Chapter 1

Fibre bundles and Classifying Spaces

Definition 1. Let B be a pointed topological space. A (locally trivial) fibre bundle over B consists of a map $p : E \to B$ such that for all $b \in B$ there exists an open neighbourhood U of b for which there is a homeomorphism $\phi : p^{-1}(U) \to p^{-1}(*) \times U$ satisfying

$$\pi'' \circ \phi = p\big|_U$$

where π'' denotes projection onto the second factor.

In this case we say that B is the base space of the fibre bundle, X the total space, p^{-1} the fibre, and p the projection. A bundle map of fibre bundles consists of maps between the total spaces and base spaces which form a commutative square with the bundle projections.

Definition 2. Let G be a topological group and B a topological space. A *principal G-bundle* over B consists of a fibre bundle $p : X \to B$ together with an action $G \times X \to X$ such that:

1. the map $G \times X \to X \times X$ given by $(g, x) \mapsto (x, g \cdot x)$ maps $G \times X$ homeomorphically to its image;
2. $B = X/G$ and $p : X \to X/G$ is the quotient map;
3. for all $b \in B$ there exists a open neighbourhood U of b such that $p : p^{-1}(U) \to U$ is G-bundle isomorphic to the trivial bundle $\pi" : G \times U \to U$. That is, there exists a homeomorphism $\phi : p^{-1}(U) \to G \times U$ satisfying $p = \pi" \circ \phi$ and $\phi(gx) = g\phi(x)$, where $g(g', u) = (gg', u)$.

Let ξ be a principal G-bundle $p : X \to G$. Given a map $f : B' \to B$, the pullback yields a principal G-bundle over B', written $f^*(\xi)$, and the pullback square becomes a bundle map from $f^*(\xi)$ to ξ.

Definition 3. A (numerable) principal G-bundle γ over a pointed space $\tilde{B}$ is called a *universal G-bundle* if

1. for any (numerable) principal G-bundle ξ there exists a map $f : B \to \tilde{B}$ from the base space B of ξ to the base space of γ such that $\xi = f^*(\gamma)$;

2. whenever f,h are two pointed maps from some space B into the base space $\tilde{B}$ of γ such that $f^*(\gamma) \cong h^*(\gamma)$ then $f \simeq h$.

In other words, a numerable principal G-bundle γ with base space $\tilde{B}$ is a universal G-bundle if, for any pointed space B, pullback induces a bijection from $[B, \tilde{B}]$ to isomorphism classes of numerable principal bundles over B.

Let G be a topological group. Let EG be the infinite join $EG = \varinjlim_{n} G^{*n}$. Explicity, as a set

$$EG = \{(g_0, t_0, g_1, t_1, \dots, g_n, t_n, \dots) \in (G \times I)^\infty\}/\sim$$

such that at most finitely many t_i are nonzero, $\sum t_i = 1$, and

$$(g_0, t_0, g_1, t_1, \dots, g_n, 0, \dots) \sim (g_0, t_0, g_1, t_1, \dots, g_n', 0, \dots).$$

A G-action on EG is given by

$$g \cdot (g_0, t_0, g_1, t_1, \cdots, g_n, t_n, \cdots) = (gg_0, t_0, gg_1, t_1, \cdots, gg_n, t_n, \cdots).$$

Let $BG = EG/G$. We also write $E_nG = G^{*(n+1)}$ and $B_n(G) = E_nG/G$, referring to the inclusions $B_0G \hookrightarrow B_1G \hookrightarrow B_2G \hookrightarrow \dots B_nG \hookrightarrow \dots$ as the Milnor filtration on BG. Since homotopy classes of maps into BG classify principal G-bundles, BG is called the classifying space of the group G.

Theorem 1. For every topological group G, the quotient map $EG \to BG$ is a (numerable) G-bundle and this bundle is a universal G-bundle.

