Title

Computer Science 604

Advanced Algorithms

Lecture 4a: Dealing with NP-completeness

Integer Linear Programming & Linear Programming

David Juedes

School of EECS

juedes@cs.ohiou.edu

NP-completeness, cont'd

Let's look at some more important NP-complete problems.

0-1 Integer Linear Programming (0-1 ILP)

Decision Version Given an (m, n) matrix A of integers, an m-vector $\vec{b}$ of integers, an n-vector $\vec{c}$ of integers and an integer B, determine whether there exists an n-vector $\vec{x} \in \{0, 1\}^n$ s.t.

$$A\vec{x} \leq \vec{b}$$

where $\leq$ is component-wise less than and

$$\vec{c} \cdot \vec{x} = \sum_{i=1}^{n} c[i] * x[i] \leq B$$

Examples

Examples for component-wise $\leq$

$$x = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 2 \\ 5 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 3 \\ 7 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \not\leq \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Optimization Version

Given $A, \vec{b}, \vec{c}$ as above, produce $\vec{x} \in \{0,1\}^n$ that minimizes $\vec{c} \cdot \vec{x}$ and $A\vec{x} \leq \vec{b}$.

An important result

0-1 Integer Linear programming is NP-complete.

Proof:

It is easy to see that 0-1 ILP is in NP. We can guess a solution $\vec{x}$ and check if it satisfies the inequalities.

Now we are going to show that Vertex Cover $\leq_m^p$ 0-1 ILP. Hence 0-1 ILP will be NP-complete by property #3 from the previous notes.

Intuition

Intuitively, we want x_i to be 0 if v is not in the vertex cover and 1 if otherwise. Let

$$V_{\vec{x}} = \{v_i | x_i = 1\},$$

then it will be the case that

$$A\vec{x} \leq \vec{b} \Leftrightarrow V_{\vec{x}} \text{ is a vertex cover of } G.$$

Cont'd

Given an instance of vertex cover $I = \langle G, k \rangle$, where $G = (V, E)$, define

$$m = |E|, \; n = |V|$$

$$A = \begin{bmatrix} a_{11} & & \\ & \dots & \\ & & a_{mn} \end{bmatrix} \text{where} a_{ij} = \begin{cases} -1 & \text{if } v_j \in e_i; \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\vec{c} = \overbrace{\langle 1, \cdots, 1 \rangle}^{n}, \; \vec{b} = \overbrace{\langle -1, \cdots, -1 \rangle}^{m}, \; B = k$$

Many-one Reduction

The many-one reduction is:

$$f(\langle G, k\rangle) \longrightarrow \langle A, \vec{b}, \vec{c}, B\rangle$$

Technically, we need to prove that G has a vertex cover of size k iff

$$\vec{x} \in \{0, 1\}^n \text{ s.t. } A\vec{x} \leq \vec{b} \text{ and } \vec{c} \cdot \vec{x} \leq B = k$$

Notice that $\vec{c} \cdot \vec{x}$ is simply the number of 1's in $\vec{x}$.

Proof, cont'd

$\Rightarrow$ If G has a vertex cover V^* of size k, define

$$x_i^* = \begin{cases} 1 & \text{if } v_i \in V^*; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\vec{c} \cdot \vec{x^*} = k \leq B$.

To see that $A\vec{x^*} \leq \vec{b}$, consider the ith value of $A\vec{x^*}$

$$= \sum_{j=1}^{n} a_{ij} x_j^* = -1 \times \vec{x_u^*} + -1 \times \vec{x_v^*} \leq -1 \text{ where } e_i = (u, v)$$

because either $\vec{x_u^*} = 1$ or $\vec{x_v^*} = 1$. Hence $A\vec{x^*} \leq \vec{b}$.

Cont'd

$\Leftarrow$ Assume that $\exists \vec{x}$ s.t. $A\vec{x^*} \leq \vec{b}, \vec{c} \cdot \vec{x} \leq B = k$.

Define $V_{\vec{x}} = \{v_i | \vec{x_i} = 1\}$, clearly $|V_{\vec{x}}| \leq k$. For each $e_i = (u, v) \in E$, $\sum_{j=1}^{n} a_{ij}\vec{x_j} \leq -1$

$$\Rightarrow -1 \times \vec{x_u^*} + -1 \times \vec{x_v^*} \leq -1$$

$$\Rightarrow \text{either } x_u = 1 \text{ or } x_v = 1$$

$$\Rightarrow \text{either } u \in V_{\vec{x}} \text{ or } v \in V_{\vec{x}}$$

$$\Rightarrow \text{edge } e_i \text{ is covered by } V_{\vec{x}}$$

Hence $V_{\vec{x}}$ is a vertex cover of size k

An more General Problem

A more general version is Integer Linear Programming (ILP)

Decision Version Given an (m, n) matrix A of integers, an m-vector $\vec{b}$ of integers, an n-vector $\vec{c}$ of integers and an integer B, determine whether there exists an n-vector $\vec{x} \in \mathbb{Z}^n$ s.t.

$$A\vec{x} \leq \vec{b}$$

where $\leq$ is component-wise less than and

$$\vec{c} \cdot \vec{x} = \sum_{i=1}^{n} c[i] * x[i] \leq B$$

.

Optimization Version Given $A, \vec{b}, \vec{c}$ as above, produce $\vec{x} \in \mathbb{Z}^n$ that minimizes $\vec{c} \cdot \vec{x}$ and $A\vec{x} \leq \vec{b}$.

