32AH CHALLENGE PROBLEM 2

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1. The Taxicab and Chebyshev Metrics

Definitions 1.1-1.2 are on the 32AH Challenge Problem Set 2 sheet.
Definition 1.3 (Properties of metric spaces). Let X be a set, and $d: X \times X \to R$ be a map that satisfies the following three properties for all $a, b \in X$:

1. Symmetry
   $d(a, b) = d(b, a)$.
2. Non-negativity and positive-definiteness
   $d(a, b) \geq 0$, and $d(a, b) = 0$ if and only if $a = b$.
3. Triangle inequality
   $d(a, b) \leq d(a, c) + d(c, b)$ for any $c \in X$.

Then we say that (X, d) is a metric space.
Definition 1.4 (Properties of absolute value) Let $x, y \in R$.

1. Multiplicativity
   $|xy| = |x||y|$. In particular, $|- x| = |x|$.
2. Non-negativity
   $|x| \geq 0$.
3. Positive-definiteness
   $|x| = 0 \Leftrightarrow x = 0$.

Theorem 1.5 (Triangle inequality for real numbers) Let $x, y \in R$. Then,

$$||x + y|| \leq ||x|| + ||y||.$$

Definition 1.6 (Open ball) Let (X , d) be a metric space and $u, v \in R^2$ where v is the center. An open ball with radius 1 around v is set:

$$B_r(v) := \{u \in X \mid d(u, v) < r\}$$

Result 1.1 We will prove that the taxicab metric on R^n is a metric. Let $u = \langle u_1, \dots, u_n \rangle$ and $v = \langle v_1, \dots, v_n \rangle \in R^n$. The taxicab metric $d_1 : R^n \times R^n \to R$ is given by the formula:

$$d_1(u, v) = \sum_{i=1}^{n} \left| u_i - v_i \right| \text{ (\textit{Definition 1.1})}.$$

For the taxicab metric on R^n,

1. Prove that $d_1(u, v) = d_1(v, u)$.

$$d_1(u, v) = d_1(v, u).$$

$$\sum_{i=1}^{n} \left| u_i - v_i \right| = \sum_{i=1}^{n} \left| v_i - u_i \right|$$

By the multicative property of absolute value,

$$\sum_{i=1}^{n} \left| u_i - v_i \right| = 1 \cdot \sum_{i=1}^{n} \left| v_i - u_i \right|$$

$$\sum_{i=1}^{n} \left| u_i - v_i \right| = \sum_{i=1}^{n} 1 \cdot \left| v_i - u_i \right|$$

$$\sum_{i=1}^{n} \left| u_i - v_i \right| = \sum_{i=1}^{n} |-1| \cdot \left| v_i - u_i \right|$$

$$\sum_{i=1}^{n} \left| u_i - v_i \right| = \sum_{i=1}^{n} \left| -1(v_i - u_i) \right|$$

$$\sum_{i=1}^{n} \left| u_i - v_i \right| = \sum_{i=1}^{n} \left| -v_i + u_i \right|$$

$$\sum_{i=1}^{n} \left| u_i - v_i \right| = \sum_{i=1}^{n} \left| u_i - v_i \right| \text{ (\textit{Definition 1.4})}.$$

The taxicab metric on R^n satisfies the metric property of symmetry. For all $u, v \in R^n$, $d_1(u, v) = d_1(v, u)$.

2. Prove that $d_1(u, v) \geq 0$, and $d_1(u, v) = 0$ if and only if $u = v$.

By *Definition 1.4*, the distance in R of vectors u and v will always be non-negative for all i unless the sum is 0. The distance between u and v can only be 0 if it's the distance calculated for the same vector where $u = v$. In summary, if

$$d_1(u, v) = 0, \text{ then}$$

$$d_1(u, v) = d_1(u, u)$$

When written out,

$$\left| u_1 - v_1 \right| + \dots + \left| u_n - v_n \right| = 0$$

is equivalent to

$$\left|u_1 - u_1\right| + \ ... \ + \ \left|u_n - u_n\right| = 0.$$

Note that the taxicab metric on R^n calculates the distance of two vectors by finding the magnitude of the difference between each of the vector components from $1 \leq i \leq n$ and adding them together. We can break down the summation of $d_1(u, v)$ and $d_1(u, u)$ after satisfying the conditions above and compare them component-wise,