Definition 4. Let $p : P \to B$ be a principal G-bundle. The gauge group Aut(P) of P is the subspace of all G-equivariant maps $u \in \mathrm{Map(P, P)}$ such that the following diagram

$$\begin{tikzcd} P \arrow[r, "u"] \arrow[d, "p"] & P \arrow[d, "p"] \\ B \arrow[r] & B \end{tikzcd}$$

commutes.

Definition 5. Let ξ be a principal $SO(n)$-bundle over an oriented manifold B. A spin structure on ξ is a pair (η, F) consisting of

1. A principal Spin(n)-bundle η over B

2. A map $f : E(\eta) \to E(\xi)$ such that the following diagram is commutative .

$$\begin{array}{ccccc} E(\eta) \times Spin(n) & \longrightarrow & E(\eta) & \longrightarrow & B \\ \downarrow{\scriptstyle f\times\lambda} & & \downarrow{\scriptstyle f} & & \| \\ E(\eta) \times SO(n) & \longrightarrow & E(\xi) & \longrightarrow & B \end{array}$$

Here λ denotes the standard homomorphism from Spin (n) to SO(n).

Chapter 2

The topology of the gauge group

Proposition 1. Let BG be the classifying space for G. Then in homotopy theory

$$B\mathscr{G}(P) = \mathrm{Map}_f(M, BG).$$

The subscript f denotes the component of $\mathrm{Map}(M, BG)$ which contains the map f that induces P.

Proof. Let

$$G \to EG \to BG$$

be a universal bundle for G, and consider the space $\mathrm{Map}(P, EG)$ of G-equivariant maps of P to EG. The group $\mathscr{G}(P)$now acts naturally on this space by composition, to yield the principal fibring

$$\mathscr{G}(P) \longrightarrow \mathrm{Map}(P, EG) \longrightarrow \mathrm{Map}_f(M, BG).$$

If BG is paracompact and locally contractible, π will be a locally trivial principal fibring. The total space $\mathrm{Map}_f(P, EG)$ is contractible so that this is a universal bundle for $\mathscr{G}(P)$, and

$$B\mathscr{G}(P) = \mathrm{Map}_f(M, BG)$$

as was asserted.

$\square$

$$\vee_{i=1}^{t}(S^3 \wedge S^4) \vee (\vee_{j=1}^{d-t} P^4(p^r) \tag{2.1}$$

Chapter 3

Counting homotopy types of gauge groups

3.1 Samelson and Whitehead products

Let G be a homotopy associative H-space with multiplication $\mu : G \times G \to G$ and homotopy inverse $\iota : G \to G$. We write

$$\mu(x, y) = xy \quad \text{and} \quad \iota(x) = x^{-1}$$

Then we write the commutator map [,]: $G \times G \to G$ as

$$[x, y] = \mu(\mu(\mu(X, y), \iota(x)), \iota(y)) = ((xy)x^{-1})y^{-1}$$

or if we ignore the homotopy associativity, as

$$[x, y] = xyx^{-1}y^{-1}$$

Since it is nullhomotopi con $G \vee G$ the commutator map factors as follows:

$$G \times G \to G \wedge G \xrightarrow{\overline{[\ ,\ ]}} G.$$

IF $f : X \to G$ and $g; Y \to G$ are maps, then the commutator

$$C(f, g) = [\ ,\ ] \circ (f \times g) = fgf^{-1}g^{-1} : X \times Y \to G \times G \to G.$$

factors up to homotopy through the map

$$[f, g] = \overline{[\ \ ,\ \ ]} \circ (f \wedge g) : X \wedge Y \to G \wedge G \to G$$

Definition 6. The map $[f, g] : X \wedge Y \to G$ is called the Samelson product of $f : X \to G$ and $g : Y \to G$.

This map is well defined up to homotopy since the sequence of cofibrations

$$X \vee Y \to X \times Y \to X \wedge Y \to \Sigma X \vee \Sigma Y \to \Sigma(X \times Y)$$

Proposition 2. The Samelson product vanishes in the range G is homotopy commutative.

