AMATH 271 Final Project

Three's a Crowd

Lana Bozanic, Benjamin Wong

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Abstract

The three-body problem is an important classical mechanics problem with many applications, yet its complexity is greater than the two-body problem we have seen in class. Many say that we have yet to completely understand the three-body problem's complexity, yet alone the complexity of higher body problems [6]. In this project, we consider a special case of the three-body problem. We consider the effects of adjusting the orbit radii of the triple star system HD-188753 and the familiar Sun-Earth-Moon. HD-188753 is composed of a binary system that orbits around a central star. Under its current conditions, we can approximate this three-body problem as a composition of two two-body problems. One two-body problem for the binary star system, and one where we consider the binary system as one body that orbits the central star. We are interested in finding the semi-major axis of the central star's orbit such that we can no longer approximate the system with two-body problems and the system descends into chaos.

Contents

1 Introduction

The three-body problem is a classical problem that describes the motion of three massive bodies under gravity given some initial conditions (such as distance, velocity, etc.). However, unlike the two-body problem, there is no general solution. Its complexity is still unknown and is vastly understudied [6].

Here, we wish to analyze a special case of the three-body problem. We investigate the triple-star system HD-188753 (informally known as the lana system) (Figure 1). This star system is composed of a binary star system (we shall refer to them as HD-188753 BC) that orbits around a larger star, HD-188753 A.

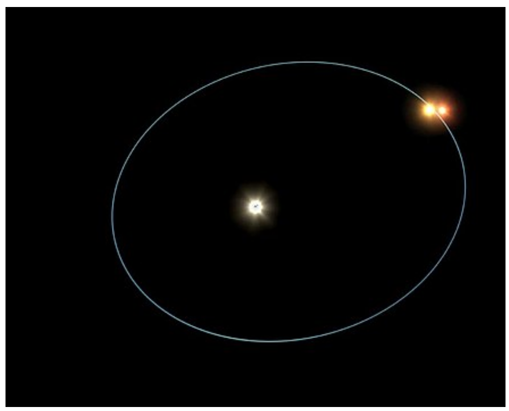

Figure 1: The orbit of HD-188753 [5]

Additionally, we shall also consider the Sun-Earth-Moon (SEM) system. In doing so, we hope to create an accurate model to describe three-body systems by first considering a familiar case. Then, we shall extend it to HD-188753 so we can better understand its behaviour under certain initial conditions.

2 Background

The three-body problem is a very complex problem that was investigated by many mathematicians over the years such as Poincaré, Newton and Kepler. Its goal is to understand the motion of three massive bodies (such as stars) subject to gravitational forces. Unlike its younger cousin, the two-body problem, it was proved by Poincaré in the 1890s that no general closed form solution exists when given only the positions, velocities and mass ratios of the three-bodies. [6]

For application's sake, there are many numerical methods that can be used to approximate the motions of three-bodies to a very high accuracy as we shall see in our analysis. However, it is yet to be shown that we can express exact solutions in terms of sophisticated mathematical functions. One big reason that many believe we won't be able to express these solutions in terms of these sophisticated functions is because of the sensitivity of initial conditions. Small changes to the initial conditions can lead to vastly different orbits.

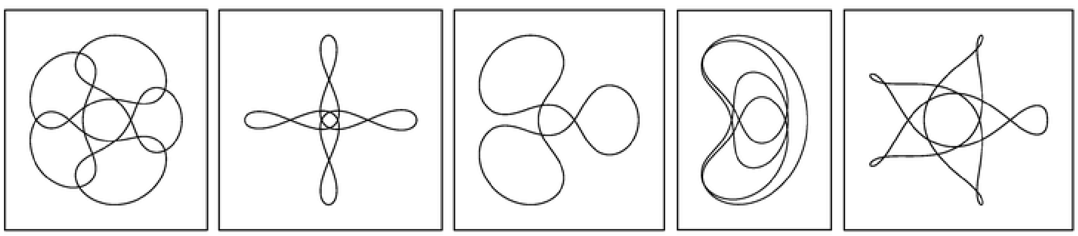

Figure 2: Sample 3-body orbits [6].

Figure 2 shows a couple of orbit patterns, discovered by Poincaré, for the three-body problem of a planet orbiting two stars. This is only a small sample of the possibilities for these orbits as they can descend into much more chaotic states. We see that even for the simple problem of a planet orbiting two stars the orbit pattern can vary widely.

We are interested in understanding the chaotic natures of the three-body problem. One of the systems we will consider, is the HD-188753 star system. This is a triple star system located about 155 million light years away.

It is composed of a binary star system consisting of two stars with masses $0.67M_{\odot}$ and $0.96M_{\odot}$[1]. These two masses orbit each other about their barycenter. We call this binary system HD-188753 BC. This system has a semi-major axis of about 0.67 AU (eccentricity 0.1), and orbit each other with a period of approximately 156 days.

HD-188753 BC orbits another star, HD-188753 A which is of mass $1.06M_{\odot}$[1]. They orbit at a distance of about 12.3 AU (eccentricity of 0.5) and have a period of about 25.7 years[1] This triple star system follows the orbit depicted in Figure 1.

