**DOUBLE COMMUTANTS OF SINGLY GENERATED OPERATOR ALGEBRAS**
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DOUBLE COMMUTANTS OF SINGLY GENERATED

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by

Thomas Rolf Turner

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of the requirements for the degree of
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**PLEASE NOTE:**

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# INTRODUCTION

This thesis is concerned with algebras of operators. Our operators will always be <u>bounded</u> linear transformations and will act on <u>complex</u> Hilbert spaces. Convergence of nets of operators will always be in the <u>weak operator topology</u> unless it is explicitly stated otherwise. We shall use an ordinary arrow, $\rightarrow$ , to indicate such weak convergence. For example, "$A_i \rightarrow A$" is always to be read "$A_i$ converges to $A$ in the weak operator topology." Arrows will also be used to indicate the action of functions, as in "X : $\mathcal{H} \rightarrow \mathcal{K}$" which is to be read "X mapping $\mathcal{H}$ into $\mathcal{K}$ ." This should cause no confusion.

In addition to our use of arrows, we introduce the notation, "wlim," to mean "limit in the weak operator topology." For example, "$A = \underset{i}{\text{wlim}}\, A_i$" means that $A$ is the limit in the weak operator topology of the operators $A_i$.

Throughout this thesis, we shall be making use of matrices with operator entries. Therefore, at this point we take a moment or two to review the elementary facts about them.

Let $\mathfrak{I}$ be an index set of arbitrary cardinality. For each i in $\mathfrak{I}$ let $\mathcal{H}_i$ be a Hilbert space. We define the <u>direct sum</u>

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$\mathcal{H} = \sum_{i \in \mathfrak{I}} \oplus \mathcal{H}_i$ of this family of Hilbert spaces to be the set of all

families $(h_i)_{i \in \mathfrak{I}}$ such that $h_i$ is in $\mathcal{H}_i$ for each $i$, and such

that $\sum_{i \in \mathfrak{I}} \|h_i\|^2$ is finite. The space $\mathcal{H}$ is made into an inner

product space by defining $( (h_i)_{i \in \mathfrak{I}}, (g_i)_{i \in \mathfrak{I}} ) = \sum_{i \in \mathfrak{I}} (h_i, g_i)$. It is

a routine matter to check that under this definition of inner product

$\mathcal{H}$ is a Hilbert space.

Now for each ordered pair $(i, j)$ in $\mathfrak{I} \times \mathfrak{I}$, let $A_{ij}$ be

an operator mapping $\mathcal{H}_j$ into $\mathcal{H}_i$. Denote for the moment by $A$

the doubly indexed family $[A_{ij}]_{i, j \in \mathfrak{I}}$. $A$ is then called a matrix

with operator entries. If there is no possibility of confusion over

the index set, we shall omit mention of it and simply write

$A = [A_{ij}]$. Suppose that for each vector $(h_i)_{i \in \mathfrak{I}}$ in $\mathcal{H}$,

$\sum_{j \in \mathfrak{I}} A_{ij} h_j$ converges absolutely to a vector $g_i$ in $\mathcal{H}_i$, and

that furthermore $\sum_{i \in \mathfrak{I}} \|g_i\|^2$ is finite. Then we may consider $A$

to be a linear transformation on $\mathcal{H}$ by defining

$A(h_i)_{i \in \mathfrak{I}} = ( \sum_{j \in \mathfrak{I}} A_{ij} h_j )_{i \in \mathfrak{I}}$. We shall, of course, be most interested

in the case in which $A$ actually turns out to be an operator.

The problem of determining whether a given matrix defines an

operator, or even a well-defined linear transformation, has in gen-

eral no really satisfactory solution. This is true even for matrices

of (one-dimensional) scalars. However, the specific examples

which one encounters usually present little difficulty. Also, there

is one general criterion which does prove useful and which we shall

appeal to later on in this thesis. The criterion may be stated as

follows: Let $\mathcal{F}$ be a finite subset of $\mathcal{I}$ . We define $A_{\mathcal{F}}$ ,

the <u>finite section</u> of A <u>associated with</u> $\mathcal{F}$ , to be the matrix

$[A_{ij}]_{i,j \in \mathcal{F}}$ . Since $\mathcal{F}$ is finite, A acts as an operator on

$$\mathcal{H}_{\mathcal{F}} = \sum_{i \in \mathcal{F}} \oplus \mathcal{H}_i .$$ We consider $\mathcal{H}_{\mathcal{F}}$ to be a subspace of $\mathcal{H}$ and

define an operator $A'_{\mathcal{F}}$ on $\mathcal{H}$ by setting $A'_{\mathcal{F}}$ equal to $A_{\mathcal{F}}$ on

$\mathcal{H}_{\mathcal{F}}$ and equal to zero on $\mathcal{H}_{\mathcal{F}}^{\perp}$ . With this terminology set down,

we can say that A defines an operator on $\mathcal{H}$ if and only if the

norms of the operators $A'_{\mathcal{F}}$ are bounded.

It is clear that if A is an operator, then each $A'_{\mathcal{F}}$ has

norm less than or equal to the norm of A. On the other hand, suppose

that the $A'_{\mathcal{F}}$ 's are bounded in norm. Order the finite subsets of

$\mathcal{I}$ by inclusion. The operators $A'_{\mathcal{F}}$ then form a net which is

easily seen to be a strongly Cauchy net on vectors having finitely

many non-zero entries. Such vectors form a dense subset of $\mathcal{H}$ ,

and since the net is bounded it therefore converges strongly to an

operator B on $\mathcal{H}$. Now take an arbitrary vector h in $\mathcal{H}$ .

When we try to apply A to h, the fact that the operators $A'_{\mathcal{F}}$

are bounded tells us that at worst we obtain a family of vectors

$(g_i)_{i \in \mathcal{J}}$ where $g_i \in \mathcal{H}_i$. This family might fail to be a vector in $\mathcal{H}$ by failing to be square summable. However, we next notice that $g_i$ must equal the ith entry of Bh for every i. Therefore, $(g_i)_{i \in \mathcal{J}}$ = Bh, which is a vector in $\mathcal{H}$. We conclude that A is equal to B and is consequently a bounded operator.

We conclude our discussion of matrices with operator entries by noting that given an arbitrary operator B on $\mathcal{H}$, we may write B as an operator matrix $[B_{ij}]$. This is done simply by setting $B_{ij}$ equal to the restriction of B to $\mathcal{H}_j$ followed by the projection from $\mathcal{H}$ onto $\mathcal{H}_i$. It is trivially checked that the action of the matrix $[B_{ij}]$, as previously defined, agrees with that of the original operator.

Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all operators on $\mathcal{H}$. If $\mathcal{S}$ is a subset of $\mathcal{L}(\mathcal{H})$, we denote by $\mathcal{S}'$ the set $\{A \in \mathcal{L}(\mathcal{H}) \mid AS = SA \text{ for all } S \text{ in } \mathcal{S}\}$. We abbreviate $(\mathcal{S}')'$ to $\mathcal{S}''$, and so on. The set $\mathcal{S}'$ is called the <u>commutant</u> of $\mathcal{S}$ and $\mathcal{S}''$ is called the <u>double commutant</u> of $\mathcal{S}$.

The main objective of this thesis is to study the question, "When is a singly generated, weakly closed algebra (with identity) of operators on Hilbert space equal to its own double commutant?" In a sense, we prove analogues of the von Neumann Double Commutant Theorem for singly generated algebras which are generated by

particular kinds of operators.

Definition: If $A$ is an operator in $\mathscr{L}(\mathcal{H})$, let $\mathcal{O}_A$ be the weakly closed algebra generated by $A$ (and the identity). If $\mathcal{O}_A = \mathcal{O}_A''$, then we shall say that $A$ is an operator in the class (dc).

In Chapter I we answer the question "When is a normal operator in the class (dc)?" using Sarason's theorem that normal operators are reflexive [18], the von Neumann Double Commutant Theorem, and the Fuglede-Putnam Theorem. We then prove a sequence of easy but useful lemmas about the double commutants of direct sums, ampliations, and adjoints. We also give sufficient conditions that $A \oplus B$ be in the class (dc) given that $A$ is. Finally, we show that in certain circumstances $A$ being in the class (dc) implies that $A \oplus 0$ is in the class (dc).

Weighted shifts are noted for being particularly tractible operators, and are resultingly much studied. True to their nature they provide a broad collection of examples of operators in the class (dc). In Chapter II we show that any one-sided weighted shift is in the class (dc) and that a two-sided weighted shift is in the class (dc) if and only if it is not invertible. The proofs make heavy use of some techniques which were developed by Shields and Wallen [20] to show that a one-sided shift with all non-zero weights generates a maximal abelian weakly closed algebra.

Chapter III is devoted to proving a theorem that serves to characterize completely the isometries in the class (dc). The theorem states that an isometry is in the class (dc) provided that its pure part does not vanish. We note that if its pure part _does_ vanish the isometry is unitary, whence the theorem about normal operators from Chapter I applies.

In their paper, "Lifting Commuting Operators" [4], Douglas, Muhly, and Pearcy ask the question: "Is there a relationship between the double commutant of a contraction and that of its minimal co-isometric extension?" A partial answer can be given as a corollary of the aforementioned theorem on isometries. If T is a contraction, its double commutant can be lifted to the double commutant of its minimal co-isometric extension _only if_ T is in the class (dc), or (trivially) if T is unitary.

In Chapter IV we generalize the well-known theorem that any linear transformation on a finite dimensional vector space generates an algebra equal to its double centralizer. "Double centralizer" is the algebraic analogue of the concept of "double commutant." On finite dimensional Hilbert spaces the two concepts agree. Our result is as follows: Let A belonging to $\mathcal{L}(\mathcal{H})$ be an algebraic operator (i.e., an operator such that there exists a polynomial p, with p(A) = 0). Then A is in the class (dc). Strangely enough, techniques from the proof of the von Neumann Double Commutant

Theorem find their way into this seemingly totally algebraic problem.

Compact n-normal operators, being closely related to compact normal operators and to operators on finite dimensional Hilbert spaces, might reasonably be expected to be in the class (dc). This expectation is heightened when it is shown (as is done in Chapter V) that every compact n-normal operator is quasi-similar to an operator in the class (dc). However, we then proceed to give an example of a compact 2-normal which is _not_ in this class. Thus we simultaneously destroy the hope that compact n-normal operators are in the class (dc) and the hope that quasi-similarity preserves this class.

In the appendix we prove a slight generalization of the result of Shields and Wallen [20] previously mentioned. This generalization is used frequently in Chapter II. In the process we find it necessary to generalize slightly a lemma of Schur [19] upon which the other proof depends. The proofs of these generalizations differ very little from the proofs of the originals. However, the generalized results are not immediately obvious from the originals, so it was deemed wise to include the proofs for the sake of completeness and rigour.

A tedious calculation needed in the proof of Proposition 2.9 was also relegated to the appendix.

In the third part of the appendix we give counter examples to some obvious conjectures about sums and products of operators in the class (dc). In the fourth and last part we list some unanswered

questions.

Finally, a word about notation. The lower case Greek letter sigma will be used throughout to denote spectrum. That is, if A is an operator, then $\sigma(A)$ denotes the spectrum of A.

# CHAPTER I

## PRELIMINARY RESULTS

We begin our discussion of the class (dc) by summarizing the elementary facts about commutants. These are stated in the following easy lemma, the proof of which is omitted.

Lemma 1.1: Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{S}$ be a subset of $\mathcal{L}(\mathcal{H})$.

(a) The set $\mathcal{S}'$ is always a weakly closed algebra with identity.

(b) $\mathcal{S} \subseteq \mathcal{S}''$ .

(c) $\mathcal{S}' = \mathcal{S}'''$ .

(d) If $\mathcal{S} \subseteq \mathcal{J}$ , then $\mathcal{J}' \subseteq \mathcal{S}'$ .

(e) If $\mathcal{S}$ is a commuting set (i.e., $\mathcal{S} \subseteq \mathcal{S}'$), then $\mathcal{S}'' \subseteq \mathcal{S}'$, whence $\mathcal{S}''$ is abelian.

Let $A \in \mathcal{L}(\mathcal{H})$. Recall that $\mathcal{O}_A$ denotes the weakly closed algebra generated by $A$ (and the identity). Note that $\mathcal{O}'_A = \{A\}'$, and that $\mathcal{O}''_A = \{A\}''$ . When it is convenient we will denote $\mathcal{O}'_A$ and $\mathcal{O}''_A$ by $(A)'$ and $(A)''$, respectively.

We now proceed to characterize normal operators in the class (dc).

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Theorem 1.2: Let $N$ be a normal operator in $\mathcal{L}(\mathcal{H})$. Then $N$ is in the class (dc) if and only if every invariant subspace for $N$ reduces $N$.

To facilitate the proof we need the following terminology.

Definition: Let $A \in \mathcal{L}(\mathcal{H})$. Then Lat $A$ denotes the set of all subspaces (closed linear manifolds) of $\mathcal{H}$ which are invariant under $A$.

Lat $A$ is easily seen to be a lattice under the usual operations of taking intersections and closed linear spans.

Definition: An operator $A$ is said to be reflexive if Lat $A \subseteq$ Lat $B$ implies that $B \in \mathcal{A}_A$.

Proof of Theorem 1.2: (a) Suppose that $N$ is in the class (dc). Let $C$ be an operator in $\mathcal{A}'_N$. This simply means that $CN = NC$. Since $N$ is normal, it follows by the Fuglede-Putnam Theorem [16, p. 9] that $N^*C = CN^*$. Thus $N^*$ commutes with $C$ for each $C$ in $\mathcal{A}'_N$ and so $N^*$ is in $\mathcal{A}''_N = \mathcal{A}_N$. Therefore, there exists a net of polynomials $p_\alpha$ such that $p_\alpha(N) \to N^*$. It follows that every subspace invariant for $N$ is invariant for $N^*$; i.e., every invariant subspace for $N$ is reducing.

(b) Suppose that every invariant subspace for $N$ reduces $N$. This says that Lat $N \subseteq$ Lat $N^*$. By a theorem of Sarason [18, p. 511] all normal operators are reflexive. It follows therefore

that $N^* \in \mathcal{A}_N$. Consequently, $\mathcal{A}_N$ is a self-adjoint algebra,

and hence by the von Neumann Double Commutant Theorem

$\mathcal{A}_N = \mathcal{A}_N''$, so N is in the class (dc).

Remark 1.3: Wermer [21, p. 275] gives a very nice

characterization of unitary operators having non-reducing invariant

subspaces. Wermer's result is that a unitary operator has a non-

reducing invariant subspace if and only if it has a direct summand

which is unitarily equivalent to the bilateral shift of multiplicity one.

We shall make use of this characterization later on.

We have now, in a sense, completely solved the problem of

determining if a normal operator is in the class (dc). In order to

investigate more intransigent operators we must build up a few tools.

To this end, let us begin considering the double commutants of direct

sums of operators.

Definition: Let $\mathcal{I}$ be an index set of arbitrary cardinality.

For each i in $\mathcal{I}$ let $\mathcal{H}_i$ be a Hilbert space, and let

$$\mathcal{H} = \sum_{i \in \mathcal{I}} \oplus \mathcal{H}_i.$$ (See page 1 of the introduction for the definition.)

For each i in $\mathcal{I}$ let $A_i$ belong to $\mathcal{L}(\mathcal{H}_i)$ and suppose that

there exists a positive constant M such that $\|A_i\| \leq M$ for each

i in $\mathcal{I}$. Then we define the direct sum $A = \sum_{i \in \mathcal{I}} \oplus A_i$ of this

family of operators by $A(h_i)_{i \in \mathcal{I}} = (A_i h_i)_{i \in \mathcal{I}}$. It is a routine

matter to check that A is in $\mathcal{L}(\mathcal{H})$.

In what follows we shall, wherever it is convenient, identify direct summands (the $\mathcal{H}_i$) with their natural embeddings in the direct sum $\mathcal{H}$.

Lemma 1.4: Let $A$ and $\mathcal{H}$ be direct sums as in the preceding definition. Let $D \in \mathcal{A}_A''$. Then $D = \sum_{i \in \mathcal{J}} \oplus D_i$ where each $D_i \in (A_i)''$.

Proof: Let $P_i$ be the projection on the ith direct summand, $\mathcal{H}_i$. Then $P_i$ is clearly in $\mathcal{A}_A'$, hence $P_i$ commutes with $D$. Therefore, each $\mathcal{H}_i$ is a reducing subspace for $D$. Let $D_i = D\big|_{\mathcal{H}_i}$ for each $i$. Then $D = \sum_{i \in \mathcal{J}} \oplus D_i$. Now let $C \in (A_j)'$ for some fixed $j$ and let $\tilde{C} = \sum_{i \in \mathcal{J}} \oplus B_i$ where

$$B_i = \begin{cases} C & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} .$$

Again $\tilde{C}$ is clearly in $\mathcal{A}_A'$, so $D\tilde{C} = \tilde{C}D$, so $D_j C = CD_j$. Since this holds for any $C$ in $(A_j)'$ we see that $D_j$ is in $(A_j)''$. This is true for arbitrary $j$.

Definition: Let $(\mathcal{A}_i)_{i \in \mathcal{J}}$ be a family of operator algebras, $\mathcal{A}_i$ acting on the space $\mathcal{H}_i$. We define the direct sum $\mathcal{A} = \sum_{i \in \mathcal{J}} \oplus \mathcal{A}_i$ to be the set of all direct sums of bounded families of operators $(A_i)_{i \in \mathcal{J}}$ with $A_i \in \mathcal{A}_i$ for all $i$.

