Chapter

Numerical Methods for the Viscid and Inviscid Burgers Equations

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Abstract

The Burgers equation is the simplest equation combining both nonlinear propagation effects and diffusive effects. It describes a weak shock phenomena in gas dynamics. This equation has been a center of interest for researchers studying various physical phenomena such as theory of shock waves, fluid dynamics, turbulent flow and gas dynamics. In this chapter, the viscid and inviscid Burgers equations are studied. Two important mathematical models described by the Burgers equations are presented. Then a very brief review of theoretical results for conservation laws are provided. In particular, the characteristic method is used to demonstrate the development of shocks. Furthermore, the notion of weak solutions, state the Rankine-Hugoniot jump condition, and introduce the notion of an entropy condition are introduced. Finally, several important numerical methods to approximate the solution to the viscid and inviscid Burgers equations are developed. For the inviscid Burgers equation, the classical finite difference and finite volume methods are presented in details including Up-wind nonconservative, Up-wind conservative, Lax-Friedrichs, Lax-Wendroff, MacCormack, Godunov methods. In addition, the discontinuous Galerkin method is described in details. For the viscid Burgers equation, a finite difference method, a Chebyshev collocation method, an ultra-weak discontinuous Galerkin method, and a local discontinuous Galerkin method are presented.

Keywords: viscid and inviscid Burgers equation, finite difference method, finite volume method, discontinuous Galerkin method, computational fluid dynamics, fluid mechanics, gas dynamics, traffic flow

1. Introduction

Burgers' equation or Bateman-Burgers equation is a fundamental partial differential equation (PDE) occurring in various areas of applied mathematics, such as fluid mechanics, nonlinear acoustics, gas dynamics, and traffic flow. The typical solutions to Burgers equations may exhibit boundary or interior layers. The design and development of accurate, efficient, and robust numerical schemes to address these difficulties has survived as a classical challenge since the earliest days of digital computation. For convection problems, the main reasons for the limitations involve the development of discontinuous solution (called shocks) in finite time even from smooth initial data. For convection-diffusion problems, the main reasons for the difficulties involve the development of sharp transition layers. Another difficulty involves the design of numerical approximations that satisfy physical principles such as the conservation of energy. An active research direction now is the search for more robust and high order numerical methods.

In this chapter, the viscid and inviscid Burgers equations are analyzed. The numerical methods presented here are applicable for other PDEs. The one-dimensional viscid Burgers equation is a nonlinear PDE

$$u_t + uu_x = \nu u_{xx}, \tag{1}$$

where $\nu > 0$ is called the viscosity. When $\nu = 0$, (1) reduces to the inviscid Burgers equation

$$u_t + uu_x = 0, \tag{2}$$

which is a prototype for conservation equations that can develop discontinuities (or shock waves). Notice that (2) can be rewritten in the following scalar hyperbolic conservation law form

$$u_t + (f(u))_x = 0, \tag{3}$$

where $f(u) = \frac{u^2}{2}$. The nonconservative form of (3) is

$$u_t + f'(u)u_x = 0. \tag{4}$$

The classical Burgers Eq. (3) models wave motion, where $u(x, t)$ is the height of the wave at point x and time t. It has been a center of interest for researchers studying various physical phenomena such as theory of shock waves, fluid dynamics, turbulent flow and gas dynamics [1, 2]. In the next section, we show that the Navier-Stokes equations reduce to the viscid Burgers equation.

2. Derivation of Burgers equations

We may obtain Burgers's equation as a simplified version of the Navier-Stokes equations. The basic equations that govern the viscid flow of simple substances like air and water under normal conditions are called the Navier-Stokes equations (nonlinear system of PDEs):

$$\rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \mu \Delta \mathbf{u} + \mathbf{F}, \tag{5}$$

where $\mathbf{u}$ is the velocity of the fluid, p is the fluid pressure, ρ is the fluid density, μ is the fluid dynamic viscosity, and $\mathbf{F}$ is an external force. The Kinematic viscosity is $\nu = \frac{\mu}{\rho}$. The term $\rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u})$ corresponds to the inertial forces, $-\nabla p$ corresponds to pressure forces, $\mu \Delta \mathbf{u}$ corresponds to viscid forces, and $\mathbf{F}$ corresponds to the external forces applied to the fluid. The Navier-Stokes equations were derived by Navier, Poisson, Saint-Venant, and Stokes between 1827 and 1845. These equations are always solved together with the continuity equation $\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0$. For incompressible flows (density of the fluid is constant), the continuity equation yields $\nabla \cdot \mathbf{u} = 0$. The

Navier-Stokes equations represent the conservation of momentum, while the continuity equation represents the conservation of mass.

If we drop the pressure term we get the Burgers' equation

$$\rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) = \mu \Delta \mathbf{u} + \mathbf{F}. \tag{6}$$

When the external force $\mathbf{F}$ is equal to zero and when ρ is a constant (for an incompressible fluid), the above Eq. (6) simplifies to

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = \nu \Delta \mathbf{u}, \tag{7}$$

where the new constant $\nu = \mu/\rho$ is called the kinematic viscosity. An even further simplification arises when we assume the viscosity is zero. Then we obtain the inviscid Burgers' equation

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{0}. \tag{8}$$

If one consider a one-dimensional problem with no pressure gradient, the Eq. (7) reduces to the viscid Burgers Eq. (1). When the viscosity ν of the fluid is zero, the Eq. (8) reduces to the inviscid Burgers Eq. (2).

Burgers equations arise in the modeling of many physical phenomena, such as the mathematical model of turbulence [1], the traffic flow, the approximate theory of flow through a shock wave traveling in viscid fluid [2], and many others. For instance, traffic flow along a long road can be described by the Burgers equation. It was developed in [3, 4] and it was discussed more extensively in [5].

3. Inviscid Burgers' equation and the development of shocks

In this section, we consider the following Cauchy problem

$$\begin{cases} u_t + (f(u))_x = 0, & x \in \mathbb{R}, \ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{9}$$

where $f(u) = u^2/2$ and u_0 is the initial data.

3.1 Characteristic method

The Cauchy problem (9) can be solved using the so-called method of characteristics. Let the initial curve Γ be parameterized by s, i.e., $x(s,0) = s$, $t(s,0) = 0$, $u(s,0) = u_0(s)$. The characteristic equations for the curves $(x(s,r), t(s,r), u(s,r))$ are

$$\frac{\partial x}{\partial r} = u, \ \frac{\partial t}{\partial r} = 1, \ \frac{\partial u}{\partial r} = 0, \ \text{ subject to } \ x(s,0) = s, \ \ t(s,0) = 0, \ \ u(s,0) = u_0(s).$$

These yield $t = r, u(s,r) = u_0(s), x = u_0(s)r + s$. We note that (s,r) can be expressed in terms of (x,t) when the Jacobian of this transformation is nonsingular i.e.,

$$J = \begin{vmatrix} x_s & x_r \\ t_s & t_r \end{vmatrix} = \begin{vmatrix} u_0'(s)r + 1 & u_0(s) \\ 0 & 1 \end{vmatrix} = u_0'(s)r + 1 \neq 0.$$

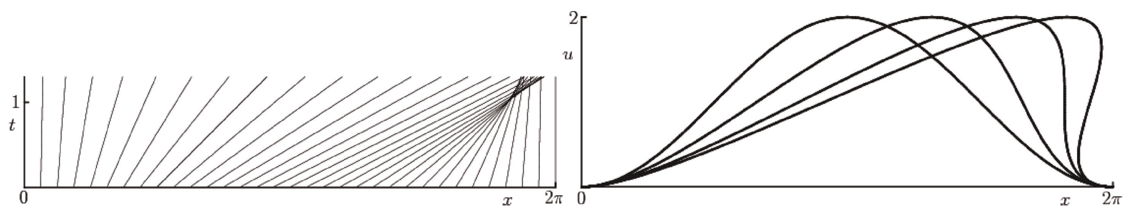

Figure 1.
Characteristics (left) and profiles at times $t = 0, .5, 1$, and 1.3 (right) for Burgers' equation with $u_0(x) = 1 - \cos(x)$.

