AM 129, Fall 2022 Final Project Type B – Numerical ODE: Fermi-Pasta-Ulam-Tsingou problem

1. Abstract:

In the summer of 1953, Enrico Fermi, John Pasta, Stanislaw Ulam, and Mary Tsingou conducted computer simulations of a vibrating string that included a non-linear term. They found that the system's behavior differed from what intuition would have led them to expect. Fermi thought that after many iterations, the system would exhibit thermalization, an ergodic behavior in which the influence of the initial modes of vibration fade, and the system becomes more or less random with all modes excited more or less equally. Instead, the system exhibited a very complicated quasi-periodic behavior. *("Fermi–Pasta–Ulam–Tsingou problem")*

In this project, we will work on the behavior of a system of point masses connected by springs, and how they move given an initial configuration. We will include a quadratic nonlinearity in the system

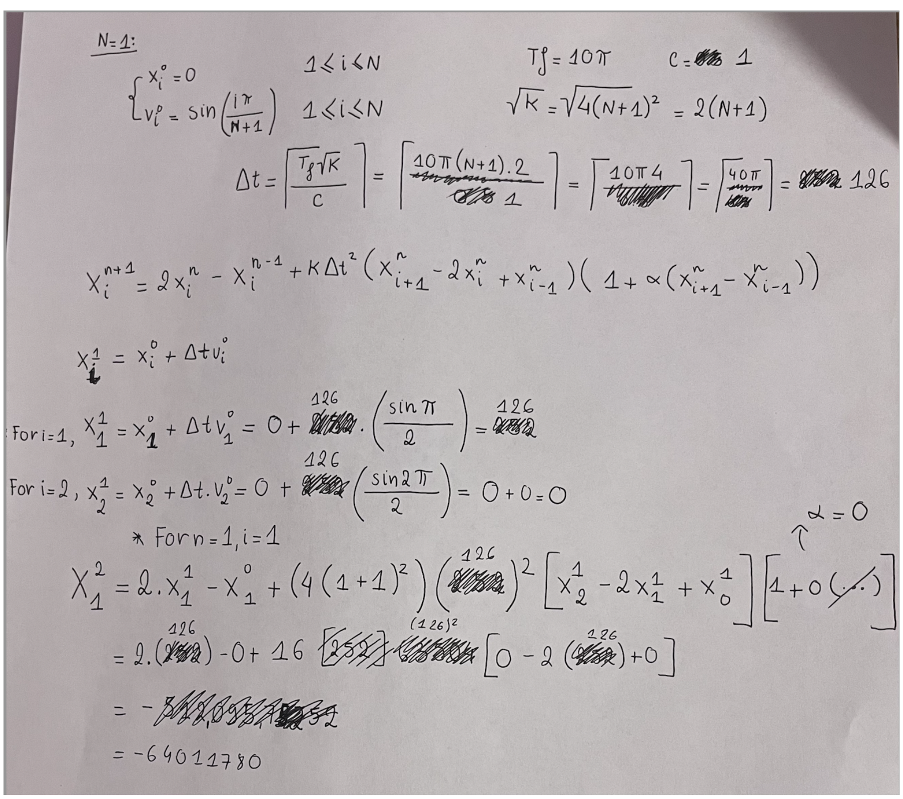
2. Body:

Methods:

Results:

Questions:

(a) Start with $N = 1$ in which case the governing equation reduces to a linear, second order, scalar equation. Find the exact solution for this equation and compare it to the solution produced by your code.



