Axiomatic formulation of Topological Quantum Field Theory

From the mathematical point of view we expect that all quantum field theories keep

- general features of quantum theory (probabilistic)
- topological framework of spacetime

Both features are embraced by the precise notion of a topological quantum field theory.

M. Atiyah (1988) gave a formal mathematical framework to develop quantum theories in physics.

Axiomatic formulation of Topological Quantum Field Theory

Definition

A topological quantum field theory (TQFT) in dimension $d + 1$ consist of the following data:

Fixed an integer d, consider smooth oriented manifolds M of dimension $d + 1$ (with boundary) and also smooth closed oriented manifolds X of dimension d.

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- *To each X we associate functorially* **a finite-dimensional complex vector space $\mathbf{Z}(X)$**.
- *To each M, whose boundary is X we associate a* **vector** $\mathbf{Z}(M) \in \mathbf{Z}(X)$.

Axiomatic formulation of Topological Quantum Field Theory

These data are subject to the following properties (**axioms**):

1) An orientation preserving diffeormorphism $f : X \to X'$ induces an isomorphism

$$\mathbf{Z}(f) : \mathbf{Z}(X) \to \mathbf{Z}(X')$$

and if $g : X' \to X''$ then

$$\mathbf{Z}(fg) = \mathbf{Z}(f)\mathbf{Z}(g)$$

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3) $\mathbf{Z}$ is multiplicative.

Thus, for disjoint unions

$$\mathbf{Z}(X_1 \cup X_2) = \mathbf{Z}(X_1) \otimes \mathbf{Z}(X_2).$$

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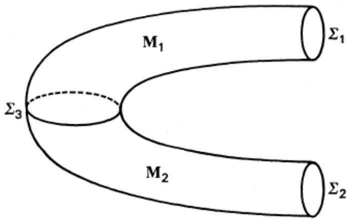
Moreover if $\partial M_1 = X_1 \cup X_3$, $\partial M_2 = X_2 \cup X_3^*$ and $M = M_1 \cup_{X_3} M_2$ is the manifold obtained by identification of the common $X_3 - \mathit{component}$.



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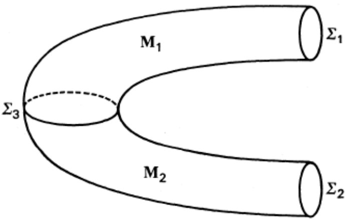

$$Z(\Sigma_1) \otimes Z(\Sigma_3) \otimes Z(\Sigma_3)^* \otimes Z(\Sigma_2) \to Z(\Sigma_1) \otimes Z(\Sigma_2)$$

Axiomatic formulation of Topological Quantum Field Theory

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Note that when M is a closed d-dimensional manifold so that $\partial M = \emptyset$, then

$$\mathbf{Z(M)} \in \mathbf{Z}(\emptyset) = \mathbb{C}$$

That means, **Z(M)** is a **numerical invariant** of M.

State sum model as a 3D TQFT of gravity

Einstein equations (movement equations) can be derived through minimizing the action S:

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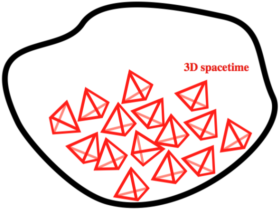
1961 Regge gave a discrete formulation of general relativity using triangulated manifolds.



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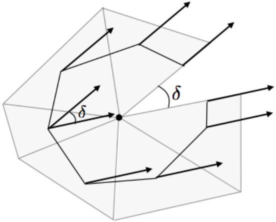
Regge showed that the curvature of a triangulated n dimensional manifold is encoded in the $(n-2)$ skeleton.



The discrete form of the action S from which Einstein field equations are derived is given by

$$S_R = \sum \left|\sigma^i(T)\right| \varepsilon_i$$

State sum model as a 3D TQFT of gravity

In quantum mechanics the action is replaced by a **partition function Z**:

$$Z = \int e^{iS} D(x)$$

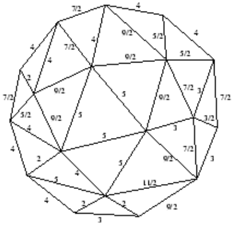
Given a manifold M, **Z**(M) obtained from the TQFT will have the physical **meaning of the partition function**.

State sum invariants

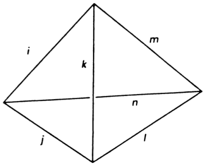
In 1991 Turaev-Viro developed a 3 dimensional TQFT of gravity called the **state sum model.**

Definition

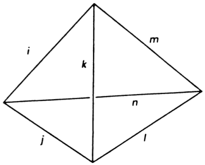
Given a 3 dimensional manifold and a triangulation, a **state** *is an assignment of an irreducible representation of $SU(2)$ to each edge of the triangulation. This are labeled by a non-half integer parameter* **j**.



State sum invariants



State sum invariants

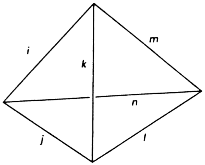


Definition

The **weight** *of the state is defined by*

$$W = \prod_{\text{interior edges}} (-1)^{2j}(2j+1) \prod_{\text{interior triangles}} (-1)^{j_1+j_2+j_3} \prod_{\text{tetrahedra}} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$

State sum invariants



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The **state sum** *(partition function) is obtained by summing over all values of the variables, subject to fixed values on the boundary.*

$$\mathbf{Z}(M) = \Sigma W$$