Provided $t = r \neq -1/u_0'(s)$ we can find x, t in terms of s, r. Since at $r = 0$ we have $J = 1 \neq 0$, which shows that the initial curve Γ is noncharacteristic. Thus, at least for small t, we can solve for x, t in terms of s, r. Since $r = t$ and $s = x - ut$, we obtain an implicit solution $u(x,t) = u_0(x - tu(x,t))$. The curves parametrized by $(x = u_0(s)r + s, t = r)$ in the xt-plane are called the *projected characteristic curves*. For fixed s, the equations of these curves are $x = s + u_0(s)t$, which are lines with slopes $\frac{dt}{dx} = \frac{1}{u_0(s)}$. Moreover $u = u_0(s)$ has the constant value $u_0(s)$ along the line $x = s + u_0(s)t$.

3.2 The development of shocks

Considering two characteristics lines $x = u_0(s_1)r + s_1$ and $x = u_0(s_2)r + s_2$ passing through $(s_1, 0)$ and $(s_2, 0)$ with $s_2 > s_1$. They intersect at some point P at positive time obtained by solving $u_0(s_1)r + s_1 = u_0(s_2)r + s_2$ for r to get $t_B = \frac{s_2 - s_1}{u_0(s_1) - u_0(s_2)}$. Depending on u_0, the characteristic lines may or may not cross. There are 3 cases:

Case 1: If $u_0(s)$ =constant, then the characteristic lines are parallel. In this case the solution will exist globally for $t > 0$.

Case 2: Suppose there is an $s_1 < s_2$ such that $u_0(s_1) < u_0(s_2)$. Then $t_B < 0$ and the projected characteristic lines fan out. In this case the solution will exist globally for $t > 0$. The solution is like a fan and it is called a *rarefaction wave*.

Case 3: Suppose there is an $s_1 < s_2$ such that $u_0(s_1) > u_0(s_2)$. In this case, $t_B > 0$ and the projected characteristic lines intersect at $(x = u_0(s_1)r + s_1, r)$, where $r = \frac{s_2 - s_1}{u_0(s_1) - u_0(s_2)} > 0$. This will result in a difficulty in defining the solution beyond the point of intersection. This phenomenon occurs as a result of wave breaking. Geometrically, even if $u(x, 0) = u_0(x)$ is sufficiently smooth, the surface $u = u(x,t)$ is folded over on itself (into a backward S-shape) at some point (x_1, t_1) and the solution $u(x,t)$ becomes multi-valued for $t > t_1$ for some critical t_1 that depends on u_0. This phenomenon is illustrated in **Figure 1** for the initial condition $u_0(x) = 1 - \cos(x)$. The implicit solution $u(x,t) = u_0(x - tu(x,t))$ is no longer valid. The solution then continues with a shock wave. The solution is not even continuous at the shock, but the solution still makes sense, because the PDE expresses a conservation law and the shock preserves conservation. We note that u_x and u_t approach infinity at the positive time as $t \to -\frac{1}{u_0'(s)}$ and the minimum time for which the solution becomes multi-valued is $t_B = -\frac{1}{\min u_0'(s)}$. The solution suffers a gradient catastrophe type of singularity (u_x and u_t become discontinuous). Such a jump discontinuities are called *shocks*. The solution fails to exist globally.

3.3 Weak solution

As we showed in the previous subsection, smooth solutions of conservation laws can blow up in finite time. After the shock, the notion of classical solution fails and a new concept is needed. The only way to establish a solution after the breaking time is to allow discontinuities of u. This type of solution is known as a weak solution which means we have to consider solutions in the sense of distributions.

Definition 3.1 We say that u is a weak solution of (9) if

$$\int_0^\infty \int_{-\infty}^\infty (uv_t + f(u)v_x)dxdt + \int_{-\infty}^\infty u_0(x)v(x,0)dx = 0, \tag{10}$$

for any smooth function $v \in C^\infty(\mathbb{R} \times [0,\infty))$ with compact support.

The idea of weak solution allows for solutions which need not be differentiable or even continuous. We would lie to mention that weak solutions have some restrictions on types of discontinuities. To illustrate this, we assume u is a weak solution of (9) and is discontinuous across a curve defined by $x = s(t)$, but u is smooth on both sides of the curve $s(t)$. We use $u^-(x,t)$ to denote the limit of $u(x,t)$ from the left. We also use $u^+(x,t)$ to denote the limit of $u(x,t)$ from the right. It turns out that the curve $x = s(t)$ cannot be any curve. We present the relation between $x = s(t)$, u^- and u^+ in the next theorem.

Theorem 1.1 If u is a weak solution of (9) such that u is discontinuous across the curve $x = s(t)$, but u is smooth on either side of $x = s(t)$, then u must satisfy the following condition, called the *Rankine-Hugoniot jump condition*, across the curve of discontinuity

$$s'(t) = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{[f(u)]}{[u]}, \tag{11}$$

where $[f(u)]$ and $[u]$ are, respectively, used to denote the jumps of $f(u)$ and u across the discontinuity curve and $s'(t)$ is the speed of the curve of discontinuity.

Consequently, if we assume that u is a weak solution with discontinuity along a curve $x = s(t)$, then the solution must satisfy the condition (11). We note that the weak solutions defined in (10) are not unique. A necessary condition to the jump discontinuity is given by the Rankine-Hugoniot condition (11). This condition can be used to give a discontinuous solution u. However, it turns out that there are still some situations in which this concept is not enough to guarantee the uniqueness of the solution after the shock. It turns out that solutions of quasilinear equations of the form (10) which satisfy an *entropy condition* are considered more physically realistic [6]. It is one of the most widely used. The entropy condition is introduced to eliminate nonphysical weak solutions. There is a number of variations in which this condition can be presented, we will mention only the simplest one.

For the Eq. (3), to select solutions which do have a physical sense, we only allow for a curve of discontinuity in our solution $u = u(x,t)$ if the wave to the left is moving faster than the wave to the right. In other words, we only allow for a curve of discontinuity between u^- and u^+ if $f'(u^+) < s' < f'(u^-)$. This condition is known as the entropy condition. A curve of discontinuity is said to be a shock curve for a solution $u(x,t)$ if the curve satisfies the Rankine-Hugoniot jump condition and the entropy condition for that solution $u(x,t)$. Thus, to select solutions which do have a physical sense and eliminate the physically less realistic solutions, we consider solutions u for which curves of discontinuity in the solution are shock curves. For the classical Burgers equation, the entropy condition reduces to the requirement that if a discontinuity is propagating with speed $s' = \frac{ds}{dt}$ then $u^+ < u^-$. As its name suggests, this entropy condition is the mathematical translation of the condition that indicates that in every physical process the entropy of the system is nondecreasing. It is well-known this is a fundamental assumption in thermodynamics.

4. Viscid Burgers' equation

In this section, we consider the following initial-value problem

$$\begin{cases} u_t + uu_x = \nu u_{xx}, & x \in \mathbb{R},\ t > 0,\ \nu > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{12}$$

This problem ban be solved using the Cole-Hopf transformation. This transformation defined by $u = -2\nu \frac{\phi_x}{\phi}$, named after Eberhard Hopf, [7], and Julian D. Cole, [2], is to transform the strongly nonlinear Burgers Eq. (12) to the linear initial-value problem for the heat equation

$$\begin{cases} \phi_t = \nu \phi_{xx}, & x \in \mathbb{R},\ t > 0,\ \nu > 0, \\ \phi(x,0) = \phi_0(x) = e^{-\int_0^x \frac{u_0(y)}{2\nu} dy}, & x \in \mathbb{R}. \end{cases} \tag{13}$$

The unique solution of (13) is $\phi(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} \phi_0(y) e^{-\frac{(x-y)^2}{4\nu t}} dy$. Consequently, we obtain the analytic solution for the problem (12)

$$u(x,t) = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{x-y}{t} \phi_0(y) e^{-\frac{(x-y)^2}{4\nu t}} dy}{\displaystyle\int_{-\infty}^{\infty} \phi_0(y) e^{-\frac{(x-y)^2}{4\nu t}} dy}.$$

5. Numerical methods for the inviscid Burgers equation

Consider the one-dimensional hyperbolic partial differential equation for $u(x,t)$ of the form (3) or (4) on the domain $x \in [a,b]$, $t \in [0,T]$ with initial condition $u(x,0) = u_0(x)$ and the boundary conditions $u(a,t) = u_1(t)$, $u(b,t) = u_2(t)$. For Burgers equation we have $f(u) = \frac{u^2}{2}$. We discretize the domain $(a,b) \times (0,T)$ to a grid with equally spaced points with a spacing of h in the x-direction and k in the t-direction, we define $u_i^n = u(x_i, t_n)$ with $x_i = a + ih$, $t_n = nk$ for $i = 0, 1, \dots, N$ and $n = 0, 1, \dots, M$, where $N = \frac{b-a}{h}$ and $M = \frac{T}{k}$ are integers representing the number of grid intervals.