Samelson products are natural with respect to maps $f_1 : X_1 \to X$, $g_1 : Y_1 \to Y$, and H-maps of H-spaces $\psi : G \to H$, that is

$$[\psi \circ f \circ f_1, \psi \circ g \circ g_1] \simeq \psi \circ [f, g] \circ (f_1 \wedge g_1).$$

Given maps $\bar{f} : \Sigma X \to Z$, $\bar{g} : \Sigma Y \to Z$ with respective adjoints

$$f : X \stackrel{\Sigma}{\to} \Omega\Sigma X \stackrel{\Omega\bar{f}}{\to} \Omega Z, \qquad g : X \stackrel{\Sigma}{\to} \Omega\Sigma X \stackrel{\Omega\bar{g}}{\to} \Omega Z$$

we define the Whitehead product $[\bar{f}, \bar{g}]_w$ to be the adjoint of the Samelson product $[f, g]$, namely,

Definition 7. The Whitehead product $[\bar{f}, \bar{g}]_w$ is the compositon

$$\Sigma(X \wedge Y) \stackrel{\Sigma[f,g]}{\to} \Sigma\Omega Z \stackrel{e}{\to} Z$$

As with Samelson products, Whitehead products are natural with respect to maps.

Theorem 2 (Crabb and Sutherland Ref)**.** Let K be a connected finite complex and let G be a compact connected Lie group. As P ranges over all principal G-bundles with base K, the number of homotopy types of $\mathscr{G}(P)$ is finite.

Proposition 3 (Spreafico Ref)**.** The homotopy type of the gauge group of all the principal $SU(2)$-bundles over S^n, with $n = 7, 8$, is the same and is the one of the trivial bundle, namely $B\mathscr{G}(S^n) \sim \Omega_0^n SU(2) \times SU(2)$.

3.2 Poincare Duality

In this section I will present the background to understand the construction of the topological spaces I will work with in the next sections. To doing so I will have to give a brief introduction to the concepts required to state the Poincare Duality Theorem.

Definition 8. A manifold of dimension n or an n-manifold is a Hausdorff space M in which each point has an open neighbourhood homeomorphic to $\mathbb{R}^n$.

The dimension of M is characterized by the fact that for $x \in M$, the homology group $H_i(M, M - \{x\})$ is nonzero only for $i = n$. A compact manifold is called closed to distinguish it from the more general notion of a compact manifold with boundary.

Definition 9. A local orientation of M at a point x is a choice of generator μ_x of the infinite cyclic group $H_n(M, M - \{x\})$.

To simplify notation I will write $H_n(X \mid A)$ for $H_n(X, X - A)$.

Definition 10. An orientation of an n-dimensional manifold M is a function $x \mapsto \mu_x$ assigning to each $x \in M$ a local orientation $\mu_x \in H_n(M \mid x)$, satisfying the local consistency condition that each $x \in M$ has a neighbourhood $\mathbb{R}^n \subset M$ containing an open ball B of finite radius about x such that all the local orientations μ_y at points $y \in B$ are the images of one generator μ_B of $H_n(M \mid B) \cong H_n(\mathbb{R}^n \mid B)$ under the natural maps $H_n(M \mid B) \to H_n(M \mid y)$.

If an orientation exists for M, then M is called orientable. One can generalise the definition of orientation by replacing the coefficient group $\mathbb{Z}$ by any commutative ring R with identity. The orientability of a closed manifold is reflected in the structure of its homology, according to the following results.

Theorem 3. Let M ne a closed connected n-manifold. Then:

a) If M is orientable, the map $H_n(M; R) \to H_n(M \mid x; R) \cong R$ is an isomorphism for all $x \in M$.

b) If M is not orientable, the map $H_n(M; R) \to H_n(M \mid x; R) \cong R$ is injective with image $\{r \in R \mid 2r = 0\}$ for all $x \in M$.

c) $H_i(M; R) = 0$ for $i > n$.