We see that, in its current state, the system behaves approximately like a two-body system. If we treat the binary system HD-188753 BC as a single body and solve the two-body problem, we can still have an accurate depiction of the motion of the system.

However, the authors establish that in these hierarchical three-body systems, when the binary system approaches the star that it is orbiting the gravitational forces lead to chaotic dynamics when the inner and outer radii are comparable. We are interested in finding the radius at which we can no longer approximate the HD-188753 system as a pair two-body systems.

3 Equations

The three-body problem is formulated in a very similar way to the two-body problem. In the two-body problem, for masses m_1 and m_2 with positions $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$, we can describe the problem using the differential equations

$$\ddot{\boldsymbol{r}}_{\boldsymbol{1}} = -Gm_2 \frac{\boldsymbol{r_1} - \boldsymbol{r_2}}{|\boldsymbol{r_1} - \boldsymbol{r_2}|^3},$$

$$\ddot{\boldsymbol{r}}_{\boldsymbol{2}} = -Gm_1 \frac{\boldsymbol{r_2} - \boldsymbol{r_1}}{|\boldsymbol{r_2} - \boldsymbol{r_1}|^3}.$$

Introducing a third mass, m_3 with position $\boldsymbol{r_3}$, we get a new set of DEs describing the three-body problem[4]:

$$\ddot{\boldsymbol{r}}_{\boldsymbol{1}} = -Gm_2 \frac{\boldsymbol{r_1} - \boldsymbol{r_2}}{|\boldsymbol{r_1} - \boldsymbol{r_2}|^3} - Gm_3 \frac{\boldsymbol{r_1} - \boldsymbol{r_3}}{|\boldsymbol{r_1} - \boldsymbol{r_3}|^3}, \tag{1}$$

$$\ddot{\boldsymbol{r}}_{\boldsymbol{2}} = -Gm_1 \frac{\boldsymbol{r_2} - \boldsymbol{r_1}}{|\boldsymbol{r_2} - \boldsymbol{r_1}|^3} - Gm_3 \frac{\boldsymbol{r_2} - \boldsymbol{r_3}}{|\boldsymbol{r_2} - \boldsymbol{r_3}|^3}, \tag{2}$$

$$\ddot{\boldsymbol{r}}_{\boldsymbol{3}} = -Gm_1 \frac{\boldsymbol{r_3} - \boldsymbol{r_1}}{|\boldsymbol{r_3} - \boldsymbol{r_1}|^3} - Gm_2 \frac{\boldsymbol{r_3} - \boldsymbol{r_2}}{|\boldsymbol{r_3} - \boldsymbol{r_2}|^3}. \tag{3}$$

In general we can extend this to any n-body problem by adding more and more terms describing the gravitational interaction between each mass to every other mass as we've shown with the previous collections of DEs.

We can also express this problem in terms of the Hamiltonian formulation[5].
The Hamiltonian formulation for the three-body problem would consist of 18 differential equation for the position $\boldsymbol{r}$ and momentum $\boldsymbol{p}$ of each mass:

$$\frac{d\boldsymbol{r_i}}{dt} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p_i}}, \qquad \frac{d\boldsymbol{p_i}}{dt} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{r_i}},$$

for $i = 1, 2, 3$, and Hamiltonian

$$\mathcal{H} = -\frac{Gm_1m_2}{|\boldsymbol{r_1} - \boldsymbol{r_2}|} - \frac{Gm_2m_3}{|\boldsymbol{r_2} - \boldsymbol{r_3}|} - \frac{Gm_3m_1}{|\boldsymbol{r_3} - \boldsymbol{r_1}|} + T,$$

where T is the kinetic energies of each planet. For this project, we will analyze our problems in terms of the Newtonian formulation.
In order to simulate the dynamics of our three body systems, the authors have use Maple to solve the nonlinear system of ODEs as above. As below, the reduction to the two-body case will allow the authors to choose suitable initial conditions.

4 Analysis

The motion of our system is given by the governing equations (1),(2),(3). Despite the analytic intractability of these equations, there is still much we can learn from Newton's law of gravitation. We analyze the system by reducing it to the two-body case in order to determine suitable initial conditions given any radius for the Sun-Earth-Moon system, or any semi-major axis for HD-188753.

4.1 Keplerian Orbits

An important task in accurately modelling a three-body system is choosing accurate initial conditions. When we simplify the problem down to a two-body system, they can be described using Keplerian orbits. As mentioned in the background, triple-body systems are very sensitive to initial conditions. We can approximate them by assuming circular orbits in most cases, however, when the eccentricity of the orbit is large, it is no longer suitable. Thus, to accurately model our system we will specify our initial conditions assuming non-circular orbits when eccentricities are non-negligible.