Next, we define the notion of conservative numerical methods.

Definition 5.1 A numerical scheme is called conservative if it has the form

$$u_i^{n+1} = u_i^n - \frac{k}{h}\left(\hat{f}(u_i^n, u_{i+1}^n) - \hat{f}(u_{i-1}^n, u_i^n)\right), \tag{14}$$

where $\hat{f}$ is Lipschitz continuous and consistent *i.e.*, $\hat{f}(u,u) = f(u)$.

Consistency is a property required for a numerical method. A numerical method is said to be consistent if the local errors introduced at each step tend to zero when the discretization steps tend to zero. A numerical method is said to be convergent when the method is stable and consistent, which means that the discrete solution tends to the exact continuous one when the time step tends to zero. However, consistency is a necessary but not sufficient condition for solution convergence.

For time and space dependent problems, the estimation of error in time and space is often independent and has the form $\left|u(x_i, t_n) - \hat{u}_i^n\right| \le C_1 h^p + C_2 k^q$, where C_1 and C_2

are constants independent of the step sizes h and k. The numbers p and q are called the orders of convergence in space and time, respectively. We say that the method is p-th order accurate in space and q-th order accurate in time.

For some time-dependent PDEs and numerical schemes, we have a constraint in the choice of h and k (called the CFL condition) in order to obtain a stable method.

Definition 5.2 Any numerical scheme of the form (14) is stable if the so-called Courant-Friedrichs-Levy (CFL) condition $\sup_{x\in[a,b],t\in[0,T]}|f'(u(x,t))|\frac{k}{h}\le 1$ is satisfied.

We note that the CFL condition is necessary but not sufficient in order to have a convergent numerical scheme.

5.1 Upwind nonconservative method

We consider the inviscid Burgers equation in the quasilinear form $u_t + uu_x = 0$. A natural finite difference method obtained by a forward discretization in time $u_t(x_i,t_n)\approx\frac{u_i^{n+1}-u_i^n}{k}$ and backward discretization in space $u_x(x_i,t_n)\approx\frac{u_i^n-u_{i-1}^n}{h}$ is

$$\frac{u_i^{n+1}-u_i^n}{k}+u_i^n\frac{u_i^n-u_{i-1}^n}{h}=0 \quad \text{or} \quad u_i^{n+1}=u_i^n-\frac{k}{h}u_i^n\left(u_i^n-u_{i-1}^n\right), \tag{15}$$

where the initial values and boundary nodes are taken from $u_i^0 = u_0(x_i)$, $u_0^n = u_1(t_n)$, and $u_N^n = u_2(t_n)$. This method is called the upwind nonconservative scheme. Although this method is consistent with (9) and is adequate for smooth solutions, it will not converge in general to a discontinuous weak solution as the grid is refine.

5.2 Upwind conservative method

Here, we consider the inviscid Burgers equation in conservation form $u_t + (f(u))_x = 0$. A finite difference method obtained by a forward in time $u_t(x_i,t_n)\approx\frac{u_i^{n+1}-u_i^n}{k}$ and backward in space $(f(u(x_i,t_n)))_x\approx\frac{f(u_i^n)-f(u_{i-1}^n)}{h}$ discretization of the derivatives is

$$\frac{u_i^{n+1}-u_i^n}{k}+\frac{f(u_i^n)-f(u_{i-1}^n)}{h}=0 \quad \text{or} \quad u_i^{n+1}=u_i^n-\frac{k}{h}\left(f(u_i^n)-f(u_{i-1}^n)\right), \tag{16}$$

where the initial values and boundary nodes are taken from $u_i^0 = u_0(x_i)$, $u_0^n = u_1(t^n)$, and $u_N^n = u_2(t^n)$. This method is called the upwind conservative scheme and is consistent and convergent.

5.3 Lax-Friedrichs method

The Lax-Friedrichs method for (3) takes the form

$$u_i^{n+1}=\frac{1}{2}\left(u_{i-1}^n+u_{i+1}^n\right)-\frac{k}{2h}\left(f(u_{i+1}^n)-f(u_{i-1}^n)\right), \tag{17}$$

where the initial values and boundary nodes are taken from $u_i^0 = u_0(x_i)$, $u_0^n = u_1(t_n)$, and $u_N^n = u_2(t_n)$. This scheme can be derived by using a finite difference method in time $u_t(x_i, t_n) \approx \frac{u_{i+1}^{n+1} - \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right)}{k}$ and a centred difference method in space $(f(u(x_i, t_n)))_x \approx \frac{f\left(u_{i+1}^n\right) - f\left(u_{i-1}^n\right)}{2h}$. Under the CFL condition, this method is explicit and first-order accurate (first order accurate in time and first order accurate in space). In addition, this method is conservative and hence quite dissipative. We note that this method can be written in conservation form:

$$u_i^{n+1} = u_i^n - \frac{k}{h}\left(\hat{f}_{i+1/2}^n - \hat{f}_{i-1/2}^n\right),$$

where $\hat{f}_{i-1/2}^n = \frac{1}{2}\left(f\left(u_{i-1}^n\right) + f\left(u_i^n\right)\right) - \frac{h}{2k}\left(u_i^n - u_{i-1}^n\right)$. Without the extra terms u_i^n and u_{i-1}^n in the discrete flux, $\hat{f}_{i-1/2}^n$, one ends up with the following Forward Time Centered Space (FTFS) scheme, which is well-known to be unconditionally unstable for hyperbolic problems

$$u_i^{n+1} = u_i^n - \frac{k}{2h}\left(f\left(u_{i+1}^n\right) - f\left(u_{i-1}^n\right)\right). \tag{18}$$

5.4 Godunov method

Godunov's scheme is a conservative numerical scheme, proposed by S. K. Godunov in 1959 [8], for solving PDEs

$$u_i^{n+1} = u_i^n - \frac{k}{h}\left(F\left(u_i^n, u_{i+1}^n\right) - F\left(u_{i-1}^n, u_i^n\right)\right), \tag{19}$$

where the numerical flux F is defined by

$$F(u_l, u_r) = \begin{cases} \min\limits_{u_l \le u \le u_r} f(u) & \text{if } u_l \le u_r, \\ \max\limits_{u_r \le u \le u_l} f(u) & \text{if } u_l > u_r. \end{cases} \tag{20}$$

This method can be viewed as a conservative finite volume method, which solves exact, or approximate Riemann problems at each inter-cell boundary. We would like to mention that Godunov's method is first-order accurate scheme in both space and time. Furthermore, it can be used as a base scheme for developing higher-order methods. We skip the details to save space.

5.5 Lax-Wendroff method

The Lax-Wendroff method to nonlinear conservation laws takes the form

$$u_i^{n+1} = u_{i+1}^n - \frac{k}{2h}\left(f\left(u_{i+1}^n\right) - f\left(u_{i-1}^n\right)\right) + \frac{k^2}{2h^2}\left(A_{i+\frac{1}{2}}\left(f\left(u_{i+1}^n\right) - f\left(u_i^n\right)\right) - A_{i-\frac{1}{2}}\left(f\left(u_i^n\right) - f\left(u_{i-1}^n\right)\right)\right), \tag{21}$$

where $A_{i\pm\frac{1}{2}} = f'\left(\frac{u_i^n + u_{i\pm1}^n}{2}\right)$. The Lax-Wendroff method is second order accurate in space and time.

5.6 MacCormack method

MacCormack method is a second-order finite difference method. It is a widely used discretization scheme for the numerical solution of hyperbolic PDEs arizing in computational fluid dynamics. The method was introduced by Robert W. MacCormack in 1969 [9]. The MacCormack method is a variation of the Lax-Wendroff scheme (21). It proceeds in two steps: a predictor step which is followed by a corrector step

$$\overline{u}_i^n = u_i^n - \frac{k}{h}\left(f\left(u_{i+1}^n\right) - f\left(u_i^n\right)\right) \tag{22}$$

$$u_i^{n+1} = \frac{1}{2}\left(u_i^n + \overline{u}_i^n\right) - \frac{k}{2h}\left(f\left(\overline{u}_i^n\right) - f\left(\overline{u}_{i-1}^n\right)\right). \tag{23}$$

This method uses first forward difference method and then backward difference method to achieve second-order accuracy. We remark that, for linear equations, the MacCormack scheme is equivalent to the Lax-Wendroff method.

