A topological three-dimensional quantum field theory of gravity.

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Motivation

Quantum gravity seeks to describe gravitation through the principles of quantum mechanics. To find an adequate formulation of such a theory hasn't been an easy task. There are two big competitors at present:

- String theory
- Loop quantum gravity

Loop suspension space

Let X be a pointed topological space. The **loop space on X** is the space of all pointed continuous maps from S^1 to X

$$\Omega X := Map_*(S^1, X).$$

- ΩX is a homotopy associative H-space, that is, there exists a map $\mu : \Omega X \times \Omega X \to \Omega X$ for which the (unique) constant map $c : \Omega X \to \Omega X$ is a homotopy identity such that the following diagrams commute up to homotopy

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d, equal] & \Omega X \times \Omega X \arrow[dl, "\mu"] \\ \Omega X & \end{tikzcd} \qquad \begin{tikzcd} \Omega X \times \Omega X \times \Omega X \arrow[r, "\mu \times \mathbb{1}"] \arrow[d, "\mathbb{1} \times \mu"] & \Omega X \times \Omega X \arrow[d, "\mu"] \\ \Omega X \times \Omega X \arrow[r, "\mu"] & \Omega X. \end{tikzcd}$$

The map μ is given by loop concatenation.

Loop suspension space

The **Moore loop space of** X is defined by

$$\Omega' X := \{(\omega, r) \in \mathrm{Map}(\mathbb{R}^+, X) \times \mathbb{R}^+ \big| \omega(0) = * \text{ and } \omega(t) = * \text{ for } t \geq r\}.$$

- Moore loop space of X is an strictly associative H-space (topological monoid) where the multiplication is given by loop concatenation.
- ΩX can be identified with the subspace of all pairs of the form $(\omega, 1)$.

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Proposition

ΩX is a deformation retract of $\Omega' X$.

Loop suspension space

The **reduced suspension** of X is defined by
$\Sigma X := S^1 \wedge X \cong [0,1] \times X / (\{0\} \times X \cup \{1\} \times X \cup [0,1] \times \{x_0\})$.

suspension-eps-converted-to.pdf

Let X be a pointed topological space. The **loop space of the suspension on** X is defined by

$$\Omega\Sigma X := Map_*(S^1, \Sigma X).$$

Loop space of a suspension

There is a natural map $E : X \to \Omega\Sigma X$ given by

$$E(x)(t) = (x, t)$$

for all $x \in X$ and $t \in [0, 1]$.

Theorem

(Bott-Samelson) Let R be a field and let X be a connected space such that $H_*(X)$ is a free R-module. Then $H_*(\Omega\Sigma X) \cong T(\tilde{H}_*(X))$ as algebras and the map $E : X \to \Omega\Sigma X$ induces the inclusion of the generating set.

The James construction

Let X be a pointed space with basepoint $*$. Let $J_n(X)$ be defined by

$$J_n(X) := X^n / \sim$$

where $(x_1, \ldots, x_{i-1}, *, x_i, \ldots, x_n) \sim (x_1, \ldots, x_{j-1}, *, x_j, \ldots, x_n)$ for any $1 \leq i, j \leq n$. The **James construction** on X is defined by

$$J(X) := \bigcup_{n=1}^{\infty} J_n(X)$$

where the *weak topology* is given to $J(X)$.

- J(X) is a topological monoid.

The James construction

Proposition

Let X be a pointed space with basepoint $*$, Y a topological monoid with identity e, and $f : X \to Y$ a map such that $f(*) = e$. Then there exists a unique homomorphism $\tilde{f} : J(X) \to Y$ of topological monoids given by

$$\tilde{f} : (x_1, x_2, \dots) \mapsto (f(x_1), f(x_2), \dots) \mapsto f(x_1)f(x_2)\cdots$$

such that the following diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & Y \\ J(X) \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

commutes.

The James construction

- The map $E : X \to \Omega'\Sigma X$, can be extended to an H-map $\tilde{E} : J(X) \to \Omega'\Sigma X$ such that the following diagram

$$\begin{tikzcd} X \arrow[r, "E"] \arrow[d, hook] & \Omega'\Sigma X \\ J(X) \arrow[ur, "\tilde{E}"'] & \end{tikzcd} \tag{1}$$

commutes.

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commutes.

Proposition

If X has the homotopy type of a connected CW-complex then

$$\Sigma(J(X)) \simeq \bigvee_{n=1}^{\infty} \Sigma(X^{\wedge n}).$$

The James construction

Theorem

The composite $J(X) \xrightarrow{\tilde{E}} \Omega' \Sigma(X) \xrightarrow{\simeq} \Omega\Sigma(X)$ *is a weak homotopy equivalence.*

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Proof.

Let assume that homology is set with coefficients over either $\mathbb{Q}$ or $\mathbb{Z}_p$, thus

$$\tilde{H}_*(\Sigma J(X)) \cong \tilde{H}_*(\bigvee_{n=1}^{\infty} \Sigma X^{\wedge n}) \cong \bigoplus_{n=1}^{\infty} \tilde{H}_*(\Sigma X^{\wedge n}) \cong \bigoplus_{n=1}^{\infty} \tilde{H}_*(\Sigma X)^{\otimes n}.$$

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Therefore as modules

$$\tilde{H}_*(J(X)) \cong \bigoplus_{n=1}^{\infty} \tilde{H}_*(X)^{\otimes n} \cong T(\tilde{H}_*(X)). \tag{2}$$

The James construction

Proof (Cont.)

Now, there exists a map $i_* : \tilde{H}_*(X) \to H_*(J(X))$ induced by the inclusion $i : X = J_1(X) \to J(X)$. Since $J(X)$ is a topological monoid, $H_*(J(X))$ is an associative algebra.

So by the universal mapping property of tensor algebras, there is an extension of i_* to an algebra homomorphism $\varphi : T(\tilde{H}_*(X)) \to H_*(J(X))$ such that the following diagram

$$\begin{tikzcd} \tilde{H}_*(X) \arrow[r, "i_*"] \arrow[d, hook, "j"'] & H_*(J(X)) \\ T(\tilde{H}_*(X)) \arrow[ur, "\varphi"'] & \end{tikzcd} \tag{3}$$

commutes.

The James construction

Proof (Cont.)

Consider the following diagram:

$$
\begin{tikzcd}
 & \tilde{H}_*(X) \arrow[dl, hook', "j"'] \arrow[d, "i_*"] \arrow[drr, "E_*"] & & & \\
T(\tilde{H}_*(X)) \arrow[r, "\varphi"] & H_*(J(X)) \arrow[rr, "\tilde{E}_*"] & & H_*(\Omega\Sigma X) \arrow[r, "\gamma", "\cong"'] & T(\tilde{H}_*(X))
\end{tikzcd}
$$

- By Bott-Salemson's Theorem γ is an algebra isomorphism.
- By (1) and (3) both triangles commute and therefore the whole diagram is commutative.
- $\gamma \circ \tilde{E}_* \circ \varphi$ restricted to $\tilde{H}_*(X)$ is the identity map.

Proof (Cont.)

- Thus φ is an injection and hence, by (2), it is an isomorphism.
- Therefore $\tilde{E}$ is an isomorphism.

This is also true for $\mathbb{Z}$ coefficients. Whitehead's theorem implies that $\tilde{E}$ is a weak homotopy equivalence. $\square$