First, we consider the initial velocity of the system. Assuming a circular orbit, it is well known that our initial velocities will be

$$V = \sqrt{\frac{GM}{r}}, \tag{4}$$

for the circular radius, r, and mass M of the body being orbited. However, if we want to consider a velocity at a given point in an elliptical orbit we must instead use the vis-viva equation [2]

$$V = \sqrt{\mu\Big(\frac{2}{r} - \frac{1}{a}\Big)}, \tag{5}$$

where μ is the standard gravitational constant (defined as $\mu = GM$ for large / small body systems, or as $\mu = Gm_1 + Gm_2$ for binary systems with masses m_1 and m_2), r is the distance of the two bodies at a given period, and a is the semi-major axis of the ellipse.

Another thing we must consider is also the period of the orbit. When assuming circular orbit, we have that our period is defined as

$$T = \frac{2\pi r}{V}, \tag{6}$$

for the circular radius r, and velocity V of the orbiting mass. However, in an elliptical orbit we define our period as

$$T = 2\pi\sqrt{\frac{a^3}{\mu}}, \tag{7}$$

where a is the semi-major axis of the ellipse, μ is the standard gravitational parameter.

Finally, we consider the initial distances between our bodies. If we assume a circular orbit (which we can do if we have a very small orbital eccentricity), we can assume that for a binary star system, the stars start at a position of $2r$ from each other, r being the distance between the stars and their barycentre.

On the other hand, if our system's eccentricity is non-negligible, we have to carefully choose an initial distance. This can once again be done using Keplerian orbital mechanics. It is known that for an elliptical orbit with eccentricity e and semi-major axis a, the distance of a smaller body orbiting a larger body, the perihelion position is located at [3]

$$R_p = a(1 - e), \tag{8}$$

and the aphelion position is located at

$$R_a = a(1 + e). \tag{9}$$

Thus, for our initial conditions, we can assume that we begin at either aphelion or perihelion to simplify our calculations, as the DEs governing the motion will take care of the rest of the mechanics.

4.2 Initial Conditions for Sun-Earth-Moon (SEM)

For the SEM system, both the orbit of the Earth-Moon subsystem and the orbit of the Earth around the sun have low eccentricity. Hence the equations of circular motion can be applied. The radius or orbit can be used to determine the initial positions. Equation 4 can be used to determine the initial velocities. These values are listed in section 5.1.

4.3 Initial Conditions for HD-188753

Having understood the SEM system, turn our attention to HD-188753. We first model the system using the physically observed conditions. Furthermore, we model the subsystems in their approximate two-body forms, where the binary system HD-188753 BC orbits around HD-188753 A.

We first consider the HD-188753 BC system on its own. This binary system of comparable masses orbit about their barycenter in an almost circular orbit. The eccentricity of this orbit is very small (compared to the eccentricity of the orbit of BC around A), so we assume that it is approximately circular.

Given our assumptions for HD-188753, we are now ready to build set initial conditions. We know that since we assume a circular orbit, it is relatively easy to find the initial velocities and distances of the two bodies. We shall assume that the two bodies are a distance of $2r$ away from each other. Given that their semi-major axis is about 0.67 AU, we are approximating that the two stars start off at a distance of 1.34 AU from each other.
Next, we must declare their initial velocities. Once again since the problem is simplified we can use Newtonian laws for the velocity, and we'll find that the initial velocities of our binary stars are defined by (6).

Using Maple to plot the motion of the binary system for 100 years. Figure 8 is the result of plotting the solution curves of the governing DEs.

We can see that the binary system orbits a centre of mass and the plot also depicts the orbital period of about 156 days, as expected. Thus, our model paired with the right initial conditions accurately portrays this binary system!

5 Nonlinear Solutions

We present nonlinear solutions to both the Sun-Earth-Moon (SEM) and HD-188753 systems. Due to the large mass of the sun relative to the earth and moon and the three orders of magnitude of distance between the inner and outer radii of the SEM system, it exhibits simpler dynamics, yet still captures the essence of the problem. On the other hand, HD-188753 has three stars of similar mass and the radii only differ by one order of magnitude. Consequently, there is a greater margin for error when choosing initial conditions, yet there is more chaotic behaviour near the critical radius ratios.

The method used to find solutions for both systems is identical. Firstly, a relation between the positions and velocities of the bodies is derived so that the initial conditions would lead to elliptical orbits of the bodies in the absence of the third body. The first set of initial conditions are chosen so that the orbits of the bodies are identical to the observed orbits. Next, several lengths of semi-major axis were chosen heuristically to demonstrate various behaviours of the system. Finally, Maple was used to solve Newton's equations of motion and animate plots of the curves. To improve visibility, the largest mass was translated to the origin in many of the animations.

5.1 Parameters

The dynamics of our system are governed by gravitational interactions. Hence the bodies, masses, positions, and velocities are the only necessary ingredients to get cooking. However, it also useful to work with some of the standard quantities when discussing orbits. For example, we frequently refer to the standard gravitational parameter μ and the semi-major axis a. Below we tabulate the values of the parameters that are present in our computations. The following table contains the mass of each body.

Body:	Sun	Earth	Moon	Star A	Star B	Star C
Mass:	$1M_\odot$	$5.97 \cdot 10^{24}$ kg	$7.35 \cdot 10^{11}$ kg	$1.06M_\odot$	$0.96M_\odot$	$0.67M_\odot$

The following table contains the standard gravitational parameter $\mu := G(m_1 + M_2)$, the semi-major axis a, and the eccentricities e for the orbits of HD-188753.

