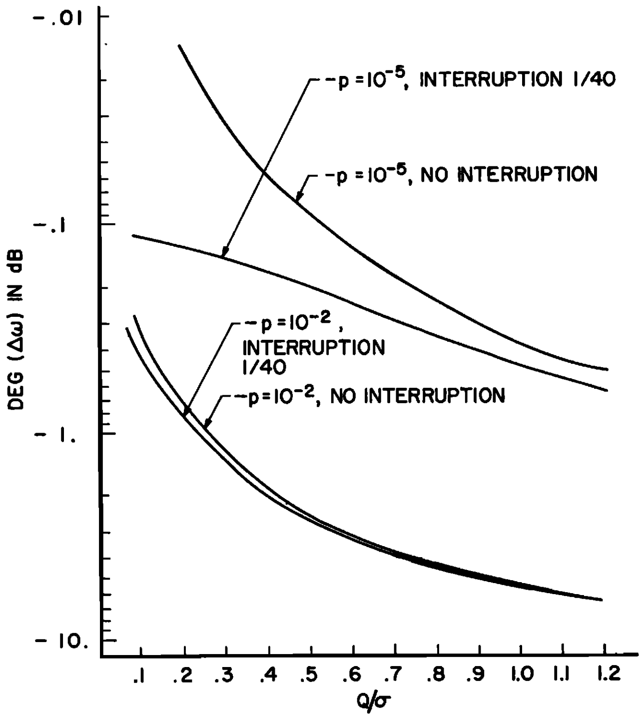

FIG. 12. Signal-to-noise ratio degradation.

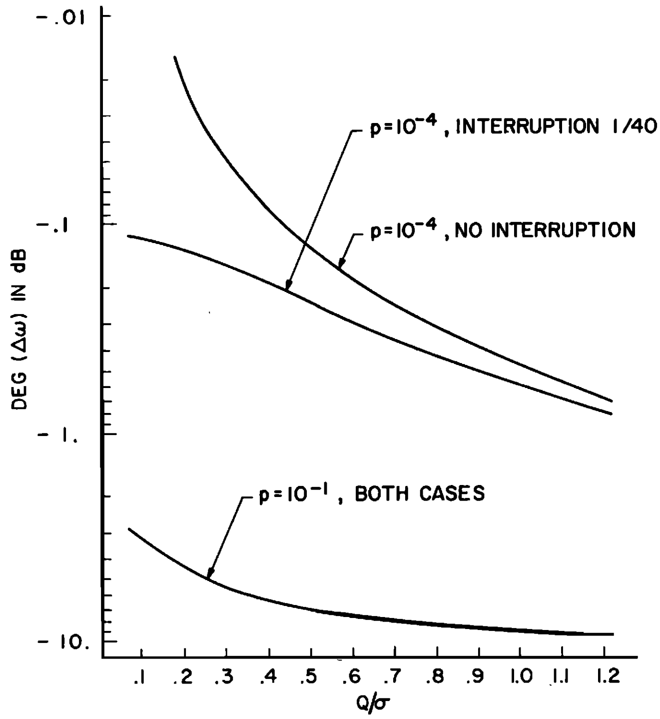

FIG. 13. Signal-to-noise ratio degradation.

over the noninterrupted case. This 0.15 dB increase holds approximately true for all values of p and Q/σ. Thus, it appears that it is possible to transmit a teletype signal on a higher rate data channel without significantly altering the quality of the reconstructed analog signal.

If the theoretical results presented herein are an accurate indication of the behavior of a real system, then it would be possible to obtain one teletype channel for every 2400-bps data channel. Such a teletype channel would not require additional bandwidth but would of course require additional hardware.

IV. SUMMARY AND CONCLUSIONS

This paper presents the results of an investigation of the signal-to-noise degradation incurred when Gaussian signals in Gaussian noise are level-quantized, time-sampled, transmitted over a discrete memoryless channel, and then reconstructed by a low-pass filter. The curves of Fig. 8 show the degradation bound as a function of the ratio of the quantizer step size to the rms noise power with the channel bit error probability as a running parameter.

A scheme for transmitting a low rate digital sequence by periodically interrupting the digital sequence formed by level and time quantizing an analog signal is discussed. A mathematical analysis of the degradation of the signal-to-noise ratio of the reconstructed analog signal is presented for the special case when the interrupting sequence is removed and replaced by an all zero sequence prior to reconstruction. It is shown that setting this sequence to all zero levels prior to reconstruction results in less degradation than if the interrupting sequence were used in the reconstruction process. It is found that, for the case in which one out of every forty of the original samples was removed, the degradation of the reconstructed analog signal-to-noise ratio increases by about 0.15 dB over that of the uninterrupted case for a wide range of quantizer-to-rms-noise-power ratios, and for a wide range of bit error probabilities. These results are shown graphically in Figs. 11, 12, and 13.

Two additional interruption schemes are proposed, and it is conjectured that these schemes would degrade the reconstructed analog signal less than the scheme that was actually analyzed. Further work is required to refute or validate these conjectures.

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