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Undetected boundary slopes and roots of unity for the character variety of a 3-manifold

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Undetected boundary slopes and roots of unity for the character variety of a 3-manifold

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Undetected boundary slopes and roots of unity for the character variety of a 3-manifold

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This dissertation is concerned with the Culler-Shalen techniques for using the $\mathrm{SL}_2(\mathbb{C})$-character variety for the fundamental group of a 3-manifold to find embedded essential surfaces in the manifold. It is known that when a boundary slope is strongly detected by the character variety the limiting eigenvalue of the slope is a root of unity. In Chapter 2, we show that every root of unity arises in this manner. Given any root of unity, we construct infinitely many hyperbolic 3-manifolds whose character varieties all detect this root. In Chapter 3, we give infinitely many hyperbolic knots whose exteriors have strict boundary slopes which are not strongly detected by the character variety.

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Chapter 1

Introduction

This thesis concerns the techniques of Culler and Shalen [8] for finding embedded essential surfaces in 3–manifolds. Embedded essential surfaces have played a critical role in the development of the theory of 3-manifolds, and the Culler-Shalen machinery provides the most general strategy for constructing essential surfaces. Their techniques play an important part in proofs of the Smith conjecture [22], the cyclic surgery theorem [7], and the finite filling conjecture [2], among others.

Let M be a compact irreducible orientable 3-manifold whose boundary consists of a disjoint union of incompressible tori. We write $X(M)$ to denote the space of all characters of representations in $\mathrm{Hom}\big(\pi_1(M), \mathrm{SL}_2(\mathbb{C})\big)$. This space has the structure of an affine algebraic set. If we assume that $X \subset X(M)$ is an irreducible affine curve and let $\widetilde{X}$ be the smooth projective model for X, then all but finitely many points of $\widetilde{X}$ correspond to characters in X. These points are referred to as *ordinary points* while the points which are not ordinary points are called *ideal points.* The set of ideal points in $\widetilde{X}$ is finite and non-empty because we have assumed that X is a curve. The Culler-Shalen machinery shows how to associate essential surfaces in M to each ideal point in $\widetilde{X}$.

Although their techniques may associate many different essential surfaces to a given ideal point, some information about these surfaces can be obtained by evaluating certain functions in the function field $\mathbb{C}(X)$ at that ideal point. Let $\gamma \in \pi_1(M)$ and define the regular function $I_\gamma \in \mathbb{C}(X)$ to be the function which evaluates characters on the element γ. That is

$$I_\gamma(\chi) \;=\; \chi(\gamma)$$

where $\chi \in X$. I_γ extends to a meromorphic function

$$I_\gamma\colon\ \widetilde{X} \longrightarrow \mathbb{C} \cup \{\infty\}.$$

Suppose there is a peripheral element $\beta \in \pi_1(\partial M)$ and an ideal point $\hat{x} \in \widetilde{X}$ such that $I_\beta(\hat{x}) = \infty$. Then there is a unique unoriented isotopy class of a simple closed curve $c \subset M$ with the property that $I_{[c]}(\hat{x})$ is finite. In fact, $I_{[c]}(\hat{x})$ is of the form $\xi + \xi^{-1}$ where ξ is a root of unity [4].

The above information translates directly into topological data about associated surfaces. Let Σ be an essential surface associated to an ideal point $\hat{x}$ as above, and take Σ to have the minimal number of boundary components among such surfaces. Then, for every connected component Σ_0 of Σ, $\partial\Sigma_0 \neq \emptyset$, the order of ξ divides the number of boundary components of Σ_0, and every boundary component of Σ_0 represents the unoriented isotopy class of c. In this situation, we say that the unoriented isotopy class of c is a *strongly detected boundary slope.*

In practice, strongly detected slopes are quite common. If the interior of M is a hyperbolic 3-manifold with finite volume, then work of Thurston [25] implies that M has at least two strongly detected boundary slopes. Also, it follows from work of Kronheimer and Mrowka that the exterior of any non-trivial knot in S^3 has at least one strongly detected boundary slope [3], [10]. Thus, we see that the Culler-Shalen techniques lend themselves particularly well to these settings.

It is known that a boundary slope which is only represented by boundary components of essential surfaces which are virtual fibers in virtual fibrations over S^1 will not be strongly detected [7]. However, in light of the discussion in the previous paragraph, it has been asked whether their techniques are sensitive enough to find all other boundary slopes for the exterior of a given non-trivial knot in S^3. A similar question has been asked for manifolds whose interiors have a finite volume structure and a single cusp. In Chapter 3, we give a negative answer to both questions by exhibiting an infinite family of hyperbolic knots with boundary slopes which are not strongly detected. (This work is joint with Stephan Tillmann.)

In [4], it is asked which roots of unity can arise when a boundary slope is strongly detected. Ohtsuki [19] shows that for the exterior of a 2-bridge knot every ideal point detects an essential surface with a connected component which has one or two boundary components. Thus, the only roots of unity which appear in these cases are ± 1. There are, however, examples of higher order roots. There are explicit examples of 3-manifolds with 4^{th}, 6^{th}, and 11^{th}

roots given in [5], [9], and [14]. In Chapter 2, we show that given any $n \in \mathbb{Z}^+$ there are infinitely many hyperbolic 3-manifolds which have a curve in their character varieties with ideal points at which n^{th} roots of unity arise in the strong detection of a boundary slope.

1.1 Preliminary Material

We begin by outlining the Culler-Shalen constructions and stating some known results that will be needed. See [23] as a reference.

1.1.1 Conventions and notation

An embedding

$$f\colon\ F \longrightarrow M$$

of a compact orientable connected surface ($F \ncong S^2, D^2$) into an irreducible 3-manifold M is called *incompressible* if the induced homomorphism

$$f_*\colon\ \pi_1(F) \longrightarrow \pi_1(M)$$

is injective. If, in addition, f is not properly homotopic into ∂M, then f is *essential* in M. We will say that the surface F is essential in M. A disconnected compact orientable surface is incompressible (essential) if each of its components is.

An essential surface F in M is a *virtual fiber* if there is a finite cover

$$p\colon\ M' \longrightarrow M$$

where M' is fibered over S^1 and the surface F lifts to a fiber in M'.

Throughout this dissertation, all 3-manifolds are assumed to be compact, connected, orientable, and irreducible. We assume further that all boundary components of 3-manifolds are incompressible and homeomorphic to tori. All surfaces are assumed to be properly embedded and orientable, but not necessarily connected.

For a topological space Y, we write $|Y|$ to denote the number of connected components of Y.

For a 3-manifold M, we will say that M is *hyperbolic* if the interior of M admits a complete finite volume hyperbolic structure. This is equivalent to the existence of a discrete faithful homomorphism

$$\pi_1(M) \longrightarrow \mathrm{PSL}_2(\mathbb{C}).$$

A *slope* on a 3-manifold M is the isotopy class of an unoriented essential simple closed curve on a component $\partial_0 M$ of ∂M. Every slope corresponds to a pair $\{\alpha, \alpha^{-1}\}$ of primitive elements of $\pi_1(\partial_0 M)$. For convenience, we will often refer to the slope $\{\alpha, \alpha^{-1}\}$ as, simply, α. Given a generating set $\{\lambda, \mu\}$ for $\pi_1(\partial_0 M)$, we can identify the set of slopes on $\partial_0 M$ with $p/q \in \mathbb{Q} \cup \{1/0\}$, where p/q corresponds to the slope $\lambda^q \mu^p$. A *boundary slope* is a slope that is represented by a boundary component of an essential surface in M. A boundary slope is *strict* if it arises as the boundary of an essential surface which is not a virtual fiber.

Let

$$G \times T \longrightarrow T$$

be a simplicial action of a group G on a simplicial tree T. The action is *non-trivial* if no point of T is fixed by the entire group. G acts *without inversions* if no element of G interchanges a pair of adjacent vertices of T.

If Y is a simplicial complex, we write $Y^{(i)}$ to denote its i-skeleton.

If L is a 1-manifold properly embedded in a 3-manifold M, then we write $U(L)$ to denote a small open tubular neighborhood of L in M. If L is a disjoint union of smooth simple closed curves in S^3, we define the *exterior* of the link L as the compact 3-manifold

$$E(L) \;=\; S^3 - U(L).$$

If G is a group, then G^{AB} denotes the abelianization of G.

1.1.2 Representation varieties

An $\mathrm{SL}_2(\mathbb{C})$-*representation* for a group G is a homomorphism

$$\rho\colon\; G \longrightarrow \mathrm{SL}_2(\mathbb{C}).$$

We say that an $\mathrm{SL}_2(\mathbb{C})$-representation ρ is *irreducible* if the only two invariant vector subspaces of $\mathbb{C}^2$ under the action of $\rho(G)$ are $\{0\}$ and $\mathbb{C}^2$. Otherwise, ρ is *reducible.*

We denote the space of all $\mathrm{SL}_2(\mathbb{C})$-representations for G by $R(G)$ and by $R(M)$ if $G = \pi_1(M)$ for a 3-manifold M. Two representations $\rho, \rho' \in R(G)$

are *equivalent* if $\rho' = J\rho$ where J is an inner automorphism of $\mathrm{SL}_2(\mathbb{C})$. The *character* of a representation $\rho \in R(G)$ is the function

$$\chi_\rho \colon G \longrightarrow \mathbb{C}$$

given by $\chi_\rho(\gamma) = \mathrm{Tr}(\rho(\gamma))$. Since the trace function on $\mathrm{SL}_2(\mathbb{C})$ is invariant under inner automorphisms, equivalent $\mathrm{SL}_2(\mathbb{C})$-representations always have the same character. A reducible representation always shares its character with an abelian representation and irreducible representations are determined up to conjugacy by their characters.

Assume that G is finitely generated and let $\langle\, \gamma_1, \ldots, \gamma_n \,|\, r_1, \ldots \rangle$ be a presentation for G. We can identify $R(G)$ with the set

$$\Big\{ \big(\rho(\gamma_1), \ldots, \rho(\gamma_n)\big) \,\Big|\, \rho \in R(G) \Big\} \;\subset\; \Big(\mathrm{SL}_2(\mathbb{C})\Big)^n \;\subset\; \mathbb{R}^{4n}.$$

$R(G)$ is an affine algebraic set in $\mathbb{R}^{4n}$ since it is defined by polynomial equations which arise from the set of relations $\{r_i\}$ and from the condition that $\det\big(\rho(\gamma_i)\big) = 1$. (If $\big|\{r_i\}\big| = \infty$, there are infinitely many defining equations, however, by the Hilbert basis theorem, we can always replace them with a finite set of polynomial equations.) Given any two such presentations for G, there is a polynomial bijection between their corresponding affine algebraic sets. Therefore, up to canonical isomorphism, $R(G)$ is well-defined as an affine algebraic set.

Let R_0 be an irreducible algebraic component of $R(G)$ and let K be the function field $\mathbb{C}(R_0)$. We construct the *tautological representation for* R_0

as follows. For $i \in \{1, \ldots, n\}$, let $a_{\gamma_i}, b_{\gamma_i}, c_{\gamma_i}$ and d_{γ_i} be the four coordinate functions for $R(G)$ which correspond to the generator γ_i. Then , by definition,

$$\rho(\gamma_i) \;=\; \begin{pmatrix} a_{\gamma_i}(\rho) & b_{\gamma_i}(\rho) \\ c_{\gamma_i}(\rho) & d_{\gamma_i}(\rho) \end{pmatrix}.$$

The map $\{\gamma_i\}_{i=1}^{n} \longrightarrow \mathrm{SL}_2(K)$ given by

$$\gamma_i \;\mapsto\; \begin{pmatrix} a_{\gamma_i} & b_{\gamma_i} \\ c_{\gamma_i} & d_{\gamma_i} \end{pmatrix}$$

extends to a homomorphism

$$\mathcal{P}\colon\ G \longrightarrow \mathrm{SL}_2(K)$$

which we refer to as the tautological representation for R_0.

1.1.3 Character varieties

Let $X(G)$ denote the space of characters of $\mathrm{SL}_2(\mathbb{C})$-characters for G. For every element $\gamma \in G$, we define the function

$$I_\gamma\colon\ R(G) \longrightarrow \mathbb{C}$$

by $I_\gamma(\rho) = \chi_\rho(\gamma)$. This map descends to a map (also denoted by I_γ) from $X(G)$ to $\mathbb{C}$.

Let

$$t\colon\ R(G) \longrightarrow X(G)$$

be the map that takes a representation to its character. It is shown in [8] that the map t can be viewed as a map into $\mathbb{C}^r$ given by

$$t(\rho) = \big(I_{\delta_1}(\rho), \ldots, I_{\delta_r}(\rho)\big) \;\in \mathbb{C}^r$$

for a finite set $\{\delta_1, \ldots, \delta_r\} \subset G$. The following result is Proposition 1.4.4 in [8].

Proposition 1.1.1. *If R_0 is an irreducible component of $R(G)$ which contains an irreducible representation, then $t(R_0)$ has the structure of an affine algebraic set.*

If $G = \pi_1(M)$ where M is hyperbolic, then we know more. Since M is hyperbolic, we have a discrete faithful representation

$$\rho_0\colon\ \pi_1(M) \longrightarrow \mathrm{PSL}_2(\mathbb{C}).$$

It is well known that there exist lifts of ρ_0 to $\mathrm{SL}_2(\mathbb{C})$. The following theorem is due to Thurston and can be found as Theorem 4.5.1 of [23].

Theorem 1.1.2. *Let $\tilde{\rho}_0\colon\ \pi_1(M) \longrightarrow \mathrm{SL}_2(\mathbb{C})$ be a lift of ρ_0. Let X_0 be an irreducible component of $X(M)$ containing $t(\tilde{\rho}_0)$. Then*

$$\dim_{\mathbb{C}}(X_0) \;=\; |\partial M|.$$

Furthermore, if $B_1, \ldots, B_m$ are the boundary components of M and $g_i \in \pi_1(M)$ lies in a conjugacy class carried by B_i, then $t(\tilde{\rho}_0)$ is an isolated point of the algebraic subset

$$\{\chi \in X_0 \,|\, I_{g_1}^2 = \cdots = I_{g_m}^2 = 4\}.$$

If X_0 is an irreducible algebraic curve in $X(G) \subset \mathbb{C}^r$, then we write $\widetilde{X_0}$ to denote its smooth projective completion. ($\widetilde{X_0}$ can be defined as the smooth projective curve which is birationally equivalent to X_0, it is unique up to birational equivalence.) The birational equivalence $\phi\colon \widetilde{X_0} \longrightarrow X_0$ is well-defined at all but a finite set of points; we call these points *ideal points* whereas the remaining points of $\widetilde{X_0}$ are called *ordinary points*. We identify the function fields of X_0 and $\widetilde{X_0}$ with the isomorphism induced by ϕ. The functions I_γ extend to meromorphic functions from $\widetilde{X_0}$ to the Riemann sphere. Because a subset of the I_γ's make up a set of coordinate functions for the affine curve X_0, for every ideal point $\hat{x} \in \widetilde{X_0}$, there is a $\gamma \in G$ with $I_\gamma(\hat{x}) = \infty$.