Orbit:	A & B-C	B & C
μ	$3.57 \cdot 10^{20}\ \frac{Nm^2}{kg^2}$	$2.16 \cdot 10^{20}\ \frac{Nm^2}{kg^2}$
a	12.3 AU	0.67 AU
e	0.5	0.1

The table below has gives the values of the semi-major axis of the outer orbit that we shall consider in our computations. We will begin by computing the systems as-is, and then shrinking their orbits to observe their behaviours. In the case of the Sun-Earth-Moon system, since the outer orbit is approximately circular we approximate the semi-major axis to be the radius r. Below a is the semi-major axis of the outer orbit of HD-188753.

	1	2	3	4	5
SEM	r	$5 \cdot 10^{-2}r$	$5 \cdot 10^{-3}r$	$5 \cdot 10^{-4}r$	
HD-188753	a	$0.75a$	$0.5a$	$0.25a$	$0.167a$

Finally we give the initial conditions which were computed using the formulae (5) and (8).

Body	Sun	Earth	Moon	Star A	Star B	Star C
$x(0)(m)$	0	$1.4730 \cdot 10^{11}$	$1.4768 \cdot 10^{11}$	0	$7.92 \cdot 10^{11}$	$8.83 \cdot 10^{11}$
$y(0)(m)$	0	0	0	0	0	0
$x'(0)(\frac{m}{s})$	0	0	0	0	0	0
$y'(0)(\frac{m}{s})$	0	$3.00 \cdot 10^{4}$	$3.10 \cdot 10^{4}$	0	$-5.62 \cdot 10^{3}$	$4.57 \cdot 10^{4}$

5.2 Sun-Earth-Moon System

Before attacking HD-188753, we warm up by analyzing nonlinear solutions of the three-body system that we call home. Under gravitational influence, the sun, earth, and moon form a hierarchy identical to HD-188753. The moon orbits the earth, and the earth-moon binary system orbits the sun. In the figures, the earth is the blue trajectory and the moon is the green trajectory.

This system exhibits simpler behaviour than HD-188753 for two reasons. Firstly, the sun's mass is large in comparison to the earth and the moon. This means that at certain distances, the forces due to the earth and the moon on each other become small compared to the force due to the sun's gravity. Secondly, the eccentricity of the earth-moon system's orbit around the sun is 0.0167 [3] meaning that the orbit is nearly circular. Hence the familiar equations of circular motion have been used to approximate initial conditions as above.

Although the solutions of the system (1),(2),(3) cannot be written in closed form, the language used to describe solutions of the two-body problem will prove useful for qualitatively describing solutions of (1),(2),(3) because the solutions analyzed below are approximately two-body solutions.

The SEM system exhibits three qualitatively different behaviours. At the original radius, the moon orbits the earth and the earth-moon binary system orbits the sun as usual as depicted in Figure 4. Here, the distance between the earth and the moon is too small to see on the plot, but is depicted in Figure 3.

When the radius is reduced to $5 \cdot 10^{-2}$ or $5 \cdot 10^{-3}$ of its usual size the force of gravity due to the sun becomes more relevant than the force of gravtiy due to the earth on the moon and both bodies orbit the sun as in Figure 5 and Figure 6. Notice that the eccentricity of the moon's orbit is more apparent when both bodies begin close to the sun. This is because the speed to radius ratio is larger when the bodies are moved nearer. This dependence will become even clearer in the final behaviour.

When the radius is reduced to $5 \cdot 10^{-4}$ of its original size, the moon follows an approximately hyperbolic path instead of orbiting the earth or the sun as before. This behaviour can be seen in Figure 7.

With the Sun-Earth-Moon system, we were able to extract several qualitatively different trajectories including an ejection on one of the bodies when the inner and outer radii become comparable as expected.

5.3 HD-188753

We are ready to turn out attention to the system HD-188753. This system is made up of three stars with similar masses. The smaller two stars B and C form a binary system which in turn orbits star A. The fact that these stars have similar masses distinguish it from the SEM system in several ways. Firstly, the initial conditions must be chosen with much more care. To accomplish this, the authors abandoned the approximation of circular motion and applied the theory of Keplerian orbits as in the analysis section. Secondly, the inner and outer radius begin only one order of magnitude apart which is much smaller than the three orders of magnitude between the inner and outer radii of the SEM system. For all of the following figures, Star A is fixed at the origin, star B is the red trajectory, and Star C is the blue trajectory.

HD-188753 exhibits three qualitatively different behaviours. Initially, the subsystem BC orbits the star A as shown in Figure 9. When the semi-major axis is taken to be 0.75 times as large, the orbit is smaller, yet the qualitative behavior is the same as in Figure 10.

As expected, chaotic behaviour emerges as the inner and outer radii become comparable. When the semi-major axis is chosen to be 0.5 or 0.25 times its original size, we have complicated behaviour as shown in Figure 11 and Figure 12.