**Lemma 1.5:** Let $(A_i)_{i \in \mathcal{J}}$ be a bounded family of operators each of which is in the class (dc). Let $A = \sum_{i \in \mathcal{J}} \oplus A_i$. Suppose that $\mathcal{O}_A = \sum_{i \in \mathcal{J}} \oplus \mathcal{O}_{A_i}$. Then $A$ is in the class (dc).

**Proof:** Let $D \in \mathcal{O}_A''$. By Lemma 1.4, $D = \sum_{i \in \mathcal{J}} \oplus D_i$ where each $D_i$ is in $(A_i)'' = \mathcal{O}_{A_i}$. Therefore, $D$ is in $\sum_{i \in \mathcal{J}} \oplus \mathcal{O}_{A_i}$ which by assumption is equal to $\mathcal{O}_A$. Thus $\mathcal{O}_A \supseteq \mathcal{O}_{A''}$. The opposite inclusion is always true (Lemma 1.1(b)).

**Lemma 1.6:** Let $(A_i)_{i \in \mathcal{J}}$ be a bounded family of operators, $A_i$ acting on the space $\mathcal{H}_i$. Let $P_i$ denote the projection on the ith summand in the direct sum of the $\mathcal{H}_i$. Let $A$ be the direct sum of the $A_i$. If $P_i$ is in $\mathcal{O}_A$ for each i, then

$$\mathcal{O}_A = \sum_{i \in} \oplus \mathcal{O}_{A_i}.$$

**Proof:** Let $X \in \sum_{i \in \mathcal{J}} \oplus \mathcal{O}_{A_i}$, $X = \sum_{i \in \mathcal{J}} \oplus X_i$ with each $X_i$ in $\mathcal{O}_{A_i}$.

Now for each fixed j, there exists a net of polynomials $\{p_\alpha\}$ such that $p_\alpha(A_j) \to X_j$. For each $\alpha$, $P_j p_\alpha(A)$ is in $\mathcal{O}_A$. Expanding $p_\alpha(A)$ as a direct sum we see that

$$P_j p_\alpha(A) = P_j \sum_{i \in \mathcal{J}} \oplus p_\alpha(A_i) = \sum_{i \in \mathcal{J}} \oplus \delta_{ij}\, p_\alpha(A_i) \quad \text{which converges}$$

weakly to $\sum_{i \in \mathfrak{f}} \oplus B_i^j$ , where $B_i^j = \begin{cases} X_j, & i = j \\ 0, & i \neq j \end{cases}$ , and $\delta_{ij}$ is the Kronecker delta.

For each $j$, $\sum_{i \in \mathfrak{f}} \oplus B_i^j$ is in $\mathcal{O}_A$. Therefore,

$$\sum_{j \in \mathfrak{f}} \left( \sum_{i \in \mathfrak{f}} \oplus B_i^j \right) = \sum_{j \in \mathfrak{f}} \oplus X_j = X \text{ is in } \mathcal{O}_A.$$

<u>Definition</u>: Let $X$ be an operator on Hilbert space. The <u>complete spectrum</u> of $X$ is the union of the spectrum of $X$ with bounded components of the complement of the spectrum of $X$.

<u>Lemma</u> 1.7: Suppose that the complete spectrum of $A_j$ is disjoint from the complete spectrum of $\sum_{i \neq j} \oplus A_i$. Then $P_j$ is in $\mathcal{O}_A$.

<u>Proof</u>: By means of a unitary equivalence we may write $A$ as $A_j \oplus B$ where $B = \sum_{i \neq j} \oplus A_i$. The complete spectrum of $A_j$ is disjoint from that of $B$, so by Theorem 5.1 of [12] $\mathcal{O}_A = \mathcal{O}_{A_j} \oplus \mathcal{O}_B$. Hence $P_j = I_j \oplus 0$ ($I_j$ = the identity on $\mathcal{H}_j$ the domain of $A_j$) is in $\mathcal{O}_A$.

<u>Lemma</u> 1.8: Let $A = \sum_{i \in \mathfrak{f}} \oplus A_i$ be an operator on $\mathcal{H} = \sum_{i \in \mathfrak{f}} \oplus \mathcal{H}_i$. Let $D = \sum_{i \in \mathfrak{f}} \oplus D_i$ be in $\mathcal{O}_A''$. Suppose

that $X : \mathcal{H}_r \rightarrow \mathcal{H}_s$ is an operator such that $A_s X = X A_r$. Then $D_s X = X D_r$.

$\underline{\text{Proof}}$: Define $\tilde{X} : \mathcal{H} \rightarrow \mathcal{H}$ by the matrix $[X_{ij}]$ where

$$X_{ij} = \begin{cases} 0 \text{ if } (i,j) \neq (s,r) \\ X \text{ if } (i,j) = (s,r) \end{cases} .$$

Then $A \tilde{X} = [A_i X_{ij}]$ and $\tilde{X} A = [X_{ij} A_j]$. The entries of both are zero except for the $(s,r)$th entry which is $A_s X$ in the first case and $X A_r$ in the second. These are equal by assumption, so $\tilde{X}$ commutes with $A$, and hence with $D$. Therefore, $[D_i X_{ij}] = D \tilde{X} = \tilde{X} D = [X_{ij} D_j]$ so in particular $D_s X_{sr} = X_{sr} D_r$ or $D_s X = X D_r$.

We now proceed to give some sufficient conditions that the direct sum of an operator in the class (dc) with another operator be in the class (dc) also.

$\underline{\text{Proposition 1.9}}$: Let $A \in \mathcal{L}(\mathcal{H}_1)$ be in the class (dc) and let $B \in \mathcal{L}(\mathcal{H}_2)$. Let $\mathcal{X} = \{ X : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \mid BX = XA \}$ and let $\mathcal{Y} = \bigcup_{X \in \mathcal{X}} \text{Range } X$. Suppose that $\underline{\text{either}}$ of the two following conditions hold:

Condition (i): Every element of $\mathcal{O}_A$ is the weak limit of a $\underline{\text{sequence}}$ of polynomials in $A$, there exists a positive constant $M$ such that $\| p(B) \| \leq M \| p(A) \|$ for any polynomial $p$, and $\mathcal{Y}$ is dense in $\mathcal{H}_2$.

Condition (ii): $\mathcal{Y} = \mathcal{H}_2$.

The conclusion then is that $A \oplus B$ is in the class (dc).

Proof: Let $D$ be in $(A \oplus B)''$. Then by Lemma 1.4 $D = E \oplus F$ with $E$ in $\alpha_A''$ and $F$ in $\alpha_B''$. Suppose (a) holds. Let $h \in \mathcal{H}_2$. Then $h = Xg$ with $g$ in $\mathcal{H}_1$ and $X$ in $\mathcal{X}$. By Lemma 1.8, $FX = XE$ so $Fh = FXg = XEg$. Since $A$ is in the class (dc), $E$ is the weak limit of a net $p_\alpha(A)$ of polynomials in $A$. Therefore,

$$XEg = X(\lim_\alpha p_\alpha(A)g) = \lim_\alpha Xp_\alpha(A)g = \lim_\alpha p_\alpha(B)Xg, \text{ since}$$

$BX = XA$ implies that $p(B)X = Xp(A)$ for any polynomial $p$.

Consequently, $Fh$ is equal to the weak limit of $p_\alpha(B)h$ for every $h$ in $\mathcal{H}_2$, so $F = \underset{\alpha}{\text{wlim}}\, p_\alpha(B)$. We conclude that

$$E \oplus F = \underset{\alpha}{\text{wlim}}\, p_\alpha(A) \oplus \underset{\alpha}{\text{wlim}}\, p_\alpha(B)$$

$$= \underset{\alpha}{\text{wlim}}\, p_\alpha(A \oplus B) \text{ which is in } \overline{A \oplus B}$$

If (b) holds then we may suppose that $E = \underset{n \to \infty}{\text{wlim}}\, p_n(A)$ where $\{p_n\}_{n=1}^\infty$ is a sequence of polynomials. Since $\{p_n(A)\}_{n=1}^\infty$ is weakly convergent, it is bounded, say, by $K$. Then $\{p_n(B)\}_{n=1}^\infty$ is bounded by $MK$. By the same reasoning as in the first part of this proof, we see that $p_n(B) \to F$ on $\mathcal{Y}$. Since $\{p_n(B)\}_{n=1}^\infty$ is bounded and $\mathcal{Y}$ is dense, we conclude that $p_n(B)$ converges to $F$. Thus as before $p_n(A \oplus B) \to E \oplus F$ and so $E \oplus F$ is in $\alpha_{A \oplus B}$.

Proposition 1.10: Let $A$ belonging to $\mathcal{L}(\mathcal{H})$ be in the class (dc). Let $\tilde{A} = \sum_{i \in \mathcal{J}} \oplus A_i$ on $\tilde{\mathcal{H}} = \sum_{i \in \mathcal{J}} \oplus \mathcal{H}_i$, where $A_i = A$ and $\mathcal{H}_i = \mathcal{H}$ for all $i$. Suppose that either

    (a) $\mathcal{J}$ is finite

or

    (b) Every element of $\mathcal{C}_A$ is the weak limit of a <u>sequence</u> of polynomials in $A$.

Then $\tilde{A}$ is in the class (dc).

Proof: Let $\tilde{D} \in (\tilde{A})''$. By Lemma 1.6 $\tilde{D} = \sum_{i \in \mathcal{J}} \oplus D_i$ with each $D_i \in \mathcal{C}_A'' = \mathcal{C}_A$. For any $i$ and $j$, $I$, the identity on $\mathcal{H}$, intertwines $A_i$ and $A_j$; i.e., $A_i I = I A_j$, since $A_i = A = A_j$. Therefore by Lemma 1.8, $D_i I = I D_j$ for any $i$ and $j$. This says that all the $D_i$'s are equal, say, to $D \in \mathcal{C}_A$. If (a) holds let $D = \operatorname*{wlim}_{\alpha} p_\alpha(A)$. Then clearly $\tilde{D} = \operatorname*{wlim}_{\alpha} p_\alpha(\tilde{A})$ and so $\tilde{D}$ is in $\mathcal{C}_{\tilde{A}}$. If (b) holds then we may write $D = \operatorname*{wlim}_{n \to \infty} p_n(A)$. Since the $p_n(A)$ are convergent they are bounded, say, by $K$. Let $M = K + \|D\|$, and let $h$ and $g$ be vectors in $\tilde{\mathcal{H}}$, where $h = (h_i)_{i \in \mathcal{J}}$ and $g = (g_i)_{i \in \mathcal{J}}$. Choose $\varepsilon > 0$ and find a finite set $\mathcal{F} \subseteq \mathcal{J}$ such that $\sum_{i \in \mathcal{J} \setminus \mathcal{F}} \|h_i\| \ \|g_i\| < \frac{\varepsilon}{2M}$.

Next find $n_0$ such that $n \geq n_0$ implies $|((D - p_n(A))h_i, g_i)| < \frac{\varepsilon}{2|\mathcal{F}|}$ for every $i$ in $\mathcal{F}$. ($|\mathcal{F}|$ denotes the number of elements in $\mathcal{F}$.)

Then $n \geq n_o$ implies that

$$|((\tilde{D} - p_n(\tilde{A}))h, g)| = |\sum_{i \in \mathfrak{I}} ((D - p_n(A))h_i, g_i)|$$

$$\leq \sum_{i \in \mathfrak{I}} |((D - p_n(A))h_i, g_i)|$$

$$+ \sum_{i \in \mathfrak{I} \backslash \mathfrak{I}} |((D - p_n(A))h_i, g_i)|$$

$$< |\mathfrak{I}| \frac{\varepsilon}{2|\mathfrak{I}|} + \sum_{i \in \mathfrak{I} \backslash \mathfrak{I}} \|D - p_n(A)\| \, \|h_i\| \, \|g_i\|$$

$$\leq \frac{\varepsilon}{2} + \sum_{i \in \mathfrak{I} \backslash \mathfrak{I}} (\|D\| + K) \|h_i\| \, \|g_i\|$$

$$< \frac{\varepsilon}{2} + M \frac{\varepsilon}{2M} = \varepsilon \quad .$$

Therefore, $\tilde{D} = \text{wlim}_{n \to \infty} p_n(\tilde{A})$ and so is in $\mathcal{O}_{\tilde{A}}$.

Remark 1.11: Note that for case (b) we have shown that every

element of $\mathcal{O}_{\tilde{A}}$ is the limit of a sequence of polynomials in $\tilde{A}$.

We now consider a question which at first glance looks as if

it ought to be simple. The question is: "Suppose that A is in the

class (dc). Is $A \oplus 0$ then in the class (dc) also? (0 denotes

a zero operator of arbitrary dimension.) As it turns out, we are

not yet able to answer this question fully. However, we can answer

it in a couple of slightly specialized cases. Before proceeding to

treat the first case, we prove an easy lemma and make one definition.

$\underline{\text{Lemma}}$ 1.12: Let $A \in \mathcal{L}(\mathcal{H})$. Then $A$ is in the class (dc) if and only if $A^*$ is in the class (dc).

$\underline{\text{Proof}}$: Clearly it suffices to prove that $A$ being in the class (dc) implies that $A^*$ is in the class (dc), since $A^{**} = A$. Now $AC = CA$ if and only if $A^*C^* = C^*A^*$, so $\mathcal{O}'_{A^*} = (\mathcal{O}'_A)^* = \{C^* \mid C \in \mathcal{O}'_A\}$

Let $D \in \mathcal{O}''_{A^*}$. Then $DC^* = C^*D$ for all $C$ in $\mathcal{O}'_A$, so $D^*C = CD^*$ for all $C$ in $\mathcal{O}'_A$. Consequently $D^*$ is in $\mathcal{O}''_A = \mathcal{O}_A$, and hence $D^* = \underset{\alpha}{w\lim}\, p_\alpha(A)$ where $\{p_\alpha\}$ is a net of polynomials. It follows that

$$D = (\underset{\alpha}{w\lim}\, p_\alpha(A))^*$$

$$= \underset{\alpha}{w\lim}\, p_\alpha(A)^* \quad,$$

since the adjoint operation is continuous with respect to the weak operator topology. Therefore, $D = \underset{\alpha}{w\lim}\, \bar{p}_\alpha(A^*)$, where if $p(z)$ is the polynomial $\gamma_0 + \gamma_1 z + \ldots + \gamma_n z^n$ then by $\bar{p}(z)$ we mean the polynomial $\bar{\gamma}_0 + \bar{\gamma}_1 z + \ldots + \bar{\gamma}_n z^n$. Thus $D$ is in $\mathcal{O}_{A^*}$.

$\underline{\text{Definition}}$: Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Let $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$. Then the operator $f \otimes g$ mapping $\mathcal{H}_1$ into $\mathcal{H}_2$ is defined by $(f \otimes g)h = (h, f) \cdot g$.

This is not quite the tensor product of two Hilbert space vectors in the usual sense. However, the notation is convenient.

Theorem 1.13: Let $A \in \mathcal{L}(\mathcal{H})$ be an operator in the class (dc). Let $0$ denote the zero operator on a Hilbert space $\mathcal{K}$. Suppose that $A$ has non-trivial kernel or co-kernel. Then $A \oplus 0$ is in the class (dc).

Proof: Applying Lemma 1.12 we may suppose that $A$ has non-trivial co-kernel, so let $f \in (\text{Range } A)^{\perp}$ be a vector such that $\|f\| = 1$. Let $g$ be an arbitrary vector in $\mathcal{K}$.

Define $X : \mathcal{H} \to \mathcal{K}$ by $X = f \otimes g$. Then for any $h$ in $\mathcal{H}$, $XAh = (Ah, f) \cdot g = 0 \cdot g = 0$, since $f \perp \text{Range } A$.

Therefore, $XA = 0 = 0X$. Note that $Xf = g$ (an arbitrary vector in $\mathcal{K}$), so that the union of the ranges of intertwinings $X$ such that $XA = 0X$ is all of $\mathcal{K}$. Hence we may apply Proposition 1.9 and conclude that $A \oplus 0$ is in the class (dc).

There is a standard term used to describe operators which do not satisfy the last hypothesis of Theorem 1.13.

Definition: Let $A$ be an operator on Hilbert space which has zero kernel and dense range (i.e., zero kernel and co-kernel). Then $A$ is called quasi-invertible.

We shall use the term quasi-invertible to include the possibility of being invertible.

Definition: Let $A$ be an operator on Hilbert space. We shall say that $A$ has the unit property if the identity is included automatically in the weakly closed algebra which $A$ generates.

More precisely let $\mathcal{B}_A$ denote the weak closure of the set of polynomials in $A$ without constant term. Then $\mathcal{B}_A$ is an algebra (not necessarily with identity) which is contained in $\mathcal{C}_A$. We shall say that $A$ has the unit property if $\mathcal{B}_A = \mathcal{C}_A$ (i.e., if the identity is in $\mathcal{B}_A$).

An equivalent formulation is to say that $A$ has the unit property if there exists a net of polynomials $p_\alpha$ without constant term (i.e., $p_\alpha(0) = 0$ for all $\alpha$) such that $p_\alpha(A) \to I$.

Clearly $A$ cannot have the unit property unless $A$ is quasi-invertible.