Definition 11. A fundamental class for a closed orientable manifold M with coefficients in R is an element of $H_n(M; R)$ whose image in $H_n(M \mid x; R)$ is a generator for all x.

Definition 12. For an arbitrary space X and coefficient ring R, define an R-bilinear cap product $\frown: C_k(X; R) \times C^l(X; R) \to C_{k-l}(X; R)$ for $k \geq l$ by setting

$$\sigma \frown \varphi = \varphi(\sigma \mid [v_0, \ldots, v_l])\sigma \mid [v_l, \ldots, v_k]$$

for $\sigma : \Delta^k \to X$ and $\varphi \in C^l(X; R)$.

Theorem 4 (Poincare duality theorem)**.** Let M be a closed and oriented n-manifold with fundamental class $[M] \in H_n(M; R)$. The map $D : H^k(M; R) \to H_{n-k}$ defined by

$$D(\alpha) = [M] \frown \alpha$$

is an isomorphism for all k.

3.3 Classification of 2-connected 7-manifolds

Definition 13. A homologically graded spectral sequence $E = \{E^r\}$ consists of a sequence of $\mathbb{Z}$-bigraded R modules $E^r = \{E^r_{p,q}\}_{r\geq 1}$ together with differentials

$$d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

such that $E^{r+1} \cong H_*(E^r)$.

Definition 14. A cohomologically graded spectral sequence $E = \{E_r\}$ consists of a sequence of $\mathbb{Z}$-bigraded R modules $E_r = \{E_r^{p,q}\}_{r\geq 1}$ together with differentials

$$d^r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

such that $E_{r+1} \cong H_*(E_r)$.

Definition 15. Let D and G be modules. An exact couple is an exact triangle

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & G \arrow[ul, "k"] & \end{tikzcd} .$$

A spectral sequence $\{E_r\}$ is obtained from an exact couple by defining $E_1 = G$, $d_1 = jk$ and defining the derived couple

$$\begin{tikzcd} i(D) \arrow[rr, "i'"] & & i(D) \arrow[dl, "j'"] \\ & E_2 \arrow[ul, "k'"] & \end{tikzcd} .$$

where $E_2 = H_*(E_1)$ with respect to d_1, i' is induced by i, k' is induced by k, and $j'(i(a)) = \{j(a)\}$. One can show these maps are well defined and the derived couple will again be exact couple. This process leads to an inductive definition of a spectral sequence.

Let C be a torsion-free chain complex over $\mathbb{Z}$. From the short exact sequence of groups

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}$$

we obtain a short exact sequence of chain complexes

$$0 \longrightarrow C \longrightarrow C \longrightarrow C \otimes \mathbb{Z}/p\mathbb{Z}.$$

By the usual argument the homology of

$$\begin{tikzcd} H_*(C) \arrow[rr, "i_*"] & & H_*(C) \arrow[dl, "j_*"] \\ & H_*(C \otimes \mathbb{Z}_p) \arrow[ul, "\partial_p"] & \end{tikzcd} . \tag{3.1}$$

Definition 16. The spectral sequence associated with the exact couple (3.1) is called the Bockstein spectral sequence of C *mod* p.

Let M be a closed 2-connected manifold. The non-zero homology groups of M are $H_0(M) \cong \mathbb{Z}$, $H_3(M)$, $H_4(M)$, $H_7(M) \cong \mathbb{Z}$ where by duality $H_3(M)$ is free abelian of the same torsion-free rank as $H_3(M)$. The cohomology groups are given by Poincare duality, thus $G = H_3(M) \cong H^4(M)$. Let T denote the torsion group subgroup of G. Thus we have a nonsingular bilinear map

$$b : T \times T \to S$$

where $S = \mathbb{Q}/\mathbb{Z}$ defined as follows

3.4 Principal SU(n)-bundles over n-manifolds, n=7,8

We start with analysing the case of a 8-manifold since the CW-complex structure is easier to handle. Analysing the 8-manifold case will be a first attempt to look into high dimensional manifolds.