Finally, when the semi-major axis is 0.1667 times its original size or smaller, the roles of the inner and outer radius interchange and now we have a binary system of star A and star B which has C orbiting that system as shown in Figure 13. This return to stable behaviour suggest that chaotic behaviour is characteristic of a specific ratio of inner and outer orbit radii and not strictly dependent on reducing the size.

6 Conclusions

Armed with a slew of Maple computations, we can finally reflect on the problem of determining which radii lead to stability and which radii lead to chaotic behaviour. For each three-body system, we began with the known stable orbits and then simulated the time-evolution of the system while varying the ratio between the inner and outer radii.

The time evolution was computed numerically in Maple by using Newton's second law to give the equations of motion and data from the physical systems as well as their stable 2-body counterparts to set appropriate initial conditions. After an appropriate relationship between semi-major axis and initial velocity. For the Sun-Earth-Moon system, this relationship came from the fact that the eccentricities of the orbits were small and the standard equations or circular motion were sufficient to find appropriate initial conditions. For HD-188753, the eccentricities were non-trivial and the equations governing Keplarian orbits were used to set the initial conditions as the semi-major axis distance was shrunk.

For the Sun-Earth-Moon system, the nonlinear solutions revealed three qualitatively different behaviours for the three-body system. At the physically observed radius, the earth-moon subsytem orbits the sun. Next, as the system is moved closer to the sun, the sun's gravitational influence becomes more prominent than that of the earth and both the earth and moon orbit the sun. Finally, when the inner and outer radii are comparable, the earth falls into a highly eccentric orbit around the sun and the moon flies off on a approximately hyperbolic trajectory. Throughout the simulations, the moon was set at a distance and initial velocity so that it would orbit the earth in the absence of the sun. It was this initial velocity that caused the moon to be ejected from the system at small radii. Interestingly, at nearly all smaller radii the moon does something other than orbiting the earth, hence it is not very easy to become a moon for the earth. At that magnitude of mass, the sun's gravitational pull would be strong enough to pull it out of earth's orbit at nearly any smaller radius than earth's current orbit radius.

The most fascinating dynamics were observed in the triple star system HD-188753. At the physical semi-major axis distance, one can see the binary system BC orbiting star A. In the case of the Sun-Earth-Moon system, the difference between the inner and outer radii is too great to see both orbits at once. With HD-188753, the orbit radii only differ by one order of magnitude so all three bodies are visible in the plot with the observed semi-major axis. When the semi-major axis is reduced to 75% of it's original size, the binary system BC still orbits A in a stable manner. However, at 50% and 25% of the original semi-major axis distance the three bodies interact chaotically. Finally at 17% and below, the roles of the inner and outer radius swap roles and the subsystem AB orbits C in a stable manner for this distance and smaller.

Since the sun's mass was much larger than the masses of the earth and the moon, the sun was responsible for the simpler dynamics of the first system. In HD-188753, all three masses are within a factor of 1.6 of each other and the radii are only different by one order of magnitude compared to the three orders of magnitude difference in the inner and outer radii of the Sun-Earth-Moon system. Both systems had stable orbits when the outer radius was much larger than the inner radius, changed behaviour as the radii became comparable and settled into a novel stable orbit as the outer radius became much smaller than the inner radius. In this case, the roles of the inner and outer radii are reversed.

References

[1] Kaplana (Ed.). (2018, May 7). *HD 188753*. Alchetron.com. Retrieved from https://alchetron.com/HD-188753.

[2] Van Hamme. (n.d.). *Orbital Mechanics*. Retrieved from https://faculty.fiu.edu/ vanhamme/ast3213/orbits.pdf.

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[4] Wikimedia Foundation. (2021, October 4). *N-body problem*. Wikipedia. Retrieved from https://en.wikipedia.org/wiki/N-body$_{p}roblem$.

[5] Wikipedia. (2021, October 31). *HD 188753*. Wikipedia. Retrieved from https://en.wikipedia.org/wiki/HD$_{1}$88753.

[6] Wolfram, S. (2019). *Chaos Theory and Randomness from Initial Conditions*. In A new kind of science. essay, Wolfram Media.

7 Appendix: Non-linear Figures

The non-linear solutions and their plots have been open sourced. The Maple files are also on the Github for the reader's viewing pleasure. The program used to determine the initial conditions from the two-body approximation and and the program used to simulate the trajectories for varied outer radius orbits of HD-188753 are attached at the end as requested as well. The equivalent programs for the Sun-Earth-Moon system are available on github as below, but are variations on the two included programs.
View at https://github.com/lanabozanic/amath271-final-project