Theorem 1.14: Let $A$ be a quasi-invertible operator. (a) If $A$ is in the class (dc) and has the unit property, then $A \oplus 0$ is in the class (dc). (b) If $A$ does not have the unit property, then $A \oplus 0$ is not in the class (dc).

Proof: (a) Suppose that $A$ is in the class (dc) and has the unit property. Let $p_\alpha$ be a net of polynomials without constant term such that $p_\alpha(A) \to I$. Then

$p_\alpha(A \oplus 0) = p_\alpha(A) \oplus p_\alpha(0) = p_\alpha(A) \oplus 0 \to I \oplus 0$. Thus the projection on the first coordinate space is in the algebra $\mathcal{C}_{(A \oplus 0)}$. The projection on the second coordinate space is simply the identity on the whole space minus the projection on the first coordinate space, so it is in the algebra $\mathcal{C}_{(A \oplus 0)}$ too. Therefore, by Lemmas 1.5 and 1.6, $A \oplus 0$ is in the class (dc).

(b) Suppose that $A$ does not have the unit property.

Then $I \oplus 0$ is <u>not</u> in the algebra $\mathcal{O}_{(A \oplus 0)}$. For, suppose that it

is. Then there exists a net of polynomials $q_\alpha$ such that

$q_\alpha(A \oplus 0) \to I \oplus 0$. For each $\alpha$, $q_\alpha(A \oplus 0) = q_\alpha(A) \oplus q_\alpha(0)$.

Therefore, we conclude that $q_\alpha(A) \to I$, and $q_\alpha(0) \to 0$. Define

a new net of polynomials $p_\alpha$ by $p_\alpha(z) = q_\alpha(z) - q_\alpha(0)$. (This

last $0$ denotes the zero scalar.) Then the $p_\alpha$ are without constant

term and $p_\alpha(A) = q_\alpha(A) - q_\alpha(0) \to I - 0 = I$, which is a contradiction.

However, $I \oplus 0$ <u>is</u> in the double commutant $(A \oplus 0)''$.

To see this, let $C \in (A \oplus 0)'$, and write $C$ as a two-by-two

operator matrix

$$\begin{bmatrix} E & F \\ G & H \end{bmatrix} .$$

If we write $A \oplus 0$ also as a two-by-two operator matrix

$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$$

then the equation $C(A \oplus 0) = (A \oplus 0) C$ becomes

$$\begin{bmatrix} EA & 0 \\ GA & 0 \end{bmatrix} = \begin{bmatrix} AE & AF \\ 0 & 0 \end{bmatrix} .$$

This says that $E$ commutes with $A$ and that $GA = 0 = AF$. Since

$A$ is quasi-invertible this says that $G = 0 = F$. Hence the

commutant of $A \oplus 0$ consists of operators of the form

$$\begin{bmatrix} E & 0 \\ 0 & H \end{bmatrix} \quad ,$$

where $E$ is in the commutant of $A$ and $H$ is arbitrary. Clearly,

$$I \oplus 0 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

commutes with all operators of this form and hence is in the double

commutant of $A \oplus 0$. Thus we have an operator which is in the

double commutant but not in the algebra, so $A \oplus 0$ is not in the

class (dc).

Theorem 1.15: Let $A$ be an invertible operator in the class

(dc). Then $A \oplus 0$ is in the class dc also.

Proof: By Theorem 1.14 it suffices to show that $A$ has the

unit property. $A$ is invertible and $AC = CA$ implies that

$A^{-1}C = CA^{-1}$, so $A^{-1}$ is in $\mathcal{C}_A'' = \mathcal{C}_A$. Therefore, there is

a net of polynomials $q_\alpha$ such that $q_\alpha(A) \to A^{-1}$. Then

$A \, q_\alpha(A) \to AA^{-1} = I$. Define $p_\alpha(z) = z \, q_\alpha(z)$ for each $\alpha$. Then

$p_\alpha(0) = 0$ and $p_\alpha(A) = A \, q_\alpha(A) \to I$, so $A$ has the unit property.

The following result is a corollary of Theorems 1.13 and 1.15.

Corollary 1.16: Any finite rank operator is in the class (dc).

Proof: Any finite rank operator may be written as $A \oplus 0$,

where $A$ is an operator on a finite dimensional space. By purely

algebraic considerations any operator on a finite dimensional space

is in the class (dc). (See for instance [12, p. 113].)

Also by finite dimensionality, A either has non-trivial kernel or co-kernel, or is invertible. In the first case Theorem 1.13 applies, and in the second case Theorem 1.15 applies to show that $A \oplus 0$ is in the class (dc).

Remark 1.7: It turns out that there exist quasi-invertible operators in the class (dc) which do not have the unit property. Thus there exist operators A such that A is in the class (dc) but $A \oplus 0$ is not. To construct examples of such · A we must turn our attention to weighted shifts.

CHAPTER II

WEIGHTED SHIFTS

Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_i\}_{i=0}^{\infty}$. Let $\{\alpha_i\}_{i=1}^{\infty}$ be a bounded sequence of scalars. Define an operator $S$ on $\mathcal{H}$ by setting $Se_i = \alpha_{i+1} e_{i+1}$ for all $i$, and extending by linearity and boundedness. $S$ is called a one-sided shift with weights $\alpha_i$. Similarly, if $\{e_i\}_{i=-\infty}^{\infty}$ is a doubly infinite orthonormal basis and $\{\alpha_i\}_{i=-\infty}^{\infty}$ is a bounded doubly infinite sequence of scalars, the operator $S$ obtained by setting $Se_i = \alpha_{i+1} e_{i+1}$ is called a two-sided shift. Finally, if $\mathcal{H}$ is finite dimensional with orthonormal basis $\{e_0, e_1, \ldots, e_n\}$, and $\alpha_1, \ldots, \alpha_n$ are scalars, the operator $S$ obtained by setting $Se_i = \alpha_{i+1} e_{i+1}$, for $i = 0, 1, \ldots, n-1$, $Se_n = 0$, is called a finite shift.

The adjoint of a (one-sided, two-sided, or finite) shift is called a backward (one-sided, two-sided, or finite) shift. We note that each finite backward shift is unitarily equivalent to a finite forward shift. Likewise each two-sided backward shift is unitarily equivalent to a two-sided forward shift. In both cases the unitary equivalence is effected simply by renumbering the basis vectors.

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Shields and Wallen [20] have proven that a one-sided shift with all weights non-zero generates a maximal abelian weakly closed algebra and hence is à fortiori in the class (dc). Motivated by this, we proceed to attack the question of whether or not a general one-sided shift (no assumptions about the weights being non-zero) is in the class (dc). To facilitate this study, we state a lemma which enables us to express shifts having some zero weights as the direct sum of shifts having no zero weights. We also state a similar lemma about two-sided shifts which will be used later on.

Lemma 2.1: Let $S$ be a one-sided weighted shift. Then $S$ is either unitarily equivalent to an operator of the form
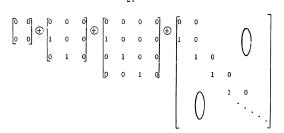
$$(a) \quad 0 \oplus \sum_{i=0}^{\infty} \oplus S_i$$

or to an operator of the form

$$(b) \quad 0 + (\sum_{i=0}^{\infty} \oplus S_i) \oplus S_{\infty} \quad ,$$

where each $S_i$ $(0 \leq i < \infty)$ is a finite shift and $S_{\infty}$ is a one-sided shift, all with non-zero weights.

For example, the shift with weights $1,1,0,1,1,1,0,0,0,1,1,1,\ldots,1,1,\ldots$ is unitarily equivalent to

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & 1 & 0 & \\ & & & 1 & 0 \\ & & & & 1 & 0 \\ & & & & & \ddots & \ddots \end{bmatrix}$$

<u>Lemma</u> 2.2: Let $S$ be a two-sided shift with at least one zero weight. Then $S$ is either unitarily equivalent to an operator of the form

$$\text{(a)} \quad 0 \oplus \sum_{i=-\infty}^{\infty} \oplus S_i$$

or to an operator of the form

$$\text{(b)} \quad 0 \oplus ( \sum_{i=0}^{\infty} \oplus S_i) \oplus S_\infty$$

or to an operator of the form

$$\text{(c)} \quad 0 \oplus ( \sum_{i=0}^{\infty} \oplus S_i) \oplus T$$

or to an operator of the form

$$\text{(d)} \quad 0 \oplus ( \sum_{i=0}^{N} \oplus S_i) \oplus S_\infty \oplus T \quad,$$

where $S_i$ $(0 \leq i < \infty)$ denotes a finite shift, $S_\infty$ denotes a one-sided shift, and $T$ denotes the adjoint of a one-sided shift, all of which have no zero weights.

Remark 2.3: For our purposes, the zero summands in the foregoing lemmas may be ignored. This is because in each case the remaining part of the direct sum always has co-kernel. Hence, if we can show that the remaining part of the direct sum is in the class (dc), it follows that the whole shift is in the class (dc) by Theorem 1.13.

Remark 2.4: We notice that form (a) of Lemma 2.2 is essentially the same as form (a) of Lemma 2.1. Also in Lemma 2.2, the adjoint of an operator of form (c) is unitarily equivalent to an operator of form (b). Applying Lemma 1.12, we see that for our purposes form (c) reduces to being the same as form (b). To summarize: Without loss of generality we may assume that a two-sided shift with at least one zero weight is either unitarily equivalent to an operator of the form

$$\text{(a)} \quad ( \sum_{i=0}^{\infty} \oplus S_i) \oplus S_\infty$$

or to an operator of the form

$$\text{(b)} \quad ( \sum_{i=0}^{N} \oplus S_i) \oplus S_\infty \oplus T \quad ,$$

where $S_i$, $S_\infty$, and $T$ are as in Lemma 2.2.

Shields and Wallen, in their paper [20] previously cited, develop a technique of which we shall make a great deal of use in what follows. Essentially what they do is to show that if an operator is matricially a formal power series (we shall define what this means shortly) in a one-sided shift with non-zero weights, then it is in the weakly closed algebra generated by the shift and the identity. Their proof rests upon obtaining a bound for the Cesaro means of the partial sums of the power series. This is done using a lemma of Schur [19].

We make their result explicit and elaborate it slightly in Lemma 2.6. Its proof is by-and-large the same as what was done by Shields and Wallen, and depends in turn upon a slight elaboration of the aforementioned lemma of Schur. The proofs of both these elaborations have been relegated to the Appendix.

<u>Definition:</u>  Let  $f(z) = \sum_{i=0}^{\infty} \gamma_i z^i$  be a formal power series. Denote its partial sums by  $f_n(z)$. Let  A  be an operator on a Hilbert space $\mathcal{H}$ , with orthonormal basis $\mathcal{E} = \{e_i\}_{i \in \mathcal{J}}$ . Let A have matrix  M  with respect to $\mathcal{E}$ . For each  i  and  j  in $\mathcal{J}$  let  $\mu_{ij}^n$  denote the  (i, j)th  entry of  $f_n(M)$. Suppose that for each pair  (i, j),  $\{\mu_{ij}^n\}_{n=0}^{\infty}$  is a convergent sequence of complex numbers. Denote its limit by  $\mu_{ij}$. Let  M'  be the matrix

$[\mu_{ij}]$. If M' is the matrix with respect to $\underset{\sim}{\Sigma}$ of an operator X on $\mathcal{H}$ , we shall say that X _is matricially equal to the formal power series_ f(A).

Remark 2.5: If A is a shift, then at least the matrix M' always exists for any formal power series f. That is to say, the sequences $\{\mu_{ij}^n\}_{n=0}^{\infty}$ are always convergent. In fact, $\mu_{ij}^n$ is constant for $n \geq |i + j|$.

In [6] Herrero discusses operators which are formal power series in a given operator in a different sense. The two senses agree where the given operator is a shift.

Lemma 2.6: Let the operator S be a direct sum, $S = \sum_{i \epsilon} \oplus S_i$, where each summand is a one-sided (backward or forward), two-sided, or finite shift. Suppose that $D = \sum_{i \epsilon \jmath} \oplus D_i$ is an operator which is matricially equal to a formal power series is S. Then D is in $\mathcal{A}_S$.

With these introductory observations out of the way, we are now in a position to prove our theorem on one-sided shifts.

Theorem 2.7: Any one-sided shift is in the class (dc).

Proof: Let S be a one-sided shift. By Lemma 2.1 and Remark 2.3 we may assume that S either has the form

$$\text{(a)} \quad S = \sum_{i=0}^{\infty} \oplus S_i$$

or

$$\text{(b)} \quad S = \left( \sum_{i=0}^{N} \oplus\, S_i \right) \oplus S_\infty \quad .$$
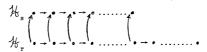
Let $S_i$ act on the space $\mathcal{H}_i$ with orthonormal basis $\left\{ e_{i0}, \ldots, e_{in_i} \right\}$ and have weights $\alpha_{i1}, \ldots, \alpha_{in_i}$. Let $S_\infty$ act on the space $\mathcal{H}$ with orthonormal basis $\left\{ e_i \right\}_{i=0}^{\infty}$ and have weights $\left\{ \alpha_i \right\}_{i=1}^{\infty}$.

Define $\pi_{ij} = \prod_{k=1}^{j} \alpha_{ik}$ for $1 \le j \le n_i$ and $\pi_{i0} = 1$, for $0 \le i < \infty$. Likewise define $\pi_j = \prod_{k=1}^{j} \alpha_k$ for $1 \le j < \infty$, and $\pi_0 = 1$. We now proceed to treat cases (a) and (b) separately.

(a) Let $D \in \mathcal{U}_S''$. By Lemma 1.4, $D = \sum_{i=0}^{\infty} \oplus\, D_i$ where $D_i$ is in $(S_i)'' = \mathcal{U}_{S_i}$. (Since $S_i$ is an operator on a finite dimensional space, it is in the class (dc). This was previously remarked upon in Corollary 1.16.) Finite dimensionality also implies that the weakly closed algebra generated by $S_i$ is just the <u>algebra</u> generated by $S_i$, so each $D_i = p_i(S_i)$ where $p_i$ is a polynomial, in fact of degree less than or equal to $n_i$. Basically what we must do is to show that the $p_i$ agree with each other wherever they can. To this end, for any $n_r \ge n_s$, define

$$E_{rs} : \mathcal{H}_r \to \mathcal{H}_s \quad \text{as follows:}$$

$$E_{rs}e_{ri} = \begin{cases} \dfrac{\pi_{si}}{\pi_{ri}} \, e_{si} & \text{for } 0 \leq i \leq n_s \\[2mm] 0 & \text{for } n_s \leq i \leq n_r \end{cases}$$

extending linearly. Then $S_s E_{rs} = E_{rs} S_r$. To see this it helps to consider the following diagram



The large dots represent the basis vectors of $\mathcal{H}_s$ and $\mathcal{H}_r$, respectively. The straight arrows represent the action of $S_s$ and $S_r$, and the curved arrows that of $E_{rs}$. It is clear that for $i \geq n_s$ we have

$$S_s E_{rs} e_{ri} = 0 = E_{rs} S_r e_{ri} \quad .$$

For $i < n_s$ we have

$$S_s E_{rs} e_{ri} = S_s \left( \frac{\pi_{si}}{\pi_{ri}} e_{si} \right)$$

$$= \frac{\pi_{si}}{\pi_{ri}} \alpha_{si+1} e_{si+1}$$

$$= \frac{\pi_{si+1}}{\pi_{ri}} e_{si+1}$$

and

$$E_{rs} S_r e_{ri} = E_{rs}(\alpha_{ri+1} e_{ri+1})$$

$$= \frac{\pi_{si+1}}{\pi_{ri+1}} \alpha_{ri+1} e_{si+1}$$

$$= \frac{\pi_{si+1}}{\pi_{ri}} e_{si+1} \qquad .$$

Thus $E_{rs}$ intertwines $S_s$ and $S_r$ as claimed. Consequently by Lemma 1.8, $D_s E_{rs} = E_{rs} D_r$. Now $E_{rs}$ maps <u>onto</u> $\mathcal{H}_s$ and so for every $h$ in $\mathcal{H}_s$ there is a $g$ in $\mathcal{H}_r$ such that $h = E_{rs} g$. Thus

$$D_s h = D_s E_{rs} g = E_{rs} D_r g = E_{rs} p_r(S_r) g = p_r(S_s) E_{rs} g = p_r(S_s) h.$$

We conclude that $p_s(S_s) = D_s = p_r(S_s)$ for $n_r \geq n_s$. We now consider two cases:

Case I: The sequence $\{n_i\}_{i=0}^{\infty}$ is bounded. Choose $r$ such that $n_r = \sup_i n_i$. Then for every $i$, $D_i = p_i(S_i) = p_r(S_i)$. Hence $D = p_r(S)$ and is in $\mathcal{A}_s$.

Case II: The sequence $\{n_i\}_{i=0}^{\infty}$ is unbounded. Define a formal power series $f(z) = \sum_{k=0}^{\infty} \gamma_k z^k$ where $\gamma_k$ is the kth coefficient of any $p_i$ with $n_i \geq k$. We must check that the numbers $\gamma_k$ are well defined. Suppose that both $n_r$ and $n_s$ are both greater than or equal to $k$. Without loss of generality we may assume that $n_r \geq n_s$. We know therefore that

$p_r(S_s) = p_s(S_s)$. Denote the coefficients of $p_r$ by $\beta_i$,
$i = 0, \ldots, n_r$, and the coefficients of $p_s$ by $\omega_i$, $i = 0, \ldots, n_s$.
The $(k, 0)$th entry of the matrix for $p_r(S_s)$ is $\pi_{sk} \beta_k$. The
$(k, 0)$th entry of the matrix for $p_s(S_s)$ is $\pi_{sk} \omega_k$. Since these
entries must be equal, we see that $\beta_k = \omega_k$, so $\gamma_k$ is well
defined.