Cont'd

Integer Linear programming is NP-complete.

Hint: Perform a reduction from 0-1 ILP or Vertex Cover.

Integer Linear Programming (0-1 ILP), Cont'd

Decision Version Given an (m, n) matrix A of integers, an m-vector $\vec{b}$ of integers, an n-vector $\vec{c}$ of integers and an integer B, determine whether there exists an n-vector $\vec{x} \in \mathbb{Z}^n$ s.t.

$$A\vec{x} \leq \vec{b}$$

where $\leq$ is component-wise less than and

$$\vec{c} \cdot \vec{x} = \sum_{i=1}^{n} c[i] * x[i] \leq B$$

Cont'd

Claim: Integer Linear programming is NP-complete.

Proof:
It is easy to see that ILP is in NP. We can guess a solution $\vec{x} \in \mathbb{Z}^n$ and check if it satisfies the inequalities.

Now we are going to show that 0-1 ILP $\leq_m{}^p$ ILP. Hence ILP will be NP-complete by property #3 from previous notes.

Intuitively, we want to add some constraints to the instance of ILP to force the solution $\vec{x}$ to be in $\{0, 1\}^n$.

Additional Constraints

To achieve this, let a row of A contain a single 1 at position i and let the same row of b be 1, hence $x_i \leq 1$; similarly, let a row of A contain a single -1 at position i and let the same row of b be 0, we get $x_i \geq 0$ from inequality $A\vec{x} \leq \vec{b}$.

Cont'd

Given an instance of 0-1 ILP $\langle A, \vec{b}, \vec{c}, B \rangle$, where A is a (m, n) matrix, define A' to be an $(m' = m + 2n, n)$ matrix where

$$
\begin{aligned}
a'_{ij} &= a_{ij} & \text{if } 1 \le i \le m,\ 1 \le j \le n \\
a'_{m+2i-1,j} &= \left\{ \begin{array}{ll} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{array} \right\} & \text{if } 1 \le i \le n \\
a'_{m+2i,j} &= \left\{ \begin{array}{ll} -1 & \text{if } i = j \\ 0 & \text{otherwise} \end{array} \right\} & \text{if } 1 \le i \le n
\end{aligned}
$$

Cont'd

and $\vec{b}'$ to be an $(m + 2n)$ vector where

$$
\begin{array}{ll}
{b_i}' = b_i & if 1 \leq i \leq m \\
b'_{m+2i-1} = 1 & if 1 \leq i \leq n \\
b'_{m+2i} = 0 & if 1 \leq i \leq n
\end{array}
$$

and $\vec{c}' = \vec{c}$, $B' = B$.

Cont'd

The many one reduction is

$$f(\langle A, \vec{b}, \vec{c}, B\rangle) \longrightarrow \langle A', \vec{b}', \vec{c}', B'\rangle$$

Technically, we need to prove that

$$\langle A, \vec{b}, \vec{c}, B\rangle \in \text{in 0-1 ILP iff} \langle A', \vec{b}', \vec{c}', B'\rangle \in \text{ILP}.$$

Iff

$\Rightarrow$ Assume we have a solution $\vec{x}$ for 0-1 ILP. I claim that this is also a solution to the new problem in ILP. (Easy to see.)

$\Leftarrow$ Assume that we have a solution $\vec{x}$ to the new problem $\langle A', \vec{b}', \vec{c}', B' \rangle$, then it must be the case that

$$\begin{bmatrix} A \\ A'' \end{bmatrix} \vec{x} \leq \vec{b}' = \begin{bmatrix} \vec{b} \\ \vec{b}'' \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A\vec{x} \\ A''\vec{x} \end{bmatrix} \leq \begin{bmatrix} \vec{b} \\ \vec{b}'' \end{bmatrix}$$

Cont'd

The bottom part $A''\vec{x} \leq \vec{b}''$ implies that $\vec{x} \in \{0, 1\}^n$ and the top part $A\vec{x} \leq \vec{b}$ implies that $\vec{x}$ satisfies the inquality. So $\langle A, \vec{b}, \vec{c}, B \rangle$ has a solution $\vec{x} \in$ 0-1 ILP.

Good News/Bad News

So the bad news is that 0-1 ILP and ILP are NP-complete. But the good news is that LP is solvable in polynomial time (GLPK GNU Linear Programming Kit).

LP is defined as: Given $A, \vec{b}, \vec{c}$ as before, find a vector $\vec{x} \in \mathbb{R}^n$, s.t. $A\vec{x} \leq \vec{b}$ and $\vec{c} \cdot \vec{x}$ is minimized.

Relaxing Integer Linear Programs

You can "relax" integer linear programming problems to form linear programming problems.

$$x_i \in 0,1 \Longrightarrow^{\text{relax}} 0 \leq x_i \leq 1 \text{ where } x_i \in \mathbb{R}$$

Let $\vec{x}^*$ be an optimal solution to the 0-1 ILP $\langle A, \vec{b}, \vec{c} \rangle$. The optimal value of the relaxed version $\langle A', \vec{b}', \vec{c}' \rangle$ is always a lower bound on the value of the optimal solution to the 0-1 ILP problem because $A\vec{x}^* \leq \vec{b}$ and hence $\vec{x}^*$ is a solution (but not necessarily the minimum one to the relaxed problem).