If R_0 is an irreducible algebraic set in $R(G)$ with $t\big(R(G)\big) = X_0$, then the regular map t induces an injection

$$t_*\colon\ \mathbb{C}(X_0) \longrightarrow \mathbb{C}(R_0).$$

Using this identification of $\mathbb{C}(X_0)$ with a subfield of $\mathbb{C}(R_0)$, we can view $\mathbb{C}(R_0)$ as an extension of $\mathbb{C}(X_0)$. Since R_0 and X_0 are finite dimensional, $\mathbb{C}(R_0)$ and $\mathbb{C}(X_0)$ are finitely generated over $\mathbb{C}$, and therefore $\mathbb{C}(R_0)$ is finitely generated over $\mathbb{C}(X_0)$.

1.2 The action associated to an ideal point

Take R_0, X_0, and $\widetilde{X_0}$ as at the end of the previous section. Let $F = \mathbb{C}(X_0)$ and $K = \mathbb{C}(R_0)$.

Because $\widetilde{X_0}$ is smooth, for any point $x \in \widetilde{X_0}$ we get a valuation

$$v_x\colon\ F^\times \longrightarrow \mathbb{Z}$$

where

$$v_x(f) \;=\; \begin{cases} \text{the order of the zero at } x & \text{if } f(x) = 0 \\ -(\text{the order of the pole at } x) & \text{if } f(x) = \infty \\ 0 & \text{if } f(x) \in \mathbb{C} - \{0\} \end{cases} .$$

We say that a valuation is *non-trivial* if there exists an element which values negatively. Recall that if $\hat{x}$ is an ideal point, we have a $\gamma \in G$ with $I_\gamma(x) = \infty$. Thus, $v_{\hat{x}}(I_\gamma) < 0$, and so the valuation $v_{\hat{x}}$ is non-trivial.

We can use the following extension theorem from [15] to "extend" the valuation $v_{\hat{x}}$ to a non-trivial valuation on $K^\times$.

Theorem 1.2.1. *Let K be a finitely generated extension of a field F. Let $\omega\colon F^\times \longrightarrow \mathbb{Z}$ be a valuation. Then there exists a valuation $v\colon K^\times \longrightarrow \mathbb{Z}$ such that $v|_{F^\times} = d \cdot \omega$ for some $d \in \mathbb{Z}^+$.*

Recalling the tautological representation for R_0, we have

$$\mathcal{P}\colon\ G \longrightarrow \mathrm{SL}_2(K)$$

where K is now equipped with a non-trivial discrete valuation given by $\hat{x}$. Work in [21] shows how use a non-trivial valuation to construct a simplicial tree T on which $\mathrm{SL}_2(K)$ acts simplicially, non-trivially, and without inversions. Using $\mathcal{P}$, we pull back this action to get an action of G on T. We say that such an action is *associated to the ideal point* $\hat{x}$. The following proposition is fundamental (see Proposition 1.2.6 of [7]).

Proposition 1.2.2. *Let $G \times T \longrightarrow T$ be an action associated to an ideal point $\hat{x}$. An element $\gamma \in G$ has a fixed point in T if and only if $I_\gamma(\hat{x}) \neq \infty$.*

Moreover, if N is a normal subgroup of $\mathcal{P}(G)$ which fixes a point of T, then N is contained in the center of $\mathrm{SL}_2(K)$ *or* $\mathcal{P}(G)$ *contains an abelian subgroup of index at most* 2.

Henceforth, as no new information comes from them, we will ignore components of $X(G)$ which correspond to tautological representations where $\mathcal{P}(G)$ contains an abelian subgroup of finite index. Notice also, that the first part of Proposition 1.2.2 implies that G acts non-trivially on T.

1.3 Ideal points and essential surfaces

For the remainder of this chapter, we assume that $G = \pi_1(M)$ for a 3-manifold M with $|\partial M| = 1$. Let

$$p\colon\ \widetilde{M} \longrightarrow M$$

be the universal cover for M. Fix a triangulation for M and give $\widetilde{M}$ the induced triangulation.

Assume that

$$\pi_1(M) \times T \longrightarrow T$$

is an action associated to an ideal point $\hat{x}$. Let E be the set of midpoints of edges of T. We explain how to use this setup to construct essential surfaces in M.

We say that a properly embedded surface $\Sigma \subset M$ is *associated to the* *action* if there exists a $\pi_1(M)$-equivariant map

$$f\colon \widetilde{M} \longrightarrow T$$

which is transverse to E and such that $f^{-1}(E) = p^{-1}(\Sigma)$. Notice that any such surface is orientable because $\pi_1(M)$ acts without inversions.

There are many such maps; one such is constructed as follows. Let $S^{(0)}$ be a complete set of orbit representatives for the action of $\pi_1(M)$ on $\widetilde{M}^{(0)}$. Define f on $S^{(0)}$ by any map into $T^{(0)}$ and extend f to all of $\widetilde{M}^{(0)}$ using $\pi_1(M)$-invariance. This map extends to a $\pi_1(M)$-invariant map on all of $\widetilde{M}$ since T is contractible. This map can be taken to be simplicial, and hence transverse to E. The surface $p\big(f^{-1}(E)\big)$ is associated to the action.

We can now state the following (see Proposition 1.3.2 of [7]).

Proposition 1.3.1. *Assume that Σ is a surface associated to the action given by an ideal point. Then, for each component D of $M - \Sigma$, the image of $\pi_1(D)$ in $\pi_1(M)$ is contained in a vertex stabilizer. Thus, the fact that the action is non-trivial implies that any associated surface is non-empty.*

Standard arguments show how f can be adjusted to make $p\big(f^{-1}(E)\big)$ essential in M (see [23] for a detailed account). This proves the following theorem.

Theorem 1.3.2. *If $X(M)$ has a component with dimension at least one, then M contains an essential surface.*

Proof. Let R_0 be an irreducible algebraic subset of $R(M)$ whose projection to $X(M)$ is a curve X_0. Since X_0 is a curve, the set of ideal points in $\widetilde{X_0}$ is non-empty. Let $\hat{x}$ be one such ideal point. Then, as discussed above, we can construct a tree T on which $\pi_1(M)$ acts without inversions. Take $f\colon \widetilde{M} \longrightarrow T$ to be any $\pi_1(M)$-invariant map which is transverse to E. Let $\Sigma = p\big(f^{-1}(E)\big)$. From the comments above, we may assume that Σ is an essential surface in M and, by Proposition 1.3.1, $\Sigma \neq \emptyset$. $\square$

We obtain more information about surfaces associated to ideal points using Proposition 1.3.8 from [7].

Proposition 1.3.3. *Assume that $\pi_1(M) \times T \longrightarrow T$ is an action on a tree given by an ideal point. If C is a connected subcomplex of ∂M such that the image of $\pi_1(C)$ in $\pi_1(M)$ is contained in a vertex stabilizer, then there is an essential surface associated to the action which is disjoint from C.*

This is especially useful in the following setting. Assume that $\hat{x}$ is an ideal point in $\widetilde{X_0}$ where X_0 is an algebraic curve in $X(M)$. Assume also that c is an oriented essential simple closed curve on ∂M with

$$I_{[c]}(\hat{x}) \in \mathbb{C}.$$

Then $[c]$ stabilizes a vertex under the action $\pi_1(M) \times T \longrightarrow T$ given by $\hat{x}$. By Proposition 1.3.3, there exists an essential surface associated to $\hat{x}$ which is disjoint from the curve c. If we assume further that there exists another oriented essential simple closed curve s on ∂M where

$$I_{[s]}(\hat{x}) = \infty$$

then, by Proposition 1.2.2, $[s] \in \pi_1(M)$ does not fix any vertex of T. Using Proposition 1.3.1, we see that any surface associated to the action on T given by $\hat{x}$ must intersect the curve s. Choose an essential surface Σ associated to $\hat{x}$ which is disjoint from c. Since $\partial\Sigma$ is a non-empty collection of essential simple closed curves parallel to c, $[c]$ is a boundary slope. In fact, by considering Propositions 1.2.2 and 1.3.1, we conclude that this boundary slope is strict. When a strict boundary slope arises in this manner we say that it is *strongly detected by the ideal point* $\hat{x}$.

If M is hyperbolic, then, by Theorem 1.1.2, we know that there is a curve X_0 in $X(M)$. Theorem 1.1.2 also implies that if α is a non-trivial primitive element of $\pi_1(\partial M)$, then I_α is non-constant on X_0. This implies that there is an ideal point $\hat{x} \in \widetilde{X_0}$ with $I_\alpha(\hat{x}) = \infty$. As above, any essential surface Σ associated to $\hat{x}$ must intersect any simple closed curve on ∂M which is parallel to a representative of α. Thus, components of $\partial\Sigma$ represent boundary slopes for M. Now choose $\beta \in \pi_1(\partial M)$ so that a representative for β is parallel to the components of $\partial\Sigma$ and repeat the above argument. We obtain a new ideal point $\hat{y}$ with $I_\beta(\hat{y}) = \infty$. Let Λ be an essential surface associated to $\hat{y}$. Then boundary components of Λ represent a new boundary slope for M. Therefore, M has at least two strongly detected boundary slopes.

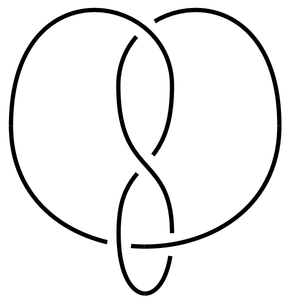


Figure 1.1: The figure-eight knot

Example (The figure-eight knot). Let K be the figure-eight knot as shown in Figure 1.1. Then $M = E(K)$ is an example of a hyperbolic manifold with one boundary component and exactly two strongly detected boundary slopes. It is worth noting that $E(K)$ actually has three boundary slopes; in addition to the two strongly detected slopes, the boundary of an essential Seifert surface gives another boundary slope for $E(K)$. However, this slope is not strict which explains why it is not strongly detected.

In light of the discussion preceding the example, it is convenient to divide the ideal points of $X(M)$ into two classes.

- An ideal point $\hat{x}$ is of *type I* if for every $\alpha \in \pi_1(\partial M)$ we have $I_\alpha(\hat{x}) \in \mathbb{C}$.
- An ideal point $\hat{x}$ is of *type II* if there exists an $\alpha \in \pi_1(\partial M)$ with $I_\alpha(\hat{x}) = \infty$.

If $\hat{x}$ is of type I, then Proposition 1.3.3 implies that there is a closed essential surfaces associated to $\hat{x}$. 3-manifolds which do not contain closed essential surfaces are called *small*. In this terminology, we see that small 3-manifolds do not have any type I ideal points.

On the other hand, if $\hat{x}$ is of type II, then $\hat{x}$ strongly detects a unique strict boundary slope. Assume that Σ is an essential surface associated to $\hat{x}$. Then $\partial\Sigma \neq \emptyset$ and the boundary components of Σ are essential closed curves all representing the same boundary slope. Assume that $\beta \in \pi_1(\partial M)$ represents this slope. By Proposition 1.3.1, β stabilizes a vertex of T. Proposition 1.2.2 implies that $I_\beta(\hat{x}) \in \mathbb{C}$. If Λ is another essential surface associated to $\hat{x}$, then let $\alpha \in \pi_1(\partial M)$ represent the boundary slope realized by Λ. Again, we have that $I_\alpha(\hat{x}) \in \mathbb{C}$. If $\alpha \notin \{\beta^{\pm 1}\}$, take C to be the 1-complex in ∂M made up of two simple closed curves representing α and β. Proposition 1.3.3 implies that we can find a closed essential surface associated to $\hat{x}$. This contradicts our assumption that $\hat{x}$ is of type II, so we must have $\alpha \in \{\beta^{\pm 1}\}$. We summarize with the following handy proposition.

Proposition 1.3.4. *If $\hat{x}$ is a type II ideal point, then there is a unique (up to inverses) primitive element $\beta \in \pi_1(\partial M)$ with the property that $I_\beta(\hat{x}) \in \mathbb{C}$. β is the unique strongly detected boundary slope detected by $\hat{x}$.*

A surface Σ associated to an ideal point is said to be *reduced* if it is essential and has the minimal number of boundary components among all essential surfaces associated to this ideal point. The following theorem gives us some additional topological information about reduced surfaces associated to type II ideal points [4].

Theorem 1.3.5. *Assume that $\beta \in \pi_1(\partial M)$ represents a boundary slope which is strongly detected by an ideal point $\hat{x}$. Then $I_\beta(\hat{x}) = \xi + \xi^{-1}$ where ξ is a root of unity. Moreover, if Σ is a reduced surface associated to β, then the order of ξ divides $|\partial\Sigma|$.*

We say that ξ *appears* as a root of unity for M.

Sometimes boundary slopes are realized by surfaces associated to type I ideal points. In other words, although surfaces associated to an ideal point may be taken to be disjoint from ∂M, there may be essential surfaces associated to the same ideal point which have non-empty boundary. We say that this ideal point *weakly detects* the boundary slope represented by the boundary of such a surface.

There are known examples of 3-manifolds with strict boundary slopes which are not strongly detected but are weakly detected. The first examples were given in [20]. In their paper, Schanuel and Zhang give a family of (non-hyperbolic) graph manifolds with strict boundary slopes which are not strongly detected. The slopes in question are, however, weakly detected [27].

As mentioned above, if an ideal point $\hat{x}$ strongly detects a boundary slope, then it detects a unique slope. The same is not known to be true if the boundary slope is weakly detected.

Question 1.3.1. If $\hat{x}$ is an ideal point which weakly detects a boundary slope β, is β the unique boundary slope detected by this ideal point?

1.4 A-polynomials

The A-polynomial $A_M(l, m)$ for M was introduced in [4]. It is helpful in identifying strongly detected boundary slopes for M.

Throughout this section, we fix a generating set $\{\lambda, \mu\}$ for $\pi_1(\partial M)$. If ρ is an $\mathrm{SL}_2(\mathbb{C})$-representation of the peripheral subgroup $\pi_1(\partial M)$ with $\mathrm{Tr}\big(\rho(\mu)\big) \neq \pm 2$, then ρ is conjugate to a diagonal representation. We are concerned with characters of $\mathrm{SL}_2(\mathbb{C})$-representations and traces are invariant under conjugation, hence we will assume that ρ is diagonal. Take $l, m \in \mathbb{C}^\times$ so that

$$\rho(\lambda) = \begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix} \quad \text{and} \quad \rho(\mu) = \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}.$$

On the other hand, since $\pi_1(\partial M) \cong \mathbb{Z} \times \mathbb{Z}$, every pair (l, m) of non-zero complex numbers determines such a representation. Two such representations ρ and ρ' will have the same character if and only if the upper left entries of the pair $\rho(\lambda)$, $\rho(\mu)$ are related to the upper left entries of the pair $\rho'(\lambda)$, $\rho'(\mu)$ by the involution $(l, m) \mapsto \big(l^{-1}, m^{-1}\big)$. Thus, we see that $X(\partial M)$ is the quotient of $\mathbb{C}^\times \times \mathbb{C}^\times$ by this involution.