Now consider $f(S)$, the formal power series in $S$. For
each $i$, $S_i$ is nilpotent of order $n_i + 1$ and the coefficients of
$f$ of index less than or equal to $n_i$ agree with the coefficients of
$p_i$. Thus $f(S_i)$ is well defined and in fact is equal to $p_i(S_i) = D_i$.
Therefore, $D$ is matricially equal to $f(S)$. By Lemma 2.6 we
conclude that $D$ is in $\alpha_s$.

(b) $S = (\sum_{i=0}^{N} \oplus S_i) \oplus S_\infty$ . For each $r = 0, 1, 2, \ldots, N$
define $E_r : \mathcal{H}_0 \oplus \ldots \oplus \mathcal{H}_{r-1} \oplus \mathcal{H} \to \mathcal{H}_r$ $(E_0 : \mathcal{H} \to \mathcal{H}_0)$
as follows:

$$
E_r e_i =
\begin{cases}
\dfrac{\pi_{ri}}{\pi_i} e_{ri} & \text{for } 0 \le i \le n_r \\[4mm]
0 & \text{for } i > n_r
\end{cases}
.
$$

extending linearly, with $E_r = 0$ on $\mathcal{H}^\perp$ . Then
$E_r(S_0 \oplus S_1 \oplus \ldots \oplus S_{r-1} \oplus S_\infty) = S_r E_r$ by arguments similar
to those used on the operators $E_{rs}$ in part (a). Note that

Range $E_r = \mathcal{H}_r$. Now we proceed inductively. We know by [20]

that $S_\infty$ is in the class (dc). Suppose that we know

$S_0 \oplus \ldots \oplus S_{r-1} \oplus S_\infty$ is in the class (dc). Since

Range $E_r = \mathcal{H}_r$ we may apply Proposition 1.9 and obtain the

fact that $S_r \oplus S_0 \oplus \ldots \oplus S_{r-1} \oplus S_\infty$ is in the class (dc).

But this operator is unitarily equivalent to $S_0 \oplus \ldots \oplus S_r \oplus S_\infty$.

We now turn to the discussion of two-sided shifts. These

operators need <u>not</u> always be in the class (dc). The familiar

bilateral shift (all weights equal to one) provides a counter-example.

If we denote this shift by $S$, then $S^* = S^{-1}$ is in the double

commutant of $S$, but not in $\mathcal{C}_S$.

The operator $S^{-1}$ is strictly upper-triangular whereas

everything in $\mathcal{C}_S$ is lower triangular.

As it turns out, this example gives us the clue to the general

situation. If a two-sided shift $S$ is invertible, its inverse is in

its double commutant but not in $\mathcal{C}_S$. If it is not invertible,

the shift is in the class (dc). We proceed to establish these facts

in the following sequence of propositions.

First we introduce a little bit of notation: If $\mathcal{S}$ is a subset

of some Hilbert space, we shall denote the closed linear span of $\mathcal{S}$

by $\langle \mathcal{S} \rangle$.

Proposition 2.8: Let $S$ be a two-sided shift with at least one zero weight. Then $S$ is in the class (dc).

Proof: By Remark 2.5, we have two cases to consider.

(i) $S = (\sum_{i=0}^{\infty} \oplus S_i) \oplus S_{\infty}$: Let $D \in \alpha_S''$. Then

$D = (\sum_{i=0}^{\infty} \oplus D_i) \oplus D_{\infty}$, where $D_i$ is in $(S_i)'' = \alpha_{S_i}$ for $0 \leq i \leq \infty$. Consequently, each $D_i = p_i(S_i)$, where the $p_i$ are polynomials, for $0 \leq i < \infty$, and $D_{\infty} = f(S_{\infty})$ where $f$ is a formal power series (see [20, p. 780]). We build operators which intertwine $S_{\infty}$ with $S_i$ (and hence $D_{\infty}$ with $D_i$) as was done in Theorem 2.7. As in Theorem 2.7, part (a), we deduce that the coefficients of $p_i$ agree with those of $f$ as far as they go. Thus $D$ is a formal power series in $S$, $f(S)$. By Lemma 2.6, $D$ is in $\alpha_S$.

(ii) $S = (\sum_{i=0}^{N} \oplus S_i) \oplus S_{\infty} \oplus T$: If we can show that $S_{\infty} \oplus T$ is in the class (dc), then an induction argument, exactly the same as was used in part (b) of Theorem 2.7 shows that $S$ is in the class (dc). We therefore turn our attention to operators of the form $R \oplus T$, where $R$ is a forward and $T$ is a backward one-sided shift, both with non-zero weights. Let $R$ act on $\mathcal{H}$

with **orthonormal basis** $\{e_i\}_{i=0}^{\infty}$ and have weights $\{\alpha_i\}_{i=0}^{\infty}$.
Let $T$ act on $\mathcal{K}$ with orthonormal basis $\{f_i\}_{i=0}^{\infty}$ and have
weights $\{\beta_i\}_{i=0}^{\infty}$.

Define $\pi_j = \prod_{k=0}^{j} \alpha_k$, for $1 \leq j < \infty$, and $\pi_0 = 1$. Define
$\kappa_j = \prod_{k=0}^{j} \beta_k$, for $1 \leq j < \infty$, and $\kappa_0 = 1$.

Let $D$ be in $(R \oplus T)''$, then $D = E \oplus F$ with $E$ in
$\mathcal{C}_R'' = \mathcal{C}_R$ and $F$ in $\mathcal{C}_T'' = \mathcal{C}_T$. By [20, p. 780] $E$ is a
formal power series $f(R)$ in $R$. Likewise $F^*$ is a formal power
series in $T^*$, so $F$ is a formal power series $g(T)$ in $T$.
If we can show that the coefficients of $g$ agree with those of $f$,
then we will know that $D = f(R \oplus T)$. By Lemma 2.6 this will
imply that $D$ is in $\mathcal{C}_{(R \oplus T)}$.

It remains to demonstrate this agreement of coefficients. To
do this, we define operators $G_r : \mathcal{H} \to \mathcal{K}$ by means of the matrix
$[\gamma_{ij}]$ where

$$\gamma_{ij} = \begin{cases} \dfrac{1}{\kappa_i \pi_j} & \text{for } |i+j| \leq r \\[2ex] 0 & \text{for } |i+j| > r \end{cases}$$

At most finitely many of the numbers $\gamma_{ij}$ are non-zero, so this
matrix defines a bounded operator.

We now assert that $G_r R = T G_r$. For $j < r$,

$$G_r R e_j = G_r \alpha_{j+1} e_{j+1} = \alpha_{j+1} \sum_{i=0}^{\infty} \gamma_{i\,j+1} e_i$$

$$= \alpha_{j+1} \sum_{i=0}^{r-j-1} \frac{1}{\kappa_i \pi_{j+1}} e_i$$

$$= \sum_{i=0}^{r-j-1} \frac{1}{\kappa_i \pi_j} e_i \quad .$$

$$T G_r e_j = T \sum_{i=0}^{\infty} \gamma_{ij} e_i = \sum_{i=0}^{r-j} \gamma_{ij} T e_i$$

$$= \sum_{i=1}^{r-j} \gamma_{ij} \beta_i e_{i-1}$$

$$= \sum_{i=0}^{r-j-1} \gamma_{i+1\,j} \beta_{i+1} e_i$$

$$= \sum_{i=0}^{r-j-1} \frac{1}{\kappa_{i+1} \pi_j} \beta_{i+1} e_i$$

$$= \sum_{i=0}^{r-j-1} \frac{1}{\kappa_i \pi_j} e_i \quad .$$

For $j = r$, $G_r R e_j = G_r \alpha_{r+1} e_{r+1} = \alpha_{r+1} G_r e_{r+1} = 0$. On the

other hand, $TG_r e_j = T\gamma_{0r} e_0 = \gamma_{0r} T e_0 = 0$.

For $j > r$ clearly $G_r Re_j = 0 = TG_r e_j$. So $G_r R = TG_r$.

Hence by Lemma 1.8 $FG_r = G_r E$. Note that for $0 \le j \le r$, $G_r e_j$ is a linear combination with non-zero coefficients of $f_0, \cdots, f_r$. This yields $r+1$ equations from which the $f_j$ may be eliminated one at a time. We conclude that each vector in $\langle f_0, \cdots, f_r \rangle$ is the image under $G_r$ of a vector in $\langle e_0, \cdots, e_r \rangle$. Now recall that $E$ is a formal power series in $R$, explicitly,

$$E = f(R) = \sum_{i=0}^{\infty} \mu_i R^i.$$ The series converges on finite linear

combinations of basis vectors. Likewise, $F = g(T) = \sum_{i=0}^{\infty} \nu_i T^i$.

Therefore,

$$Ff_r = FG_r h \quad \text{for some} \quad h \quad \text{in} \quad \langle e_0, \cdots, e_r \rangle$$

$$= G_r Eh = G_r f(R)h = G_r \sum_{i=0}^{\infty} \mu_i R^i h$$

$$\therefore \qquad = \sum_{i=0}^{\infty} \mu_i G_r R^i h = \sum_{i=0}^{\infty} \mu_i T^i G_r h$$

$$= \sum_{i=0}^{\infty} \mu_i T^i f_r = \sum_{i=0}^{r} \mu_i T^i f_r$$

$$= \sum_{i=0}^{r} \mu_i \frac{\kappa_r}{\kappa_i} f_{r-i} \quad .$$

On the other hand,

$$Ff_r = g(T)f_r$$

$$= \sum_{i=0}^{\infty} \nu_i T^i f_r$$

$$= \sum_{i=0}^{r} \nu_i \frac{\kappa_r}{\kappa_i} f_{r-i} \quad .$$

Comparing coefficients we see that $\nu_i = \mu_i$, for $i = 0, \ldots, r$.

Since $r$ was arbitrary, we see that $\nu_i = \mu_i$ for all $i$, so we

are done.

Proposition 2.9: Let $S$ be a two-sided shift with non-zero

weights $\alpha_i$. Suppose that $\inf_i |\alpha_i| = 0$. Then $S$ is in the class

(dc); in fact, $S$ generates a maximal abelian weakly closed

algebra.

Proof: Let $C \in \mathcal{O}_S^!$ with matrix $[\gamma_{ij}]$. Computing

$SCe_j = CSe_j$ for arbitrary $j$ and comparing coefficients in the

resulting series, we obtain the following characterization for the

numbers $\gamma_{ij}$:

$$\gamma_{ij} = \begin{cases} \dfrac{\alpha_{j+1} \times \ldots \times \alpha_0}{\alpha_{i+1} \times \ldots \times \alpha_{i-j}} \, \gamma_{i-j\,0} & \text{for } j \leq 0 \\[3ex] \dfrac{\alpha_{i-j+1} \times \ldots \times \alpha_i}{\alpha_1 \times \ldots \times \alpha_j} \, \gamma_{i-j\,0} & \text{for } j > 0 \end{cases} \quad .$$

An easy but tedious case argument which is given in Appendix B shows that this reduces to

$$\gamma_{ij} = \begin{cases} \dfrac{\alpha_{i+1} \times \ldots \times \alpha_i}{\alpha_1 \times \ldots \times \alpha_{i-j}} \, \gamma_{i-j\,0} & \text{for } i \geq j \\[3ex] \dfrac{\alpha_{i-j+1} \times \ldots \times \alpha_0}{\alpha_{i+1} \times \ldots \times \alpha_j} \, \gamma_{i-j\,0} & \text{for } i < j \end{cases} \quad .$$

Now fix a positive integer $k$ and consider the $k$th super-diagonal $i = j - k$. Along this super-diagonal

$$\gamma_{ij} = \dfrac{\alpha_0 \times \ldots \times \alpha_{-k+1}}{\alpha_{j-k+1} \times \ldots \times \alpha_j} \, \gamma_{-k\,0} \quad .$$

Now

$$\inf_j |\alpha_{j-k+1} \times \ldots \times \alpha_j| \leq \inf_j \|S\|^{k-1} |\alpha_j|$$

$$= \|S\|^{k-1} \inf |\alpha_j|$$

$$= 0 \quad .$$

Therefore, the numbers $\gamma_{ij}$ are unbounded along this super-diagonal unless $\gamma_{-k\,0} = 0$. Since $C$ is a bounded operator, we must indeed have $\gamma_{-k\,0} = 0$, whence the whole $k$th super-diagonal is zero. This holds for $k = 1, 2, 3, \ldots$, and so $C$ is lower-triangular. We conclude that

$$\gamma_{ij} = \begin{cases} \dfrac{\alpha_{j+1} \times \ldots \times \alpha_i}{\alpha_1 \times \ldots \times \alpha_{i-j}} \gamma_{i-j\,0} & \text{for } i \geq j \\[2mm] 0 & \text{for } i < j \end{cases} .$$

Consequently, $C$ is matricially a formal power series in $S$,

$$\sum_{i=0}^{\infty} \mu_i S^i \quad \text{where} \quad \mu_i = \frac{\gamma_{i0}}{\alpha_1 \times \ldots \times \alpha_i} \quad \text{for } i > 0, \quad \mu_0 = \gamma_{00}.$$

Applying Lemma 2.6, we see that $C$ is in $\mathcal{Q}_S$.

Proposition 2.10: If $S$ is an invertible two-sided shift, then $S$ is not in the class (dc).

Proof: Let $S$ have weight sequence $\{\alpha_i\}_{i=-\infty}^{\infty}$. Then $S^{-1}$ is the backward shift with weight sequence $\{1/\alpha_i\}_{i=-\infty}^{\infty}$. Since $S^{-1}$ is a backward shift, it is strictly upper-triangular and hence cannot be in $\mathcal{Q}_S$, since $S$ is lower-triangular. However, $S^{-1}$ is clearly in $\mathcal{Q}_S''$.

We summarize these results in the following theorem:

Theorem 2.10: A two-sided shift is in the class (dc) if and only if it is not invertible.

Proof: We note that a two-sided shift is invertible if and only if the infimum of the moduli of its weights is greater than zero. The proof is now immediate from Propositions 2.8, 2.9, and 2.10.

Finally we can now give a more-or-less complete answer to the question posed in Chapter I about the direct sum of an operator in the class (dc) with the zero operator.

Theorem 2.11: Let $A$ be an operator in the class (dc).
If either (a) $A$ has non-trivial kernel or co-kernel, or (b) $A$
is invertible, then $A \oplus 0$ is also in the class (dc). However,
if $A$ is quasi-invertible but not invertible, then $A \oplus 0$ need
not be in the class (dc).

Proof: The first part of this theorem is immediate from
Theorems 1.13 and 1.15.

To get a counter-example to demonstrate the second part
of the theorem, we take $A$ to be a two-sided shift with all non-
zero weights, but with the infimum of the absolute values of its
weights equal to zero. Then $A$ is in the class (dc) by
Proposition 2.9, $A$ is clearly quasi-invertible, so to show that
$A \oplus 0$ is not in the class (dc) it suffices, by Theorem 1.14,
to show that $A$ does not have the unit property. This is obvious,
since anything in $\mathcal{B}_A$ (see page 21) is strictly lower triangular,
so $\mathcal{B}_A$ cannot contain the identity.

Remark 2.12: This shows that the direct sum of operators
in the class (dc) need not be in the class (dc), since $0$ is
blatantly in the class. If we take the weights of $A$ to tend to zero
in both directions, then $A$ is a compact operator, hence so is
$A \oplus 0$. Thus we have an example of a compact operator which is
not in the class (dc). Since any finite rank operator is in the
class (dc) by Corollary 1.16, this shows that the class (dc) is

not preserved under norm limits. That is, there exists a sequence of operators in the class (dc) (i.e., finite rank operators) converging in norm to an operator not in the class (dc) (the previously mentioned compact operator).

We notice that although weighted shifts themselves are remarkably well behaved with respect to the class (dc), they have just served to produce a breath-taking amount of pathology.

## ISOMETRIES

It is well known (see for instance [9, prob. 118] that any isometry $V$ is unitarily equivalent to an operator of the form $U \oplus W$ where $U$ is a unilateral shift of some multiplicity and $W$ is unitary. $U$ acts on a space $\mathbb{H}_{\mathfrak{D}} = \mathfrak{D} \oplus \mathfrak{D} \oplus \mathfrak{D} \cdots$ , where $\mathfrak{D}$ is a Hilbert space of dimension equal to the multiplicity of $U$. $U$ is called the pure part of $V$ and $W$ is called the unitary part. Either part may, of course, be absent.

A unitary operator may be decomposed into the direct sum of two other unitary operators. To describe this decomposition we must make some definitions.

Let $W \in \mathcal{L}(\mathcal{H})$ be unitary with spectral measures $E$. Let $f$ and $g$ be vectors in $\mathcal{H}$ , and define Borel measure $\mu_{fg}$ as usual by $\mu_{fg}(\sigma) = (E(\sigma)f, g)$. Let $m$ be Lebesgue measure on $\mathbb{T}$ (the unit circle) restricted to Borel sets. If $\mu_{ff} << m$ for all $f$ in $\mathcal{H}$ then we say that $W$ is absolutely continuous. If $\mu_{ff} \perp m$ for all $f$ in $\mathcal{H}$ then we say that $W$ is singular.