$$\left|u_1 - v_1\right| = \left|u_1 - u_1\right|$$

$$\cdots$$

$$\left|u_n - v_n\right| = \left|u_n - u_n\right|$$

$v_1 = u_1, \ldots, v_n = u_n$. u and v are made up of the same components, so they are the same vector. That is, $u_i = v_i$ for all i. This proves that taking the distance taken of the same vector equals 0 on the taxicab metric. **The taxicab metric on R^n satisfies the metric property of non-negativity and positive-definiteness. For all $u, v \in R^n$, $d_1(u, v)$ is always ≥ 0, and $d_1(u, v) = 0$ if and only if $u = v$.**

3. Prove that $d_1(u, v) \leq d_1(u, w) + d_1(w, v)$.

   Let $w = \langle w_1, \ ... \ , w_n\rangle \in R^n$.

$$\sum_{i=1}^{n}\left|u_i - v_i\right| \leq \sum_{i=1}^{n}\left|u_i - w_i\right| + \sum_{i=1}^{n}\left|w_i - v_i\right|$$

We can add and subtract the same vector, w, to the left-hand side in a way so it can be written as

$$\text{Add 0: } \sum_{i=1}^{n}\left|u_i - w_i + w_i - v_i\right| \leq \sum_{i=1}^{n}\left|u_i - w_i\right| + \sum_{i=1}^{n}\left|w_i - v_i\right|.$$

$$\sum_{i=1}^{n}\left|(u_i - w_i) + (w_i - v_i)\right| \leq \sum_{i=1}^{n}\left(\left|u_i - w_i\right| + \left|w_i - v_i\right|\right).$$

Note that the taxicab metric takes two vectors in R^n and outputs the distance between them. Distance is a scalar quantity or real number in R, so the triangle inequality for real numbers still holds for distance in R. Let

$$\sum_{i=1}^{n}\left|(u_i - w_i) + (w_i - v_i)\right| = ||x + y||$$

and

$$\sum_{i=1}^{n}\left(\left|u_i - w_i\right| + \left|w_i - v_i\right|\right) = ||x|| + ||y||.$$

Then,

$$\sum_{i=1}^{n}\left|(u_i - w_i) + (w_i - v_i)\right| \le \sum_{i=1}^{n}\left(\left|u_i - w_i\right| + \left|w_i - v_i\right|\right)$$

$$\Leftrightarrow$$

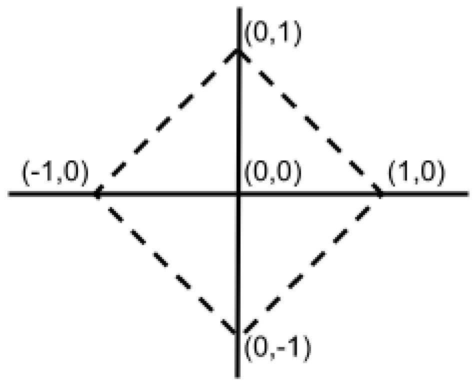
$$||x + y|| \le ||x|| + ||y|| \text{ (\textit{Theorem 1.5})}.$$

The taxicab metric on R^n satisfies the metric property for triangle inequality. For all $u, v, w \in R^n$, $d_1(u, v) \le d_1(u, w) + d_1(w, v)$.

We proved the taxicab metric on R^n is a metric where R^n is a set and $d_1: R^n \times R^n \to R$ is a map that satisfies the three properties of metric spaces for all $u, v \in R^n$.