The inclusion

$$i\colon\ \partial M \longrightarrow M$$

induces a regular map

$$i^*\colon\ X(M) \longrightarrow X(\partial M)$$

by restricting characters to the boundary subgroup of $\pi_1(M)$. Let $\{X_j\}_{j=1}^k$ be the set of irreducible components of $X(M)$ such that, for every $j \in \{1, \ldots, k\}$,

X_j contains the character of an irreducible representation and $\dim_{\mathbb{C}}\big(i^*(X_j)\big) = 1$. Define

$$D_M = \bigcup_{j=1}^{k} i^*(X_j)$$

and $\overline{D_M}$ as the inverse image of D_M under the quotient map $\mathbb{C}^\times \times \mathbb{C}^\times \longrightarrow X(\partial M)$. If $A_M(l,m) \in \mathbb{Q}[l,m]$ is a defining polynomial for $\overline{D_M}$, it can be normalized to have relatively prime $\mathbb{Z}$-coefficients. This is the A-polynomial for M corresponding to the generating set $\{\lambda, \mu\}$.

The *Newton polygon* $\mathcal{N}_M$ for the A-polynomial $A_M = \sum a_{ij} l^i m^j$ is the convex hull of the set

$$\big\{(i,j) \,|\, a_{ij} \neq 0\big\} \subset \mathbb{R}^2.$$

From [4], we have the famous "boundary slopes are boundary slopes" theorem.

Theorem 1.4.1. *There exists an edge of $\mathcal{N}_M$ with slope p/q if and only if p/q is a strongly detected boundary slope for M.*

We give a rough description of why this is true. Assume that e is an edge of $\mathcal{N}_M$ with slope p/q. Let $\sigma = \lambda^a \mu^b$ and $\tau = \lambda^q \mu^p$ where a and b are chosen so that $ap - bq = 1$. Then $\{\sigma, \tau\}$ gives a new generating set for $\pi_1(\partial M)$. We have $\lambda = \sigma^p \tau^{-b}$ and $\mu = \sigma^{-q} \tau^a$. To compute the A-polynomial which corresponds to this new generating set, we set $l = s^p t^{-b}$ and $m = s^{-q} t^a$ in $A_M(l,m)$ and normalize as before. The result is of the form

$$A_M(s,t) = f(t) + r(s,t)$$

where f has at least one non-zero root ξ and $r(0,t)=0$ for every $t \in \mathbb{C}$. The *edge polynomial for* e is defined as

$$f_e(x) \;=\; x^{-n} \cdot f(x)$$

where $x \in \mathbb{Z}^+$ is chosen so that $f_e \in \mathbb{Z}[x]$ and $f_e(0) \neq 0$.

Notice that the solution $(0,\xi)$ to $A_M(s,t)=0$ does not correspond to an image of a character in $X(M)$. However, in the set of solutions to this equation, the points that do correspond to characters of irreducible representations are dense. Hence, there is a sequence $\{\chi_i\}$ of such characters with

$$\lim_{i\to\infty} \chi_i(\tau) \;=\; \xi + \xi^{-1} \quad \text{and} \quad \lim_{i\to\infty} \chi_i(\sigma) \;=\; \infty.$$

Therefore, there is a type II ideal point corresponding to the limit of these characters. $\tau = \lambda^q \mu^p$ is a strongly detected boundary slope and ξ is the associated root of unity.

Chapter 2

Roots of unity and the character variety

In this chapter, we prove the following theorem.

Theorem 2.0.2. *Let n be a positive integer. There exist infinitely many hyperbolic 3-manifolds $\{\mathcal{M}_i\}$ with $|\partial\mathcal{M}_i| = 1$ such that if ξ is any n^{th} root of unity, then ξ appears as a root root of unity for every $\mathcal{M}_i$.*

The idea of the proof is this. For each n, we construct a 3-manifold M_n for which every n^{th} root of unity appears. The manifolds M_n contain essential tori. It is well known that hyperbolic 3-manifolds do not contain essential tori, hence these manifolds are never hyperbolic. However, we can use standard techniques to construct hyperbolic 3-manifolds that admit degree-one maps onto M_n. The root of unity property is inherited by the hyperbolic manifolds through the induced epimorphism on fundamental groups.

2.1 Construction of the manifolds M_n

This section relies on the existence of 3-manifolds M with $|\partial M| = 1$ and a strongly detected boundary slope with root of unity $+1$. First, we show that the exterior of the figure-eight knot satisfies these conditions. We follow this discussion with the general construction for M_n.

Let $K \subset S^3$ be the figure-eight knot and $M = E(K)$. We have a standard 2-bridge presentation for $\pi_1(M)$ given by

$$\pi_1(M) \;=\; \langle\, \mu, y \,|\, \mu W = W y \,\rangle$$

where $W = y^{-1}\mu y \mu^{-1}$, and μ is a meridian for M. A longitude for μ can be computed to be $\lambda = WW^*$ where $W^* = \mu^{-1} y \mu y^{-1}$. We prove the following well-known proposition.

Proposition 2.1.1. *$\lambda\mu^4$ is a strongly detected boundary slope for M. The associated root of unity is* $+1$.

Proof. The A-polynomial for M with respect to the basis $\{\lambda, \mu\}$ can be computed from the above presentation of $\pi_1(M)$. As in [4], it is given by

$$A_M(l,m) \;=\; -l + lm^2 + m^4 + 2lm^4 + l^2m^4 + lm^6 - lm^8. \tag{2.1}$$

The Newton polygon for this polynomial is pictured in Figure 2.1. Since the polygon has an edge with slope 4, we apply Theorem 1.4.1 to see that $\lambda\mu^4$ is a strongly detected boundary slope. Let $\sigma = \lambda\mu^3$ and $\tau = \lambda\mu^4$.

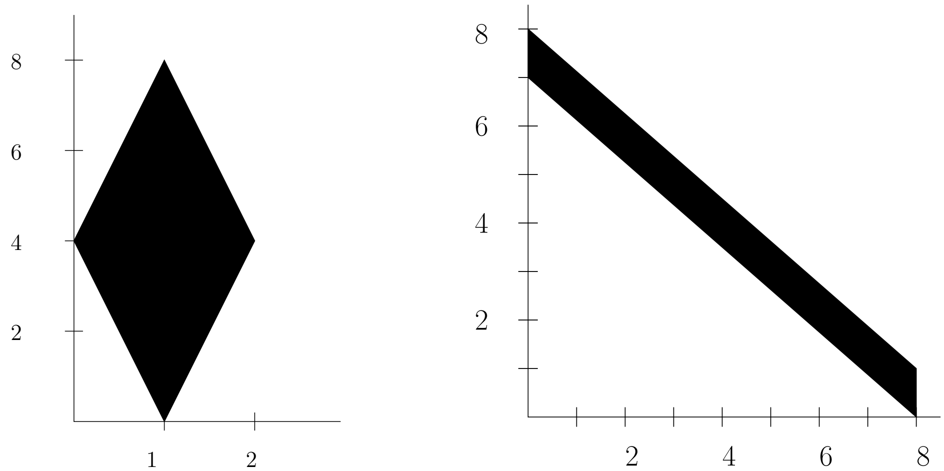
We compute the A-polynomial for M with respect to the basis $\{\sigma, \tau\}$, as in the discussion following Theorem 1.4.1. We obtain

$$A_M(s,t) \;=\; (t^7 - t^8) + s^2(-s^6 + s^4t^2 + 2s^2t^4 + s^6t + t^6).$$

Therefore, the edge polynomial which corresponds to the boundary slope τ is

$$f(x) \;=\; 1 - x.$$

$\square$



(a) With respect to $\{\lambda, \mu\}$

(b) With respect to $\{\sigma, \tau\}$

Figure 2.1: Newton polygons for the figure-eight knot

Let M be any 3-manifold with $|\partial M| = 1$ and $\hat{x}$ an ideal point of $X(M)$ which strongly detects a boundary slope $\tau \in \pi_1(\partial M)$ with root of unity $+1$. Choose $\sigma \in \pi_1(\partial M)$ so that $\{\sigma, \tau\}$ form a generating set for $\pi_1(\partial M)$. Then there is a sequence of irreducible characters $\{\chi_i\}$ lying on a curve in $X(M)$ which approach the ideal point $\hat{x}$. Proposition 1.3.4 implies that

$$\chi_i(\tau) \to 2 \quad \text{and} \quad \chi_i(\sigma) \to \infty.$$

We can choose irreducible representations in the fibers over these characters which restrict to diagonal representations on $\pi_1(\partial M)$. That is, we have a sequence of irreducible representations

$$\rho_i \colon \pi_1(M) \longrightarrow \mathrm{SL}_2(\mathbb{C})$$

with

$$\rho_i(\sigma) \;=\; \begin{pmatrix} s_i & 0 \\ 0 & {s_i}^{-1} \end{pmatrix} \quad \text{and} \quad \rho_i(\tau) \;=\; \begin{pmatrix} t_i & 0 \\ 0 & {t_i}^{-1} \end{pmatrix}$$

and

$$\lim_{i\to\infty} s_i \;=\; 0 \quad \text{and} \quad \lim_{i\to\infty} t_i \;=\; 1.$$

(We may need to replace $\{\sigma, \tau\}$ with $\{\sigma^{-1}, \tau^{-1}\}$ to arrange this.)

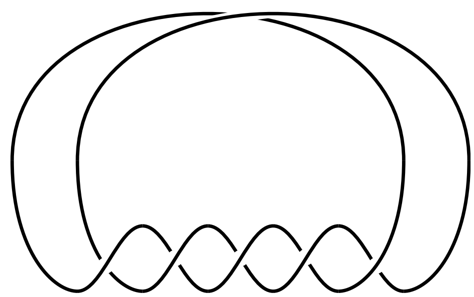


Figure 2.2: The $(2,6)$-torus link

The manifold M_n will be the union of the manifold M and a two component link exterior as follows. Fix a positive integer n, and let L_n be the exterior of a two component link $J \cup K \subset S^3$ where the linking number of J and K is $\pm n$. (For example, take $J \cup K$ to be the $(2,2n)$-torus link. The $(2,6)$-torus link is pictured in Figure 2.2.) For $A \in \{J, K\}$, let $\{\lambda_A, \mu_A\}$ be a longitude/meridian pair for the exterior $E(A)$. We have the Hurewicz epimorphism

$$h\colon\; \pi_1(L_n) \longrightarrow \mathrm{H}_1(L_n) \;\cong\; \mathbb{Z} \times \mathbb{Z}.$$

The condition on the linking number of J and K gives the equalities

$$h(\lambda_J) \;=\; h(\mu_K^{\pm n}) \quad \text{and} \quad h(\lambda_K) \;=\; h(\mu_J^{\pm n}).$$

Notice that $\mathrm{H}_1(L_n)$ is generated by $h(\mu_J)$ and $h(\mu_K)$. We form the manifold M_n as the union $M \cup_f L_n$ where $f\colon\ \partial M \longrightarrow \partial U(K)$ is a homeomorphism with

$$f_*(\sigma) \;=\; \mu_K \quad \text{and} \quad f_*(\tau) \;=\; \lambda_K.$$

Hence, $|\partial M_n| = 1$ and the set $\{\lambda_J, \mu_J\}$ is a generating set for $\pi_1(\partial M_n)$.

2.2 M_n has n^{th} roots

Theorem 2.2.1. *Fix a positive integer n, and let ξ be any n^{th} root of unity. Then there exists a sequence of irreducible representations*

$$\left\{\rho_i\colon\ \pi_1(M_n) \longrightarrow \mathrm{SL}_2(\mathbb{C})\right\}_{i=1}^{\infty}$$

such that

$$\rho_i(\lambda_J) \;=\; \begin{pmatrix} b_i & 0 \\ 0 & {b_i}^{-1} \end{pmatrix} \quad \text{and} \quad \rho_i(\mu_J) \;=\; \begin{pmatrix} a_i & 0 \\ 0 & {a_i}^{-1} \end{pmatrix}$$

where as $i \to \infty$, $b_i \to 0$ and $a_i \to \xi$. In particular, ξ appears at an ideal point for M_n.

Proof. To prove the theorem, we show that the representations

$$\rho_i\colon\ \pi_1(M) \longrightarrow \mathrm{SL}_2(\mathbb{C})$$

from the previous section extend to $\pi_1(M_n)$. As such, we will see that they satisfy the conclusion of the theorem.

Take any sequence of non-zero complex numbers $\{a_i\}$ such that $a_i \to \xi$ and $a_i^{\pm n} = t_i$. Let $b_i = s_i^{\pm n}$. Define a homomorphism

$$\rho_i'\colon\ \pi_1(L_n) \longrightarrow \mathrm{SL}_2(\mathbb{C})$$

so that each ρ_i' factors through h and

$$\rho_i'(\mu_J) \;=\; \begin{pmatrix} a_i & 0 \\ 0 & {a_i}^{-1} \end{pmatrix} \quad \text{and} \quad \rho_i'(\mu_K) \;=\; \begin{pmatrix} s_i & 0 \\ 0 & {s_i}^{-1} \end{pmatrix}.$$

ρ_i' is a homomorphism because $\mathrm{H}_1(L_n)$ is free abelian with generators $h(\mu_J)$ and $h(\mu_K)$. Since ρ_i and ρ_i' agree on the amalgamating subgroup $\langle \sigma, \tau \rangle$, we can use ρ_i' to extend ρ_i to all of $\pi_1(M_n)$. We observe that

$$\mathrm{Tr}\big(\rho_i(\mu_J)\big) \;=\; a_i + {a_i}^{-1} \;\rightarrow \xi + \xi^{-1}$$

and

$$\mathrm{Tr}\big(\rho_i(\lambda_J)\big) \;=\; s_i^n + s_i^{-n} \;\rightarrow \infty.$$

Hence, there is an ideal point for M_n which strongly detects the boundary slope μ_J with associated root of unity ξ. $\square$

As usual, the root of unity behavior of M_n can be seen in the A-polynomial A_{M_n}.

Corollary 2.2.2. *If $A_{M_n}(y,z)$ is the A-polynomial for M_n, with respect to the generating set $\{\lambda_J, \mu_J\}$ for $\pi_1(\partial M_n)$, then the Newton polygon for A_{M_n} has a vertical edge e such that $1 - x^n$ divides the edge polynomial $f_e(x)$.*

Proof. Given any n^{th} root of unity, let $\{\rho_i\}_{i=1}^{\infty}$ be a sequence of representations as given by Theorem 2.2.1. Let $\chi_i = t(\rho_i)$ and recall that

$$\lim_{i\to\infty} \big|\chi_i(\lambda_J)\big| \;=\; \infty.$$

Therefore, the set $\left\{i^*(\chi_i)\right\}_{i=1}^{\infty} \subset D_{M_n}$ contains an infinite number of distinct points. Hence, by passing to a subsequence, we may assume $\left\{i^*(\chi_i)\right\}_{i=1}^{\infty}$ is a set of distinct points on a single curve component $\mathcal{C}$ of D_{M_n}. Recall that the choice of generating set $\{\lambda_J, \mu_J\}$ induces a quotient $\mathbb{C}^\times \times \mathbb{C}^\times \longrightarrow X(\partial M_n)$. Let $\overline{\mathcal{C}} \subset \overline{D_{M_n}} \subset \mathbb{C}^\times \times \mathbb{C}^\times$ be the inverse image of $\mathcal{C}$ under this quotient map. Let $P(y, z)$ be the irreducible polynomial that defines the curve $\overline{\mathcal{C}}$.