Now the result about decomposition of unitary operators can be stated as follows: If $W$ is a unitary operator then $W$ is

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unitarily equivalent to a direct sum $W_a \oplus W_s$ where $W_a$ and $W_s$ are respectively absolutely continuous and singular unitary operators. Either summand may be absent. (See [5, p. 55] and [14, p. 19].)

We now prove a lemma about absolutely continuous unitaries.

Lemma 3.1: Let $T \in \mathcal{L}(\mathcal{H})$ be an absolutely continuous unitary operator with a cyclic vector. Then $T$ is unitarily equivalent to $M_z$ (multiplication by $z$) on $L^2(\nu)$, where $\nu$ is Lebesgue measure $(m)$ restricted to a Borel subset of $\mathbb{T}$, the unit circle.

Proof: Let $f$ be a cyclic vector and let $E$ be the spectral measure for $T$. Let $\mu = \mu_{ff}$. Then, since $T$ is absolutely continuous, $\mu \ll m$. Therefore, by the Radon-Nikodym Theorem, there exists a positive Borel measurable function $g$ such that $d\mu = g\,dm$. Choose any representative of the equivalence class of $g$ and let $\gamma = \{ z : g(z) > 0 \}$. Let $\nu = m\big|_\gamma$ and define $Y : \mathcal{H} \rightarrow L^2(\nu)$ by $Y(\phi(T)f) = \phi\sqrt{g}$, for any bounded Borel function $\phi$. Then

$$\| Y(\phi(T)f) \|^2 = \| \phi \sqrt{g} \|^2$$

$$= \int_\gamma |\phi|^2 g d\nu$$

$$= \int_\gamma |\phi|^2 g dm$$

$$= \int_{\mathbb{T}} |\phi|^2 g dm$$

$$= \int_{\mathbb{T}} |\phi|^2 d\mu$$

$$= \| \phi(T)f \|^2 \qquad .$$

Thus $Y$ is well defined and isometric. Since $Y$ is densely defined, it extends to an isometry on $\mathcal{H}$. Let us show that $Y$ is onto $L^2(\nu)$.

Define $g_n$ by

$$g_n(z) = \begin{cases} g(z) & \text{if } g(z) \geq 1/n^2 \\ 1/n^2 & \text{if } g(z) < 1/n^2 \end{cases} \qquad .$$

Let $\phi_n = \dfrac{\phi}{\sqrt{g_n}}$. Then $\phi_n$ is bounded by $n \| \phi \|_\infty$. Now

$$Y(\phi_n(T)f) = \phi_n \sqrt{g}$$

$$= \phi \sqrt{\frac{g}{g_n}} \qquad .$$

The sequence $\sqrt{\dfrac{g}{g_n}}$ converges to $1$ a.e. and is bounded by $1$.

Therefore, $\phi\sqrt{\dfrac{g}{g_n}}$ converges to $\phi$ in $L^2(\nu)$. Since $Y$ is an isometry it has closed range, whence $\phi$ is in Range $Y$. Thus $L^\infty(\nu)$ is contained in Range $Y$. Since $L^\infty(\nu)$ is dense in $L^2(\nu)$, $Y$ is onto, and hence unitary. Now

$$YT(\phi(T)f) = Y(T\phi(T)f)$$

$$= z \ \phi\sqrt{g}$$

$$= M_z(\phi \ \sqrt{g})$$

$$= M_z Y(\phi(T)f) \quad .$$

This shows that $YT = M_z Y$ on a dense subset of $\mathcal{H}$, and therefore $T = Y^* M_z Y$ .

Now we return to consideration of an isometry $V = U \oplus W_a \oplus W_s$. Let the space on which $W_a$ acts be $\mathcal{H}_a$, and that on which $W_s$ acts be $\mathcal{H}_s$.

Lemma 3.2: Let $U$ be a pure isometry acting on $\mathbb{H}_{\mathfrak{D}}$, and let $W_a$ be an absolutely continuous unitary operator acting on $\mathcal{H}_a$. Then $U \oplus W_a$ acting on $\mathbb{H} \oplus \mathcal{H}_a$ is in the class (dc).

Proof: The pure isometry $U$ is unitarily equivalent to a direct sum $\displaystyle\sum_{i \in \mathfrak{J}} \oplus \ U_i$, where each $U_i$ is equal to $U_1$, the unilateral shift of multiplicity 1. (The index set $\mathfrak{J}$ is of cardinality equal to the orthogonal dimension of $\mathfrak{D}$ .). By [20, p. 777] each element of $\alpha_{U_1}$ is the limit of a <u>sequence</u> of polynomials

in $U_1$. Thus by Proposition 1.10 $U$ is in the class (dc). Note
that by Remark 1.11, each element of $\mathcal{C}_U$ is also the limit of a
sequence of polynomials in $U$. Now let

$$\mathcal{X} = \{X : \mathbb{H}_{\mathcal{D}} \to \mathcal{H}_a \mid W_a X = XU\}, \quad \text{and} \quad \mathcal{Y} = \bigcup_{X \in \mathcal{X}} \text{Range } X.$$

We claim that $\mathcal{Y}$ is dense in $\mathcal{H}_a$. (This is actually proven in
[3], although the result is not explicitly stated. Since the argument
is buried inside the proof of a different theorem, we reproduce it
here for the sake of completeness.) Let $h \in \mathcal{H}_a$ and let $\mathcal{M}$ be
the cyclic reducing subspace for $W_a$ generated by $h$. Let
$T = W_a\big|_{\mathcal{M}}$. $T$ is absolutely continuous, and has a cyclic vector $h$,
so by Lemma 3.1, $T$ is unitarily equivalent to $M_z$ on $L^2(\nu)$,
where $\nu$ is Lebesgue measure restricted to some Borel subset
of $\mathbb{T}$. Let $Y : \mathcal{M} \to L^2(\nu)$ be the unitary operator effecting
the unitary equivalence.

Let $B$ be the bilateral shift on

$$\mathbb{L}_{\mathcal{D}} = \ldots \oplus \mathcal{D} \oplus \mathcal{D} \oplus \mathcal{D} \oplus \ldots \, .$$ The operator $B$ is the minimal
unitary extension of the isometry $U$. Choose $f$ in $\mathbb{H}_{\mathcal{D}}$
(considered as a subspace of $\mathbb{L}_{\mathcal{D}}$) such that $f \perp \text{Range } U$ and
$\|f\| = 1$. Let $\mathcal{F}$ be the cyclic reducing subspace for $B$ generated
by $f$. Note that $\{B^i f\}_{i=-\infty}^{\infty}$ is an orthonormal basis for $\mathcal{F}$.
Finally, let $k \in L^{\infty}(\nu)$, and define $Z_k : \mathbb{L}_{\mathcal{D}} \to \mathcal{M}$ by

.

$$Z_k \left( \sum_{i=-\infty}^{\infty} \alpha_i B^i f \right) = Y^*(k \sum_{i=-\infty}^{\infty} \alpha_i e_i) \qquad ,$$

where $e_i(z) = z^i$, and $Z_k = 0$ on $\mathcal{J}^\perp$.  Now

$$\left\| Z_k (\sum_{i=-\infty}^{\infty} \alpha_i B^i f) \right\|^2 = \left\| k \sum_{i=-\infty}^{\infty} \alpha_i e_i \right\|_2^2$$

$$= \int \left| k(z) \sum_{i=-\infty}^{\infty} \alpha_i e_i(z) \right|^2 d\nu$$

$$\leq \| k \|_\infty^2 \int_{\mathbb{T}} \left| \sum_{i=-\infty}^{\infty} \alpha_i e_i(z) \right|^2 dm$$

$$= \| k \|_\infty^2 \left\| \sum_{i=-\infty}^{\infty} \alpha_i e_i \right\|_2^2$$

$$= \| k \|_\infty^2 \sum_{i=-\infty}^{\infty} | \alpha_i |^2$$

$$= \| k \|_\infty^2 \left\| \sum_{i=-\infty}^{\infty} \alpha_i B^i f \right\|^2 \qquad .$$

Thus $Z_k$ is bounded; indeed $\| Z_k \| \leq \| k \|_\infty$.

We now assert that $Z_k B = W_a Z_k$.  For if $g \perp \mathcal{J}$, then

$Bg \perp \mathcal{J}$, so $Z_k Bg = 0 = W_a Z_k g$.  So let $g = \displaystyle\sum_{i=-\infty}^{\infty} \alpha_i B^i f$.  Then

$$Z_k Bg = Z_k \sum_{i=-\infty}^{\infty} \alpha_i B^{i+1} f$$

$$= Z_k \sum_{i=-\infty}^{\infty} \alpha_{i-1} B^i f$$

$$= Y^*(k \sum_{i=-\infty}^{\infty} \alpha_{i-1} e_i)$$

On the other hand,

$$W_a Z_k g = W_a Y^*(k \sum_{i=-\infty}^{\infty} \alpha_i e_i)$$

$$= T Y^*(k \sum_{i=-\infty}^{\infty} \alpha_i e_i)$$

$$= Y^* M_z (k \sum_{i=-\infty}^{\infty} \alpha_i e_i)$$

$$= Y^*(k \sum_{i=-\infty}^{\infty} \alpha_i e_{i+1})$$

$$= Y^*(k \sum_{i=-\infty}^{\infty} \alpha_i e_{i+1})$$

$$= Y^*(k \sum_{i=-\infty}^{\infty} \alpha_{i-1} e_i)$$

$$= Z_k Bg$$

Let $X_k = Z_k\big|_{\mathbb{H}_{\mathfrak{H}}}$ . Then for $g$ in $\mathbb{H}_{\mathfrak{H}}$

$X_k U g = Z_k U g = Z_k B g = W_a Z_k g = W_a X_k g$. Hence $X_k U = W_a X_k$.

We notice that $X_k f = Y^* k$. Thus $\mathcal{Y}$ contains $Y^*(L^\infty(\nu))$.

Since $L^\infty(\nu)$ is dense in $L^2(\nu)$ and $Y^*$ is unitary, $Y^*(L^\infty(\nu))$

is dense in $\mathcal{M}$. But $\mathcal{M}$ was an arbitrary cyclic reducing

subspace, so we conclude that $\mathcal{Y}$ is dense in $\mathcal{H}_a$.

Now for any polynomial p,

$$\|p(W_a)\| \leq \|p\|_\infty = \|p(U_1)\| = \|p(U)\| \qquad . \text{ Therefore, since}$$

any element of $\mathcal{A}_U$ is the limit of a sequence of polynomials in

U, we may apply Proposition 1.9 and conclude that $U \oplus W_a$

is in the class (dc).

Lemma 3.3: A singular unitary operator is in the class (dc).

Proof: Let $W$ be a singular unitary operator. Since $W$

is normal, it is in the class (dc) if and only if every invariant

subspace for it is reducing. (Theorem 1.2) If $W$ has a non-reducing

invariant subspace, then by Wermer's result [21, p. 275], $W$

is unitarily equivalent to a direct sum $B_1 \oplus R$, where $B_1$ is

the bilateral shift of multiplicity 1. Let $E$ be the spectral

measure for $B_1$ and let $F$ be the spectral measure for $R$.

Define $G = E \oplus F$ by $G(\sigma) = E(\sigma) \oplus F(\sigma)$ for any Borel set $\sigma$.

Then it is easy to check that $G$ is the spectral measure for $W$.

Let $B_1$ act on the space $\mathcal{H}$, and let $R$ act on $\mathcal{K}$.

Let $f$ be a vector in $\mathcal{H}$ and identify it with its embedding in

$\mathcal{H} \oplus \mathcal{K}$. As usual, define $\mu_{ff}(\sigma) = (G(\sigma)f, f)$. But then $\mu_{ff}(\sigma)$ also is equal to $(E(\sigma)f, f)$. Since $B_1$ is absolutely continuous, $\mu_{ff} \ll m$. Since $W$ is singular, $\mu_{ff} \perp m$. Therefore, $\mu_{ff} = 0$ for each $f$ in $\mathcal{H}$. Consequently, $E$ is $0$, so $B_1 = 0$, which is a contradiction.

$\underline{\text{Theorem}}$ 3.4: Let $V$ be an isometry such that the pure part is not absent. Then $V$ is in the class (dc).

$\underline{\text{Proof}}$: Decompose $V$ as $U \oplus W_a \oplus W_s$. By Lemma 3.2, $U \oplus W_a$ is in the class (dc). By Lemma 3.3, $W_s$ is in the class (dc). By the corollary in [2], $\mathcal{O}_V = \mathcal{O}_{(U+W_a)} \oplus \mathcal{O}_{W_s}$. Hence by Lemma 1.5, $V$ is in the class (dc).

In [4] Douglas, Muhly, and Pearcy show that if $T \in \mathcal{L}(\mathcal{H})$ is a contraction, and $V^* \in \mathcal{L}(\mathcal{K})$ $(\mathcal{K} \supseteq \mathcal{H})$ is its minimal co-isometric extension, then the commutant of $T$ can be "lifted" to that of $V^*$. That is, if $C$ is in $\mathcal{O}_T'$ then there exists $\tilde{C}$ in $\mathcal{O}_{V^*}'$ such that $\mathcal{H}$ is in Lat $\tilde{C}$, $\tilde{C}\big|_{\mathcal{H}} = C$, and $\|\tilde{C}\| = \|C\|$. They then ask (p. 394), "Is there in general a relationship between $\mathcal{O}_T''$ and $\mathcal{O}_{V^*}''$?" We shall show as a corollary of Theorem 3.4 that at least $\mathcal{O}_T''$ $\underline{\text{cannot}}$ be lifted to $\mathcal{O}_{V^*}''$ in general.

$\underline{\text{Theorem}}$ 3.5: $\mathcal{O}_T''$ can be lifted to $\mathcal{O}_{V^*}''$ only if $T$ is in the class (dc) or (trivially) if $T$ is unitary.

Proof: Assume that $\mathcal{O}_T''$ can be lifted and that $T$ is not unitary. If $T$ is an isometry, then $T$ is in the class (dc) by Theorem 3.4. If $T$ is not an isometry, then $V^*$ is not unitary. (The restriction of a unitary operator to an invariant subspace is always isometric.) By Theorem 3.4 and Lemma 1.12, $V^*$ is in the class (dc).

Write $\mathcal{K}$ as $\mathcal{H}_0 \oplus \mathcal{H}_0^\perp$, and represent $V^*$ with respect to this decomposition as

$$\begin{bmatrix} T & R \\ 0 & S \end{bmatrix} \quad .$$

Let $D \in \mathcal{O}_T''$, and let $\tilde{D}$ be its lifting to $\mathcal{O}_{V^*}''$;

$$\tilde{D} = \begin{bmatrix} D & E \\ 0 & F \end{bmatrix} \quad .$$

Since $V^*$ is in the class (dc), $\tilde{D} = \text{wlim}_\alpha p_\alpha(V^*)$ for some net of polynomials $p_\alpha$. But $p_\alpha(V^*) =$

$$p_\alpha(V^*) = \begin{bmatrix} p_\alpha(T) & * \\ 0 & p_\alpha(S) \end{bmatrix} \quad ,$$

whence $D = \text{wlim}_\alpha p_\alpha(T)$, whence $D \in \mathcal{O}_T$. Therefore, $T$ is in the class (dc).

CHAPTER IV

ALGEBRAIC OPERATORS

Definition: Let  $A$  be a linear transformation on a vector space.  If there is a polynomial  $p$  such that  $p(A) = 0$,  we shall say that  $A$  is algebraic.

Any linear transformation on a finite dimensional vector space is algebraic in this sense.

If $\mathcal{V}$ is a vector space, denote the algebra of all linear transformations on $\mathcal{V}$ by $\mathcal{J}(\mathcal{V})$.  Since this is a purely algebraic construct (no topology involved) we shall refer to the commutant of a set in $\mathcal{J}(\mathcal{V})$ as the set's centralizer.

Now as was previously mentioned (page 21), the following fact is well known:

(1)  If $\mathcal{V}$ is a finite dimensional vector space and  $A$  is in $\mathcal{J}(\mathcal{V})$,  then  $A$  generates an algebra equal to its double centralizer in  $\mathcal{J}(\mathcal{V})$.

Less well known but nonetheless true is a generalization of this fact:

(2)  If $\mathcal{V}$ is a vector space and  $A \in \mathcal{J}(\mathcal{V})$  is algebraic, then  $A$  generates an algebra equal to its double centralizer in $\mathcal{J}(\mathcal{V})$.   (Cf. [13, p. 72].)

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Fact (1) specializes to the Hilbert space situation immediately and says that any operator on a finite dimensional Hilbert space is in the class (dc). This is because any linear transformation on a finite dimensional space is bounded and so we know that $\mathcal{J}(\mathcal{H}) = \mathcal{L}(\mathcal{H})$ in this situation. Also the algebra generated by A in $\mathcal{L}(\mathcal{H})$ is finite dimensional and hence closed in all the usual topologies. Therefore, the <u>algebra</u> generated by A is the same as the <u>weakly closed</u> <u>algebra</u> generated by A.

We run into difficulties, however, in trying to specialize (2) to Hilbert space. The algebra generated by an algebraic operator A is still finite dimensional, so $\mathcal{O}_A$ is just the algebra generated by A. The trouble is that in general $\mathcal{J}(\mathcal{H})$ is far larger than $\mathcal{L}(\mathcal{H})$. Thus if $\mathcal{S} \subseteq \mathcal{L}(\mathcal{H}) \subseteq \mathcal{J}(\mathcal{H})$, the centralizer of $\mathcal{S}$ usually contains many elements of $\mathcal{J}(\mathcal{H})$ which are not in $\mathcal{L}(\mathcal{H})$ and hence not in $\mathcal{S}'$. As a result, an operator in the double commutant of $\mathcal{S}$ is not necessarily in the double centralizer of $\mathcal{S}$.