5.7 The DG method for hyperbolic conservation laws

Here, we present the discontinuous Galerkin (DG) method for hyperbolic conservation laws. DG methods are a class of finite element methods. They use discontinuous basis functions (usually chosen as piecewise polynomials). Unlike the basis functions for finite element methods, the basis functions for DG methods can be discontinuous. Therefore, these methods have the flexibility which is not shared by typical finite element methods including high order accuracy, geometric flexibility, suitability for $h-$ and $p-$ adaptivity, extremely local data structure, high parallel efficiency and a good theoretical foundation for stability and error estimates. These methods have found their way into the main stream of computational fluid dynamics and other areas of applications. The first DG method was introduced in 1973 by Reed and Hill [10] for neutron transport equations, and since that time there has been an active development of DG methods for many PDEs. A major development of the DG method is the Runge-Kutta DG (RKDG) framework introduced for solving nonlinear hyperbolic conservation laws containing first-order spatial derivatives in a series of papers by Cockburn et al. [11–15]. The proceeding review article of Shu [16–18] contain a more complete and current survey of the DG methods and their applications.

Multiplying (3) by a test function v, integrating over an arbitrary element $I_i = (x_{i-1}, x_i)$, and using integration by parts, we obtain the week form

$$\int_{I_i} u_t v\, dx - \int_{I_i} f(u) v_x\, dx + f(u_i) v_i - f(u_{i-1}) v_{i-1} = 0. \tag{24}$$

where $v_i = v(x_i)$ and $u_i = u(x_i, t)$. We introduce the following discontinuous finite element approximation space $V_h^p = \{v : v\big|_{I_i} \in P^p(I_i),\ \ i = 1, 2, \dots, N\}$. Here, $P^p(I_i)$ is the space of polynomials of degree at most p on I_i. Let u_i^+ and u_i^-, respectively, denote the values of u at $x = x_i$ from the right cell I_{i+1} and from the left cell I_i.

Next, we replace the exact solution $u(\cdot, t)$ by $u_h(\cdot, t) \in V_h^p$. We also replace the boundary terms $f(u_{i-1})$ and $f(u_i)$ by single-valued numerical fluxes $\hat{f}_{i-1} = \hat{f}\left(u_{i-1}^-, u_{i-1}^+\right)$

and $\hat{f}_i = \hat{f}(u_i^-, u_i^+)$, respectively. These numerical fluxes in general depend both on the left limit and on the right limit. The idea is to treat these terms by an upwinding mechanism (information from characteristics), borrowed from successful high resolution finite volume schemes. For the Eq. (3), the flux $\hat{f}_i$ is taken as a monotone numerical flux, *i.e.*, it is Lipschitz continuous in both arguments u^- and u^+, consistent *i.e.*, $\hat{f}(u,u) = f(u)$, non-decreasing in the first argument and non-increasing in the second argument. Some examples of monotone fluxes which are suitable for DG methods can be found in Ref. [13]. For instance, we could use the simple Lax-Friedrichs flux

$$\hat{f}(u^-, u^+) = \frac{1}{2}\left(f(u^-) + f(u^+) - \alpha(u^+ - u^-)\right), \quad \text{where } \alpha = \max|f'(u)|.$$

Here the maximum is taken over a relevant range of u. Finally, we replace the test function v at the boundaries x_i and x_{i-1} by v_i^- and v_{i-1}^+. The resulting DG scheme consists of finding $u_h \in V_h^p$ such that $\forall\ i = 1, 2, \dots, N$,

$$\int_{I_i} (u_h)_t v\, dx - \int_{I_i} f(u_h) v_x\, dx + \hat{f}_i v_i^- - \hat{f}_{i-1} v_{i-1}^+ = 0. \tag{25}$$

The resulting system of ODEs (26), (27), (28) is then discretized by the nonlinearly stable high order Runge-Kutta time discretizations [19]. The most popular scheme in this classis the third-order Runge-Kutta method for solving $\mathbf{u}_t = \mathbf{F}(t, \mathbf{u})$

$$\mathbf{u}^{(1)} = \mathbf{u}^n + k\mathbf{F}(t_n, \mathbf{u}^n), \tag{26}$$

$$\mathbf{u}^{(2)} = \frac{3}{4}\mathbf{u}^n + \frac{1}{4}\mathbf{u}^{(1)} + \frac{k}{4}\mathbf{F}\left(t_n + k, \mathbf{u}^{(1)}\right), \tag{27}$$

$$\mathbf{u}^{n+1} = \frac{1}{3}\mathbf{u}^n + \frac{2}{3}\mathbf{u}^{(2)} + \frac{2k}{3}\mathbf{F}\left(t_n + \frac{k}{2}, \mathbf{u}^{(2)}\right), \tag{28}$$

To complete the definition of the DG scheme, we still need to define the discrete initial condition $u_h(x, 0) \in V_h^p$. One could use a special projection of the exact initial condition $u_0(x)$ as

$$u_h(x, 0) = P_h u_0(x), \quad x \in I_i, \quad i = 1, 2, \dots, N, \tag{29}$$

where P_h is the standard L^2-projection.

5.8 A numerical example

Consider the one-dimensional Burgers Eq. (3) on the domain $x \in [-2, 5]$, $t \in [0, 10]$ with initial condition $u(x, 0) = e^{-(x-1)^2}$ and the boundary conditions $u(-2, t) = u(5, t) = 0$. We use $N = 20$ and $k = 0.01$ and display the numerical solutions using the Upwind non conservative, Upwind conservative, Lax-Friedrichs, and Godunov schemes in **Figure 2**.

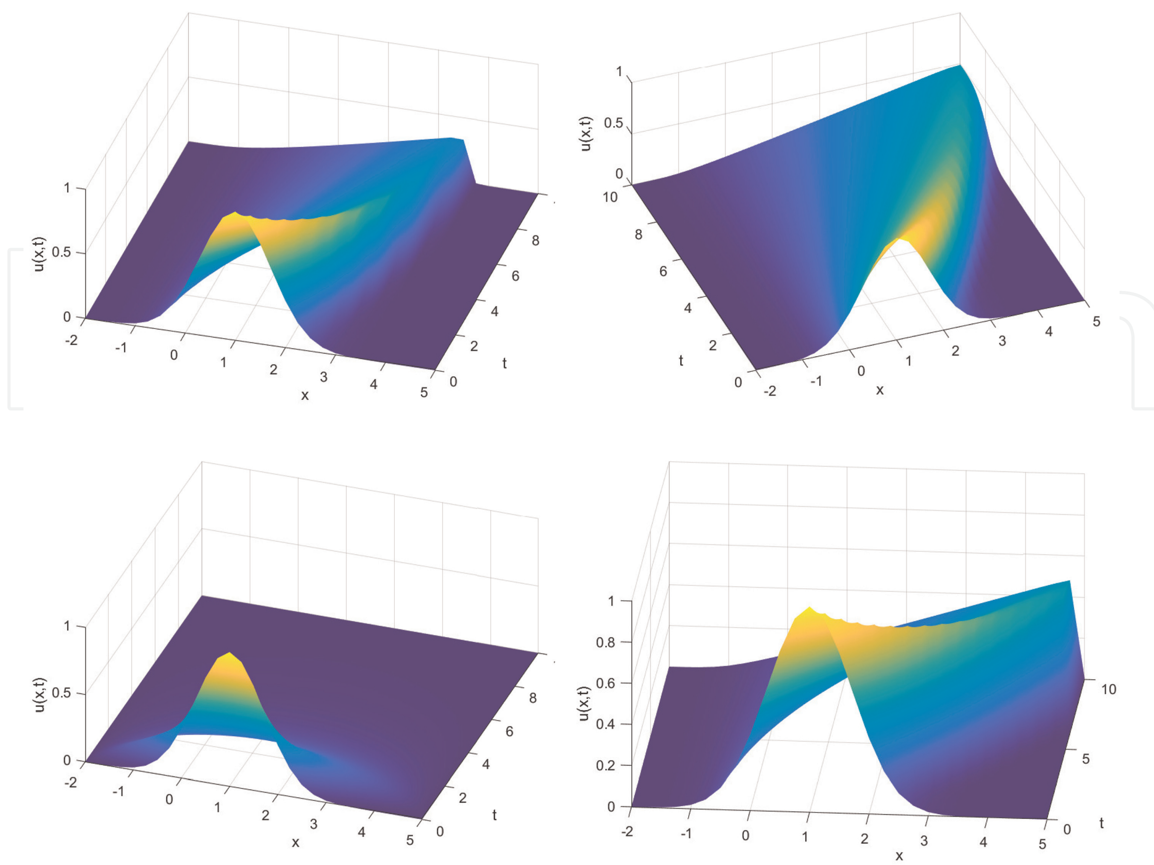

Figure 2.
Upwind non conservative, Upwind conservative, Lax-Friedrichs, and Godunov respectively.