By construction, $P(0, \xi) = 0$. However, $P(0, z)$ is not identically zero because the points $(0, z) \in \overline{\mathcal{C}}$ always correspond to ideal points in $X(M_n)$ and $\overline{\mathcal{C}}$ has only finitely many such points. Therefore, $P(0, z)$ is a non-trivial single variable polynomial. Thus, we see that the Newton polygon $\mathcal{N}_P$ has a vertical edge with edge polynomial $x^{-m} \cdot P(0, x)$ for some integer m. ξ is a root of this edge polynomial. Also, by definition of the A-polynomial, $P(y, z)$ divides $A_{M_n}(y, z)$. So $P(0, z)$ divides $A_{M_n}(0, z)$. This implies that the Newton polygon for A_{M_n} also has a vertical edge e and $f_e(\xi) = 0$. Since this is true for any n^{th} root of unity, we conclude that $1 - x^n$ divides f_e. $\square$

2.3 Hyperbolic Examples

Here, we extend the result to hyperbolic examples and complete the proof of Theorem 2.0.2.

Before we prove the theorem, we give a definition and state two lemmas. A *null-homotopic knot* in a 3-manifold M is an embedded circle $k \subset M$, such that k is homotopic to a point. Now, as a special case of the main theorem in [17], we have

Lemma 2.3.1. *For each $n \in \mathbb{Z}^+$, M_n contains infinitely many null-homotopic hyperbolic knots.*

Our second lemma is also standard but we give a proof for completeness (cf. Proposition 3.2 in [1]).

Lemma 2.3.2. *Let k be a null homotopic knot in M_n and M be a 3-manifold obtained by a Dehn surgery on k. Then there is a degree-one map $f\colon M \longrightarrow M_n$.*

Proof. Let $M' = M_n - U(k)$. From our assumptions, we have a slope s on $\partial M'$ so that M is obtained from M' by identifying the boundary of a solid torus V to $\partial M'$ so that a simple closed curve representing the slope s bounds a disk $D \subset V$.

Define $f\colon M' \longrightarrow M' \subset M_n$ to be the identity. If we can extend the map f over V then it will be a degree one map as needed. First, f extends over a regular neighborhood $U(D)$ of the disk D since $f(\partial D)$ is homotopic to k which is homotopic to a point. Next, f extends over the ball $V - U(D)$ because every immersed 2-sphere in M_n is homotopic to a point. $\square$

Armed with these lemmas, we are ready to prove Theorem 2.0.2.

Proof. Let k be a null-homotopic knot in M_n as provided by Lemma 2.3.1. By Thurston's hyperbolic Dehn surgery theorem [24], for all but finitely many slopes s, the manifold obtained by doing Dehn surgery on k with surgery slope

s is hyperbolic. This gives us an infinite set of distinct hyperbolic manifolds $\{\mathcal{M}_i\}_{i=1}^{\infty}$.

Furthermore, for every $i \in \mathbb{Z}^+$, Lemma 2.3.2 gives us a degree-one map

$$f_i \colon \mathcal{M}_i \longrightarrow M_n.$$

Since f_i is degree-one, the induced map $f_i^* \colon \pi_1(\mathcal{M}_i) \longrightarrow \pi_1(M_n)$ is an epimorphism and the restriction to the peripheral subgroup of $\pi_1(\mathcal{M}_i)$ is an isomorphism onto the peripheral subgroup of $\pi_1(M_n)$.

Let α be a primitive element of $\pi_1(\partial\mathcal{M}_i)$ so that $f_i^*(\alpha) = \mu_J$. Similarly, take β to be a primitive element of $\pi_1(\partial\mathcal{M}_i)$ so that $f_i^*(\beta) = \lambda_J$. Pick ξ to be any n^{th} root of unity. Then Theorem 2.2.1 gives us a sequence of irreducible representations $\left\{\rho_j \colon \pi_1(M_n) \longrightarrow \mathrm{SL}_2(\mathbb{C})\right\}_{j=1}^{\infty}$ such that

$$\rho_i(\lambda_J) \;=\; \begin{pmatrix} b_i & 0 \\ 0 & b_i^{-1} \end{pmatrix} \quad \text{and} \quad \rho_i(\mu_J) \;=\; \begin{pmatrix} a_i & 0 \\ 0 & a_i^{-1} \end{pmatrix}$$

where $b_i \to 0$ and $a_i \to \xi$. Composing these representations with f_i^*, we get the sequence $\{\rho_j \circ f_i^*\}_{j=1}^{\infty}$ of irreducible representations for $\pi_1(\mathcal{M}_i)$. Furthermore,

$$\lim_{j\to\infty} \mathrm{Tr}\big(\rho_j \circ f_i^*(\alpha)\big) \;=\; \lim_{j\to\infty} \left(a_j + a_j^{-1}\right) \;=\; \xi + \xi^{-1}$$

and

$$\lim_{j\to\infty} \left|\mathrm{Tr}\big(\rho_j \circ f_i^*(\beta)\big)\right| \;=\; \lim_{j\to\infty} \left|b_j + b_j^{-1}\right| \;=\; \infty.$$

Therefore, α represents a detected boundary slope and the root of unity which appears at the associated ideal point is ξ. $\square$

2.4 Computations for $n = 5$

In this section, we give explicit calculations for the case $n = 5$. In the construction of M_n, take M to be the exterior of the figure eight knot. We compute the factor of the A-polynomial for M_5 which corresponds to the representations that are irreducible on M and abelian on L_5. Take σ and τ as in Proposition 2.1.1.

We want to replace the variables l and m in the A-polynomial (2.1) with the variables b and a which represent the eigenvalues of $\rho(\lambda_J)$ and $\rho(\mu_J)$ respectively. Here, we take resultants instead of directly substituting expressions in terms of s and a. (If we substitute directly, we will get an expression with non-integral exponents.) Our resultant calculations are done with a computer.

Recall that, under representations of $\pi_1(M_5)$ that are abelian on L_5, we have

$$\rho(\mu_J^5) = \rho(\tau) \quad \text{and} \quad \rho(\mu_K) = \rho(\sigma).$$

As in the discussion following 1.4.1, and by using the equation $a^5 = t$, we have

$$m = \frac{t}{s} = \frac{a^5}{s} \quad \text{and} \quad l = \frac{s^4}{t^3} = \frac{s^4}{a^{15}}.$$

Starting with $A_M(l, m)$, we take a resultant with respect to $ms - a^5$ to eliminate the indeterminant m. Next, we take a resultant with respect to $la^{15} - s^4$ to eliminate L, and finally with respect to $b - s^5$ to eliminate s. We conclude that the polynomial equation

$$\begin{aligned}
0 \;=\; & -a^{175}+5a^{180}-10a^{185}+10a^{190}-5a^{195}+a^{200}+5a^{135}b^2-35a^{140}b^2+\\
& +60a^{145}b^2-26a^{150}b^2-5a^{155}b^2-5a^{90}b^4-45a^{95}b^4+\\
& +98a^{100}b^4-45a^{105}b^4-5a^{110}b^4-5a^{45}b^6-26a^{50}b^6+\\
& +60a^{55}b^6-35a^{60}b^6+5a^{65}b^6+b^8-5a^5b^8+10a^{10}b^8-\\
& -10a^{15}b^8+5a^{20}b^8-a^{25}b^8
\end{aligned}$$

holds true for all such representations. Therefore, the right hand side is a factor of the A-polynomial for M_5 with respect to the generating set $\{\lambda_J, \mu_J\}$. Evaluating at $b = 0$, we get

$$\begin{aligned}
0 &= -a^{175}+5a^{180}-10a^{185}+10a^{190}-5a^{195}+a^{200}\\
&= a^{175}(a^5-1)^5.
\end{aligned}$$

We conclude that there is a vertical edge e in the Newton polygon for the A-polynomial of M_5. Furthermore, if $f_e(x)$ is the associated edge polynomial, $(x^5-1)^5$ divides $f_e(x)$.

2.5 Final Remarks

It should be noted that, although the manifolds constructed in Section 2.3 are hyperbolic, the ideal points at which n^{th} roots of unity appear are not on components of the character varieties that correspond to the complete hyperbolic structure of the manifold. This is because the components considered here arise from epimorphisms with non-trivial kernels and so the characters on these components are never characters of discrete faithful representations. The question of whether all roots of unity appear at ideal points on hyperbolic components remains unanswered.

Chapter 3

Not all boundary slopes are strongly detected

In this chapter, we give an infinite family of hyperbolic knots with strict boundary slopes that are not strongly detected.

As mentioned in Section 1.3, Schanuel and Zhang give examples of graph manifolds with strict boundary slopes that are not strongly detected. Their examples left open the following interesting questions.

Question 3.0.1 (Cooper-Long 1996). Is every strict boundary slope for the complement of a knot in S^3 strongly detected?

Question 3.0.2 (Schanuel-Zhang 2001). Let M be an orientable hyperbolic 3-manifold with $|\partial M| = 1$. Is every strict boundary slope of M strongly detected?

Question 3.0.3. In the setting of either of the previous two questions, is every strict boundary slope either strongly or weakly detected? (Recall the definition of weak detection following Theorem 1.3.5.)

We answer the first two questions with Theorem 3.0.3, the main theorem of this chapter. We first note that Question 3.0.1 can be answered with non-hyperbolic examples by using the more elementary Theorem 3.0.1. Corollary 3.0.2 will show that a connect sum of two small knots (Figure 3.1) provides an answer to this question.

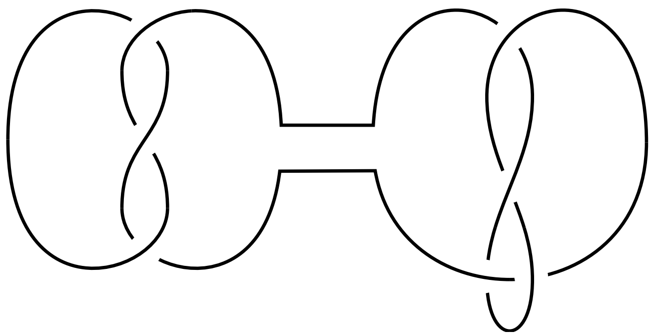


Figure 3.1: A connect sum of two small knots

Let K be the connect sum of two knots K_L and K_R and write $M = E(K)$, $M_L = E(K_L)$, and $M_R = E(K_R)$. Let μ be a meridian for K. We can view $\pi_1(M)$ as the free product with amalgamation

$$\pi_1(M) \;=\; \pi_1(M_L) *_{\langle\mu\rangle} \pi_1(M_R).$$

This decomposition allows us to view μ also as a meridian for both K_L and K_R. Let λ, λ_L, and λ_R be longitudes for μ in $\pi_1(M)$, $\pi_1(M_L)$, and $\pi_1(M_R)$. Observe that $\lambda = \lambda_L\lambda_R$. By construction, M contains an essential annulus with boundary slope μ. We can prove the following theorem.

Theorem 3.0.1. *If the strict boundary slope μ is strongly detected for M then μ is also a strongly detected boundary slope for either M_L or M_R.*

Our proof uses the following elementary facts about $\mathrm{SL}_2(\mathbb{C})$.

(1) If A, B, and C are elements of $\mathrm{SL}_2(\mathbb{C})$ such that A and B commute and also B and C commute, then A and C commute.

(2) Let $\{A_i\}$ and $\{B_i\}$ be subsets of $\mathrm{SL}_2(\mathbb{C})$ with the property that A_i commutes with B_i for each i. If both $\mathrm{Tr}(A_i)$ and $\mathrm{Tr}(B_i)$ are bounded then $\mathrm{Tr}(A_iB_i)$ is also bounded.

Proof. Assume that μ is strongly detected for M. Then Proposition 1.3.4 provides a curve $X \subset X(M)$ and an ideal point $\hat{x} \in \widetilde{X}$ such that

$$I_\mu(\hat{x}) \;=\; z \in \mathbb{C} \qquad \text{and} \qquad I_\lambda(\hat{x}) \;=\; \infty.$$

This implies that there is a sequence of representations $\{\rho_i\}_{i+1}^{\infty} \subset R(M)$ whose characters lie on X and such that

$$\lim_{x\to\infty} \mathrm{Tr}\big(\rho_i(\mu)\big) \;=\; z \qquad \text{and} \qquad \lim_{x\to\infty} \mathrm{Tr}\big(\rho_i(\lambda)\big) \;=\; \infty.$$

Since μ commutes with both λ_L and λ_R we can apply fact (1) to conclude that the sequences $\{\rho_i(\lambda_L)\}$ and $\{\rho_i(\lambda_R)\}$ satify the hypotheses of fact (2). Because the traces $\mathrm{Tr}\big(\rho_i(\lambda)\big)$ are unbounded, fact (2) allows us, without loss of generality, to assume that

$$\lim_{x\to\infty} \mathrm{Tr}\big(\rho_i(\lambda_L)\big) \;=\; \infty.$$

We conclude that μ is a strongly detected boundary slope for M_L. $\square$

Corollary 3.0.2. *Let K be the connect sum of two small knots K_L and K_R. Then $E(K)$ has the strict boundary slope $1/0$ which is not strongly detected.*

Proof. Since K_L and K_R are small, [7] shows that neither M_L nor M_R have the boundary slope $1/0$. $\square$

If K is as in Corollary 3.0.2 then it is true that the character variety for K has ideal points that detect the essential annulus in $E(K)$. These ideal must, by definition, be of type I. Hence, the strict boundary slope $1/0$ of $E(K)$ is weakly and not strongly detected.

For the remainder of this chapter, we devote our attention to exhibiting knots which answer Question 3.0.2. Let $M_n = E(K_n)$ where K_n is the $(3, 5, 2n + 1, 2)$-pretzel knot pictured in Figure 3.2.

Theorem 3.0.3. *Let $n > 1$ be an integer. Then M_n is hyperbolic and has the strict boundary slope $4(n + 4)$ which is not strongly detected.*

We do not know if these boundary slopes are weakly detected or not. In either case, there are interesting ramifications. If the slope is not weakly detected, then these would be the first known examples of strict boundary slopes that are not detected by any ideal point, thus, giving an answer to Question 3.0.3. If it is weakly detected, then using [18], one can show that there is an ideal point which weakly detects two distinct boundary slopes: $4(n + 4)$ and $1/0$. This second scenario would provide an answer to Question 1.3.1.

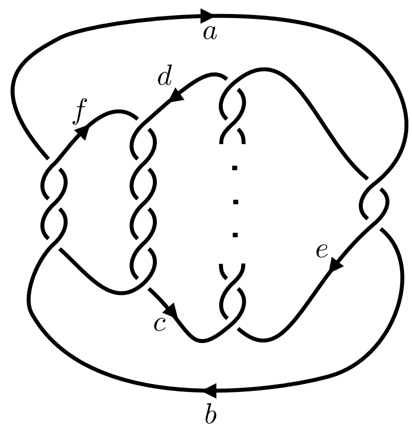


Figure 3.2: The pretzel knot K_n

Notice that weak detection implies the existence of a closed essential surface. This inspires the following question.

Question 3.0.4. Is there a small knot exterior with a strict boundary slope that is not strongly (and hence not weakly) detected?

3.1 Proof of Theorem 3.0.3

This section gives an outline for the proof of Theorem 3.0.3. Missing details are given in Sections 3.2 and 3.3.