We are left with the interesting question: "Is every algebraic operator on Hilbert space necessarily in the class (dc)?" The answer, as we shall show, turns out to be, "Yes."

<u>Definition:</u> Let $\mathcal{H}$ be a Hilbert space. The nth <u>ampliation</u> <u>of</u> $\mathcal{H}$, denoted by $\mathcal{H}^{(n)}$ is defined to be the direct sum

$$\mathcal{H} \oplus \mathcal{H} \oplus \ldots \oplus \mathcal{H}$$ of n copies of $\mathcal{H}$. If A is in $\mathcal{L}(\mathcal{H})$,

the nth <u>ampliation</u> <u>of</u> A, denoted by $A^{(n)}$, is defined to be the direct sum of n copies of A acting on $\mathcal{H}^{(n)}$.

    <u>Theorem</u> 4.1: Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Let $\mathcal{K} \subseteq \mathcal{L}(\mathcal{H})$ be a set such that:

    (1) For each $K \in \mathcal{K}$ and for any n, $K^{(n)}$ is unitarily

        equivalent to some element of $\mathcal{K}$.

    (2) For each $K \in \mathcal{K}$, Lat K $\subseteq$ Lat D for every D

        in $\mathcal{A}_K''$.

Then every operator in $\mathcal{K}$ is in the class (dc).

    <u>Proof:</u> Let K be in $\mathcal{K}$ and let $D \in \mathcal{A}_K''$. Let $\mathcal{N}$ be an arbitrary basic strong neighborhood of D given by

$$= \left\{ X \in \mathcal{L}(\mathcal{H}) \mid \| (D - X)h_i \| < 1, \ i = 1, \ldots, n \right\}$$

where the $h_i$ are vectors in $\mathcal{H}$.

    Now consider $K^{(n)}$ and $D^{(n)}$. It is easily seen that $D^{(n)}$ is in $(K^{(n)})''$. Also $K^{(n)}$ is unitarily equivalent to some $H \in \mathcal{K}$. The same unitary equivalence will carry $D^{(n)}$ to $E \in \mathcal{L}(\mathcal{H})$ such that E is in $\mathcal{A}_H''$. By hypothesis (2) Lat H $\subseteq$ Lat E, hence Lat $K^{(n)} \subseteq$ Lat $D^{(n)}$. In particular, the cyclic invariant subspace $\mathcal{M}$ for $K^{(n)}$ generated by $h = (h_1, \ldots, h_n)$ is invariant under $D^{(n)}$. Therefore, $D^{(n)}h$ is in $\mathcal{M}$, and so there is a sequence $\{p_i\}$ of polynomials such that $p_i(K^{(n)})h$ converges in norm to $D^{(n)}h$. Choose i such

that $\| p_i (K^{(n)} h - D^{(n)} h \| < 1$.  This says

$\| ((p_i(K) - D)h_1, \ldots, (p_i(K) - D)h_n) \| < 1$.  We conclude that

$\| (p_i(K) - D)h_j \| < 1$  for  $j = 1, \ldots, n$,  whence  $p_i(K)$  is in  $\mathcal{N}$ .

  We have shown that each basic strong neighborhood of  D

contains a polynomial in  K.  As a result,  D  must be in the

strong closure of the algebra generated by  K,  hence in  $\mathcal{O}_K$.

  Remark 4.2:  In [7] Herrero and Salinas introduce some

terminology which is relevant here.  They say that a subspace is

bi-invariant for an operator if it is invariant under the double

commutant of that operator.  Thus hypothesis (2) of Theorem 4.1

can be restated as "for each  K  in  $\mathcal{K}$,  every invariant subspace

of  K  is bi-invariant."

  Theorem 4.3:  Let  $\mathcal{H}$  be a Hilbert space.  Then any

nilpotent operator in  $\mathcal{L}(\mathcal{H})$  is in the class  (dc).

  Proof:  Without loss of generality we may assume that

$\mathcal{H}$  is infinite dimensional.  Therefore, by Theorem 4.1 it suffices

to show that if  A  is nilpotent and  $D \in \mathcal{O}_A''$,  then  Lat A $\subseteq$ Lat D.

(If  $\mathcal{H}$  is infinite dimensional and  $A \in \mathcal{L}(\mathcal{H})$  is nilpotent, then

$A^{(n)}$  is clearly nilpotent and acts on a space of the same dimension

as  $\mathcal{H}$ .)

  Moreover, it is sufficient to show that any cyclic invariant

subspace for  A  is invariant for  D.  (This is equivalent to

Lat A $\subseteq$ Lat D,  since any invariant subspace is the span of cyclic

invariant subspaces. So let $e \in \mathcal{H}$ and suppose that $A^k e = 0$, but $A^j e \neq 0$ for $j < k$. Let $\mathcal{H}_1$ be the cyclic invariant subspace for $A$ generated by $e$. We may write $\mathcal{H}_1 = \langle e, Ae, \ldots, A^{k-1}e \rangle$. We shall denote $A^j e$ by $e_{j+1}$ $(0 \leq j \leq k-1)$, for the sake of simplifying notation later on.

Let $\mathcal{H}_2 = \mathcal{H}_1^{\perp}$, write $\mathcal{H}$ as $\mathcal{H}_1 \oplus \mathcal{H}_2$, and write $A$ as

$$\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}$$

with respect to this decomposition. We now use the decomposition to construct some operators in the commutant of $A$. Suppose $X : \mathcal{H}_2 \to \mathcal{H}_1$ intertwines $A_1$ with $A_3$, i.e., $A_1 X = X A_3$. Then

$$Y = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$$

is in $\mathcal{Q}_A'$:

$$YA = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} = \begin{bmatrix} 0 & XA_3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & A_1 X \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} = AY \quad .$$

Therefore, $Y$ commutes with $D$. When $D$ is written as

$$\begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}$$

with respect to our decomposition of $\mathcal{H}$ , we see that to show $\mathcal{H}_1$

to be invariant under $D$ it is necessary and sufficient to show that

$D_3 = 0$. The equation $YD = DY$ translates to

$$\begin{bmatrix} 0 & D_1X \\ 0 & D_3X \end{bmatrix} = \begin{bmatrix} XD_3 & XD_4 \\ 0 & 0 \end{bmatrix}$$

which implies that $D_3X = 0$ and $XD_3 = 0$. Our task is now to

construct sufficiently many $X$'s so that these two equations force

$D_3$ to be zero. We first determine what "sufficiently many" is.

Let $\mathcal{X} = \{X : \mathcal{H}_2 \to \mathcal{H}_1 \mid XA_3 = A_1X\}$ , let

$\mathcal{S} = \bigcup_{X \in \mathcal{X}} \text{Range } X$, and let $\mathcal{J} = \bigcap_{X \in \mathcal{X}} \text{Ker } X$. If $\mathcal{S} = \mathcal{H}_1$

there are sufficiently many $X$'s, for then $D_3X = 0$ for all $X$

in $\mathcal{X}$ implies $D_3h = 0$ for all $h$ in $\mathcal{H}_1$ which implies that

$D_3 = 0$. If $\mathcal{J} = \{0\}$, then there are sufficiently many $X$'s

also: Let $h \in \mathcal{H}_1$. The fact that $XD_3 = 0$ for every $X$ in

implies that $XD_3h = 0$ for every $X$ in $\mathcal{X}$, whence $D_3h$ is

in $\text{Ker } X$ for every $X$ in $\mathcal{X}$, so $D_3h \in \mathcal{J} = \{0\}$. Thus

$D_3h = 0$ for each $h$ in $\mathcal{H}_1$, which is to say $D_3 = 0$.

Consequently, it suffices to show that either $\mathcal{S} = \mathcal{H}_1$, or

$\mathcal{J} = \{0\}$.

We note that $A_1$ and $A_3$ are both nilpotent, $A_1$ of order k. Let $A_3$ be nilpotent of order r.

Case (i): $r \geq k$.

The order of nilpotence of $A_3$ is also the order of nilpotence of $A_3^*$. Since $r \geq k$, we may find f in $\mathcal{H}_2$ such that $A_3^{*k}f = 0$, $A_3^{*i}f \neq 0$ for $i < k$. Let

$$f_k = f$$
$$f_{k-1} = A_3^* f$$
$$\vdots$$
$$f_1 = A_3^{*k-1} f \qquad .$$

Note that $A_3^* f_i = f_{i-1}$ for $i = 2, \ldots, k$, and that $A_3^* f_1 = 0$.
Now define $X : \mathcal{H}_2 \to \mathcal{H}_1$ by

$$Xg = (g, f_1)e_1 + \ldots + (g, f_k)e_k \qquad .$$

Then

$$A_1 Xg = (g, f_1)A_1 e_1 + \ldots + (g, f_k)A_1 e_k$$

$$= (g, f_1)Ae_1 + \ldots + (g, f_k)Ae_k$$

$$= (g, f_1)e_2 + \ldots + (g, f_{k-1})e_k + 0 \qquad .$$

On the other hand,

$$XA_3 g = (g, A_3^* f_1)e_1 + \ldots + (g, A_3^* f_k)e_k$$

$$= 0 + (g, f_1)e_2 + \ldots + (g, f_{k-1})e_k \qquad .$$

Therefore, $A_1 X = X A_3$, i.e., $X$ is in $\mathcal{X}$ .

Now $f_1, \ldots, f_k$ are linearly independent, so we may choose $g$ in $\mathcal{H}_2$ such that $g \perp \langle f_2, \ldots, f_k \rangle$ and $(g, f_1) = 1$. Then $Xg = e_1$, $XA_3^i g = A_1^i X g = A_1^i e_1 = e_{i+1}$, for $i = 0, \ldots, k-1$. Hence Range $X = \mathcal{H}_1$, so $\mathcal{S} = \mathcal{H}_1$.

Case (ii): $r < k$.

Let $f \neq 0$ be a vector in $\mathcal{H}_2$. Let $j$ be the first positive integer such that $A_3^{*j} f = 0$. Note that $j \leq r < k$. Let

$$f_j = f$$

$$f_{j-1} = A_3^* f$$

$$\vdots$$

$$f_1 = A_3^{*j-1} f \qquad .$$

Note as before that $A_3^* f_i = f_{i-1}$, for $i = 2, \ldots, j$, and that $A_3^* f_1 = 0$.

Let $s = k - j$ and define $X : \mathcal{H}_2 \to \mathcal{H}_1$ by

$$Xg = (g, f_1)e_s + (g, f_2)e_{s+1} + \ldots + (g, f_j)e_k \qquad .$$

As in case (i), $A_1 X = X A_3$. Note that $Xf \neq 0$; otherwise, we would have $\alpha_1 e_s + \alpha_2 e_{s+1} + \ldots + \alpha_j e_k = 0$, with $\alpha_j = \| f \|^2 \neq 0$, which is impossible, since $e_1, \ldots, e_k$ are linearly independent.

Thus we have shown that for each non-zero $f$ in $\mathcal{H}_2$ there is an $X$ in $\mathcal{X}$ such that $f \notin \mathrm{Ker}\ X$. Consequently, $f \neq 0$ implies $f \notin \bigcap_{X \in \mathcal{X}} \mathrm{Ker}\ X = \mathcal{J}$. So $\mathcal{J} = \{0\}$.

We are now almost in a position to prove that all algebraic operators are in the class (dc). The only further tool that we need is the following slight generalization of a theorem of Rosenblum.

Lemma 4.4: Let $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$ have disjoint spectra. Let $C : \mathcal{K} \to \mathcal{H}$ be an operator. Then there exists an operator $X$ mapping $\mathcal{K}$ into $\mathcal{H}$ such that $AX - XB = C$.

Proof: (a) Dim $\mathcal{K} \leq$ Dim $\mathcal{H}$. Embed $\mathcal{K}$ as a subspace of a space $\tilde{\mathcal{K}}$ having the same dimension as $\mathcal{H}$. Let $U : \tilde{\mathcal{K}} \to \mathcal{H}$ be a unitary operator. Let $\tilde{A} = U^* A U$. Then $\tilde{A}$ is in $\mathcal{L}(\tilde{\mathcal{K}})$. Let $\lambda \in \sigma(B)$ and define $\tilde{B}$ in $\mathcal{L}(\tilde{\mathcal{K}})$ by

$$\tilde{B} f = \begin{cases} Bf & \text{for}\ f \in \mathcal{K} \\ \lambda f & \text{for}\ f \in \mathcal{K}^\perp \end{cases}.$$

Define $\tilde{C} : \tilde{\mathcal{K}} \to \mathcal{H}$ by

$$\tilde{C} f = \begin{cases} Cf & \text{for}\ f \in \mathcal{K} \\ 0 & \text{for}\ f \in \mathcal{K}^\perp \end{cases}.$$

Then $\sigma(\tilde{A}) = \sigma(A)$ and $\sigma(\tilde{B}) = \sigma(B)$, so $\sigma(\tilde{A})$ and $\sigma(\tilde{B})$ are disjoint. Therefore, by Theorem 3.1 of [17], there is a solution $\tilde{X}$ to the equation $\tilde{A} \tilde{X} - \tilde{X} \tilde{B} = U^* \tilde{C}$.

Let $X = U \tilde{X} \big|_{\mathcal{K}}$. Then for $f \in \mathcal{K}$,

$$(AX - XB)f = AU\tilde{X}f - U\tilde{X}Bf$$

$$= U\tilde{A}U^*U\tilde{X}f - U\tilde{X}\tilde{B}f$$

$$= U\tilde{A}\tilde{X}f - U\tilde{X}\tilde{B}f$$

$$= U(\tilde{A}\tilde{X} - \tilde{X}\tilde{B})f$$

$$= UU^*\tilde{C}f$$

$$= \tilde{C}f$$

$$= \tilde{C}f \quad .$$

So $AX - XB = C$.

(b) Dim $\mathcal{K}$ > Dim $\mathcal{H}$ . Consider the equation $B^*X - XA^* = -C^*$. By part (a) we can find a solution $X$ to this equation. Then $X^*$ is a solution to $AX - XB = C$.

Theorem 4.5: All algebraic operators are in the class (dc).

Proof: Let $A$ be an algebraic operator on $\mathcal{H}$ and let $p$ be its minimal polynomial. Factor $p(z)$ as $(z - \lambda_1) \times \ldots \times (z - \lambda_n)$. Let $\mathcal{M}_k = \text{Ker}((A - \lambda_1) \times \ldots \times (A - \lambda_k))$ for $k = 1, \ldots, n$. Let

$$\mathcal{H}_k = \begin{cases} \mathcal{M}_1 & \text{for } k = 1 \\ \mathcal{M}_k \cap \mathcal{M}_{k-1}^{\perp} & \text{for } k = 2, \ldots, n \end{cases} .$$

Then by a result in [10], $\mathcal{H}$ decomposes as $\mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n$ and with respect to this decomposition $A$ has a matrix $[A_{ij}]$ which is upper triangular. Moreover, the diagonal entries of this matrix are scalars, specifically $A_{ii} = \lambda_i$ .

Now assume that the roots of  p  (the  $\lambda_i$'s)  are ordered so that equal roots are grouped together.  Let the distinct roots of p  be  $\nu_1, \ldots, \nu_r$.  Subdivide the matrix of  A  into blocks to obtain a new matrix  $[B_{ij}]$   $(1 \leq i,j \leq r)$  as follows:  Suppose that $\lambda_s, \ldots, \lambda_t$  are equal to  $\nu_i$  and  $\lambda_m, \ldots, \lambda_q$  are equal to  $\nu_j$.  Then

$$B_{ij} = \begin{bmatrix} A_{sm} \cdots\cdots\cdots A_{sq} \\ \vdots \qquad\qquad \vdots \\ \vdots \qquad\qquad \vdots \\ \vdots \qquad\qquad \vdots \\ A_{tm} \cdots\cdots\cdots A_{tq} \end{bmatrix}.$$

We notice that each diagonal block,  $B_{ii}$,  is equal to  $\nu_i$  plus a nilpotent operator.

Finally, we subdivide this new matrix for  A  once more, dividing off the last column and the last row.  We write this last subdivision as a two-by-two matrix

$$\begin{bmatrix} B & C \\ 0 & D \end{bmatrix},$$

where  $D = B_{rr}$.  We claim that

$$\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

is similar to

$$\begin{bmatrix} B & 0 \\ 0 & D \end{bmatrix}.$$

To show this, we observe that $\sigma(B) = \{\nu_1, \ldots, \nu_{r-1}\}$, and

$\sigma(D) = \{\nu_r\}$. Since the $\nu_i$'s are distinct, the spectra of B and

D are disjoint. Therefore, by Lemma 4.4, there is a solution X

to the equation $BX - XD = C$. For such an X

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & D \end{bmatrix} ,$$

and

$$\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}$$

is the inverse of

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} ,$$
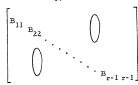
so the two matrices are similar as claimed.