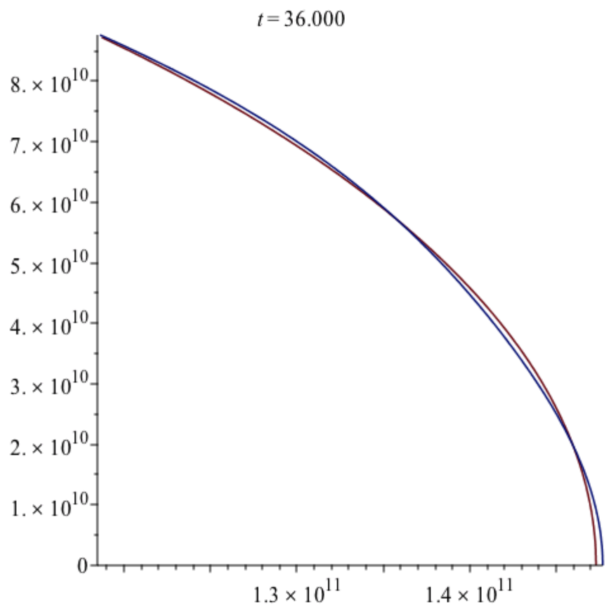

Figure 3: Quarter of the Earth (red) and moon (blue) orbit around the sun. Animation can be found here

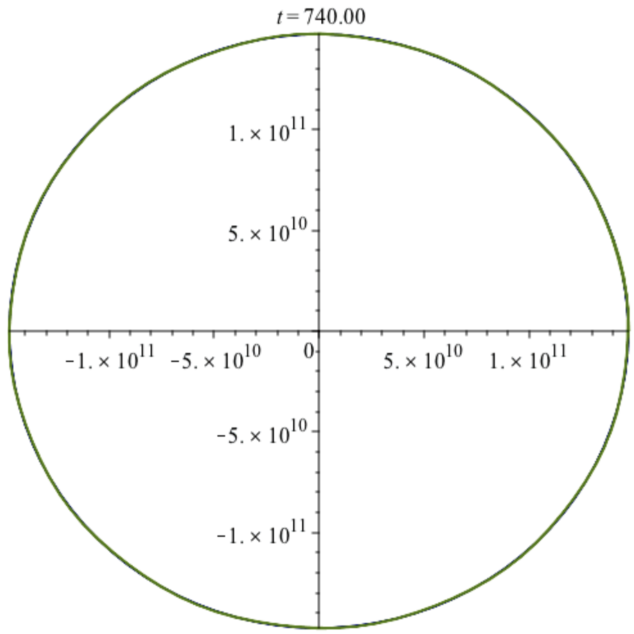

Figure 4: Earth (red) and moon (blue) orbit around the sun. (Full orbit) Animation can be found here

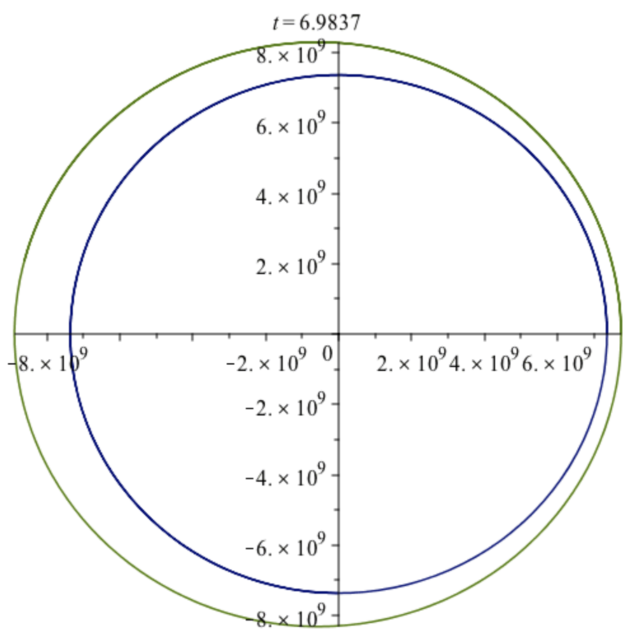

Figure 5: Earth (red) and moon (blue) orbit around the sun, with $R' = R/20$ Animation can be found here

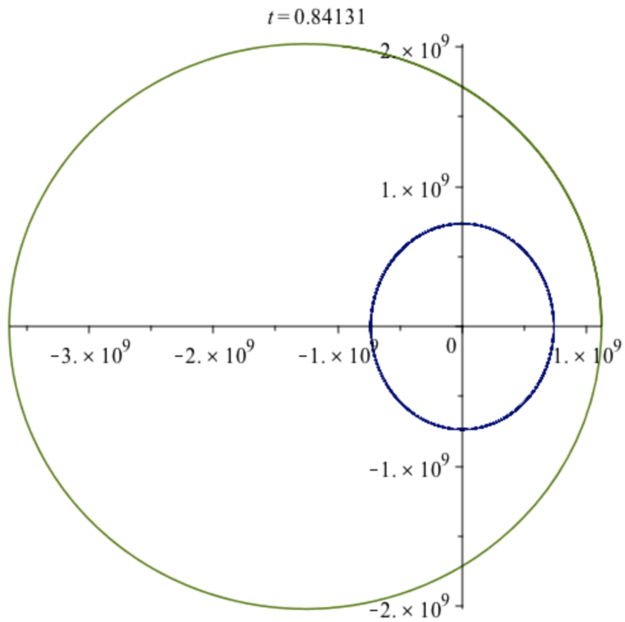

Figure 6: Earth (red) and moon (blue) orbit around the sun, with $R' = R/200$ Animation can be found here