One obvious approach to proving that a slope is not strongly detected is to calculate the A-polynomial and use the Newton polygon to find the strongly detected boundary slopes. This is computationally difficult and is unlikely to give a general argument for an infinite collection of manifolds. We take a different approach – the key idea is to use a relationship between the character varieties of mutants.

We define a *Conway sphere* in a knot exterior M to be a separating essential surface in M which is homeomorphic to a 2-sphere with four open disks removed and whose boundary components represent meridians for the knot.

If S_4 is a Conway sphere in M and

$$\tau\colon\ S_4 \longrightarrow S_4$$

is an orientation preserving involution, the *mutant* M^τ is defined as the manifold formed by cutting M along S_4 and re-glueing with the homeomorphism τ.

It follows from work of Oertel [18] that the manifolds M_n are not only hyperbolic, but they also contain Conway spheres. In particular, there is a Conway sphere in M_n which separates the first two tangles of K_n from the second two. A mutation in this Conway sphere results in a manifold M_n^τ which is homeomorphic to the exterior of the $(5, 3, 2n+1, 2)$-pretzel knot. We will write K_n^τ to denote this knot.

It will sometimes be convenient for us to think about mutation in a slightly different way. Because $[\partial S_4] = 0 \in \mathrm{H}_1(M)$, the components of ∂S_4 come in two oppositely oriented pairs. Therefore, ∂S_4 bounds two annuli in ∂M. Let G_2 be the genus-2 surface formed by attaching these annuli to S_4 and pushing the resulting surface slightly into the interior of M. The involution

$\tau\colon S_4 \longrightarrow S_4$ extends to an involution $\tau\colon G_2 \longrightarrow G_2$. If we cut along G_2 and re-glue using τ, we get a manifold which is homeomorphic to M^τ.

The following proposition is the starting point for the ideas in this chapter. We outline the proof below, complete details can be found in [26].

Proposition 3.1.1. *Suppose that M and M^τ are knot exteriors related to each other by a mutation τ on a Conway sphere S_4. If $X \subset X(M)$ is an algebraic curve containing the character of a representation whose restriction to $\pi_1(S_4)$ is irreducible, then there exists a curve $X^\tau \subset X(M^\tau)$ and a birational map $f\colon X \longrightarrow X^\tau$. Furthermore, using standard longitude/meridian coordinates for slopes, the set of boundary slopes strongly detected for M by X is identical to the set of boundary slopes strongly detected by X^τ for M^τ.*

Proof. Because the peripheral subgroup of M is carried by one component of $M - G_2$, it will be helpful to thing of M^τ as being formed using the involution $\tau\colon G_2 \longrightarrow G_2$. (Recall the notation preceding the proposition.)

The surface G_2 splits both M and M^τ into the same two submanifolds M_1 and M_2. In particular, we have the following commutative diagram of inclusion induced homomorphisms.

$$\begin{array}{ccc} \pi_1(G_2) & \xrightarrow{g_1} & \pi_1(M_1) \\ {\scriptstyle g_2}\downarrow & & \downarrow{\scriptstyle i_1} \\ \pi_1(M_2) & \xrightarrow[i_2]{} & \pi_1(M) \end{array}$$

We also have the corresponding diagram for M^τ.

$$\begin{tikzcd} \pi_1(G_2) \arrow[r, "g_1"] \arrow[d, "g_2\circ\tau_*"'] & \pi_1(M_1) \arrow[d, "j_1"] \\ \pi_1(M_2) \arrow[r, "j_2"'] & \pi_1(M^\tau) \end{tikzcd}$$

We label M_1 and M_2 so that M_1 is the submanifold with two boundary components.

Suppose that

$$\rho\colon\ \pi_1(M) \longrightarrow \mathrm{SL}_2(\mathbb{C})$$

is a representation whose restriction to $\pi_1(S_4)$ is irreducible. Then since $\pi_1(S_4)$ is contained in the image of $\pi_1(G_2)$ in $\pi_1(M)$, we see that

$$\rho \circ i_1 \circ g_1\colon\ \pi_1(G_2) \longrightarrow \mathrm{SL}_2(\mathbb{C})$$

is also irreducible. Hence, this representation for $\pi_1(G_2)$ is determined, up to conjugacy, by its character. A calculation (see [26] for example) shows that

$$\mathrm{Tr} \circ \rho \circ i_1 \circ g_1 \ = \ \mathrm{Tr} \circ \rho \circ i_2 \circ g_2 \circ \tau_*.$$

Thus, we have an inner automorphism $\mathcal{I}\colon\ \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{SL}_2(\mathbb{C})$ such that

$$\mathcal{I} \circ (\rho \circ i_2 \circ g_2 \circ \tau_*) \ = \ \rho \circ i_1 \circ g_1.$$

We have another commutative diagram.

$$\begin{tikzcd} \pi_1(G_2) \arrow[r, "g_1"] \arrow[d, "g_2\circ\tau_*"'] & \pi_1(M_1) \arrow[d, "\rho\circ i_1"] \\ \pi_1(M_2) \arrow[r, "\mathcal{I}\circ\rho\circ i_2"'] & \mathrm{SL}_2(\mathbb{C}) \end{tikzcd}$$

By applying Van Kampen's theorem to $M^\tau = M_1 \cup_{G_2} M_2$ ([12]), there exists a unique homomorphism

$$\rho^\tau\colon\ \pi_1(M^\tau) \longrightarrow \mathrm{SL}_2(\mathbb{C})$$

with

$$\rho^\tau \circ j_1 = \rho \circ i_1 \quad \text{and} \quad \rho^\tau \circ j_2 = \mathfrak{I} \circ \rho \circ i_2.$$

Let f be the map which takes the character of ρ to the character of ρ^τ.

The characters in X that correspond to reducible representations on $\pi_1(S_4)$ form a codimension-1 subvariety of X. Hence, f is well-defined on all but a codimension-1 subvariety of X. It is shown in [26] that f is a rational map. Finally, to see that f is a birational equivalence, we simply define an inverse map g from the irreducible characters in X^τ to X in exactly the same way as we defined f.

It remains to prove that f preserves the set of strongly detected boundary slopes. Let B be the genus-1 boundary component of M_1 and take $\{\lambda, \mu\} \subset \pi_1(B)$ so that $\{i_1(\lambda), i_1(\mu)\}$ is a longitude/meridian pair for M. Let $\langle \cdot, \cdot \rangle$ denote oriented intersection number. Since

$$\big\langle i_1(\lambda), i_1(\mu) \big\rangle \;=\; \langle \lambda, \mu \rangle \;=\; \big\langle j_1(\lambda), j_1(\mu) \big\rangle,$$

we will be done if we can show that the image of λ is trivial in $\mathrm{H}_1(M^\tau)$.

To see this, we may choose a presentation

$$\langle\, a, b, c, d \,|\, aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 \,\rangle$$

for $\pi_1(G_2)$ such that

(1) both a and d have trivial images in $\mathrm{H}_1(M_2)$ and

(2) b and c are meridians for M with the same orientation.

Note that the order-2 automorphism $\tau_*\colon \pi_1(G_2) \longrightarrow \pi_1(G_2)$ is either given by $\tau_*(\alpha) = \alpha^{-1}$ for every $\alpha \in \{a, b, c, d\}$, by $\tau_*(a) = d$ and $\tau_*(b) = c$, or by the composition of the two. In any case, properties (1) and (2) are preserved under precomposition with τ_*. This implies that

$$\ker(h \circ i_1 \circ g_1) \;=\; \ker(h \circ j_2 \circ g_2 \circ \tau_*)$$

where h denotes the Hurewicz epimorphism from fundamental group to first homology. We have shown that the image of λ in $\mathrm{H}_1(M^\tau)$ is trivial, which completes the proof of the proposition. $\square$

Thus, from Proposition 3.1.1, we can conclude that if the set of boundary slopes for our manifolds M_n and M_n^τ are different, then any boundary slope which is strongly detected and not contained in their intersection must be detected by a curve of characters whose restriction to the Conway sphere is reducible.

Section 3.3 shows that, for the manifolds $\{M_n\}$, the only boundary slope which could possibly be strongly detected on such a curve is the slope $1/0$. In Section 3.2, we show that $4(n+4)$ is a boundary slope for M_n but not for M_n^τ. Thus, Proposition 3.1.1 completes the proof that $4(n+4)$ is a boundary slope for M_n which is not strongly detected.

3.2 Mutants with distinct slope sets

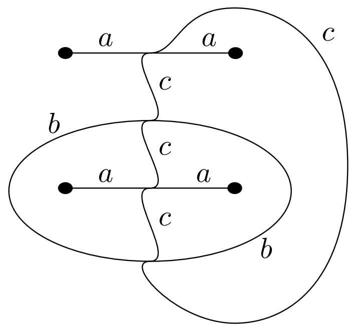
In this section, the algorithm of [13] is used to show that the pretzel knot K_n has the boundary slope $4(n+4)$ whereas K_n^τ does not.

3.2.1 Setup from [13]

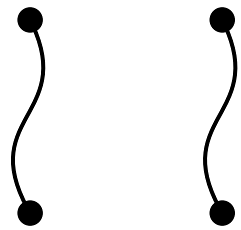
The constructions from [13] which apply to our setting are reviewed in this section. In particular, it is assumed throughout that K denotes a four-tangle pretzel knot.

The 3-sphere S^3 can be decomposed into the union of four 3-balls, $\{B_i\}_{i=0}^3$, where $\bigcap_0^3 B_i \cong S^1$ (the *axis* for K), and each B_i contains exactly one of the four given tangles. If F is a properly embedded surface in $M = E(K)$, we may assume that it is transverse to ∂B_i for every i. Hence, $F_i = F \cap B_i$ is properly embedded in B_i for all i and $P_i = \partial B_i \cap K$ consists of four points. In particular, ∂F_i is a curve system on the 4-punctured sphere $S_i = \partial B_i - U(P_i)$. We will denote the i^{th} tangle $K \cap B_i$ by T_i.

Curve systems on the 4-punctured sphere are either carried by the train track shown in Figure 3.3(a) or by its mirror image. Usually we will consider curve systems up to projective class, so rational projective coordinates $[a, b, c] \in \mathbb{Q}P^2$ represent a projective curve system according to the figure. We also define uv-coordinates for projective curve systems, where the curve system $[a, b, c]$ corresponds to the point $\left(\frac{b}{a+b}, \frac{c}{a+b}\right)$ in uv-coordinates. The slope of the curve system is the v-coordinate.



(a) The traintrack with projective weights a, b, and c



(b) The ∞-tangle

Figure 3.3: Curve systems and tangles

A *p/q-tangle*, denoted by $\langle \frac{p}{q} \rangle$, is a projective curve system $[1, q-1, p]$, or equivalently $((q-1)/q,\ p/q)$ in uv-coordinates. A *p/q-circle*, denoted by $\langle \frac{p}{q} \rangle^\circ$, is a projective curve system $[0, q, p]$, or equivalently $(1,\ p/q)$. The *∞-tangle*, denoted by $\langle \infty \rangle$, is the projective class of the pair of vertical arcs shown in Figure 3.3(b), and will be represented by $(-1, 0)$ in uv-coordinates.

We now use the above definitions to define a graph $\mathcal{D}$ in the uv-plane. The vertices of $\mathcal{D}$ are the uv-coordinates of the p/q-tangles and p/q-circles for every $p/q \in \mathbb{Q}$ together with the point $\langle \infty \rangle = (-1, 0)$. There are four types of edges in $\mathcal{D}$, *non-horizontal edges, horizontal edges, vertical edges* and *infinity edges*. Two vertices $\langle \frac{p}{q} \rangle$ and $\langle \frac{r}{s} \rangle$ are connected by a non-horizontal edge if $|ps - qr| = 1$, or equivalently, if $\langle \frac{r}{s} \rangle$ can be obtained from $\langle \frac{p}{q} \rangle$ by surgery on an arc. The horizontal edges connect the vertices $\langle \frac{p}{q} \rangle^\circ$ to $\langle \frac{p}{q} \rangle$. The vertical edges connect $\langle n \rangle$ to $\langle n+1 \rangle$ for every $n \in \mathbb{Z}$. Finally, the infinity edges connect the integer vertices $\langle n \rangle$ to $\langle \infty \rangle$. Note that, for all but the horizontal edges, two vertices are connected by an edge if and only if the projective curve systems given by the vertices can be obtained from each other by surgery on an arc. If v and w are vertices of $\mathcal{D}$ that are connected by an edge, we will denote the edge by $[v, w]$. The subgraph $\mathcal{S} \subset \mathcal{D}$ is defined as the portion of $\mathcal{D}$ with u-coordinate in the interval $[0, 1]$. Figure 3.4 shows part of the the graph $\mathcal{D}$.

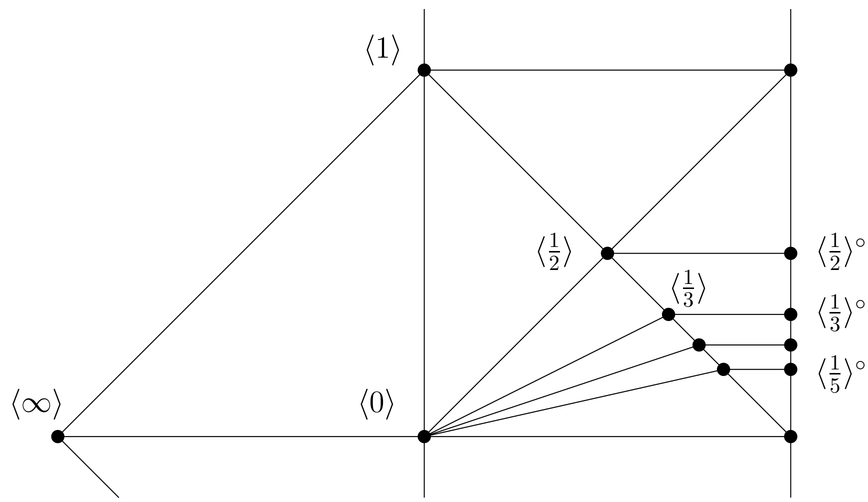


Figure 3.4: A piece of $\mathcal{D}$

Given an edge $[v, w]$ in $\mathcal{D}$, we subdivide it as follows. For each $m \in \mathbb{Z}^+$ and $k = 1, \ldots, m - 1$, let

$$\frac{k}{m} \cdot v + \frac{m-k}{m} \cdot w$$

denote the point on $[v, w]$ corresponding to the curve system consisting of k parallel copies of the pair of arcs representing v, together with $m - k$ copies of the pair of arcs representing w.

Let $\partial B_i \times [0,1]$ be a collar on ∂B_i inside B_i with $\partial B_i = \partial B_i \times \{1\}$. Let $1/q_i$ be the rational number corresponding to the tangle T_i. Then T_i is isotopic (rel ∂B_i) to the two component representative for $\langle \frac{1}{q_i} \rangle$ at the level $S_i \times \{0\}$ together with the four arcs $P_i \times [0,1]$. Our goal is to associate a surface in $B_i - U(T_i)$ to certain paths in $\mathcal{D}$.

An *edgepath system* $\gamma = (\gamma_0, \ldots, \gamma_3)$ is an 4-tuple of edgepaths in $\mathcal{D}$ which start and end at rational points of $\mathcal{D}$. An *admissible* path system is a path system with the following four properties:

(E1) The starting point of γ_i lies on the horizontal edge connecting $\langle \frac{1}{q_i} \rangle^\circ$ to $\langle \frac{1}{q_i} \rangle$ and if the starting point is not the vertex $\langle \frac{1}{q_i} \rangle$, then the path γ_i is constant.