Now by induction we may assume that

$$B = \begin{bmatrix} B_{11} & & & & \\ & B_{22} & & \text{\Large\bigstar} & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & B_{r-1\ r-1} \end{bmatrix}$$

is similar to

$$\begin{bmatrix} B_{11} & & & & \\ & B_{22} & & \bigcirc & \\ & & \ddots & & \\ & \bigcirc & & \ddots & \\ & & & & B_{r-1\ r-1} \end{bmatrix} \qquad .$$

Thus $A$ is similar to

$$\tilde{A} = \begin{bmatrix} B_{11} & & & & \\ & B_{22} & & \bigcirc & \\ & & \ddots & & \\ & \bigcirc & & \ddots & \\ & & & & B_{rr} \end{bmatrix} \qquad ,$$

which may be written as $B_{11} \oplus B_{22} \oplus \ldots \oplus B_{rr}$. Now each $B_{ii}$ is a nilpotent plus a scalar (namely $\nu_i$) and therefore in the class (dc) by Theorem 4.3. (Any translate of an operator in the class (dc) is clearly in the class (dc) also.)

Since the spectra of the $B_{ii}$ are distinct points, Lemma 1.7 applies to show that the projection on the ith coordinate space is in $\mathcal{O}_{\tilde{A}}$ for each i. Therefore, $\mathcal{O}_{\tilde{A}} = \mathcal{O}_{B_{11}} \oplus \ldots \oplus \mathcal{O}_{B_{rr}}$ by Lemma 1.6. By Lemma 1.5, $\tilde{A}$ is in the class (dc), and then so is $A$.

CHAPTER V

## COMPACT n-NORMAL OPERATORS
## AND QUASI-SIMILARITY

We follow [9, p. 23] in our definition of n-normal operator.

Thus we say that an operator $A$ is n-normal if it may be

represented as an $n \times n$ operator matrix $[A_{ij}]$ acting on $\mathcal{H}^{(n)}$,

where the $A_{ij} \in \mathcal{L}(\mathcal{H})$ are commuting normal operators. In

what follows we shall, to avoid extraneous complications, assume

that $\mathcal{H}$ is separable.

If $A$ is compact and n-normal, it follows that each of the

entries $A_{ij}$ must be compact. We can therefore show that they

may be simultaneously diagonalized.

Lemma 5.1: Let $C_1, \ldots, C_r$ be a finite family of commuting

compact normal operators on $\mathcal{H}$. Then there is an orthonormal

basis of $\mathcal{H}$ with respect to which each $C_j$ is diagonal.

Proof: We proceed by induction on $r$. The result is well

known for $r = 1$. (See, for instance, [1, p. 181] or [9, p. 86].)

Assume the result true for $r - 1$. Consider $C_1$: We may write

$\mathcal{H}$ as $\sum_{i=0}^{\infty} \oplus \mathcal{H}_i$, where $\mathcal{H}_i$ is the eigenspace for $C_1$

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corresponding to the eigenvalue $\lambda_i$. (Assume $\lambda_0 = 0$.) Each $\mathcal{H}_i$ is finite dimensional, except possibly for $\mathcal{H}_0$. For each $C_j$, $2 \leq j \leq r$, all the spaces $\mathcal{H}_i$ are invariant under $C_j$, since $C_j$ commutes with $C_1$: Let $f \in \mathcal{H}_i$. Then $C_1 C_j f = C_j C_1 f = C_j \lambda_i f = \lambda_i C_j f$, so $C_j f$ is in $\mathcal{H}_i$. Since <u>all</u> of the direct summands are invariant for $C_j$, they are reducing for $C_j$, $2 \leq j \leq r$.

Let $C_j^i = C_j|_{\mathcal{H}_i}$, $j = 1, \ldots, r$, and $i = 0, 1, 2, \ldots$. Then $C_1^i, \ldots, C_r^i$ form a finite commuting family of normal operators on the <u>finite</u> <u>dimensional</u> Hilbert space $\mathcal{H}_i$ for $i = 1, 2, 3, \ldots$. Therefore (see, for instance, [8, p. 172]), there exists an orthonormal basis $\mathcal{X}_i$ for $\mathcal{H}_i$ with respect to which $C_1^i, \ldots, C_r^i$ are diagonal.

We also note that $C_1^0, \ldots, C_r^0$ form a commuting family of normal operators on $\mathcal{H}_0$. By our induction hypothesis there exists an orthonormal basis $\mathcal{X}_0$ for $\mathcal{H}_0$ such that $C_2^0, \ldots, C_r^0$ are diagonal with respect to $\mathcal{X}_0$. Clearly $C_1^0$ is diagonal with respect to $\mathcal{X}_0$ since $C_1^0 = 0$. Identify the vectors in each $\mathcal{X}_i$ with their embeddings in $\mathcal{H}$. Let $\mathcal{X} = \bigcup_{i=0}^{\infty} \mathcal{X}_i$. Then $\mathcal{X}$ is an orthonormal basis for $\mathcal{H}$ with respect to which $C_1, \ldots, C_r$ are diagonal.

The point of the foregoing lemma is that a compact n-normal operator may be represented as an $n \times n$ operator matrix

each of whose entries is diagonal. Such an operator is unitarily
equivalent (by re-ordering the basis for $\mathcal{H}^{(n)}$) to a direct sum
of $n \times n$ scalar matrices. We shall show that any compact operator
of the form $\sum_{i=0}^{\infty} \oplus A_i$, with each $A_i$ an operator on $n$-dimensional
Hilbert space, is quasi-similar to an operator in the class (dc).
We need a lemma to facilitate the proof of this fact.

Lemma 5.2: Let $\{A_i\}_{i=0}^{\infty}$ be a norm bounded sequence
of operators, where each $A_i$ acts on an $n$-dimensional Hilbert
space. (The number $n$ is fixed.) Let $A = \sum_{i=0}^{\infty} \oplus A_i$. Let $K$
denote the closure of the set $\bigcup_{i=0}^{\infty} \sigma(A_i)$. Then $\sigma(A) = K$.

Proof: The fact that $K$ is contained in $\sigma(A)$ is clear.
To prove the reverse inclusion, let $\lambda_0 \notin K$. Then the distance
from $\lambda_0$ to $K = \delta > 0$, since $K$ is compact. Thus $A_i - \lambda_0$
is invertible for every $i$, and the distance from $0$ to
$\sigma(A_i - \lambda_0)$ is greater than or equal to $\delta > 0$. Let us denote our
$n$-dimensional Hilbert space by $\mathcal{H}$ and the invertible operators
on $\mathcal{H}$ by $\mathcal{I}$. Now we notice that if $\mathcal{F} \subseteq \mathcal{I}$ is such that
the distance from $0$ to $\sigma(F)$ is greater than or equal to $\delta > 0$
for every $F$ in $\mathcal{F}$, then $\mathcal{F}^-$, the norm closure of $\mathcal{F}$,
is also contained in $\mathcal{I}$. For let $F \in \mathcal{F}^-$; then there is a

sequence $\left\{ F_i \right\}_{i=0}^{\infty} \subseteq \mathfrak{F}$ , such that $F_i$ converges in norm

to $F$. Therefore, $\det F_i$ converges to $\det F$. But

$$\det F_i = \left| \prod_{\lambda \in \sigma(F_i)} \lambda \right|$$

$$= \prod_{\lambda \in \sigma(F_i)} |\lambda|$$

$$\geq \prod_{\lambda \in \sigma(F_i)} \delta$$

$$= \delta^n \qquad \text{for each } i \qquad .$$

Thus $|\det F| \geq \delta^n > 0$, so $F$ is invertible. Now let

$\mathfrak{F} = \left\{ (A_i - \lambda_0) \mid i = 0, 1, 2, \ldots \right\}$ . By the foregoing remarks,

$\mathfrak{F}^- \subseteq \mathfrak{U}$ . $\mathfrak{F}$ is bounded by $|\lambda_0| + \sup_i \|A_i\|$, hence so is

$\mathfrak{F}^-$ . Thus $\mathfrak{F}^-$ is a closed, bounded subset of a finite $(n^2)$

dimensional Banach space, and is consequently compact. The

function $X \mapsto X^{-1}$ which maps $\mathfrak{U}$ into $\mathfrak{L}(\mathcal{H})$ is continuous

and therefore bounded on compact sets, in particular on $\mathfrak{F}^-$ .

Consequently, the sequence $\left\{ \|(A_i - \lambda_0)^{-1}\| \right\}_{i=0}^{\infty}$ is bounded.

We conclude that $B = \sum_{i=0}^{\infty} \oplus (A_i - \lambda_0)^{-1}$ is a bounded operator.

Clearly, $B(A - \lambda_0) = I = (A - \lambda_0)B$, so $\lambda_0 \notin \sigma(A)$.

Theorem 5.3: Any compact n-normal operator is quasi-similar to an operator in the class (dc).

Proof: Let $A$ be compact and n-normal. By the remarks preceding Lemma 5.2, we may assume that $A = \sum_{i=0}^{\infty} \oplus A_i$, where each $A_i$ is an operator on n-dimensional Hilbert space. We shall assume that this n-dimensional Hilbert space is the canonical one, $\mathbb{C}^n$, and shall identify each $A_i$ with its matrix.

For each $A_i$ let $J_i$ be the Jordan Canonical Form of $A_i$, and let $S_i$ be the similarity transform taking $A_i$ to $J_i$. That is, $A_i = S_i J_i S_i^{-1}$. For each $i$, $\|J_i\| \leq 1 + \rho(A)$ where $\rho(A)$ denotes the spectral radius of $A$, so $J = \sum_{i=0}^{\infty} \oplus J_i$ is a bounded operator. Let

$$S = \sum_{i=0}^{\infty} \oplus S_i / \|S_i\|$$

and

$$T = \sum_{i=0}^{\infty} \oplus S^{-1} / \|S_i^{-1}\| \qquad .$$

Then $S$ and $T$ are quasi-invertible, $AS = SJ$ and $TA = JT$, so $A$ and $J$ are quasi-similar. We shall show that $J$ is in the class (dc). By means of a unitary equivalence, group together the Jordan blocks of $J$ having the same eigenvalues. Thus we

re-write J as $\sum_{i=0}^{\infty} \oplus B_i$, where each $B_i$ is a direct sum of

Jordan blocks having the single eigenvalue $\lambda_i$. Each block is, of

course, of size $n \times n$ or smaller.

For $\lambda_i \neq 0$, $B_i$ is an operator on a finite dimensional

space and is therefore in the class (dc). (See page 21.) The

fact that A is compact precludes the infinite repetition of non-zero

eigenvalues. For $\lambda_i = 0$, $B_i$ is either finite or a one-sided

weighted shift. In either case it is in the class (dc). (In the latter

case this follows by Theorem 2.7.) Consequently by Lemmas 1.5

and 1.6, it suffices to show that $P_i$, the projection on the space

on which $B_i$ acts, is in $\mathcal{O}_J$ for each i. By Lemma 5.2,

$\sigma(J)$ is equal to the closure of $\bigcup_{i=0}^{\infty} \sigma(J_i)$, which is equal to

the closure of $\bigcup_{i=0}^{\infty} \sigma(A_i)$, which is equal to $\sigma(A)$. This equals

$\{\lambda_0, \lambda_1, \lambda_2, \ldots\} \cup \{0\}$ with each $\lambda_i \neq 0$ an isolated

point, since A is compact. Hence for $\lambda_j \neq 0$ the complete

spectrum of $B_j$ ($= \{\lambda_j\}$) is disjoint from the complete spectrum

of $\sum_{i \neq j} \oplus B_i$, and so by Lemma 1.7, $P_j$ is in $\mathcal{O}_J$. For

$\lambda_j = 0$, $P_j = I - \sum_{i \neq j} P_i$, so it is in $\mathcal{O}_J$ too.

Therefore, J is in the class (dc).

The point of the foregoing theorem was not to make any particular use of the fact that compact n-normal operators are quasi-similar to operators in the class (dc), but rather to show that quasi-similarity does not help too much in the investigation of this class. To demonstrate this assertion, we shall construct a compact 2-normal operator which is _not_ in the class (dc). This will show that the class (dc) is not preserved under quasi-similarity. I am grateful to Professor Paul Federbush of the University of Michigan for a conversation concerning this example.

Proposition 5.4: Let

$$A_i = \begin{bmatrix} 1/2^i & 1/i \\ 0 & 1/2^{i+1} \end{bmatrix} \quad ,$$

for $i = 1, 2, 3, \ldots$ . Each of these $2 \times 2$ matrices is considered to be an operator on the space $\mathbb{C}^2$. Let

$$A = \sum_{i=1}^{\infty} \oplus A_i.$$ Then $A$ does not have the unit property.

Proof: For any polynomial $p$ and any $2 \times 2$ matrix

$$M = \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix} \quad ,$$

with $\alpha \neq \gamma$,

$$p(M) = \begin{bmatrix} p(\alpha) & \beta \dfrac{p(\alpha) - p(\gamma)}{\alpha - \gamma} \\ \\ 0 & p(\gamma) \end{bmatrix} .$$

(See [13, p. 140].)

Thus $p(A) = \sum_{i=1}^{\infty} \oplus p(A_i)$, where each $p(A_i)$ has the

above form.

Let $w$ be the vector

$$\left( \begin{bmatrix} \sqrt{i/2^{i+1}} \\ \\ 0 \end{bmatrix} \right)_{i=1}^{\infty} ,$$

and let $v$ be the vector

$$\left( \begin{bmatrix} 0 \\ \\ \sqrt{i/2^{i+1}} \end{bmatrix} \right)_{i=1}^{\infty} .$$

Note that

$$\sum_{i=1}^{\infty} \sqrt{i/2^{i+1}}^{\,2} = \sum_{i=1}^{\infty} i/2^{i+1} < \infty$$

(by the ratio test), so that $v$ and $w$ are legitimate vectors.

Consider the inner product of $p(A)v$ with $w$:

$$(p(A)v, w) = \left(\left(\begin{bmatrix} \frac{1}{i} \dfrac{p(1/2^i) - p(1/2^{i+1})}{1/2^i - 1/2^{i+1}} \sqrt{\dfrac{i}{2^{i+1}}} \\[2ex] p(1/2^{i+1})\sqrt{\dfrac{i}{2^{i+1}}} \end{bmatrix}\right)_{i=1}^{\infty}, \left(\begin{bmatrix} \sqrt{i/2^{i+1}} \\[2ex] O \end{bmatrix}\right)_{i=1}^{\infty}\right)$$

$$= \sum_{i=1}^{\infty} \left(\frac{1}{i} \frac{p(1/2^i) - p(1/2^{i+1})}{1/2^{i+1}} \frac{i}{2^{i+1}}\right)$$

$$= \sum_{i=1}^{\infty} \left(p(1/2^i) - p(1/2^{i+1})\right)$$

$$= \lim_{n \to \infty} \left(p(1/2) - p(1/2^{n+1})\right)$$

$$= p(1/2) - p(0) \quad .$$

Now suppose that there exists a net of polynomials $p_\alpha$ without constant term such that $p_\alpha(A) \to I$. Then

$$(p_\alpha(A)v, w) = p_\alpha(1/2) - p_\alpha(0) = p_\alpha(1/2) \to (Iv, w)$$

$$= (v, w) = 0 \quad .$$

On the other hand, since $p_\alpha(1/2)$ is the first entry on the main diagonal of $p^\alpha\{\alpha\}$, the fact that $p_\alpha(A) \to I$ implies that the numbers $p_\alpha(1/2)$ converge to 1. This is a contradiction, so $A$ does not have the unit property.

Theorem 5.5: There exists a compact 2-normal operator which is not in the class (dc).

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Proof: Let $A$ be as in Proposition 5.4 and let $0$ denote a $2 \times 2$ zero matrix. Then $A \oplus 0$ is 2-normal. $\|A_i\| \leq 1/2^i + 1/i$, which tends to zero as $i$ tends to infinity. Therefore, $A$ is compact and hence so is $A \oplus 0$.

Each direct summand of $A$ is invertible, and so $A$ is quasi-invertible. Since by Proposition 5.4 $A$ does not have the unit property, $A \oplus 0$ is not in the class (dc) by Theorem 1.14.

APPENDIX A

THE PROOF OF LEMMA 2.6

What follows is a slight elaboration of work done by Schurr [19] and by Shields and Wallen [20]. Before actually proving Lemma 2.6 we make some definitions, and we prove a couple of sub-lemmas.

Definition 1: Let $A = [\alpha_{ij}]$ and $B = [\beta_{ij}]$ be matrices of the same size. We define the Hadamard product of $A$ and $B$, denoted by $A*B$, to be the matrix $[\alpha_{ij}\beta_{ij}]$.

Definition 2: Let $A$ be a matrix. Let $Y \in \mathcal{L}(\mathcal{H})$ have matrix $B$ of the same size as $A$ with respect to the orthonormal basis $\mathcal{E}$. If $A*B$ is the matrix (with respect to $\mathcal{E}$) of a bounded operator on $\mathcal{H}$ we shall call this operator the Hadamard product of $A$ with $Y$ with respect to $\mathcal{E}$ and denote it by $A *_{\mathcal{E}} Y$.

Definition 3: Let $A = [\alpha_{ij}]_{i,j \in \mathcal{J}}$ be a matrix. A finite section of $A$ is a finite matrix $A_{\mathcal{J}} = [\alpha_{ij}]_{i,j \in \mathcal{J}}$, where $\mathcal{J}$ is a finite subset of $\mathcal{J}$.

Definition 4: Let $A$ be a matrix. We shall say that $A$ is positive if each finite section of $A$ is positive in the usual sense.

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It is easy to check that if $A$ is the matrix of a bounded operator then this operator is positive in the usual sense if and only if $A$ is positive in the sense just defined.