6. Numerical methods for the viscid Burgers equation

We consider the following problem with $f(u) = u^2/2$.

$$\begin{cases} u_t + (f(u))_x = \nu u_{xx}, \quad x \in [a,b], \quad t \in [0,T], \ \ \nu > 0, \\ u(x,0) = u_0(x), \quad x \in [a,b], \\ u(a,t) = u_a(t), \quad u(c,t) = u_b(t), \quad t \in [0,T]. \end{cases} \tag{30}$$

6.1 A finite difference method

We approximate $u_t(x_i, t_n)$ with the forward in time $u_t(x_i, t_n) \approx \frac{u_i^{n+1} - u_i^n}{k}$, we approximate $(f(u(x_i, t_n)))_x$ with the central difference method $(f(u(x_i, t_n)))_x \approx \frac{f(u_{i+1}^n) - f(u_{i-1}^n)}{2h}$, and we approximate $u_{xx}(x_i, t_n)$ with the finite difference method $u_{xx}(x_i, t_n) \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}$. Thus, we get the explicit method

$$u_i^{n+1} = u_i^n + k\left(\nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} + \frac{f(u_{i+1}^n) - f(u_{i-1}^n)}{2h}\right), \tag{31}$$

where the initial values and boundary nodes are taken from $u_i^0 = u_0(x_i)$, $u_0^n = u_1(t_n)$, and $u_N^n = u_2(t_n)$.

6.2 The Chebyshev collocation method

Spectral methods are one of the most powerful solution techniques for ordinary and partial differential equations [20]. Here, we present an efficient and accurate numerical method for solving (30). The method is based on the Chebyshev collocation technique in space and the fourth-order Runge-Kutta in time. This proposed scheme is robust, fast, flexible, and easy to implement using modern mathematical software such as MATLAB.

We first consider the viscid Burgers' equation on the reference interval $[-1, 1]$

$$u_t + uu_\xi = \nu u_{\xi\xi}, \quad \xi \in [-1, 1], \quad t > 0, \tag{32}$$

$$u(\xi, 0) = u_o(\xi), \quad u(-1, t) = u_a(t), \quad u(1, t) = u_a(t). \tag{33}$$

A numerical solution of the problem (32) and (33) using the Chebyshev pseudospectral method consists of the following steps:

1. Let $N \geq 1$ be any integer. We introduce the Chebyshev extreme points (also called the Gauss-Lobatto-Chebyshev points)

$$\xi_i = \cos\left(\frac{i\pi}{N}\right), \ i = 0, 1, \dots, N. \tag{34}$$

These collocation points are the extreme points in $[-1, 1]$ of the Chebyshev polynomial $T_N(\xi) = \cos(N\cos^{-1}(\xi)), \ \xi \in [-1, 1]$. For simplicity we will call them the Chebyshev points.

1. Let $P^N(-1, 1)$ be the set of polynomials of degree $\leq N$ on the interval $[-1, 1]$. We choose $L_i(\xi), \ i = 0, \dots, N$, as a basis of $P^N(-1, 1)$, where $L_i(\xi) = \prod_{j=0, j\neq i}^{N} \frac{\xi - \xi_j}{\xi_i - \xi_j}$ is the polynomial of degree N satisfying $L_i\left(\xi_j\right) = \delta_{ij}$. Here δ_{ij} is the Kronecker symbol.

2. We assume that the numerical solution at fixed time t is a linear combination of the basis functions $\{L_i\}_{i=0}^{i=N}$. In other words, we approximate the exact solution $u(\xi, t)$ by $u_N(\xi, t) = \sum_{i=0}^{N} c_i(t) L_i(\xi)$ and require that u_N satisfies the PDE at the collocation points ξ_i, *i.e.*,

$$\frac{\partial u_N(\xi_i, t)}{\partial t} + u_N(\xi_i, t)\frac{\partial u_N(\xi_i, t)}{\partial \xi} = \nu \frac{\partial^2 u_N(\xi_i, t)}{\partial \xi^2}, \quad i = 1, \dots, N-1,$$

and $u_N(\xi_i, 0) = u_0(\xi_i)$, $u_N(-1, t) = u_a(t)$, $u_N(1, t) = u_b(t)$. This yields a system of ODEs for the coefficients $c_i(t)$.

1. We use an ODE solver to approximate the unknown coefficients $c_i(t)$.

A simple way to implement the Chebyshev pseudospectral method is via the so-called differentiation matrices. It turns out that when the polynomials $L_i(\xi)$ are used as basis functions and the collocation points are chosen to be the Chebyshev points, the entries of the differentiation matrix are explicitly known.

Given the approximate values $u_N(\xi_i, t),\ \ i = 0, \dots, N$ to $u(\xi_i, t)$ at a fixed time t. Let $u_N(\xi, t) = \sum_{j=0}^{N} u_N\left(\xi_j, t\right) L_j(\xi)$. Then, we obtain highly accurate approximations to $u_\xi(\xi_i, t)$ and $u_{\xi\xi}(\xi_i, t)$ at fixed time t using [21].

$$\frac{\partial u_N(\xi_i, t)}{\partial \xi} = \sum_{j=0}^{N} u_N\left(\xi_j, t\right) L_j'(\xi_i) = \sum_{j=0}^{N} d_{ij} u_N\left(\xi_j, t\right), \quad \text{where } \ d_{ij} = L_j'(\xi_i), \tag{35}$$

$$\frac{\partial^2 u_N(\xi_i, t)}{\partial^2 \xi} = \sum_{j=0}^{N} u_N\left(\xi_j, t\right) L_j''(\xi_i) = \sum_{j=0}^{N} b_{ij} u_N\left(\xi_j, t\right), \quad \text{where } \ b_{ij} = L_j''(\xi_i). \tag{36}$$

Letting $\mathbf{u} = [u_N(\xi_0, t), \dots, u_N(\xi_N, t)]^t$. It turns out that a highly accurate approximation to $\mathbf{u}_\xi = \left[\frac{\partial u_N(\xi_0, t)}{\partial \xi}, \dots, \frac{\partial u_N(\xi_N, t)}{\partial \xi}\right]^t$ is given by $\mathbf{u}_\xi = D_N \mathbf{u}$, where D_N is the first-order spectral differentiation matrix $D_N = \left(d_{ij}\right)_{0 \le i,j \le N}$. The entries of D_N are given by [28, Theorem 7]

$$\begin{aligned} d_{00} &= -d_{NN} = \frac{2N^2 + 1}{6}, \\ d_{ii} &= \frac{\xi_i}{2(\xi_i^2 - 1)}, \ \ i = 1, \dots, N-1, \\ d_{ij} &= \frac{c_i (-1)^{i+j}}{c_j\left(\xi_i - \xi_j\right)}, \ \ i \neq j, \ \ i,j = 0, \dots, N, \end{aligned}$$

where $c_0 = c_N = 2$ and $c_i = 1, \ \ i = 1, \dots, N-1$.

The construction of the differentiation matrix D_N does not explicitly include the boundary conditions. In order to satisfy the boundary conditions, we must manually set $u_N(\xi_0, t) = u_N(1, t) = u_b(t)$ and $u_N(\xi_N, t) = u_N(-1, t) = u_a(t)$.