Let (R^2, d_1) be a metric space and $u, v \in R^2$ where v is the 0 vector (origin). An open ball with radius 1 around v is set:

$$\begin{aligned} B_1(v) &:= \left\{u \in R^2 \mid d_1(u, v) < 1\right\} \\ &= \left\{\left|u_1 - v_1\right| + \left|u_2 - v_2\right| < 1\right\} \\ &= \left\{\left|u_1 - 0\right| + \left|u_2 - 0\right| < 1\right\} \\ &= \left\{\left|u_1\right| + \left|u_2\right| < 1\right\} \end{aligned}$$



(*Definition 1.6*) An open ball with radius 1 and center at the origin in the Taxicab space is represented as a diamond in R^2 with vertices at (1,0), (0,1), (-1,0), (0,-1). The open set does not include the outline made by the diamond. Distances existing in the open ball are confined to a diamond shape because the absolute value sum of the two vector components in R^2 must be less than the radius 1. Anything outside of the diamond or on the diamond's outline is ≥ r.

Result 1.2 We will prove that the Chebyshev metric on R^n is a metric. Let $u = \langle u_1, \dots, u_n\rangle$ and $v = \langle v_1, \dots, v_n\rangle \in R^n$. The Chebyshev metric $d_2: R^n \times R^n \to R$ is given by the formula:

$$d_2(u, v) = max_i\left|u_i - v_i\right| \text{ (\textit{Definition 1.2})}.$$

For the Chebyshev metric on R^n,

1. Prove that $d_2(u, v) = d_2(v, u)$.

$$d_2(u, v) = d_2(v, u)$$

$$max_i\left(\left|u_1 - v_1\right|, \dots, \left|u_n - v_n\right|\right) = max_i\left(\left|v_1 - u_1\right|, \dots, \left|v_n - u_n\right|\right)$$

(*Definition 1.4*) By the multicative property of absolute value,

$$max_i\left(\left|u_1 - v_1\right|, \dots, \left|u_n - v_n\right|\right) = max_i\left(|-1|\left|v_1 - u_1\right|, \dots, |-1|\left|v_n - u_n\right|\right)$$

$$max_i\left(\left|u_1 - v_1\right|, \dots, \left|u_n - v_n\right|\right) = max_i\left(\left|-1(v_1 - u_1)\right|, \dots, \left|-1(v_n - u_n)\right|\right)$$

$$max_i\left(\left|u_1 - v_1\right|, \ldots, \left|u_n - v_n\right|\right) = max_i\left(\left|- v_1 + u_1\right|, \ldots, \left|- v_n + u_n\right|\right)$$
$$max_i\left(\left|u_1 - v_1\right|, \ldots, \left|u_n - v_n\right|\right) = max_i\left(\left|u_1 - v_1\right|, \ldots, \left|u_n - v_n\right|\right).$$

The Chebyshev metric on R^n satisfies the metric property of symmetry. For all $u, v \in R^n$, $d_2(u, v) = d_2(v, u)$.

2. Prove that $d_2(u, v) \geq 0$, and $d_2(u, v) = 0$ if and only if $u = v$.

$$d_2(u, v) = max_i\left(\left|u_1 - v_1\right|, \ldots, \left|u_n - v_n\right|\right)$$

By the absolute value, $max_i\left(\left|u_1 - v_1\right|, \ldots, \left|u_n - v_n\right|\right)$ is always positive unless $u = v$, then $max_i\left|u_i - v_i\right| = 0$. Note that the Chebyshev metric on R^n takes the magnitude of the difference of the vector components in *u* and *v* from and chooses the largest value for the distance. If $d_2(u, v) = 0$, then the max is 0. If 0 is the largest magnitude after subtracting each component of *v* from each component of *u*, then $max_i\left|u_i - v_i\right| = 0$ for all *i*. $u_i = v_i$ for all *i*. $d_2(u, v) = 0$ if and only if $u = v$. **The Chebyshev metric on R^n satisfies the metric property of non-negativity and positive-definiteness. For all $u, v \in R^n$, $d_2(u, v) \geq 0$, and $d_2(u, v) = 0$ if and only if $u = v$.**

3. Prove that $d_2(u, v) \leq d_2(u, w) + d_2(w, v)$.

$$max_i\left|u_i - v_i\right| \leq max_i\left|u_i - w_i\right| + max_i\left|w_i - v_i\right|$$

Add 0: $max_i\left|u_i - w_i + w_i - v_i\right| \leq max_i\left|u_i - w_i\right| + max_i\left|w_i - v_i\right|$

Note that the Chebyshev metric is on R^n, so the triangle inequality for real numbers still holds (*Theorem 1.5*).