(E2) γ_i is *minimal.* That is, it never stops and retraces itself and it never travels along two sides of a triangle in $\mathcal{D}$ in succession.

(E3) The ending points of the γ_i's all have the same u-coordinates and their v-coordinates sum to zero.

(E4) Each γ_i proceeds monotonically from right to left, in the sense that traversing vertical edges is permitted.

Admissible edgepath systems are divided into the following three types:

- A *type I system* is an edgepath system γ where each γ_i stays in $\mathcal{S}$ and has no vertical edges.

- A *type II system* is the same as a type I system except that at least one γ_i has a vertical edge.

- A *type III system* is a system where the γ_i's end to the left of $\mathcal{S}$.

Hatcher and Oertel show how to associate *candidate surfaces* to each of these types of edgepaths. It is also shown in [13] that every essential surface in M with non-empty boundary of finite slope is isotopic to such a surface. For the purpose of this paper, it suffices to describe the construction of the candidate surfaces for type II and III systems with no constant paths and with the endpoints of the γ_i's on vertices of $\mathcal{D}$. Let γ be such a system. We will write γ_i as $[v_n, \ldots, v_0]$ where the v_j's are the vertices of γ_i and the vertex v_j is followed by the vertex v_{j+1} as we move along the path. A complete list of candidate surfaces for γ is described in the following paragraphs.

For any edge $[v_{j+1}, v_j]$ of γ_i and $a < b$, we can build a 1-sheeted surface in $S_i \times [a, b]$ as follows. Let α and β be the curve systems with two arc components in S_i that represent v_j and v_{j+1} respectively. The surface is

$$\alpha \times \Big[a,\, (a+b)/2\Big) \;\cup\; \beta \times \Big((a+b)/2,\, b\Big] \;\cup\; D,$$

where D is a regular neighborhood (saddle) of a surgery arc in $S_i \times \big\{(a+b)/2\big\}$ given by the existence of the the edge $[v_{j+1}, v_j]$. Up to level preserving isotopy there are two choices for each saddle. One of the two possible surfaces for $\big[\langle 1\rangle, \langle \frac{1}{2}\rangle\big]$ is shown in Figure 3.5. To each edgepath γ_i, we can now associate a surface Γ_i which is properly embedded in $B_i - U(T_i)$ by simply stacking the surfaces associated to the edges of γ_i in the collar $S_i \times [0,1]$. Then $\Gamma_i \cap \big(S_i \times \{1\}\big)$ is the two component representative of the vertex v_n and $\Gamma_i \cap \big(S_i \times \{0\}\big)$ is the two component representative of the vertex v_0 and so lies on the tangle T_i. The condition (E3) guarantees that the surfaces $\{\Gamma_i\}$ fit together to give a surface in M.

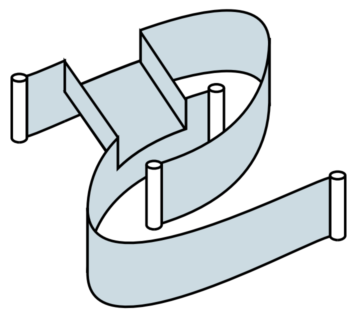


Figure 3.5: A surface corresponding to the edge $\big[\langle 1\rangle, \langle \frac{1}{2}\rangle\big]$

We have described the construction of a 1-sheeted candidate surface. If m is a positive integer, we can also construct an m-sheeted surface for γ in a similar way. Again, for every edge $[v_j, v_{j+1}]$ in γ_i, we construct a surface in $S_i \times [a,b]$. This time we further subdivide the edge into m edges separated by the points $\frac{k}{m} \cdot v_j + \frac{m-k}{m} \cdot v_{j+1}$ where $k = 1, \ldots, m-1$. Starting with the product surface $\alpha \times \big[a, (a+b)/2m\big)$, we add a saddle at the level $(a+b)/2m$ to pass to the curve system $\frac{1}{m} \cdot v_j + \frac{m-1}{m} \cdot v_{j+1}$. An m-sheeted surface for $[v_j, v_{j+1}]$ is completed by adding saddles to pass between the product surfaces that correspond to each point $\frac{k}{m} \cdot v_j + \frac{m-k}{m} \cdot v_{j+1}$. Just as before, we can build a surface for γ by stacking the surfaces for each edge γ_i and finally gluing everything together to give a surface in M. The meridian of the knot will intersect an m-sheeted surface m times. It is worth noticing that, because every time we pass a point $\frac{k}{m} \cdot v_j + \frac{m-k}{m} \cdot v_{j+1}$ we have a choice between two possible saddles, there can be many possible m-sheeted candidate surfaces associated to a given path system.

The question whether a candidate surface is essential or not can be studied via the path systems which carry them. This motivates the following terminology. An admissible path system γ is *incompressible* if every candidate surface associated to γ is incompressible, and it is *compressible* if every associated candidate surface is compressible. If there are both compressible and incompressible candidate surfaces associated to γ, it is said to be *indeterminant.*

For a path γ_i in γ, let e_+ be the number of leftward directed edges of γ_i which increase slope, and e_- be the number of slope decreasing edges. The numbers e_+ and e_- are independent of the behavior of γ_i to the left of $u = 0$. The *twist number* of γ_i is

$$t(\gamma_i) = 2(e_- - e_+),$$

and the twist number of γ is

$$t(\gamma) = \sum t(\gamma_i).$$

Hatcher and Oertel show that if S is the Seifert surface for a knot and s a path system which carries S, then the boundary slope for any surface carried by γ is $t(\gamma) - t(s)$.

In order to decide whether an admissible path system is compressible, incompressible, or indeterminant, Hatcher and Oertel define the notions of r-values and completely reversible paths. The *r-value* of a leftward directed non-horizontal edge is the denominator of the v-coordinate of the point where the extension of the edge meets the right hand edge of $\mathcal{D}$. The r-value is taken to be positive if the edge travels upwards and negative if it travels downwards. If γ_i is a path in $\mathcal{D}$ then its *final r-value* is the r-value for the last edge in the path. If γ is an admissible path system, then its *cycle of final r-values* is the 4-tuple of final r-values for the four paths $\{\gamma_i\}$. The cycle of final r-values for γ is defined up to cyclic permutation. A path γ_i is *completely reversible* if each pair of successive edges in γ_i lies in triangles that share a common edge.

3.2.2 Boundary slopes for K_n and K_n^τ

The goal of this section is to apply the machinery of the previous section to prove that K_n has the boundary slope $4(n+4)$ whereas K_n^τ does not.

Consider the admissible path system $s = (s_i)$ given by

$$\begin{aligned}
s_0 &= [\langle\infty\rangle,\, \langle 1\rangle,\, \langle 1/2\rangle,\, \langle 1/3\rangle] \\
s_1 &= [\langle\infty\rangle,\, \langle 1\rangle,\, \langle 1/2\rangle,\, \langle 1/3\rangle,\, \langle 1/4\rangle,\, \langle 1/5\rangle] \\
s_2 &= [\langle\infty\rangle,\, \langle 1\rangle,\, \langle 1/2\rangle,\, \ldots,\, \langle 1/(2n+1)\rangle] \\
s_3 &= [\langle\infty\rangle,\, \langle 1\rangle,\, \langle 1/2\rangle]\,.
\end{aligned}$$

Also let $s^\tau = (s_1,\, s_0,\, s_2,\, s_3)$.

Lemma 3.2.1. *A Seifert surface Σ for K_n is carried by the path system s. Similarly, a Seifert surface Σ^τ for K_n^τ is carried by s^τ. Furthermore, the twist numbers $t(\Sigma)$ and $t(\Sigma^\tau)$ are both $-(14+4n)$.*

Proof. While building a 1-sheeted candidate surface for s, according to the construction from Section 3.2.1, we may choose saddles so that the resulting surface is is the same as the Seifert surface obtained by applying Seifert's algorithm to the projection of K_n shown in Figure 3.2. We calculate the twist number using the formula $t(\gamma) = \sum t(\gamma_i)$. We have

$$\begin{aligned} t(s_0) &= 2(0-2) = -4, \\ t(s_1) &= 2(0-4) = -8, \\ t(s_2) &= 2(0-2n) = -4n, \\ t(s_3) &= 2(0-1) = -2, \end{aligned}$$

and hence

$$t(s) = \sum t(s_i) = -4-8-4n-2 = -(14+4n).$$

The same procedure works for K_n^τ. □

Lemma 3.2.2. $4(n+4)$ *is not a boundary slope of K_n^τ.*

Proof. It suffices to show that every candidate surface with twist number 2 is compressible; this will be proved using three claims which will be established below.

Claim 1. If δ is a type III path system with $t(\delta) = 2$ then

$$\begin{aligned}
\delta_0 &= [\langle\infty\rangle,\ \langle 0\rangle,\ \langle 1/5\rangle] \\
\delta_1 &= [\langle\infty\rangle,\ \langle 1\rangle,\ \langle 1/2\rangle,\ \langle 1/3\rangle] \\
\delta_2 &= [\langle\infty\rangle,\ \langle 0\rangle,\ \langle 1/(2n+1)\rangle] \\
\delta_3 &= [\langle\infty\rangle,\ \langle 0\rangle,\ \langle 1/2\rangle]
\end{aligned}$$

Assume δ is as claimed. Proposition 2.5 of [13] implies that since the sum of integer vertices of δ is 1, if two of the paths δ_i are completely reversible then δ is a compressible path system. It can be verified that δ_0 is completely reversible by noticing that the triangle $[\langle 1/5\rangle, \langle 0\rangle, \langle 1\rangle]$ shares an edge with the triangle $[\langle 0\rangle, \langle\infty\rangle, \langle 1\rangle]$. The same argument applies to δ_2 and δ_3, hence δ is a compressible path system.

Claim 2. If δ is a type II path system with $t(\delta) = 2$ then (up to adding or removing vertical edges on the individual paths)

$$\begin{aligned}
\delta_0 &= [\langle 0\rangle,\ \langle 1/5\rangle] \\
\delta_1 &= [\langle 0\rangle,\ \langle 1\rangle,\ \langle 1/2\rangle,\ \langle 1/3\rangle] \\
\delta_2 &= [\langle 0\rangle,\ \langle 1/(2n+1)\rangle] \\
\delta_3 &= [\langle 0\rangle,\ \langle 1\rangle,\ \langle 1/2\rangle]
\end{aligned}$$

If δ is as in Claim 2, the cycle of final r-values for δ is $(-4,\ 1,\ -2n,\ 1)$. Proposition 2.9 of [13] shows that there is no incompressible surface associated to δ.

Claim 3. There are no type I path systems with twist number 2.

Assuming the claims, the proof of the lemma is completed. $\square$

We now work on establishing the claims.

A *basic path* is a path starting at a vertex $\langle \frac{p}{q} \rangle$ that proceeds monotonically to the left (without vertical edges) ending at the left edge of $\mathcal{S}$. A *basic path system* is a path system made up of basic paths. Note that a basic path δ_i from $\langle \frac{1}{r} \rangle$ is either

$$[\langle 1 \rangle, \langle 1/2 \rangle, \langle 1/3 \rangle, \ldots, \langle 1/r \rangle]$$

or

$$[\langle 0 \rangle, \langle 1/r \rangle]\,.$$

Hence, any such path ends at either 0 or 1. If it ends at 0 then

$$t(\delta_i) = 2(1-0) = 2.$$

If it ends at 1 then

$$t(\delta_i) = 2\big(0-(r-1)\big) = 2-2r.$$

If δ is a basic path system for K_n^τ that satisfies (E1) and (E2), then δ_0 starts at $\langle \frac{1}{5} \rangle$, δ_1 starts at $\langle \frac{1}{3} \rangle$, δ_2 starts at $\langle \frac{1}{2n+1} \rangle$, and δ_3 starts at $\langle \frac{1}{2} \rangle$. So we have

$$\begin{aligned}
t(\delta_0) &= \begin{cases} 2 & \text{if } \delta_0 \text{ ends at } 0 \\ -8 & \text{if } \delta_0 \text{ ends at } 1 \end{cases} \\
t(\delta_1) &= \begin{cases} 2 & \text{if } \delta_1 \text{ ends at } 0 \\ -4 & \text{if } \delta_1 \text{ ends at } 1 \end{cases} \\
t(\delta_2) &= \begin{cases} 2 & \text{if } \delta_2 \text{ ends at } 0 \\ -4n & \text{if } \delta_2 \text{ ends at } 1 \end{cases} \\
t(\delta_3) &= \begin{cases} 2 & \text{if } \delta_3 \text{ ends at } 0 \\ -2 & \text{if } \delta_3 \text{ ends at } 1 \end{cases}
\end{aligned}$$

Proof of Claim 1. Let δ be a type III path system with $t(\delta) = 2$. Notice that (E2) implies that δ has no vertical edges. Let δ' be the basic path system $\delta \cap \mathcal{S}$. Since extending paths to $\langle\infty\rangle$ doesn't affect twist number, we have $2 = t(\delta) = t(\delta')$. There are two choices for each path and four paths, so we have a total of 2^4 possible path systems δ'. The corresponding twist numbers are in the set

$$\{-12, -8, -6, -2, 2, 4, 8, 6 - 4n, 2 - 4n, -4n,$$
$$-4 - 4n, -8 - 4n, -10 - 4n, -14 - 4n.\}$$

Since n is an integer, $2 \notin \{-4n, -4 - 4n, -8 - 4n\}$ and $n \geq 2$, so $2 \notin \{6 - 4n, 2 - 4n, -10 - 4n, 14 - 4n\}$. We must have $t(\delta') = 2$ for every n. There is only one such path system; it extends to the system δ given in the claim. $\square$

Proof of Claim 2. Assume that δ is a type II path system with $t(\delta) = 2$. Then $t(\delta)$ is determined by the basic path system δ' obtained by deleting all vertical edges of δ. There are infinitely many extensions of δ' which satisfy (E3) formed by adding vertical edges to the end of the individual paths. However, even if the extensions do not satisfy the minimality condition (E2), any two such extensions will always have the same twist number. Thus, we may chose to work with the extension $\tilde{\delta}$ where all paths end at $\langle 0\rangle$. Then

$$t(\tilde{\delta}_i) = \begin{cases} 2 & \text{if } \delta'_i \text{ ends at } 0 \\ t(\delta'_i) + 2 & \text{if } \delta'_i \text{ ends at } 1 \end{cases}.$$

Again there are 2^4 possibilities. The corresponding twist numbers are in the set

$$\{-6,\, -4,\, -2,\, 0,\, 2,\, 4,\, 6,\, 8,\, 8-4n,\, 6-4n,\, 4-4n,\\ 2-4n,\, -4n,\, -2-4n,\, -4-4n,\, -6-4n.\}$$

As before, we know that $2 \notin \{-4-4n, -4n, 4-4n, 8-4n\}$ since n is an integer. Also, $n \geq 2$ implies $2 \notin \{-6-4n, -2-4n, 2-4n, 6-4n\}$. Again, the only remaining possibility satisfies the claim. □

Proof of Claim 3. Assume that δ is a type I path system. Let $(t_0, \ldots, t_3)$ be the 4-tuple of endpoints of the paths δ_i. Since δ is an admissible path system, we know that the sum of the vertical coordinates of the t_i's is zero, but every point of every δ_i has vertical coordinate greater than or equal to zero. Therefore, $t_i = 0$ for every i. We now know that

$$\begin{aligned}
\delta_0 &= [\langle 0\rangle, \langle 1/3\rangle] \\
\delta_1 &= [\langle 0\rangle, \langle 1/5\rangle] \\
\delta_2 &= [\langle 0\rangle, \langle 1/(2n+1)\rangle] \\
\delta_3 &= [\langle 0\rangle, \langle 1/2\rangle]\,.
\end{aligned}$$

However, for this path system, we have

$$t(\delta) \;=\; 2+2+2+2 \;=\; 8.$$

We have now established claim 3. □

In order to show that K_n has boundary slope $4(n + 4)$, consider the path system $\gamma = (\gamma_i)$ in $\mathcal{S}$ given by

$$\begin{aligned}
\gamma_0 &= [\langle 1\rangle, \langle 1/2\rangle, \langle 1/3\rangle] \\
\gamma_1 &= [\langle 0\rangle, \langle 1/5\rangle] \\
\gamma_2 &= [\langle 0\rangle, \langle 1/(2n+1)\rangle] \\
\gamma_3 &= [\langle 1\rangle, \langle 1/2\rangle]
\end{aligned}$$

Note that γ is not an admissible path system, but by adding vertical edges to the ends of the paths it extends to an admissible system.