To calculate higher derivatives, simply raise the matrix to the appropriate power. For instance, to approximate the m-th derivative, simply raise the matrix D_N to the mth power. In particular, a highly accurate approximation to $\mathbf{u}_{\xi\xi} = \left[\frac{\partial^2 u_N(\xi_0, t)}{\partial^2 \xi}, \dots, \frac{\partial^2 u_N(\xi_N, t)}{\partial^2 \xi}\right]^t$ is given by $\mathbf{u}_{\xi\xi} = D_N^2 \mathbf{u}$. Now, let us discretize problem (32) and (33) using the Chebyshev pseudospectral collocation method. If we denote by $u_i(t) = u_N(\xi_i, t)$ the approximation to $u(\xi_i, t)$, then the semi-discrete Chebyshev method is given by the following system of nonlinear ODEs

$$\frac{du_i(t)}{dt} = \sum_{j=0}^{N} \left(\nu b_{ij} - u_i(t) d_{ij}\right) u_j(t), \quad u_i(0) = f(\xi_i), \quad i = 1, \dots, N-1, \tag{37}$$

where $u_0(t) = u_b(t)$, and $u_N(t) = u_b(t)$. The system (37) can be written as

$$\frac{d\mathbf{u}(t)}{dt} = \mathbf{F}(t, \mathbf{u}), \quad \mathbf{u}(0) = \mathbf{u}_0, \tag{38}$$

where $\mathbf{u}(t) = [u_1, \ldots, u_{N-1}]^t$, $\mathbf{u}_0 = [u_0(\xi_1), \ldots, u_0(\xi_{N-1})]^t$, $\mathbf{F}(t, \mathbf{u}) = [F_1, \ldots, F_{N-1}]^t$ with

$$F_i = \sum_{j=0}^{N} \left(\nu b_{ij} - u_i(t) d_{ij}\right) u_j(t), \quad i = 1, \ldots, N-1. \tag{39}$$

This system of ODEs (38) (39) can be approximated using time-stepping schemes for ODEs. Here, we employ the classical Runge-Kutta (RK4) method of fourth-order to discretize the system as follows: First, we use $h = T/M$ to denote the time step and we set $t_n = nh, \ n = 0, \ldots, M$. Then the RK4 method consists of finding approximations $\mathbf{u}^{(n)}$ to $\mathbf{u}(t_n)$ using the following algorithm: Given $\mathbf{u}^{(0)} = \mathbf{u}_0$. For $n = 0,1,2, \ldots, M-1$, we compute

$$\begin{aligned} &\mathbf{K}_1 = \mathbf{F}\left(t_n, \mathbf{u}^{(n)}\right), \qquad \mathbf{K}_2 = \mathbf{F}\left(t_n + \frac{h}{2}, \mathbf{u}^{(n)} + \frac{h}{2}\mathbf{K}_1\right), \\ &\mathbf{K}_3 = \mathbf{F}\left(t_n + \frac{h}{2}, \mathbf{u}^{(n)} + \frac{h}{2}\mathbf{K}_2\right), \qquad \mathbf{K}_4 = \mathbf{F}\left(t_n + h, \mathbf{u}^{(n)} + h\mathbf{K}_3\right), \\ &\mathbf{u}^{(n+1)} = \mathbf{u}^{(n)} + \frac{h}{6}(\mathbf{K}_1 + 2\mathbf{K}_2 + 2\mathbf{K}_3 + \mathbf{K}_4). \end{aligned} \tag{40}$$

In our numerical examples, the time step Δt is chosen so that temporal errors are small relative to spatial errors. After obtaining the approximations $u_N(\xi_i, t_n), \ i = 0, \ldots, N, \ n = 0, \ldots, M$ to $u(\xi_i, t_n)$, the approximation $u_N(\xi, t_n), \ \xi \in [-1, 1]$ at fixed time t_n can be found using interpolation as $u_N(\xi, t_n) = \sum_{i=0}^{N} u_N(\xi_i, t_n) L_i(\xi), \quad n = 0, 1, \ldots, M$. Next, we extend the above method to approximate the Burgers' Eq. (30) on the arbitrary interval $[a, b]$. We note that the shifted Chebyshev points in the interval $[a, b]$ are defined by

$$x_i = \frac{b-a}{2}\xi_i + \frac{a+b}{2} = \frac{b-a}{2}\cos\left(\frac{i\pi}{N}\right) + \frac{a+b}{2}, \quad i = 0, 1, \ldots, N. \tag{41}$$

Next, we illustrate how to approximate the solution $u(x, t)$ to the one-dimensional viscid Burgers' Eq. (30) at the locations $x_i, \ i = 1, 2, \ldots, N-1$ and at a fixed time t. Suppose that $u_N(x_i, t)$ approximates $u(x_i, t)$. Then $u_x(x_i, t)$ and $u_{xx}(x_i, t)$ can be approximated by

$$\frac{\partial u_N(x_i, t)}{\partial x} = \frac{2}{b-a}\sum_{j=0}^{N} d_{ij} u_N(x_i, t), \qquad \frac{\partial^2 u_N(x_i, t)}{\partial^2 x} = \frac{4}{(b-a)^2}\sum_{j=0}^{N} b_{ij} u_N(x_i, t), \tag{42}$$

for $i = 0, \ldots, N$. Evaluating (30) at $x = x_i$, approximating $u(x_i, t)$ by $u_i(t) = u_N(x_i, t)$, and using (42), we obtain the following system of nonlinear ODEs

$$\frac{du_i(t)}{dt} + \frac{2u_i(t)}{b-a}\sum_{j=0}^{N} d_{ij} u_j(t) = \frac{4\nu}{(b-a)^2}\sum_{j=0}^{N} b_{ij} u_j(t), \quad u_i(0) = f(x_i), \quad i = 1, \ldots, N-1, \tag{43}$$

where $u_N(t) = u_a(t)$ and $u_0(t) = u_b(t)$. As before, d_{ij} and b_{ij} are, respectively, the entries of the differentiation matrices D_N and D_N^2.

The system (43) can be written as

$$\frac{d\mathbf{u}(t)}{dt} = \mathbf{F}(t, \mathbf{u}), \quad \mathbf{u}(0) = \mathbf{u}_0, \tag{44}$$

where $\mathbf{u}(t) = [u_1, \ldots, u_{N-1}]^t$, $\mathbf{u}_0 = [f(x_1), \ldots, f(x_{N-1})]^t$, $\mathbf{F}(t, \mathbf{u}) = [F_1, \ldots, F_{N-1}]^t$ with

$$F_i = \frac{2}{b-a} \sum_{j=0}^{N} \left(\frac{2\nu b_{ij}}{b-a} - d_{ij} u_i(t) \right) u_j(t), \quad i = 1, \ldots, N-1. \tag{45}$$

We would like to mention that the Chebyshev collocation method reduces any PDE to a system of ODEs, which can be approximated using RK4 (40). The proposed numerical schemes produce accurate results and are quite easy to implement.

Remark 1 The classical RK4 method to solve systems of ODEs is not the most efficient method for the temporal integration of the proposed Chebyshev collocation scheme. There are other schemes which are of higher order or are of low storage available in the literature (see for example [17] and the references therein). In practice, the time step Δt should be chosen so that temporal errors are small relative to spatial errors.

6.3 An ultra-weak DG method

For equations with higher-order spatial derivatives, such as (30), the DG method cannot be directly applied due to the discontinuous finite element space which is not regular enough to handle higher order derivatives. However, there are different types of DG methods proposed and analyzed in the literature. The most known schemes in the literature include the symmetric interior penalty DG (SIPG) methods [22, 23], non-symmetric interior penalty DG (NIPG) methods [24], direct DG (DDG) methods [25], local DG (LDG) methods [26], and ultra-weak DG (UWDG) methods [27].

Next, we apply the UWDG to solve (30). UWDG methods are based on repeated integration by parts so that all spatial derivatives are shifted from the exact solution to the test function in the weak formulation.