$$d_2(u, v) \leq d_2(u, w) + d_2(w, v)$$

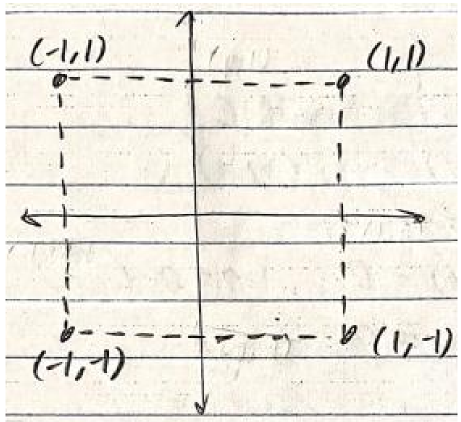
The Chebyshev metric on R^n satisfies the metric property for triangle inequality. For all $u, v, w \in R^n$**,** $d_2(u, v) \leq d_2(u, w) + d_1(w, v)$.

We proved the Chebyshev metric on R^n is a metric where R^n is a set and $d_2: R^n \times R^n \to R$ is a map that satisfies the three properties of metric spaces for all $u, v \in R^n$.

Let (R^2, d_2) be a metric space and $u, v \in R^2$ where *v* is the 0 vector (origin). An open ball with radius 1 around *v* is set:

$$B_1(v) := \left\{u \in R^2 \mid d_2(u, v) < 1\right\}$$
$$= \left\{max_2\left(\left|u_1 - v_1\right|, \left|u_2 - v_2\right|\right) < 1\right\}$$
$$= \left\{max_2\left(\left|u_1\right|, \left|u_2\right|\right) < 1\right\}$$

This means that the largest vector component of u must be less than 1. The open ball of radius 1 with the Chebyshev metric can be represented as a square with vertices at (1,1), (1,-1), (-1,-1), and (-1,1). The open set does not include the outline made by the square. The distance metric calculates the maximum magnitude between the x-axis component and y-axis component in the R^2 plane, so the vertices of the square define the extremes where both magnitudes equal 1.



2. The Discrete and SNCF Metrics

Definitions 2.1, 2.3, and **Remark 2.2** are on the 32AH Challenge Problem Set 2 sheet.

Result 2.1 We will prove that the discrete metric on R^n is a metric. Let $u = \langle u_1, \dots, u_n \rangle$ and $v = \langle v_1, \dots, v_n \rangle \in R^n$. The discrete metric $d_3 \colon R^n \times R^n \to R$ is given by the formula:

$$d_3(\boldsymbol{u}, \boldsymbol{v}) = \begin{cases} 0 & \text{if } \boldsymbol{u} = \boldsymbol{v} \\ 1 & \text{otherwise} \end{cases}$$ (*Definition 2.1*)

1. Prove that $d_3(u, v) = d_3(v, u)$.

   There are 2 cases we must observe:

   If $u = v$

   $$d_3(u, v) = 0$$
   $$d_3(v, u) = 0$$
   Then, $0 = 0$.

   Otherwise

   $$d_3(u, v) = 1$$
   $$d_3(v, u) = 1$$
   Then, $1 = 1$.

   The discrete metric on R^n satisfies the metric property of symmetry. For all $u, v \in R^n$, $d_3(u, v) = d_3(v, u)$.

2. Prove that $d_3(u, v) \geq 0$, and $d_3(u, v) = 0$ if and only if $u = v$.

By the first case of the discrete metric, the distance between two vectors is always 0 if they are the same vector or if $u = v$. In all other cases where $u \neq v$, the distance between two vectors is always 1 and 1 > 0. **Therefore, the discrete metric on R^n satisfies the metric property of non-negativity and positive-definiteness. For all $u, v \in R^n$, $d_3(u, v)$ is always ≥ 0, and $d_3(u, v) = 0$ if and only if $u = v$.**

3. Prove that $d_3(u, v) \leq d_3(u, w) + d_3(w, v)$.

   There are 5 cases we must examine:

