# A topological three-dimensional quantum field theory of gravity.

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- Axiomatic formulation of a topological quantum field theory
- State sum invariants
- State sum invariants as a TQFT

## Motivation

Quantum gravity seeks to describe gravitation through the principles of quantum mechanics. To find an adequate formulation of such a theory hasn't been an easy task. There are two big competitors at present:

- String theory
- Loop quantum gravity

Let X be a pointed topological space. The **loop space on** X is the space of all pointed continuous maps from  $S^1$  to X

$$\Omega X := Map_*(S^1, X).$$

•  $\Omega X$  is a homotopy associative H-space, that is, there exists a map  $\mu: \Omega X \times \Omega X \to \Omega X$  for which the (unique) constant map  $c: \Omega X \to \Omega X$  is a homotopy identity such that the following diagrams commute up to homotopy



The map  $\mu$  is given by loop concatenation.

# The **Moore loop space of** *X* is defined by

$$\Omega'X:=\{(\omega,r)\in\operatorname{Map}(\mathbb{R}^+,X)\times\mathbb{R}^+\big|\omega(0)=*\text{ and }\omega(t)=*\text{ for }t\geq r\}.$$

- Moore loop space of X is an strictly associative H-space (topological monoid) where the multiplication is given by loop concatenation.
- $\Omega X$  can be identified with the subspace of all pairs of the form  $(\omega,1)$ .

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## Proposition

 $\Omega X$  is a deformation retract of  $\Omega' X$ .



The **reduced suspension** of X is defined by

$$\Sigma X := S^1 \wedge X \cong [0,1] \times X/(\{0\} \times X \cup \{1\} \times X \cup [0,1] \times \{x_0\}).$$

suspension-eps-converted-to.pdf

Let X be a pointed topological space. The **loop space of the suspension** on X is defined by

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# Loop space of a suspension

There is a natural map  $E: X \to \Omega \Sigma X$  given by

$$E(x)(t) = (x, t)$$

for all  $x \in X$  and  $t \in [0,1]$ .

#### **Theorem**

(Bott-Samelson) Let R be a field and let X be a connected space such that  $H_*(X)$  is a free R-module. Then  $H_*(\Omega\Sigma X)\cong T(\tilde{H}_*(X))$  as algebras and the map  $E:X\to\Omega\Sigma X$  induces the inclusion of the generating set.

Let X be a pointed space with basepoint \*. Let  $J_n(X)$  be defined by

$$J_n(X) := X^n/\sim$$

where  $(x_1,\ldots,x_{i-1},*,x_i,\ldots,x_n)\sim (x_1,\ldots,x_{j-1},*,x_j,\ldots,x_n)$  for any  $1\leq i,j\leq n$ . The **James construction** on X is defined by

$$J(X):=\bigcup_{n=1}^{\infty}J_n(X)$$

where the weak topology is given to J(X).

• J(X) is a topological monoid.



## Proposition

Let X be a pointed space with basepoint \*, Y a topological monoid with identity e, and  $f: X \to Y$  a map such that f(\*) = e. Then there exists a unique homomorphism  $\tilde{f}: J(X) \to Y$  of topological monoids given by

$$\tilde{f}:(x_1,x_2,\dots)\mapsto (f(x_1),f(x_2),\dots)\mapsto f(x_1)f(x_2)\cdots$$

such that the following diagram



commutes.



• The map  $E: X \to \Omega' \Sigma X$ , can be extended to an H-map  $\tilde{E}: J(X) \to \Omega' \Sigma X$  such that the following diagram



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#### Proposition

If X has the homotopy type of a connected CW-complex then

$$\Sigma(J(X)) \simeq \bigvee_{n=1}^{\infty} \Sigma(X^{\wedge n}).$$

#### Theorem

The composite  $J(X) \xrightarrow{\tilde{E}} \Omega' \Sigma(X) \xrightarrow{\simeq} \Omega \Sigma(X)$  is a weak homotopy equivalence.

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#### Proof.

Let assume that homology is set with coefficients over either  $\mathbb{Q}$  or  $\mathbb{Z}_p$ , thus

$$\tilde{H}_*(\Sigma J(X)) \cong \tilde{H}_*(\bigvee_{n=1}^{\infty} \Sigma X^{\wedge n}) \cong \bigoplus_{n=1}^{\infty} \tilde{H}_*(\Sigma X^{\wedge n}) \cong \bigoplus_{n=1}^{\infty} \tilde{H}_*(\Sigma X)^{\otimes n}.$$

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Therefore as modules

$$\tilde{H}_*(J(X)) \cong \bigoplus_{n=1}^{\infty} \tilde{H}_*(X)^{\otimes n} \cong T(\tilde{H}_*(X)).$$
 (2)

# Proof (Cont.)

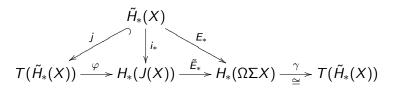
Now, there exists a map  $i_*: \tilde{H}_*(X) \to H_*(J(X))$  induced by the inclusion  $i: X = J_1(X) \to J(X)$ . Since J(X) is a topological monoid,  $H_*(J(X))$  is an associative algebra.

So by the universal mapping property of tensor algebras, there is an extension of  $i_*$  to an algebra homomorphism  $\varphi: T(\tilde{H}_*(X)) \to H_*(J(X))$  such that the following diagram

commutes.

# Proof (Cont.)

Consider the following diagram:



- ullet By Bott-Salemson's Theorem  $\gamma$  is an algebra isomorphism.
- By (1) and (3) both triangles commute and therefore the whole diagram is commutative.
- $\gamma \circ \tilde{E}_* \circ \varphi$  restricted to  $\tilde{H}_*(X)$  is the identity map.

# Proof (Cont.)

- Thus  $\varphi$  is an injection and hence, by (2), it is an isomorphism.
- Therefore  $\tilde{E}$  is an isomorphism.

This is also true for  $\mathbb Z$  coefficients. Whitehead's theorem implies that  $\tilde E$  is a weak homotopy equivalence.  $\square$