Multiplying the equation in (30) by a test function v, integrating over I_i, and using integration by parts twice, we get the UWDG weak formulation

$$\begin{aligned} &\int_{I_i} \left(u_t v - v' f(u) - \nu u \;\; v'' \right) dx \\ &+ \left(f(u_i) - \nu u_i' \right) v_i - \left(f(u_{i-1}) - \nu u_{i-1}' \right) v_{i-1} + \nu u_i v_i' - \nu u_{i-1} v_{i-1}' = 0. \end{aligned} \tag{46}$$

Next, we use the weak formulation (46) to define the UWDG scheme. Find $u_h \in V_h^p = \{ v : v|_{I_i} \in \mathbb{P}^p(I_i), \;\; i = 1, 2, \ldots, N \}$ such that for all test function $v \in V_h^p$,

$$\begin{aligned} &\int_{I_i} \left((u_h)_t v - v' f(u_h) - \nu u_h \;\; v'' \right) dx \\ &+ \left(\hat{f}_i - \nu \tilde{u}_{h,i}' \right) v_i^- - \left(f(u_{i-1}) - \nu \tilde{u}_{h,i-1}' \right) v_{i-1}^+ + \nu \hat{u}_{h,i} \left(v_i^- \right)' - \nu \hat{u}_{h,i-1} \left(v_{i-1}^+ \right)' = 0, \end{aligned} \tag{47}$$

for all $i = 1, 2, \ldots, N$. Next, we must choose the numerical fluxes, which should be designed based on different guiding principles for different equations to guarantee stability and convergence. Here, we take the numerical fluxes: The numerical flux $\hat{f}$ associated with the convection is taken as the upwind flux (Godunov flux): for $i = 0, 1, \dots, N$,

$$\hat{f}\left(u_h^-, u_h^+\right) = \begin{cases} \min\limits_{u_h^- \le u \le u_h^+} f(u), & \text{if } u_h^- < u_h^+, \\ \max\limits_{u_h^+ \le u \le u_h^-} f(u), & \text{if } u_h^- \ge u_h^+. \end{cases} \tag{48}$$

The numerical fluxes $\hat{u}_h$ and $\tilde{u}_h'$ associated with the diffusion terms are taken as the alternating fluxes

$$\hat{u}_{h,i} = u_{h,i}^-, \quad \tilde{u}_{h,i}' = \left(u_{h,i}^+\right)', \quad i = 0, 1, \dots, N. \tag{49}$$

It is crucial that we take $\hat{u}_h$ and $\tilde{u}_h'$ from the opposite directions. In practice, we could use the backward Euler time discretization to solve the resulting system of ODEs. However, in order for the time error not to dominate, we still use a small time step $\Delta t = \mathcal{O}\left(h^{p+1}\right)$ for the V_h^p space.

6.4 A local DG method

The local DG (LDG) method is an extension of the DG method aimed at solving differential equations containing higher than first-order spatial derivatives. The first LDG method was introduced by Cockburn and Shu in [26] for solving convection-diffusion problems. LDG methods are extremely flexible in the mesh-design and can easily handle irregular meshes and elements of various types. They further can achieve stability without slope limiters, and are locally (element-wise) mass-conservative. The latter property is very useful in the area of computational fluid dynamics, especially in situations where there are shocks, steep gradients or boundary layers. In addition, LDG methods exhibit strong superconvergence that can be used to estimate the discretization errors. LDG schemes have been successfully applied to elliptic, parabolic, and hyperbolic PDEs [28–30].

In order to construct the LDG scheme for (30), we introduce an auxiliary variable $q = u_x$ and convert the equation in (30) into the following first-order system

$$u_t + (f(u))_x - \nu q_x = 0, \qquad q - u_x = 0. \tag{50}$$

Multiplying the equations in (50) by v and w, integrating over I_i, and using integration by parts, we obtain

$$\begin{aligned} &\int_{I_i} u_t v\,dx + \int_{I_i} (\nu q - f(u)) v_x\,dx + \left(f(u_i) - \nu q_i\right) v_i - \left(f(u_{i-1}) - \nu q_{i-1}\right) v_{i-1} = 0, \\ &\int_{I_i} q w\,dx + \int_{I_i} u w_x\,dx - u_i w_i + u_{i-1} w_{i-1} = 0. \end{aligned}$$

Define the discontinuous finite element space $V_h^p = \{v : v\big|_{I_i} \in P^p(I_i),\ \ i = 1, \dots, N\}$, where $P^p(I_i)$ is the space of polynomials of degree at most p on I_i. Then the semi-discrete LDG method is: Find $u_h \in V_h^p$ and $q_h \in V_h^p$ such that

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$$\int_{I_i} (u_h)_t v\, dx + \int_{I_i} (\nu q_h - f(u_h)) v_x\, dx + \left(\hat{f}_i - \nu \hat{q}_{h,i}\right) v_i^- - \left(\hat{f}_{i-1} - \nu \hat{q}_{h,i-1}\right) v_{i-1}^+ = 0, \tag{51}$$

$$\int_{I_i} q_h w\, dx + \int_{I_i} u_h w_x\, dx - \hat{u}_{h,i} w_i^- + \hat{u}_{h,i-1} w_{i-1}^+ = 0. \tag{52}$$

The numerical flux $\hat{f}$ associated with the convection is taken as the upwind flux (Godunov flux): for $i = 0, 1, \ldots, N$,

$$\hat{f}(u_h^-, u_h^+) = \begin{cases} \min\limits_{u_h^- \le u \le u_h^+} f(u), & \text{if } u_h^- < u_h^+, \\ \max\limits_{u_h^+ \le u \le u_h^-} f(u), & \text{if } u_h^- \ge u_h^+. \end{cases} \tag{53}$$

The numerical fluxes $\hat{u}_h$ and $\hat{q}_h$ associated with the diffusion terms are taken as the alternating fluxes

$$\hat{u}_{h,i} = u_{h,i}^-, \qquad \hat{q}_{h,i} = q_{h,i}^+, \qquad i = 0, 1, \ldots, N. \tag{54}$$

It is crucial that we take $\hat{u}_h$ and $\hat{q}_h$ from the opposite directions. The initial condition of the LDG scheme is

$$u_h(x, 0) = P_h u_0(x), \quad x \in I_i, \quad i = 1, 2, \ldots, N,$$

where $P_h u$ is L^2-projection. Finally, we use the fourth-order strong stability preserving (SSP) Runge-Kutta method for the discretization in time. A time step is chosen so that temporal errors are small relative to spatial errors.

6.5 A numerical example

Consider the one-dimensional viscid Burgers Eq. (30) on the domain $x \in [-2, 5]$, $t \in [0, 10]$ with initial condition $u(x, 0) = e^{-(x-1)^2}$ and the boundary conditions $u(-2, t) = u(5, t) = 0$. We use $N = 100$ and $k = 0.001$ and display the numerical solutions using the finite difference method in **Figure 3** with $\nu = 1$ and $\nu = 0.01$.

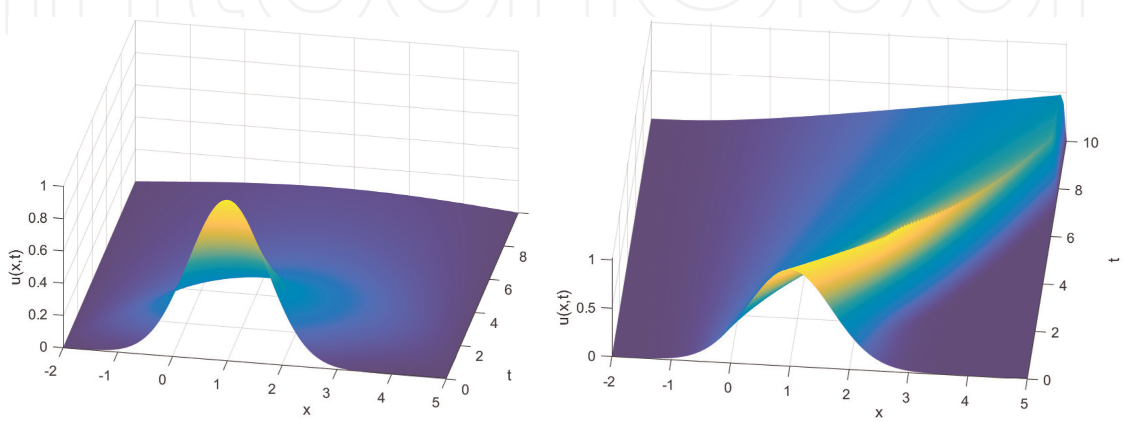

Figure 3.
The numerical solution of the viscid Burgers equation using the finite difference method (31) with $\nu = 1$ (left) and $\nu = 0.01$ (right).

Comparison and Challenges

- *Inviscid burgers equation*: The main challenge is dealing with shock formation and discontinuities. Numerical methods need to be robust enough to handle these features without introducing non-physical oscillations.

- *Viscid burgers equation*: The primary challenge is balancing accuracy and computational efficiency, especially for problems with small viscosity where solutions can exhibit sharp gradients.