   $u, v, w\ are\ the\ same$

   $$d_3(u, v) = 0,\ d_3(u, w) = 0,\ d_3(w, v) = 0$$

   $$0 \leq 0 + 0, \text{ inequality holds.}$$

   $u = v,\ w\ is\ different$

   $$d_3(u, v) = 0,\ d_3(u, w) = 1,\ d_3(w, v) = 1$$

   $$0 \leq 1 + 1, \text{ inequality holds.}$$

   $u = w,\ v\ is\ different$

   $$d_3(u, v) = 1,\ d_3(u, w) = 0,\ d_3(w, v) = 1$$

   $$1 \leq 0 + 1, \text{ inequality holds.}$$

   $v = w,\ u\ is\ different$

   $$d_3(u, v) = 1,\ d_3(u, w) = 1,\ d_3(w, v) = 0$$

   $$1 \leq 1 + 0, \text{ inequality holds.}$$

   $u, v, w\ are\ different$

   $$d_3(u, v) = 1,\ d_3(u, w) = 1,\ d_3(w, v) = 1$$

   $$1 \leq 1 + 1, \text{ inequality holds}$$

   The discrete metric on R^n satisfies the metric property for triangle inequality. For all $u, v, w \in R^n$, $d_3(u, v) \leq d_3(u, w) + d_3(w, v)$.

We proved the discrete metric on R^n is a metric where R^n is a set and $d_3: R^n \times R^n \to R$ is a map that satisfies the three properties of metric spaces for all $u, v \in R^n$.

Let (R^2, d_3) be a metric space and $u, v \in R^2$ where *v* is the 0 vector (origin). An open ball with radius 1 around *v* is set:

$$B_1(v) := \left\{ u \in R^2 \mid d_3(u, v) < 1 \right\}$$

(*Definition 1.6*) An open ball of radius 1 is an empty set in R^2 with the discrete metric. The set only includes distances less than the radius, but the discrete metric only consists of distances 1 and 0. The only distance that can fit in the open ball is no distance at all where $u \neq v$.

Alternatively, let $(R^2,\ d_3)$ be a metric space and $u,\ v \in R^2$ where v is the 0 vector (origin). An open ball with radius 2 around v is set:

$$B_2(v) := \left\{u \in R^2 \mid d_3(u,\ v) < 2\right\}$$

(*Definition 1.6*) An open ball of radius 2 contains all vectors in the metric space. For all $u \in R^2$, $d_3(u,\ v)$ is always < 2.

Result 2.2 We will prove that the SNCF metric on R^n is a metric. Let $u = \langle u_1, \dots, u_n \rangle$ and $v = \langle v_1, \dots, v_n \rangle \in R^n$. The discrete metric $d_4 : R^n \times R^n \to R$ is given by the formula:

$$d_4(\boldsymbol{u}, \boldsymbol{v}) = \begin{cases} ||\boldsymbol{u} - \boldsymbol{v}|| & \text{if } \boldsymbol{u}, \boldsymbol{v} \text{ lie on the same ray from the origin.} \\ ||\boldsymbol{u}|| + ||\boldsymbol{v}|| & \text{otherwise} \end{cases}$$

(*Definition 2.3*)

1. Prove that $d_4(u,\ v) = d_4(v,\ u)$.

   There are 2 cases we must examine:
   $u,\ v\ lie\ on\ the\ same\ ray\ from\ the\ origin$

   $$d_4(u,\ v) = d_4(v,\ u)$$
   $$||u - v|| = ||v - u||$$

   By the multicative property of absolute value,

   $$||u - v|| = ||-1|| \cdot ||v - u||$$
   $$||u - v|| = ||-1(v - u)||$$
   $$||u - v|| = ||-v + u)||$$
   $$||u - v|| = ||u - v|| \text{ (\textit{Definition 1.4}).}$$

   $Otherwise$

   $$d_4(u,\ v) = d_4(v,\ u)$$
   $$||u|| + ||v|| = ||v|| + ||u||$$
   $$||u|| + ||v|| = ||u|| + ||v||$$

   The SNCF metric on R^n satisfies the metric property of symmetry. For all $u,\ v \in R^n$, $d_4(u,\ v) = d_4(v,\ u)$.