N=1:

$$\begin{cases} x_i^0 = 0 & 1 \leq i \leq N \\ v_i^0 = \sin\left(\frac{i\pi}{N+1}\right) & 1 \leq i \leq N \end{cases}$$

$T_f = 10\pi$ $\quad c = 1$

$\sqrt{K} = \sqrt{4(N+1)^2} = 2(N+1)$

$$\Delta t = \left\lceil \frac{T_f\sqrt{K}}{c} \right\rceil = \left\lceil \frac{10\pi(N+1).2}{1} \right\rceil = \left\lceil 10\pi 4 \right\rceil = \left\lceil 40\pi \right\rceil = 126$$

$$x_i^{n+1} = 2x_i^n - x_i^{n-1} + K\Delta t^2 (x_{i+1}^n - 2x_i^n + x_{i-1}^n)(1 + \alpha(x_{i+1}^n - x_{i-1}^n))$$

$x_i^1 = x_i^0 + \Delta t v_i^0$

For $i=1$, $x_1^1 = x_1^0 + \Delta t v_1^0 = 0 + 126 \cdot \left(\frac{\sin\pi}{2}\right) = 126$

For $i=2$, $x_2^1 = x_2^0 + \Delta t . v_2^0 = 0 + 126\left(\frac{\sin 2\pi}{2}\right) = 0 + 0 = 0$

* For $n=1, i=1$ $\quad \alpha = 0$

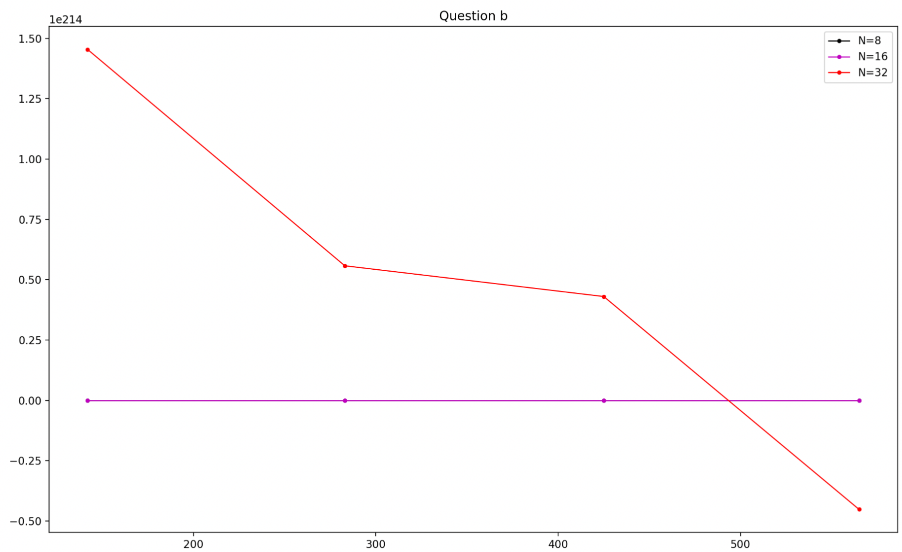
$$X_1^2 = 2.x_1^1 - x_1^0 + (4(1+1)^2)(126)^2\left[x_2^1 - 2x_1^1 + x_0^1\right]\left[1 + 0(\cancel{\dots})\right]$$

