Chapter 1

Steenrod algebra (Masher and Tangora)

1.1 Introduction

A cohomology operation is a natural transformation $\theta : H^m(X;G) \to H^n(X;H)$, that is for all maps $f : X \to Y$ the following diagram

$$\begin{tikzcd} H^m(X;G) \arrow[r, "\theta_X"] \arrow[d, "f^*"] & H^m(X;G') \arrow[d, "f^*"] \\ H^m(Y;G) \arrow[r, "\theta_Y"] & H^m(Y;G') \end{tikzcd}$$

commutes.

Definition 1. We denote by $K(G,n)$ any space which has only one non-trivial homotopy group, namely $\pi_n(K(G,n)) = G$.

The Hurewicz homomorphism $h : \pi_i(X) \to H_i(X)$ is defined for any X and any i by choosing a generator u of $H_i(S^i)$ and putting $h : [h] \to f_*(u)$ where $f : S^i \to X$.

Definition 2. A space X is said to be n-connected if $\pi_i(X)$ is trivial for all $i \leq n$.

The Hurewicz theorem states that if X is $(n-1)$-connected, then the Hurewicz homomorphism is an isomorphism in dimensions $i \leq n$ and is still an epimorphism in dimension $n+1$. This theorem is modified if $n = 1$; in this case the epimorphism $h : \pi_1(X) \to H_1(X)$ has a kernel the commutator subgroup of $\pi(X)$ and h does

not necessarily maps $\pi_2(X)$ onto $H_2(X)$.
The universal coefficient theorem for cohomology gives an exact sequence

$$0 \to \mathrm{Ex}t(H_{n-1}(X), G) \to H^n(X; G) \to \mathrm{Hom}(H_n(X), G) \to 0$$

for any space X. If X is $(n-1)$-connected, this becomes an isomorphism between the last two terms of the sequence, since $\mathrm{Ext}(0, G) = 0$. Now if $G = \pi_n(X)$ then the group $\mathrm{Hom}(H_n(X), \pi_n(X))$ contains h^{-1}, the inverse of the Hurewicz homomorphism, which is also an isomorphism.

Definition 3. Let X be $(n-1)$-connected. The fundamental class of X is the cohomology class $\iota \in H^n(X; \pi_n(X))$ which correspond to h^{-1} under the above isomorphism.We write sometimes ι_n or ι_X instead.

Theorem 1. There is a one-to-one correspondence $[X, K(G, n)] \to H^n(X; G)$, given by $[f] \to f^*(\iota_n)$.

Notation: let's write $H^m(G, n; G')$ for $H^m(K(G, n); G')$.

Theorem 2. Let denote $O(n, G; m, G')$ the set of all cohomolgy operations. There is a one-to one correspondence

$$O(G, n; G', m) \to H^m(G, n; G')$$

given by $\theta \to \theta(\iota_n)$.

Read about Obstruction theory!!!

1.2 Steenrod squares

Steenrod squares are cohomology operation of type $(\mathbb{Z}_2, n; \mathbb{Z}_2, n-i)$. Given an abelian group G it is possible to construct a CW-complex $K(G, n)$ where $n \geq 2$. In particular, we can construct a complex $K(\mathbb{Z}_2, 1)$.

Definition 4. For each integer $i \geq 0$, define a *cup-i product*

$$C^p(K) \otimes C^q \to C^{p+q-i}(K) : (u, v) \to u \smile_i v$$

where $C^p(X)$.

Chapter 2

Fibre bundles and Classifying Spaces

Definition 5. Let B be a pointed topological space. A (locally trivial) fibre bundle over B consists of a map $p : E \to B$ such that for all $b \in B$ there exists an open neighbourhood U of b for which there is a homeomorphism $\phi : p^{-1}(U) \to p^{-1} \times U$ satisfying

$$\pi'' \circ \phi = p\big|_U$$

where π'' denotes projection onto the second factor.

Definition 6. Let G be a topological group B and B a topological space. A *principal G-bundle* over B consists of a fibre bundle $p : X \to B$ together with an action $G \times X \to X$ such that:

- the map $G \times X \to X \times X$ given by $(g, x) \mapsto (x, g \cdot x)$ maps $G \times X$ homeomorphically to its image;
- $B = X/G$ and $p : X \to X/Gisthequotientmap$;