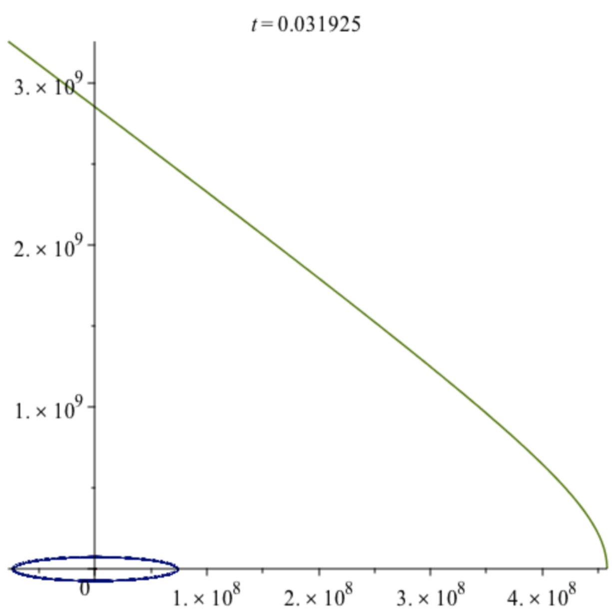

Figure 7: Earth (red) and moon (blue) orbit around the sun, with $R' = R/2000$ Animation can be found here

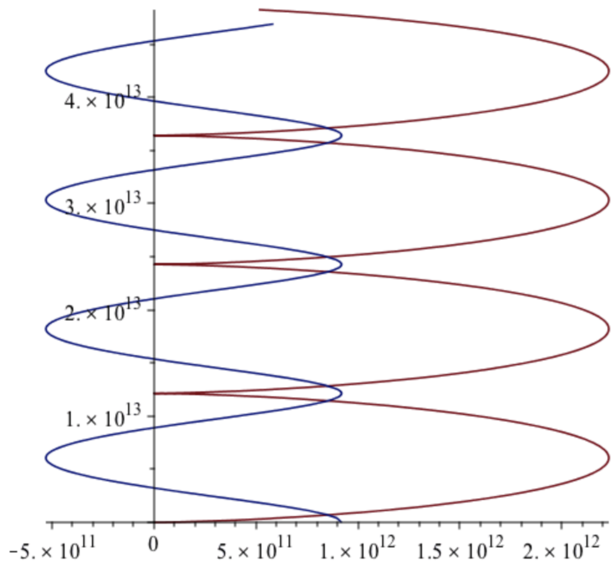

Figure 8: Motion of HD-188753 BC independent of HD-188753 A. The blue solution represents the smaller mass, and the red the larger mass in the binary system. Animation of the plot can be also found here.

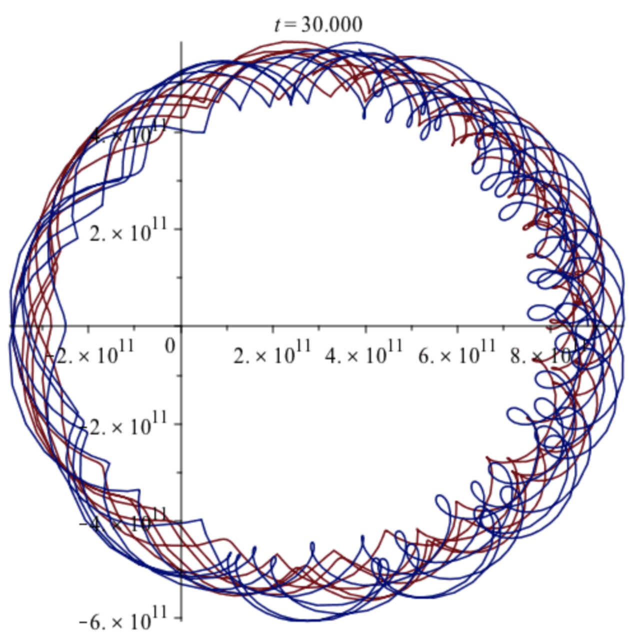

Figure 9: Orbit of HD-188753 BC around HD-166753 A. Animation of the plot can be also found here.

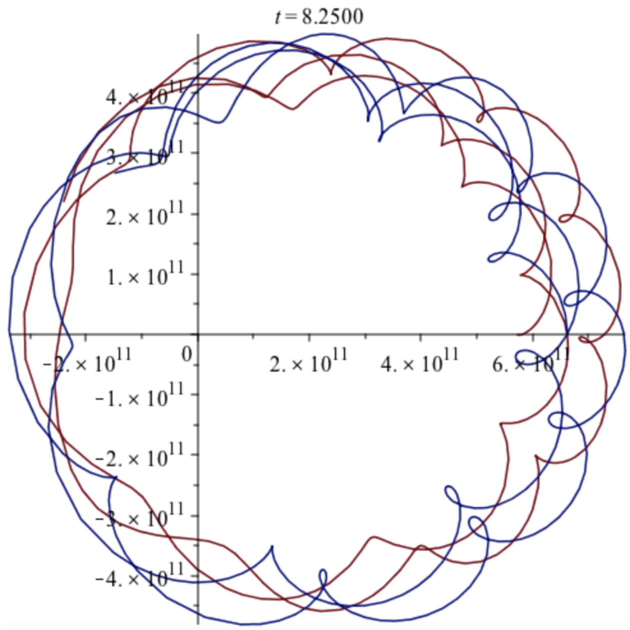

Figure 10: Orbit of HD-188753 BC around HD-166753 A, where the semi-major axis $a' = 3a/4$. Animation of the plot can be also found here.