Lemma 1: Let $A$ be a positive matrix with bounded diagonal entries. Let $\rho$ be the supremum of the absolute values of these entries. Let $Y \in \mathcal{L}(\mathcal{H})$ have matrix $B$ of the same size as $A$ with respect to the orthonormal basis $\mathcal{E} = \{e_i\}_{i \in \mathcal{J}}$. Then $A *_\mathcal{E} Y$ exists and has norm less than or equal to $\rho \|Y\|$.

Proof: Let $A = [\alpha_{ij}]$ and let $B = [\beta_{ij}]$. Let $\mathcal{F}$ be a finite subset of $\mathcal{J}$. Since $A_\mathcal{F}$ is positive, it has a unique positive square root. Denote this square root by $[\gamma_{ij}]_{i,j \in \mathcal{F}}$.

Since $A_\mathcal{F} * B_\mathcal{F}$ is a finite matrix, it defines an operator on the subspace $\mathcal{H}_\mathcal{F}$ of $\mathcal{H}$, where $\mathcal{H}_\mathcal{F} = \langle \{e_i\}_{i \in \mathcal{F}} \rangle$. We shall continue to denote this operator by $A_\mathcal{F} * B_\mathcal{F}$.

Let $f = \sum_{i \in \mathcal{F}} \nu_i e_i$ and $g = \sum_{i \in \mathcal{F}} \mu_i e_i$ be unit vectors in $\mathcal{H}_\mathcal{F}$. Then

$$(A_{\mathcal{J}} * B_{\mathcal{J}} \, f, g) = (\sum_{j \in \mathcal{J}} \nu_j \sum_{i \in \mathcal{J}} \alpha_{ij}\beta_{ij}e_i, \quad \sum_{i \in \mathcal{J}} \mu_i e_i)$$

$$= \sum_{i \in \mathcal{J}} \sum_{j \in \mathcal{J}} \bar{\mu}_i \nu_j \alpha_{ij}\beta_{ij}$$

$$= \sum_{i \in \mathcal{J}} \sum_{j \in \mathcal{J}} \bar{\mu}_i \nu_j \beta_{ij} (\sum_{k \in \mathcal{J}} \gamma_{ik}\gamma_{kj})$$

$$= \sum_{k \in \mathcal{J}} (\sum_{i \in \mathcal{J}} \sum_{j \in \mathcal{J}} \nu_j \beta_{ij}\gamma_{ij}\bar{\mu}_i \gamma_{ik})$$

$$= \sum_{k \in \mathcal{J}} (\sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{J}} \nu_j \gamma_{kj}\beta_{ij}\bar{\mu}_i \bar{\gamma}_{ki})$$

(Since we are considering the positive square root of a positive operator $\gamma_{ik} = \bar{\gamma}_{ki}$.)

$$= \sum_{k \in \mathcal{J}} (B_{\mathcal{J}} f^{(k)}, g^{(k)}) \qquad,$$

where

$$f^{(k)} = \sum_{j \in \mathcal{J}} \nu_j \gamma_{kj}e_j$$

and

$$g^{(k)} = \sum_{i \in \mathcal{J}} \mu_i \gamma_{ki}e_i \quad.$$

Now

$$| (B_{\mathcal{I}} f^{(k)}, g^{(k)}) | \leq \| B_{\mathcal{I}} \| \cdot \| f^{(k)} \| \cdot \| g^{(k)} \|$$

$$= \| B_{\mathcal{I}} \| ( \sum_{i \in \mathcal{I}} | \nu_i \gamma_{ki} |^2 )^{\frac{1}{2}} ( \sum_{i \in \mathcal{I}} | \mu_i \gamma_{ki} |^2 )^{\frac{1}{2}}$$

$$\leq \| B_{\mathcal{I}} \| \sum_{i \in \mathcal{I}} \frac{| \nu_i \gamma_{ki} |^2 + | \mu_i \gamma_{ki} |^2}{2}$$

$$(\text{Since } ab \leq \frac{a^2 + b^2}{2}$$
$$\text{for any real } a \text{ and } b)$$

$$= \frac{\| B_{\mathcal{I}} \|}{2} \sum_{i \in \mathcal{I}} | \gamma_{ki} |^2 ( | \nu_i |^2 + | \mu_i |^2 ) \quad .$$

Therefore,

$$| (A_{\mathcal{I}} * B_{\mathcal{I}} f, g) | \leq \sum_{k \in \mathcal{I}} \frac{\| B_{\mathcal{I}} \|}{2} \sum_{i \in \mathcal{I}} | \gamma_{ki} |^2 ( | \nu_i |^2 + | \mu_i |^2 )$$

$$= \frac{\| B_{\mathcal{I}} \|}{2} ( \| f \|^2 + \| g \|^2 ) \sum_{k \in \mathcal{I}} \overline{\gamma}_{ki} \gamma_{ki}$$

$$= \| B_{\mathcal{I}} \| \sum_{k \in \mathcal{I}} \gamma_{ik} \gamma_{ki}$$

$$(\text{Since } \| f \|^2 = \| g \|^2 = 1)$$

$$= \| B_{\mathcal{I}} \| \cdot \alpha_{ii} \leq \| B_{\mathcal{I}} \| \rho \quad .$$

Therefore, $\|A_{\mathfrak{J}} * B_{\mathfrak{J}}\| \leq \rho \|B_{\mathfrak{J}}\| \leq \rho \|Y\|$ for any finite subset $\mathfrak{J}$ of $\mathfrak{J}$. Thus the norms of all the finite sections of $A*B$ are bounded by $\rho \|Y\|$, whence $A*B$ defines a bounded operator of norm $\leq \rho \|Y\|$. We conclude that $A *_E Y$ exists and has norm less than or equal to $\rho \|Y\|$.

Lemma 2: Let $S \in \mathcal{L}(\mathcal{H})$ be a weighted shift (backward or forward, one-sided, two-sided, or finite). Let $D \in \mathcal{L}(\mathcal{H})$ be matricially a formal power series in $S$, $f(S)$. Then the Cesaro means of the partial sums of $f(S)$ are bounded in norm by $\|D\|$.

Proof: Let $\mathfrak{J}$ denote any of the index sets $\mathfrak{J}_1 = \{0, 1, \ldots, n\}$, $\mathfrak{J}_2 = \{0, 1, 2, \ldots\}$, $\mathfrak{J}_3 = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Let $\Sigma = \{e_i\}_{i \in \mathfrak{J}}$ be the orthonormal basis for $\mathcal{H}$ with respect to which $S$ is represented as shift. Define a matrix $A_k = [\alpha_{ij}]_{i, j \in \mathfrak{J}}$ by

$$\alpha_{ij} = \begin{cases} 1 - \dfrac{|i - j|}{k + 1} & \text{for } |i - j| \leq k \\[2ex] 0 & \text{for } |i - j| > k \end{cases}.$$

In the case $\mathfrak{J} = \mathfrak{J}_3$ the matrix $A_k$ is the matrix of the Laurent operator $L_\phi$, where

$$\phi(e^{i\theta}) = \sum_{r=-k}^{k} (1 - \frac{|r|}{k+1}) e^{ir\theta} .$$

By [20, p. 88],

$$\phi(e^{i\theta}) = \frac{1}{k+1} \left[ \frac{\sin \frac{(k+1)\theta}{2}}{\sin \frac{\theta}{2}} \right]^2 \quad ,$$

which is clearly a non-negative function.

Thus in this case $A_k$ is the matrix of a positive operator. In the cases $\mathcal{J} = \mathcal{J}_1$ or $\mathcal{J}_2$ the $A_k$'s are matrices of compressions of $L_\phi$ and hence are also matrices of positive operators. We note that the entries on the diagonal (in fact, all of the entries) of $A_k$ are always bounded by 1. Therefore, by Lemma 1, $A_k *_{\mathcal{E}} D$ exists and has norm less than or equal to $\|D\|$. But $A_k *_{\mathcal{E}} D$ is the kth Cesaro mean of the partial sums of $f(S)$.

Proof of Lemma 2.6: Since $D$ is matricially equal to the formal power series $f(S)$, each $D_i$ is matricially equal to the formal power series $f(S_i)$. Let $e_j$ be the jth basis vector in the space on which $S_i$ and $D_i$ act.

Note that the partial sums of $f(S_i)e_j$ are simply the partial sums for the Fourier expansion of $D_i e_j$ and hence converge to $D_i e_j$.

Thus $f(S_i)$ converges to $D_i$ at each basis vector for each i. Consequently, $f(S)$ converges to $D$ on a dense set. Let $c_n(S)$ denote the nth Cesaro mean of the partial sums of $f(S)$. Then

$c_n(S)$ converges to $D$ on the same dense set.

By Lemma 2, $\|c_n(S_i)\| \leq \|D_i\| \leq \|D\|$, for all $i$, so $\|c_n(S)\| = \sup_i \|c_n(S_i)\| \leq \|D\|$, for all $n$. The $c_n(S)$ are therefore bounded and convergent to $D$ on a dense set. Hence, $c_n(S) \to D$. Since each $c_n(S)$ is a polynomial in $S$, $D$ is in $\mathcal{O}_S$.

## A CALCULATION NEEDED FOR THE
## PROOF OF PROPOSITION 2.9

We are given that

$$\gamma_{ij} = \begin{cases} \dfrac{\alpha_{j+1} \times \ldots \times \alpha_0}{\alpha_{i+1} \times \ldots \times \alpha_{i-j}} & \gamma_{i-j\ 0} \quad \text{for} \quad j \leq 0 \\[3ex] \dfrac{\alpha_i \times \ldots \times \alpha_{i-j+1}}{\alpha_1 \times \ldots \times \alpha_j} & \gamma_{i-j\ 0} \quad \text{for} \quad j > 0 \end{cases} \quad .$$

We wish to show that

$$\gamma_{ij} = \begin{cases} \dfrac{\alpha_{j+1} \times \ldots \times \alpha_i}{\alpha_1 \times \ldots \times \alpha_{i-j}} & \gamma_{i-j\ 0} \quad \text{for} \quad i \geq j \\[3ex] \dfrac{\alpha_{i-j+1} \times \ldots \times \alpha_0}{\alpha_{i+1} \times \ldots \times \alpha_j} & \gamma_{i-j\ 0} \quad \text{for} \quad i < j \end{cases} \quad .$$

We proceed by cases. In what follows, round parentheses will indicate cancellation of terms in order to bring the expression for $\gamma_{ij}$ to the right form. Square brackets will indicate insertion of terms for the same purpose.

(a) For $i \geq j$, $\gamma_{ij} = \dfrac{\alpha_{j+1} \times \ldots \times \alpha_i}{\alpha_1 \times \ldots \times \alpha_{i-j}} \gamma_{i-j\ 0}$ :

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Case (i): $\quad j \leq i \leq 0$ .

$$\gamma_{ij} = \frac{\alpha_{j+1} \times \ldots \times \alpha_i (\alpha_{i+1} \times \ldots \times \alpha_0)}{(\alpha_{i+1} \times \ldots \times \alpha_0)\alpha_1 \quad \times \ldots \times \alpha_{i-j}} \quad \gamma_{i-j\,0}$$

Case (ii): $\quad j \leq 0 < i$ .

$$\gamma_{ij} = \frac{\alpha_{j+1} \times \ldots \times \alpha_0 [\alpha_1 \times \ldots \times \alpha_i]}{[\alpha_1 \times \ldots \times \alpha_i]\alpha_{i+1} \times \ldots \times \alpha_{i-j}} \quad \gamma_{i-j\,0}$$

Case (iii): $\quad 0 < j \leq i < 2j$ .

$$\gamma_{ij} = \frac{(\alpha_{i-j+1} \times \ldots \times \alpha_j)\alpha_{j+1} \times \ldots \times \alpha_i}{\alpha_1 \times \ldots \times \alpha_{i-j}(\alpha_{i-j+1} \times \ldots \times \alpha_j)} \quad \gamma_{i-j\,0}$$

Case (iv): $\quad 0 < j < 2j \leq i$ .

$$\gamma_{ij} = \frac{[\alpha_{j+1} \times \ldots \times \alpha_{i-j}]\alpha_{i-j+1} \times \ldots \times \alpha_i}{\alpha_1 \times \ldots \times \alpha_j [\alpha_{j+1} \times \ldots \times \alpha_{i-j}]} \quad \gamma_{i-j\,0}$$

(b) For $\; i < j$, $\quad \gamma_{ij} = \frac{\alpha_{i-j+1} \times \ldots \times \alpha_0}{\alpha_{i+1} \quad \times \ldots \times \alpha_j} \gamma_{i-j\,0}$ :

Case (i): $\quad i < 2j \leq j \leq 0$ .

$$\gamma_{ij} = \frac{[\alpha_{i-j+1} \times \ldots \times \alpha_j]\alpha_{j+1} \times \ldots \times \alpha_0}{\alpha_{i+1} \times \ldots \times \alpha_{i-j}[\alpha_{i-j+1} \times \ldots \times \alpha_j]} \quad \gamma_{i-j\,0}$$

Case (ii): $\quad 2j \leq i < j \leq 0$ .

$$\gamma_{ij} = \frac{(\alpha_{j+1} \times \ldots \times \alpha_{i-j})\alpha_{i-j+1} \times \ldots \times \alpha_0}{\alpha_{i+1} \times \ldots \times \alpha_j (\alpha_{j+1} \times \ldots \times \alpha_{i-j})} \quad \gamma_{i-j\,0}$$

.

Case (iii):     $i \leq 0 < j$ .

$$\gamma_{ij} = \frac{\alpha_{i-j+1} \times \dots \times \alpha_i [\alpha_{i+1} \times \dots \times \alpha_0]}{[\alpha_{i+1} \times \dots \times \alpha_0] \alpha_1 \times \dots \times \alpha_j} \gamma_{i-j\ 0}$$

Case (iv):     $0 < i \leq j$ .

$$\gamma_{ij} = \frac{\alpha_{i-j+1} \times \dots \times \alpha_0 (\alpha_1 \times \dots \times \alpha_i)}{(\alpha_1 \times \dots \times \alpha_i) \alpha_{i+1} \times \dots \times \alpha_j} \gamma_{i-j\ 0}$$

APPENDIX C

COUNTER EXAMPLES

We have already seen (cf. Remark 2.12) the direct sum of operators in the class (dc) needs not be in the class (dc). We shall now show that the sum and product of commuting operators in the class (dc) need not be in the class (dc).

To give an example of the latter situation, let $A$ be a quasi-invertible operator in the class (dc) which does not have the unit property. Let $\lambda$ be outside the complete spectrum of $A$, and let $I$ denote the identity operator on the space on which $A$ acts. Then $I \oplus 0$ and $A \oplus \lambda$ are commuting operators in the class (dc). Their product is $A \oplus 0$, which is not in the class (dc).

Turning to sums, we shall actually show that we can perturb a non-compact operator in the class (dc) by a compact operator in the class (dc) and obtain an operator not in the class (dc). Let $A$ be a two-sided shift with positive weights having infimum zero but not tending to zero. Let $\lambda$ have absolute value greater than the spectral radius of $A$. Then $\lambda$ is not in the spectrum (complete spectrum) of $A$. Let $\lambda$ and $-\lambda$ act as scalar operators on a

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finite dimensional Hilbert space. Then $0 \oplus -\lambda$ is compact, in the class (dc), and commutes with $A \oplus \lambda$ which is <u>non</u>-compact and in the class (dc). Their sum is $A \oplus 0$, which is not in the class (dc).

If one wishes, one can construct examples of two operators not in the class (dc) whose sum, or product as the case may be, <u>is</u> in the class (dc). Likewise, there are examples of operators in the class (dc) and operators not in the class (dc) whose sum, product, or direct sum are, or are not, in the class (dc) as desired. Only one of all the possible combinations does not immediately yield up a counter example. We as yet have no example of two operators not in the class (dc) whose direct sum <u>is</u> in the class (dc). Nor have we been able to prove that this cannot happen.

## APPENDIX D

## SOME OPEN QUESTIONS

The following are some questions which we have thought about to one extent or another and have been unable to answer. Some of them may turn out to be easy, of course, but they all seem to be fairly interesting.

(a)  If $A$ is in the class (dc) and invertible, must $A^{-1}$ be in the class (dc)? Suppose that $A$ is normal. The question then boils down to: "If $A$ has a non-reducing invariant subspace, need $A^{-1}$ have one?

(b)  If $A$ is in the class (dc), need $A^n$ be in it also, for all positive integers $n$?

(c)  Are all Toeplitz operators in the class (dc)? How about Analytic Toeplitz operators? Powers of $U_1$ (the unilateral shift of multiplicity 1) are all right; they are simply shifts of higher multiplicities. When one starts taking linear combinations of these powers, the situation immediately becomes unclear.

(d)  Does every operator in the class (dc) have an invariant subspace? A weaker question: Does every operator generating

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a maximal abelian weakly closed algebra have an invariant subspace?

(e) For fixed $\mathcal{H}$ , are the operators in the class (dc) norm dense in $\mathcal{L}(\mathcal{H})$?

(f) Are the operators in the class (dc) connected in $\mathcal{L}(\mathcal{H})\setminus\{\lambda \mid \lambda \in \mathbb{C}\}$ ? Are they pathwise connected? They are clearly pathwise connected in $\mathcal{L}(\mathcal{H})$ with the scalars included, since translation and multiplication by scalars leaves the class (dc) invariant.

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