A topological three-dimensional quantum field theory of gravity.

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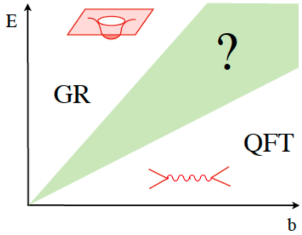
Plan

1. Motivation and background
2. Axiomatic formulation of a topological quantum field theory
3. State sum invariants as a TQFT of gravity

1) Motivation and background

Quantum gravity (QG) seeks to describe gravitational field through the principles of quantum mechanics.



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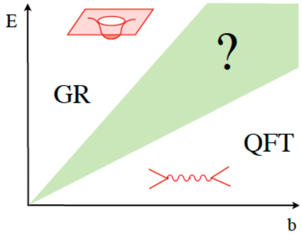
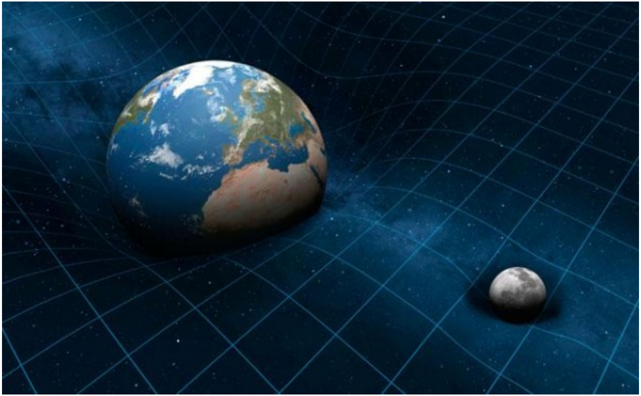


1) Motivation and background

General relativity is the discovery that spacetime and the gravitational field are the same physical entity. All fields we know exhibit quantum properties at some scale, therefore it is believe spacetime to have quantum properties as well.



Definition

A differential manifold of dimension n is a topological space M with a family of bijective maps $\phi_\alpha : U_\alpha \subset M \to V_\alpha \subset \mathbb{R}^n$ such that

i) $\bigcup_\alpha U_\alpha = M$

ii) *for all α, β such that $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$, the sets $\phi_\alpha(U_{\alpha\beta})$ and $\phi_\beta(U_{\alpha\beta})$ are open in $\mathbb{R}^n$ an the maps $\phi_\beta \circ \phi_\alpha^{-1}$ are smooths.*

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The pairs (U_α, ϕ_α) are called coordinate charts. The set of all charts is called an atlas. We will also require that any two points have disjoint neighborhoods and the atlas of M is countable.

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Let M and N be differentiable manifolds. A continuous map $f : M \to N$ is called differentiable in p if for every chart (V, ϕ) with $f(p) \in V$ there exists a chart (U, ψ) with $p \in U$ such that $f(U) \subset V$ and $\psi \circ f \circ \phi^{-1}$ is differentiable in $\phi(p)$.

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Definition

A differentiable bijective map $f : M \to N$ with f^{-1} differentiable is called a diffeomorphism.

2) Axiomatic formulation of Topological Quantum Field Theory

From the mathematical point of view we expect that a QG theory keeps

- general features of quantum theory (probabilistic)
- topological framework of spacetime

Both features are embraced by the precise notion of a topological quantum field theory.

Atiyah (1988) gave a formal mathematical framework to develop quantum theories in physics.

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These data are subject to the following axioms:

1) Z is functorial with respect to orientation preserving diffeomorphisms of Σ and M. That is, given an orientation preserving diffeormorphism $f : \Sigma \to \Sigma'$ induces an isomorphism

$$Z(f) : Z(\Sigma) \to Z(\Sigma')$$

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If f extends to an o. p. diff. $M \to M'$, with $\partial M = \Sigma$, $\partial M' = \Sigma'$, then $Z(f)$ take $Z(M)$ to $Z'(M)$.

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3) Z is multiplicative.

Thus, for disjoint unions

$$Z(\Sigma_1 \cup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2).$$

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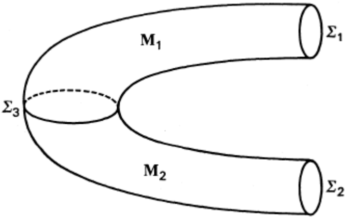
Moreover if $\partial M_1 = \Sigma_1 \cup \Sigma_3$, $\partial M_2 = \Sigma_2 \cup \Sigma_3^*$ and $M = M_1 \cup_{\Sigma_3} M_2$ is the manifold obtained by identification of the common $\Sigma_3 - \textit{component}$.



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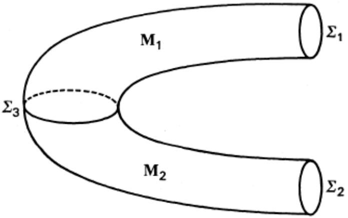

Then we require

$$Z(M) =< Z(M_1), Z(M_2) >$$

where $<,>$ denotes the natural pairing

$$Z(\Sigma_1) \otimes Z(\Sigma_3) \otimes Z(\Sigma_3)^* \otimes Z(\Sigma_2) \rightarrow Z(\Sigma_1) \otimes Z(\Sigma_2)$$

2) Axiomatic formulation of Topological Quantum Field Theory

The multiplicative axiom shows that when $\Sigma = \emptyset$ the vector space $Z(\Sigma)$ is idempotent. So therefore it is zero or isomorphic to the ground ring R.

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Similarly when $M = \emptyset$, $Z(M) \in R$ is an idempotent element.

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Note that when M is a closed d-dimensional manifold so that $\partial M = \emptyset$, then

$$Z(M) \in Z(\emptyset) = R$$

3) State sum invariants as a TQFT of gravity

Einstein equations (movement equations) can be derived through minimizing a function known as the action S:

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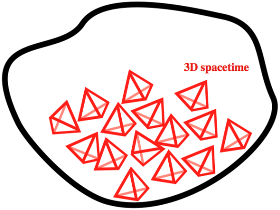
1961 Regge gave a discrete formulation of general relativity using triangulated manifolds.



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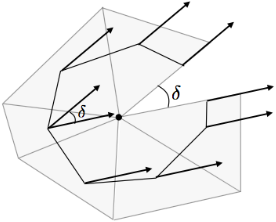
Regge showed that the curvature of a triangulated n dimensional manifold is encoded in the $(n-2)$ skeleton.



The discrete form of the action S from which Einstein field equations are derived is given by

$$S_R = \sum \left|\sigma^i(T)\right| \varepsilon_i$$

3) State sum invariants as a TQFT of gravity

In quantum mechanics the action is replaced by a partition function Z:

$$Z = \int e^{iS} D(x)$$

Given a manifold M, Z(M) obtained from the TQFT will have the physical meaning of the partition function.