The choice of numerical method for solving the Burgers equation depends on the specific form of the equation and the nature of the problem. For the inviscid form, methods that effectively capture discontinuities are crucial. For the viscid form, methods that provide a good balance between stability and accuracy are preferred. Understanding the characteristics of each method and the nature of the problem is essential for selecting the most appropriate numerical approach.

7. Numerical methods for the 2-dimensional Burgers equations

7.1 The 2-dimensional Burgers equations

The 2-dimensional Burgers equations extend the complexity of the 1-dimensional case by incorporating interactions in two spatial dimensions. These equations are used to model various physical phenomena such as fluid flow, turbulence, and gas dynamics. The equations can be written in both inviscid and viscid forms.

The 2-dimensional inviscid Burgers equations are given by:

$$\begin{aligned} u_t + uu_x + vu_y &= 0, \\ v_t + uv_x + vv_y &= 0. \end{aligned}$$

The 2-dimensional viscid Burgers equations include viscosity terms:

$$\begin{aligned} u_t + uu_x + vu_y &= \nu(u_{xx} + u_{yy}), \\ v_t + uv_x + vv_y &= \nu(v_{xx} + v_{yy}), \end{aligned}$$

where u and v are the velocity components in the x and y directions, respectively, and ν is the viscosity coefficient.

7.2 Numerical methods

Numerical methods for solving the 2-dimensional Burgers equations must account for the increased complexity due to interactions between the two spatial dimensions. Below are some commonly used methods:

Finite difference methods (FDM).

Explicit schemes

- *Upwind scheme*: Accounts for the direction of the wave propagation and helps handle discontinuities.

- *Lax-Friedrichs scheme*: Introduces numerical dissipation to stabilize the solution.

Implicit schemes

- *Crank–Nicolson scheme*: Provides second-order accuracy in both time and space and is unconditionally stable.

Finite volume methods (FVM).

Finite volume methods are suitable for conservation laws and ensure that fluxes across cell boundaries are accurately captured. These methods can handle complex geometries and boundary conditions more flexibly than finite difference methods.

- *Godunov's method*: Utilizes the exact or approximate Riemann solver to deal with discontinuities effectively.

- *MUSCL (monotonic upwind scheme for conservation laws)*: Enhances the accuracy of Godunov's method by using higher-order reconstruction techniques.

Finite element methods (FEM).

Finite element methods allow for flexible handling of complex geometries and boundary conditions. They are particularly useful for problems where adaptivity and local refinement are needed.

- *Standard galerkin method*: Suitable for problems with smooth solutions.

- *Discontinuous galerkin method*: Handles discontinuities and sharp gradients effectively, making it suitable for problems with shocks or strong nonlinearities.

Spectral methods.

Spectral methods use global basis functions such as Fourier series or Chebyshev polynomials to represent the solution. These methods are highly accurate for problems with smooth initial conditions and periodic domains.

- *Fourier spectral method*: Suitable for problems with periodic boundary conditions.

- *Chebyshev spectral method*: Suitable for problems defined on finite domains with non-periodic boundary conditions.

7.3 Handling boundary conditions

For both the inviscid and viscid Burgers equations, proper handling of boundary conditions is crucial. Common boundary conditions include:

- *Dirichlet boundary conditions*: Specify the values of the solution at the boundaries.

- *Neumann boundary conditions*: Specify the values of the derivative of the solution at the boundaries.

- *Periodic boundary conditions*: Assume the solution is periodic, simplifying the treatment of boundaries in spectral methods.

Challenges and considerations

- *Shock formation*: The inviscid Burgers equation can develop shocks, requiring robust numerical methods to handle discontinuities without introducing non-physical oscillations.

- *Stability and accuracy*: For the viscid Burgers equation, balancing stability and accuracy is crucial, especially for problems with small viscosity where solutions can exhibit sharp gradients.

- *Computational efficiency*: High-dimensional problems require efficient numerical methods and possibly parallel computing to handle large-scale computations effectively.

Numerical methods for solving the 2-dimensional Burgers equations must address the complexities arising from interactions in two spatial dimensions. Methods such as finite difference, finite volume, finite element, and spectral methods offer various advantages and challenges. Understanding the characteristics of each method and the specific nature of the problem is essential for selecting the most appropriate numerical approach. Proper handling of boundary conditions and addressing stability and accuracy are crucial for obtaining reliable solutions.

8. Numerical methods for the 3-dimensional Burgers equations

The 3-dimensional Burgers equations extend the complexity of the 2-dimensional case by incorporating interactions in three spatial dimensions. These equations are used to model various physical phenomena such as fluid flow, turbulence, and gas dynamics. The equations can be written in both inviscid and viscid forms.

8.1 The 3-dimensional Burgers equations

The 3-dimensional inviscid Burgers equations are given by:

$$\begin{aligned} u_t + uu_x + vu_y + wu_z &= 0, \\ v_t + uv_x + vv_y + wv_z &= 0, \\ w_t + uw_x + vw_y + ww_z &= 0. \end{aligned}$$

The 3-dimensional viscid Burgers equations include viscosity terms:

$$\begin{aligned} u_t + uu_x + vu_y + wu_z &= \nu\left(u_{xx} + u_{yy} + u_{zz}\right), \\ v_t + uv_x + vv_y + wv_z &= \nu\left(v_{xx} + v_{yy} + v_{zz}\right), \\ w_t + uw_x + vw_y + ww_z &= \nu\left(w_{xx} + w_{yy} + w_{zz}\right), \end{aligned}$$

where u, v, and w are the velocity components in the x, y, and z directions, respectively, and ν is the viscosity coefficient.

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8.2 Numerical methods

Numerical methods for solving the 3-dimensional Burgers equations must account for the increased complexity due to interactions between the three spatial dimensions. Here are some commonly used methods:

Finite difference methods (FDM).

Explicit schemes

- *Upwind scheme*: Accounts for the direction of the wave propagation and helps handle discontinuities.
- *Lax-Friedrichs scheme*: Introduces numerical dissipation to stabilize the solution.

Implicit schemes

- *Crank-Nicolson scheme*: Provides second-order accuracy in both time and space and is unconditionally stable.

Finite volume methods (FVM).

Finite volume methods are suitable for conservation laws and ensure that fluxes across cell boundaries are accurately captured. These methods can handle complex geometries and boundary conditions more flexibly than finite difference methods.

- *Godunov's method*: Utilizes the exact or approximate Riemann solver to deal with discontinuities effectively.
- *MUSCL (monotonic upwind scheme for conservation laws)*: Enhances the accuracy of Godunov's method by using higher-order reconstruction techniques.

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Spectral methods use global basis functions such as Fourier series or Chebyshev polynomials to represent the solution. These methods are highly accurate for problems with smooth initial conditions and periodic domains.

- *Fourier spectral method*: Suitable for problems with periodic boundary conditions.
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Challenges and considerations

- *Shock formation*: The inviscid Burgers equation can develop shocks, requiring robust numerical methods to handle discontinuities without introducing non-physical oscillations.
- *Stability and accuracy*: For the viscid Burgers equation, balancing stability and accuracy is crucial, especially for problems with small viscosity where solutions can exhibit sharp gradients.
- *Computational efficiency*: High-dimensional problems require efficient numerical methods and possibly parallel computing to handle large-scale computations effectively.

Numerical methods for solving the 3-dimensional Burgers equations must address the complexities arising from interactions in three spatial dimensions. Methods such as finite difference, finite volume, finite element, and spectral methods offer various.

9. Conclusions

In this chapter, we provided a very brief review of theoretical results for the viscid and inviscid Burgers equations. Then we presented several efficient numerical methods to approximate the solution to the viscid and inviscid Burgers equations. For the inviscid Burgers equation, we presented the classical finite difference and finite volume methods in details including Up-wind nonconservative, Up-wind conservative, Lax-Friedrichs, Lax-Wendroff, MacCormack, and Godunov methods. We also introduced the discontinuous Galerkin method. For the viscid Burgers equation, we present a finite difference method, a Chebyshev collocation method, an ultra-weak discontinuous Galerkin method, and a local discontinuous Galerkin method. We would like to mention that the procedures described above are very efficient and may be generalized without any problem to other nonlinear equations. In particular, most of these numerical methods are suitable for other two- and three-dimensional problems arising in fluid dynamics and other areas.

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