Lemma 3.2.3. *For the knot K_n, the path system γ extends to an admissible system that only carries incompressible surfaces with boundary slope $4(n+4)$. Also, every path system that carries this slope is a vertical extension of γ.*

Proof. The cycle of final r-values for the path system γ is $(1, -4, -2n, 1)$. Furthermore, the final slopes of the γ_i have positive sum, γ satisfies (E2), and each γ_i ends on the left edge of $\mathcal{S}$. Therefore, by Proposition 2.9 of [13], the γ_i's can be extended by vertical edges to form a system that carries an incompressible surface. As in the proof of Claim 2 above, we can calculate the twist number of any such path system by calculating the twist number of the vertical extension γ' given by adding the vertical edge $[\langle 0\rangle, \langle 1\rangle]$ to both γ_0 and γ_3.

We calculate the twist number as above; one has:

$$\begin{aligned} t(\gamma_0') &= 2(1-2) = -2, \\ t(\gamma_1') &= 2(1-0) = 2, \\ t(\gamma_2') &= 2(1-0) = 2, \\ t(\gamma_3') &= 2(1-1) = 0, \end{aligned}$$

giving

$$t(\gamma') = \sum t(\gamma_i) = -2+2+2 = 2.$$

Therefore, the boundary slope of any surface carried by a vertical extension of γ is

$$t(\gamma') - t(s) = 2 + 14 + 4n = 4(n+4).$$

The uniqueness of γ follows exactly as in Lemma 3.2.2. $\square$

Combining Lemma 3.2.2 and Lemma 3.2.3 yields the following result:

Proposition 3.2.4. *For every $n > 1$, $4(n+4)$ is a boundary slope of K_n but not of K_n^τ. Furthermore, every path system that carries this slope is a vertical extension of γ.*

3.3 The slope $4(n+4)$ of K_n is not strongly detected

The previous section has established that $4(n+4)$ is not a boundary slope of K_n^τ. As stated in the introduction to this chapter, if $4(n+4)$ is strongly detected by a curve in $X(K_n)$, then the restriction of every character to the

Conway sphere S_4 must be reducible. This observation greatly simplifies the problem of showing that this boundary slope is not strongly detected.

3.3.1 Lemmas

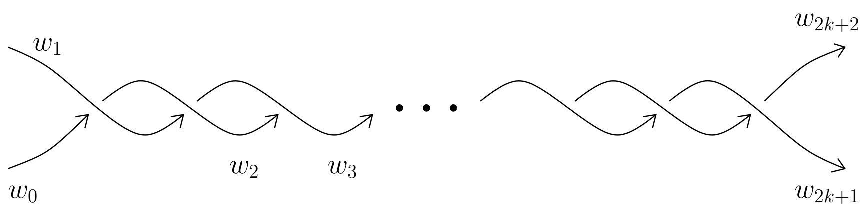


Figure 3.6: The diagram for Lemma 3.3.1

Lemma 3.3.1. *Assume that a twist region has a diagram as in Figure 3.6 with Wirtinger generators* $\{w_0, w_1, \ldots, w_{2k+2}\}$ *as shown. Then for every* $k \geq 0$, *we have*

$$w_{2k+2} \;=\; (w_0 w_1)^{-k} w_1^{-1} (w_0 w_1)^{k+1}$$

and

$$w_{2k+1} \;=\; (w_0 w_1)^{-k} w_1 (w_0 w_1)^{k}.$$

Proof. For $k = 0$, the relations are the Wirtinger relation and the trivial relation respectively. The conclusion follows by induction. $\square$

Lemma 3.3.2. *Assume that* x *and* y *are elements of a subgroup* $G < \pi_1(M_n)$ *where* x *and* y *have identical images in the abelianization of* G. *If* $\rho \in R(M_n)$ *such that* $\rho(G)$ *is a group of upper triangular matrices, then* $\rho(x)$ *and* $\rho(y)$ *are identical along their diagonals.*

Proof. Let $\Delta < \mathrm{SL}_2(\mathbb{C})$ be the subgroup of upper-triangular matrices and $D < \mathrm{SL}_2(\mathbb{C})$ be the abelian subgroup of diagonal matrices. Then we have an epimorphism $\delta\colon \Delta \longrightarrow D$ given by

$$\delta\begin{pmatrix} \alpha & \beta \\ 0 & 1/\alpha \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix}.$$

Since x and y have the same image in G^{AB} we have $\delta\rho(x) = \delta\rho(y)$. □

Lemma 3.3.3. *If*

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \alpha & 1 \\ 0 & 1/\alpha \end{pmatrix}$$

then, for every $n \in \mathbb{Z}^+$,

$$(AB)^n = \begin{pmatrix} \alpha^{2n} & p_n(\alpha) \\ 0 & \alpha^{-2n} \end{pmatrix}$$

where $p_n(x) \in \mathbb{C}(x)$ *and* $p_n(x) \neq \frac{1}{(1-x^2)x^{2n-1}}$.

Proof. First we establish an inductive formula for $p_n(x)$.

Claim 1. $p_1(x) = x$ and $p_n(x) = x^{3-2n} + x^2\, p_{n-1}(x)$.

For $n = 1$ we calculate

$$AB = \begin{pmatrix} \alpha^2 & \alpha \\ 0 & \alpha^{-2} \end{pmatrix}$$

to see that $p_1(x) = x$ as claimed. Now assume that

$$(AB)^{n-1} = \begin{pmatrix} \alpha^{2n-2} & p_{n-1}(\alpha) \\ 0 & \alpha^{2-2n} \end{pmatrix}.$$

Multiplying this by AB we see that

$$(AB)^n \;=\; \begin{pmatrix} \alpha^{2n} & \alpha^2\, p_{n-1}(\alpha) + \alpha^{3-2n} \\ 0 & \alpha^{-2n} \end{pmatrix}.$$

Hence, claim 1 is true.

Claim 2. $\qquad x^{2n-3}\, p_n(x) \in \mathbb{C}[x].$

Again, we establish this by induction. For $n = 1$

$$x^{-1}\, p_1(x) \;=\; x^{-1} \cdot x \;=\; 1 \;\in \mathbb{C}[x].$$

Now assume that $x^{2n-5}\, p_{n-1}(x) \in \mathbb{C}[x]$. Then using claim 1, we have

$$\begin{aligned} x^{2n-3}\, p_n(x) \;&=\; x^{2n-3}\big(x^{3-2n} + x^2\, p_{n-1}(x)\big) \\ &=\; 1 + x^{2n-1}\, p_{n-1}. \end{aligned}$$

This is in $\mathbb{C}[x]$ by the inductive assumption.

Finally, we see that $p_n(x) \neq \frac{1}{(1-x^2)x^{2n-1}}$ because

$$x^{2n-3} \cdot \frac{1}{(1-x^2)x^{2n-1}} \;=\; \frac{1}{(1-x^2)x^2} \;\notin\; \mathbb{C}[x].$$

$\square$

A Wirtinger presentation for $\pi_1(M_n)$ can be obtained from Figure 3.2 with generating set $\{a, b, c, d, e, f\}$ given by the labels in the figure. We single out the following relations which will be used repeatedly in the arguments that follow:

(R1) $ae = eb$, and

(R2) $d = ab^{-1}c$.

Also, by applying Lemma 3.3.1 to T_0 and T_1, we get

(R3) $b^{-1} = faf^{-1}a^{-1}f^{-1}$ ($\Leftrightarrow$ $faf = bfa$), and

(R4) $c = (fd)^{-2}d^{-1}(fd)^3$ ($\Leftrightarrow$ $d(fd)^2c = (fd)^3$).

Before moving on to the calculations, we prove one more elementary lemma.

Lemma 3.3.4. *If $\rho \in R(M_n)$ with $\rho(a) = \rho(b^{-1})$ and $\chi_\rho(a) \neq \pm 2$, then $\chi_\rho(a) = 0$.*

Proof. Since $\chi_\rho(a) \neq \pm 2$ we may conjugate ρ to assume that $\rho(a)$ is diagonal. Let

$$\rho(a) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \quad \text{and} \quad \rho(e) = \begin{pmatrix} w & x \\ y & z \end{pmatrix}.$$

The lemma follows from the relation (R1), $ae = eb$. We have

$$\rho(ae) = \begin{pmatrix} \alpha w & \alpha x \\ y/\alpha & z/\alpha \end{pmatrix}$$

and

$$\rho(eb) = \rho(ea^{-1}) = \begin{pmatrix} w/\alpha & \alpha x \\ y/\alpha & \alpha z \end{pmatrix}.$$

Then $\alpha w = w/\alpha$. This implies that either $\alpha^2 - 1 = 0$ or $w = 0$. The first possibility is ruled out by assumption, hence $w = 0$. Similarly we have $z = 0$. Therefore $\chi_\rho(a) = \chi_\rho(e) = 0$. $\square$

3.3.2 Calculations

The Conway sphere S_4 separates M_n into two submanifolds N_1 and N_2. Let N_1 be the piece that contains the tangles T_0 and T_1. Choose a basepoint in S_4 for $\pi_1(M_n)$ and write $\Gamma_i = \pi_1(N_i)$ and $H = \pi_1(S_4)$.

If the boundary slope $4(n+4)$ is a strongly detected by an ideal point on a curve $X \subset X(M_n)$ then Proposition 1.3.4 implies that there are characters $\chi \in X$ with $|\chi(a)|$ taking arbitrarily large values. Also, Propositions 3.1.1 and 3.2.4 imply that every character on X is the character of a representation which is reducible when restricted to H. These observations motivate the following proposition.

Proposition 3.3.5. *There exists a finite set $\Lambda \subset \mathbb{C}$ such that if ρ is an irreducible representation in $R(M_n)$ where $\rho|_H$ is reducible, then $\chi_\rho(a) \in \Lambda$. In particular, the boundary slope $4(n+4)$ is not strongly detected by $X(M_n)$.*

Proof. We will calculate all conjugacy classes of irreducible representations ρ that are reducible on H and show that the possible values for $\chi_\rho(a)$ lie in a finite set. We start with $\Lambda = \{0, \pm 2\}$ and add finitely many values to this set as we proceed through several cases.

We may conjugate ρ to assume that its restriction to H is upper triangular. That is, the matrices $\rho(a)$, $\rho(b)$, $\rho(c)$, and $\rho(d)$ are all upper triangular. Furthermore, since we are looking for new values to add to Λ we may assume that $\mathrm{Tr}\big(\rho(a)\big) \notin \{0, \pm 2\}$.

To simplify notation in the remainder of this section, given $g \in \pi_1(M_n)$, $\rho(g)$ will also be denoted by g. The proof breaks into several cases and subcases.

Case 1. $\rho|_H$ is abelian.

Since $\mathrm{Tr}(a) \neq \pm 2$, we can conjugate ρ to assume that a is diagonal. Since $\rho(H)$ is abelian we know that $b, c, d \in \{a^{\pm 1}\}$. By Lemma 3.3.4, we know that $b = a$. Now using the relation (R1), we have $ae = ea$. Hence, e is also diagonal. Since Γ_2 is generated by $\{a, b, c, d, e\}$, we conclude that $\rho(\Gamma_2)$ is diagonal. The representation ρ is assumed to be irreducible, so $\rho|_{\Gamma_1}$ must be irreducible.

The relation (R2) implies that $d = c$ and so (R3) becomes $faf = afa$ and (R4) becomes $cfcfc = fcfcf$.

We have already established that $c \in \{a^{\pm 1}\}$.

- Subcase A. $c = a$.

  The relation (R4) becomes $afafa = fafaf$. Using (R3) we have

$$\begin{aligned} afafa &= fa(faf) \\ &= fa(afa) \\ af &= fa. \end{aligned}$$

  Then f is diagonal, which contradicts that $\rho|_{\Gamma_1}$ is irreducible.

- Subcase B. $c = a^{-1}$.

Since $\rho|_{\Gamma_1}$ is irreducible we can re-conjugate the representation to assume that

$$a = \begin{pmatrix} \alpha & 1 \\ 0 & 1/\alpha \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} \beta & 0 \\ r & 1/\beta \end{pmatrix}$$

where $\beta \in \{\alpha^{\pm 1}\}$ and $r \neq 0$.

If $\beta = \alpha$, then (R3) implies that $1 = \alpha^2 + r + \frac{1}{\alpha^2}$ and (R4) implies that $r^2 - 3r + 1 = 0$. There are at most eight simultaneous solutions to these two equations. We add the corresponding values of $\mathrm{Tr}(a)$ to Λ and note that Λ remains finite.

Otherwise, we have $\beta = 1/\alpha$. This time (R3) forces $r = -1$. Together with (R4), this means that α must satisfy the equation $\alpha^8 + \alpha^6 + \alpha^4 + \alpha^2 + 1 = 0$. Again, we need only add a finite number of values to Λ to account for these representations.

We may now assume that $\rho(H)$ is not abelian.

Case 2. $\rho|_{\Gamma_2}$ is reducible.

In this case, we may conjugate ρ to assume that a is diagonal and b, c, d, and e are all upper triangular.

- Subcase A. $[a, b] = I$.

  Then $b \in \{a^{\pm 1}\}$. Lemma 3.3.2 shows that we must have $b = a$. Then (R1) gives $[a, e] = I$ and e must also be diagonal. Also, (R2) implies $d = c$. Since $\rho(H)$ is non-abelian, c cannot be diagonal.

Let

$$e \;=\; \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix}.$$

By conjugating by diagonal matrices if necessary, we may assume that

$$c \;=\; \begin{pmatrix} \beta & 1 \\ 0 & 1/\beta \end{pmatrix}$$

where $\beta \in \{\alpha^{\pm 1}\}$. Note that in Γ_2^{AB} we have $c = e$. Since $\rho|_{\Gamma_2}$ is reducible we may apply Lemma 3.3.2 to see that c and e are identical on their diagonals. Hence $\beta = \alpha$.