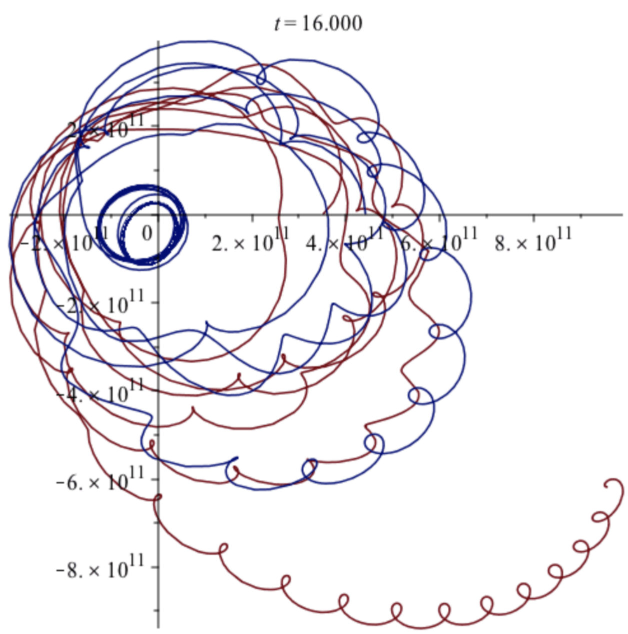

Figure 11: Orbit of HD-188753 BC around HD-166753 A, where the semi-major axis $a' = a/2$. Animation of the plot can be also found here.

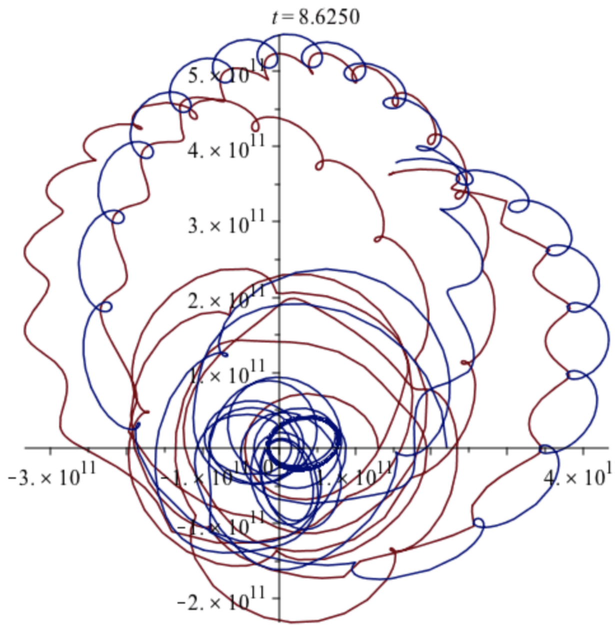

Figure 12: Orbit of HD-188753 BC around HD-166753 A, where the semi-major axis $a' = a/4$. Animation of the plot can be also found here.

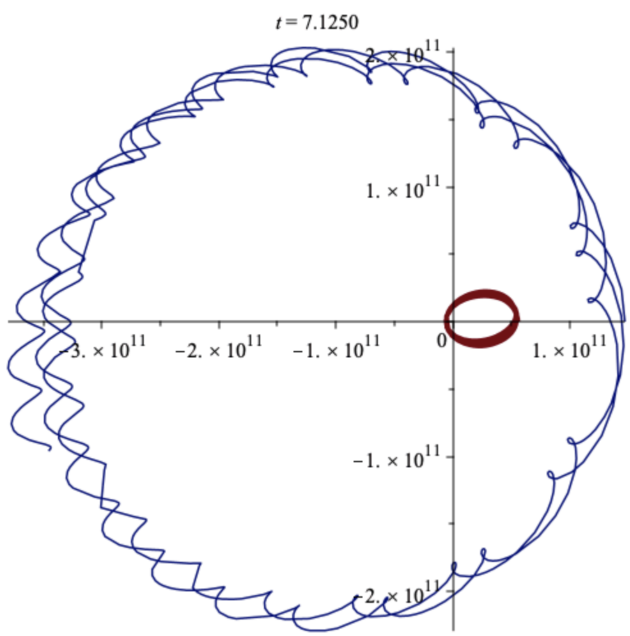

Figure 13: Orbit of HD-188753 BC around HD-166753 A, where the semi-major axis $a' = a/6$. Animation of the plot can be also found here.