2. Prove that $d_4(u,\ v) \geq 0$, and $d_4(u,\ v) = 0$ if and only if $u = v$.

   There are 2 cases we must examine:
   $u,\ v\ lie\ on\ the\ same\ ray\ from\ the\ origin$

   $$||u - v|| \geq 0$$

   The Euclidean distance is always positive unless $u = v$, then $||u - v|| = 0$. A distance of 0 is possible in this case because u and v lie on the same ray.
   $Otherwise$

   $$||u|| + ||v|| > 0$$

This will always be positive because the sum of the magnitudes of two different vectors is always positive. $d_4(u, v) = 0$ only when $u = v$, so a distance of 0 is not possible in this case because *u* and *v* lie on different rays. **Therefore, the SNCF metric on R^n satisfies the metric property of non-negativity and positive-definiteness. For all $u, v \in R^n$, $d_4(u, v)$ is always ≥ 0, and $d_4(u, v) = 0$ if and only if $u = v$.**

3. Prove that $d_4(u, v) \leq d_4(u, w) + d_4(w, v)$.

There are 5 cases we must examine:

$u,\ v,\ w\ lie\ on\ the\ same\ ray$

$$d_4(u, v) \leq d_4(u, w) + d_4(w, v)$$

$$||u - v|| \leq ||u - w|| + ||w - v||$$

$$\text{Add 0: } ||u - w + w - v|| \leq ||u - w|| + ||w - v||$$

Note that the SNCF metric is on R^n, so the triangle inequality for real numbers still holds. Let

$$||u - w + w - v|| = ||x + y||$$

and

$$||u - w|| + ||w - v|| = ||x|| + ||y||.$$

Then,

$$||u - w + w - v|| \leq ||u - w|| + ||w - v||$$

$$\Leftrightarrow$$

$$||x + y|| \leq ||x|| + ||y|| \text{ (\textit{Theorem 1.5})}.$$

$u,\ v\ lie\ on\ the\ same\ ray,\ w\ does\ not$

$$d_4(u, v) \leq d_4(u, w) + d_4(w, v)$$

$$||u - v|| \leq ||u|| + ||w|| + ||w|| + ||v||$$

$$||u - v|| \leq ||u|| + ||v|| + 2||w||$$

By the triangle inequality,

$$||u - v|| \leq ||u + v|| \leq ||u|| + ||v|| \leq ||u|| + ||v|| + 2||w||$$

Then the inequality

$$||u - v|| \leq ||u|| + ||v|| + 2||w||$$

holds.

$u,\ w\ lie\ on\ the\ same\ ray,\ v\ does\ not$

$$d_4(u, v) \leq d_4(u, w) + d_4(w, v)$$

$$||u|| + ||v|| \leq ||u - w|| + ||w|| + ||v||$$

$$\text{Add 0: } ||u - w + w|| \leq ||u - w|| + ||w||$$

Note that the SNCF metric is on R^n, so the triangle inequality for real numbers still holds. Let

$$||u - w|| + ||w|| = ||x|| + ||y||$$

and

$$||u - w + w|| = ||x + y||.$$

Then,

$$||u - w + w|| \le ||u - w|| + ||w||$$
$$\Leftrightarrow$$
$$||x + y|| \le ||x|| + ||y|| \text{ (\textit{Theorem 1.5})}.$$

v, w *lie on the same ray,* u *does not*

$$d_4(u, v) \le d_4(u, w) + d_4(w, v)$$
$$||u|| + ||v|| \le ||u|| + ||w|| + ||w - v||$$
$$\text{Add 0: } ||v + w - w|| \le ||w|| + ||w - v||$$

** HELP

u, v, w *lie on different rays*

$$d_4(u, v) \le d_4(u, w) + d_4(w, v)$$
$$||u|| + ||v|| \le ||u|| + ||w|| + ||w|| + ||v||$$
$$0 \le ||w|| + ||w||, \text{ inequality holds.}$$