Using Lemma 3.3.1, we have $(ec)^n e = c(ec)^n$. Also, by Lemma 3.3.3 we have

$$(ec)^n \;=\; \begin{pmatrix} \alpha^{2n} & p_n(\alpha) \\ 0 & \alpha^{-2n} \end{pmatrix}$$

where $p_n(x) \in \mathbb{C}(x)$ and $p_n(x) \neq \frac{1}{(1-x^2)x^{2n-1}}$.

Equating the upper right entries of the matrices $(ec)^n e$ and $c(ec)^n$ we get

$$\alpha\, p_n(\alpha) + \alpha^{-2n} \;=\; \frac{1}{\alpha}\, p_n(\alpha).$$

We have assumed that $\alpha \notin \{0, \pm 1\}$ so we have

$$p_n(\alpha) \;=\; \frac{1}{(1-\alpha^2)\alpha^{2n-1}}.$$

This equation has finitely many solutions because $p_n(x) \neq \frac{1}{(1-x^2)x^{2n-1}}$. We add the corresponding values $\alpha + \alpha^{-1}$ to Λ and move on.

- Subcase B. $[a, b] \neq I$.

Recall that we have conjugated ρ so that a is diagonal and $\rho(\Gamma_2)$ is upper triangular. Since $[a, b] \neq I$, we know that the upper right entry of b is non-zero, so we can conjugate by diagonal matrices to assume this entry is 1. As before, Lemma 3.3.2 implies that a and b are identical along their diagonals. We have

$$a \;=\; \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \quad \text{and} \quad b \;=\; \begin{pmatrix} \alpha & 1 \\ 0 & 1/\alpha \end{pmatrix}.$$

We have assumed that ρ is irreducible. This implies that the lower left entry of f is non-zero. We also know that $\mathrm{Tr}(f) = \mathrm{Tr}(a) = \alpha + 1/\alpha$. Putting this together, we know that f is of the form

$$f \;=\; \begin{pmatrix} x & \frac{1}{\alpha y}\left(-\alpha x^2 + \alpha^2 x + x - \alpha\right) \\ y & \frac{1}{\alpha}\left(-\alpha x + \alpha^2 + 1\right) \end{pmatrix}$$

where $y \neq 0$.

If ρ is a representation, then we must have $faf = bfa$ by (R3). Looking at the lower left entries of these matrices we get the equation

$$\frac{y(\alpha^3 x - \alpha x + 1)}{\alpha^2} \;=\; 0.$$

Then, because $\alpha, y \neq 0$, we must have

$$x \;=\; \frac{-1}{\alpha(\alpha^2 - 1)}.$$

Making this substitution and equating the upper left entries of faf and bfa, we arrive at the conclusion that $y\alpha = 0$, which is not possible. Therefore, there are no such representations.

Case 3. $\rho|_{\Gamma_2}$ is irreducible.

In this case, when we conjugate so that a is diagonal and b, c, and d are upper triangular, we know that the lower left entry of e must be non-zero.

First, note that $[a, b] \neq I$. This follows from (R1) and Lemma 3.3.4. Since a is diagonal and not parabolic, if a and b commute then $b = a^{-1}$ or $b = a$. Lemma 3.3.4 shows that the first possibility doesn't occur. On the other hand, if $a = b$, then (R1) implies that a and e commute. This is not possible because the lower left entry of e is non-zero.

We may now assume that

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} \beta & 1 \\ 0 & 1/\beta \end{pmatrix}$$

where $\beta \in \{\alpha^{\pm 1}\}$. Also, as argued for f in Case 2: Subcase B, we have

$$e = \begin{pmatrix} x & \frac{1}{\alpha y}\left(-\alpha x^2 + \alpha^2 x + x - \alpha\right) \\ y & \frac{1}{\alpha}\left(-\alpha x + \alpha^2 + 1\right) \end{pmatrix}$$

where $y \neq 0$.

We claim that $\beta \neq \alpha$. This follows from (R1). Equating the lower right entries of ae and eb we get

$$\frac{1}{\alpha^2}\left(-\alpha x + \alpha^2 + 1\right) = y + \frac{1}{\alpha^2}\left(-\alpha x + \alpha^2 + 1\right)$$

which implies that $y = 0$, a contradiction. Thus we must have

$$b = \begin{pmatrix} 1/\alpha & 1 \\ 0 & \alpha \end{pmatrix}.$$

Looking at the upper right entries of ae and eb, we see that $x = 0$. Substituting for x and checking $ae = eb$ in the lower right entries, we conclude that $y = \alpha^{-2}(1 - \alpha^4)$. This implies that

$$e = \begin{pmatrix} 0 & \frac{\alpha^2}{\alpha^4 - 1} \\ \frac{1-\alpha^4}{\alpha^2} & \frac{\alpha^2+1}{\alpha} \end{pmatrix}.$$

We know that c is upper triangular and $\mathrm{Tr}(c) = \mathrm{Tr}(a)$, thus,

$$c = \begin{pmatrix} \beta & r \\ 0 & 1/\beta \end{pmatrix}$$

where $\beta \in \{\alpha^{\pm 1}\}$ and $r \in \mathbb{C}$. Formally, we assign a matrix $\Theta \in \mathrm{SL}_2(\mathbb{C})$ to represent the product $(ec)^n$.

Applying Lemma 3.3.1 to the tangle T_2, we get the relation $w_{2n-1} = (ec)^{-n}c(ec)^n$ and from T_3 we have that $w_{2n-1} = aea^{-1}$. Putting these together, we conclude that the matrix Θ should satisfy the relation

$$\Theta^{-1} c\, \Theta = aea^{-1}. \quad \text{(R5)}$$

The matrix Θ should also commute with ec, so we have

$$\Theta\, ec = ec\, \Theta. \quad \text{(R6)}$$

Using (R5) and $\det(\Theta) = 1$, we can express Θ in terms of α, β, r, and a new independent variable z. By comparing the matrices in (R6), we can express r as a function in α, β, and z. If $\beta = \alpha$, then the upper right entries of the matrices from (R5) gives the equality $\alpha^6 - \alpha^4 - \alpha^2 + 1 = 0$. Thus we may assume that $\beta = 1/\alpha$ at the price of adding $\{\, \alpha + \alpha^{-1} \mid \alpha^6 -$

$\alpha^4 - \alpha^2 + 1 = 0\,\}$ to Λ. Both c and Θ are now expressed as functions of α and z:

$$c \;=\; \begin{pmatrix} 1/\alpha & r \\ 0 & \alpha \end{pmatrix} \quad \text{and} \quad \Theta \;=\; \begin{pmatrix} \frac{\alpha^4 - z^2}{\alpha^4 z} & \frac{\alpha z}{\alpha^4 - 1} \\ \frac{z(1-\alpha^4)}{\alpha^5} & z \end{pmatrix}$$

where $r = \frac{\alpha^4 + \alpha^2 z^2 - z^2}{z^2(\alpha^4 - 1)}$.

Write

$$f \;=\; \begin{pmatrix} p & q \\ s & t \end{pmatrix}$$

and consider the consequences of (R3). Equating the lower right entries of faf and bfa, we see that $q(\alpha^3 p + \alpha t - 1) - \alpha t = 0$. If $q = 0$, this equation implies that $t = 0$. Then $\det(f) = 0$, a contradiction. Hence $q \neq 0$ and we solve $\det(f) = 1$ for s to get $s = \frac{pt-1}{q}$. Substituting this for s and looking at the lower right entries again, we get an expression for p in terms of α and t. Compare the lower left entries to see that $(\alpha - t)(\alpha^2 t - t - \alpha) = 0$. There are two subcases.

- Subcase A. $\quad t = \alpha. \qquad$ (This is the case where $\rho|_{\Gamma_1}$ is reducible.)

  At this point we have f in terms of α and q. Using (R3) once again, we are able to solve for q to get

  $$f \;=\; \begin{pmatrix} 1/\alpha & \frac{\alpha^2}{2\alpha^2 - 1} \\ 0 & \alpha \end{pmatrix}.$$

  Checking (R4) gives the polynomial equation

  $$(6\alpha^8 - 17\alpha^6 + 13\alpha^4 + 2\alpha^2 - 4)z^2 - 6\alpha^8 + 7\alpha^6 - 2\alpha^4 \;=\; 0.$$

By adding $\{\alpha+\alpha^{-1}\,|\,6\alpha^8-17\alpha^6+13\alpha^4+2\alpha^2-4 \;=\; 0\}$ to the set Λ, we can solve for z^2 to get

$$z^2 \;=\; \frac{\alpha^4\,(6\alpha^4-7\alpha^2+2)}{6\alpha^8-17\alpha^6+13\alpha^4+2\alpha^2-4}.$$

We would like to use this equality to replace z in our only remaining relation, $\Theta=(ec)^n$. Although our expression for Θ in terms of α and z does contain odd powers of z, when we look at the resulting expression for Θ^2 in terms of α and z we see that it contains only even powers of z. We replace our expression for z^2 into $\mathrm{Tr}(ec)$ and $\mathrm{Tr}(\Theta^2)$ to write these quantities as rational functions in α only,

$$\mathrm{Tr}(ec) \;=\; \frac{(\alpha^2+1)(10\alpha^4-15\alpha^2+6)}{\alpha^4\,(2\alpha^2-1)(3\alpha^2-2)}$$

and

$$\mathrm{Tr}(\Theta^2) \;=\; \frac{(\alpha^2+1)(72\alpha^{12}-288\alpha^{10}+446\alpha^8-278\alpha^6-25\alpha^4+108\alpha^2-36)}{\alpha^4\,(2\alpha^2-1)(3\alpha^2-2)(6\alpha^6-11\alpha^4+2\alpha^2+4)}.$$

In light of this, instead of using $\Theta=(ec)^n$, we focus on the relation

$$\mathrm{Tr}(\Theta^2)=\mathrm{Tr}\big((ec)^{2n}\big). \tag{3.1}$$

Using standard trace formulas, $\mathrm{Tr}(\Theta^2)=P\big(\mathrm{Tr}(ec)\big)$ where $P(x)\in\mathbb{Z}[x]$ is a monic polynomial of degree $2n$. We wish to rule out the possibility that there are infinitely many solutions for α that will satisfy this equation. This will happen precisely when the rational functions $\mathrm{Tr}(\Theta^2)$ and $P\big(\mathrm{Tr}(ec)\big)$ are identically equal. However, $\mathrm{Tr}(\Theta^2)$ has a pole at the roots of the polynomial $6x^6-11x^4+2x^2+4$

while $P\big(\mathrm{Tr}(ec)\big)$ takes a finite value at each of these roots, since the poles of $P\big(\mathrm{Tr}(ec)\big)$ are the same as the poles of $\mathrm{Tr}(ec)$. Therefore, these rational functions are not identically equal and there are only finitely many values for α that satisfy equation (3.1). We add the corresponding traces to Λ and move to the second subcase.

- Subcase B. $\quad t = \frac{\alpha}{\alpha^2-1}. \qquad (\rho|_{\Gamma_1}$ is irreducible.)

Just as in subcase A, we can use (R3) to get

$$f \;=\; \begin{pmatrix} \frac{\alpha^4-\alpha^2-1}{\alpha(\alpha^2-1)} & \frac{\alpha^2}{(\alpha^2+1)(\alpha^2-1)^2} \\ \frac{\alpha^4-\alpha^2-2}{\alpha^2} & \frac{\alpha}{\alpha^2-1} \end{pmatrix}.$$

After adding $\pm\big(2^{1/2}+2^{-1/2}\big)$ to Λ, the relation (R4) implies that

$$\begin{aligned}(-\alpha^{12}+8\alpha^{10}-22\alpha^8+23\alpha^6-3\alpha^4-9\alpha^2+4)z^4+\\ +(2\alpha^{12}-12\alpha^{10}+22\alpha^8-11\alpha^6-3\alpha^4+2\alpha^2)z^2-\\ -\alpha^{12}+4\alpha^{10}-4\alpha^8 \;=\; 0. \quad (3.2)\end{aligned}$$

Let $\mathcal{C} \subset \mathbb{C}^2$ be the zero set for this polynomial. Setting the variables

$$T \;=\; \mathrm{Tr}(\Theta) \;=\; \frac{\alpha^4 z^2+\alpha^4-z^2}{\alpha^4 z} \qquad (3.3)$$

and

$$E \;=\; \mathrm{Tr}(ec) \;=\; \frac{\alpha^4 z^2-\alpha^4+z^2}{\alpha^2 z^2} \qquad (3.4)$$

we have a corresponding projection

$$\phi\colon\, \mathcal{C} \longrightarrow \big\{\, (T,E) \in \mathbb{C}^2 \,\big\}.$$

We take resultants to get a polynomial $Q(E,T) \in \mathbb{Z}[E,T]$, where $Q = 0$ on the image of ϕ. (This polynomial is quite large, so we do not include it here.) As before, we can express T as a monic degree n polynomial $r(E) \in \mathbb{Z}[E]$. The one variable polynomial equation $Q\big(E, r(E)\big) = 0$ holds on $\mathrm{Im}(\phi)$. As long as this polynomial is not identically zero, the image of ϕ is 0-dimensional. We show that this is true by showing that its degree is bigger than zero.

Writing Q as a polynomial in $\mathbb{Z}[E][T]$, we get

$$Q(E,T) \;=\; \sum_{i=0}^{5} \, p_i(E) \cdot T^{2i}$$

where

$$\begin{aligned}
\deg\big(p_0(E)\big) &= 12 \implies \deg\big(p_0(E)\cdot r(E)^0\big) = 12\\
\deg\big(p_1(E)\big) &= 11 \implies \deg\big(p_1(E)\cdot r(E)^2\big) = 11+2n\\
\deg\big(p_2(E)\big) &= 10 \implies \deg\big(p_2(E)\cdot r(E)^4\big) = 10+4n\\
\deg\big(p_3(E)\big) &= 8 \implies \deg\big(p_3(E)\cdot r(E)^6\big) = 8+6n\\
\deg\big(p_4(E)\big) &= 7 \implies \deg\big(p_4(E)\cdot r(E)^8\big) = 7+8n\\
\deg\big(p_5(E)\big) &= 4 \implies \deg\big(p_5(E)\cdot r(E)^{10}\big) = 4+10n.
\end{aligned}$$

Since $n > 1$, the degree $4+10n$ term in $p_5(E)\cdot r(E)^{10}$ is the unique term of highest degree in $Q\big(E, r(E)\big)$. Therefore,

$$\deg\big(Q\big(E, r(E)\big) \;=\; 4+10n \; > 0.$$

We have now established that $\mathrm{Im}(\phi)$ is a finite set of points. The proof will be complete if we show that the fibers of ϕ are finite.

Using equations (3.3) and (3.4),

$$z = \frac{\alpha^2 T}{2\alpha^2 - E}.$$

This, together with equation (3.4), implies that

$$0 = (T^2 - 4)\alpha^4 + (4E - T^2 E)\alpha^2 - E^2 + T^2.$$

Therefore, the fibers are finite over every (E, T) unless $T^2 = 4$ and $E^2 = 4$. It is easy to see that these points are not in the image of ϕ.

This completes the proof that $4(n+4)$ is not strongly detected. $\square$

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Vita

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