$= 2.(126) - 0 + 16\ (126)^2\ [0 - 2(126) + 0]$

$= -64011780$

The exact solution is different compare to the solution produced by the code

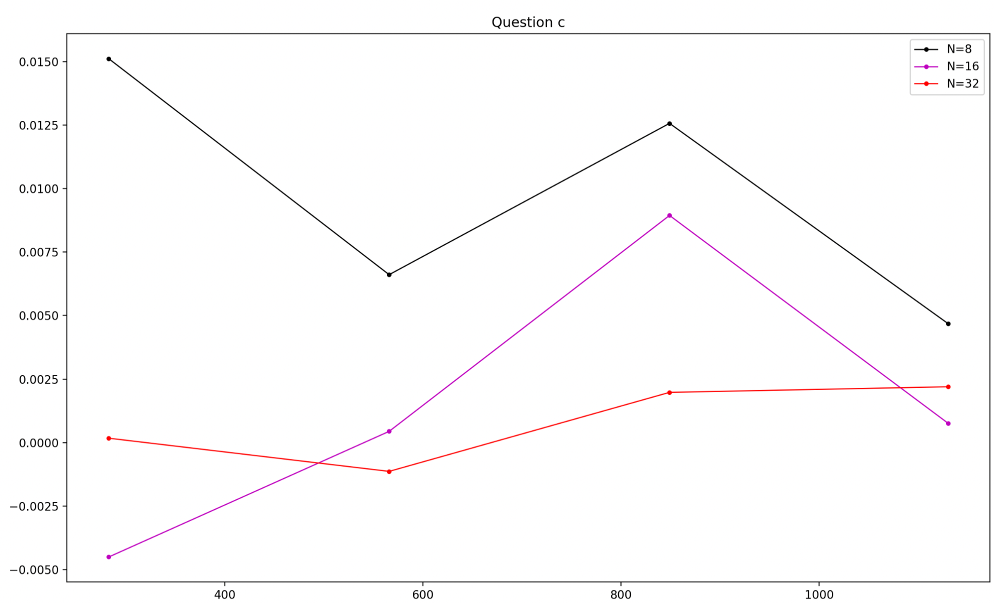
(b) Run your code for problems of size $N = \{8,\ 16,\ 32\}$. Save the solution for the the mass $i = N/2$ through all time steps, and save the solution for all masses at times of $t = \frac{T_f}{4}, \frac{T_f}{2}, \frac{3T_f}{4}, T_f$.



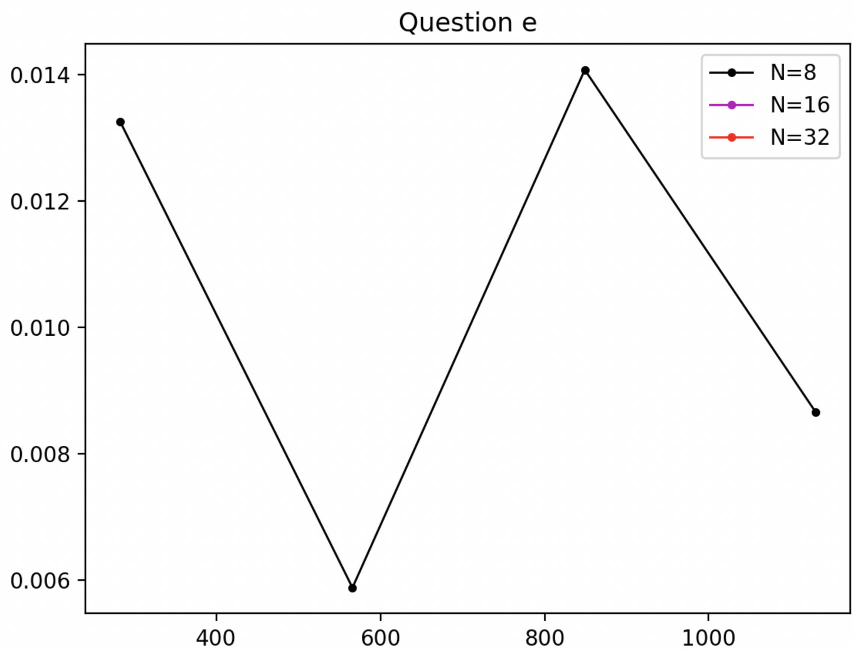
1.1.2. Nonlinear oscillators Now set $\alpha = N/10$. You will now need to use equation (11) in the reading material to advance the solution through time, and equation (12) to fix the time step size.

Questions:

(c) Use $C = 0.5$ and run with the same resolutions as in question (b), and save the same data out. What happens to the solution as time advances? How is it different from before?



(e) Investigate what happens for $\alpha = -N/10$. Save data files to support your findings.



Work Cited

- Wikipedia contributors. "Fermi–Pasta–Ulam–Tsingou problem." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 9 Dec. 2022. Web. 9 Dec. 2022.