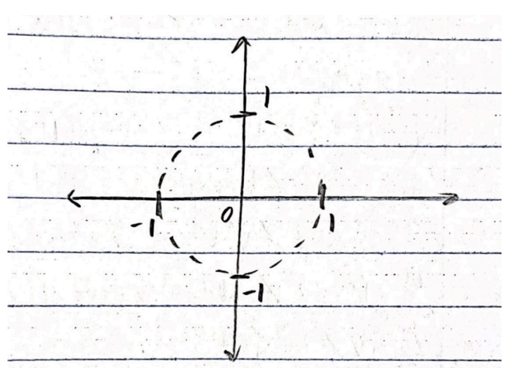
The SNCF metric on R^n satisfies the metric property for triangle inequality. For all $u, v, w \in R^n$, $d_4(u, v) \le d_4(u, w) + d_4(w, v)$.

We proved the SNCF metric on R^n is a metric where R^n is a set and $d_4: R^n \times R^n \to R$ is a map that satisfies the three properties of metric spaces for all $u, v \in R^n$.

Let (R^2, d_4) be a metric space and $u, v \in R^2$ where *v* is the 0 vector (origin). Note that the 0 vector lies on the same ray as all vectors. An open ball with radius 1 around *v* is set:

$$B_1(v) := \left\{u \in R^2 \mid d_4(u, v) < 1\right\}$$
$$= \left\{\sqrt{u_1^2 + u_2^2} < 1\right\}$$

(*Definition 1.6*) An open ball of radius 1 in R^2 with the SNCF metric can be represented graphically as:



Open ball of radius 1 centered at point (0.5,0), so $v = \langle 0.5, 0\rangle$.

$$B_1(v) := \left\{ u \in R^2 \mid d_4(u,\ v) < 1 \right\}$$

There are 2 cases we must examine:

$u\ and\ v\ lie\ on\ the\ same\ ray\ from\ the\ origin$

Note that if u and v lie on the same ray from the origin, their y-axis components in the R^2 plane must be 0. u and v lie along the x-axis.

$$d_4(u,\ v) = \left|\left| \langle u_1, 0\rangle - \langle 0.5, 0\rangle \right|\right| < 1$$

$$= \sqrt{\left(u_1 - .5\right)^2} < 1$$

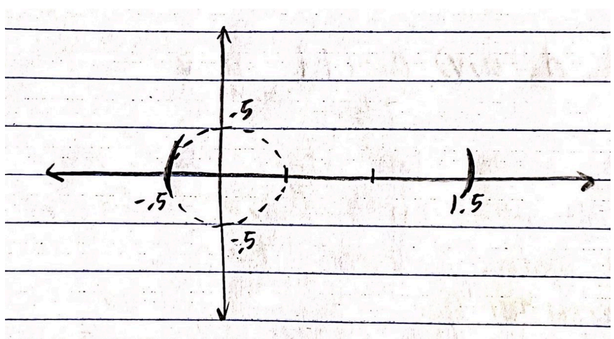
$$u_1 \in (-.5,\ 1.5),\ u_2 = 0$$

$Otherwise$

$$\sqrt{u_1^2 + u_2^2} + 0.5 < 1$$

$$\sqrt{u_1^2 + u_2^2} < 0.5$$

This is an open ball of radius 0.5 centered at the origin. Combining both cases, we can represent the values of components of u residing in the open ball of radius 1 centered at point (0.5,0) as an open ball of radius 0.5 centered at the origin and an open interval from -0.5 to 1.5.



Open ball of radius 1 centered at point (5,0), so $v = \langle 5, 0\rangle$.

$$B_1(v) := \left\{ u \in R^2 \mid d_4(u,\ v) < 1 \right\}$$

There are 2 cases we must examine:

$u\ and\ v\ lie\ on\ the\ same\ ray\ from\ the\ origin$

Note that u and v only lie along the x-axis.

$$d_4(u,\ v) = \left|\left| \langle u_1, 0\rangle - \langle 5, 0\rangle \right|\right|$$

$$= \sqrt{\left(u_1 - 5\right)^2} < 1$$

$$u_1 \in (4,\ 6),\ u_2 = 0$$

$Otherwise$

When $u \neq v$, there is no open ball in this case because (5,0) is too far from the origin. We can represent values of components of u residing in the open ball of radius 1 centered at point (5,0) as an open interval from 4 to 6.