32AH CHALLENGE PROBLEM 1

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1. Orthogonal and Orthonormal Vectors

Preliminary 1.1. By *Proposition 1.1.*, let θ be the angle between two non-zero vectors $\boldsymbol{u}, \boldsymbol{v} \in R^n$, and suppose $\boldsymbol{u}\cdot\boldsymbol{v} = a$. Then $cos(\theta) = \frac{u \cdot v}{||u||\,||v||}$. Notice the magnitude of non-zero vectors $||\boldsymbol{u}||$ and $||\boldsymbol{v}||$ is always positive. Therefore, the function $cos(\theta)$ in the geometric interpretation of the dot product is dependent on whether or not the numerator is positive or negative, where the numerator is a. We can determine whether the angle between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is acute, obtuse, or orthogonal to each other by defining the values of a. The angle between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is

Orthogonal when	$\theta = \frac{\pi}{2}$
Acute when	$0 < \theta < \frac{\pi}{2}$
Obtuse when	$\frac{\pi}{2} < \theta < \pi$

Result 1.1
The angle between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is acute when $a > 0$ because $cos(\theta)$ will be positive.
The angle between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is obtuse when $a < 0$ because $cos(\theta)$ will be negative.
The angle between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is orthogonal when $a = 0$ because $cos(\theta) = cos\left(\frac{\pi}{2}\right) = 0$.

Result 1.2. Let $B = \{v_1, v_2, \dots, v_k\}$be an orthonormal basis of V, then for any $\boldsymbol{v} \in V$, prove

$$v = \langle v, v_1\rangle v_1 +\dots \langle v, v_n\rangle v_n$$

That is, the coordinate vector $\boldsymbol{v}$ with respect to B has $\langle v, v_i\rangle$ as the i-th entry. By definition of a basis, $B$is an ordered set of vectors $\{v_1, v_2, \dots, v_k\}$ that is linearly independent and spans V . Any $\boldsymbol{v} \in V$can be written as a unique linear combination of basis vectors where $c_i \in R$ are scalars.

$$v = c_1 v_1 +\dots c_n v_n$$

Let's take the inner product of the coordinate vector $\langle v, v_i\rangle$, such that $1 \le i \le n$. The inner product can be written as

$$\langle v, v_i\rangle = \langle c_1 v_1 +\dots c_n v_n, v_i\rangle$$

By definition of bilinearity of the inner product,

$$\langle c_1 v_1 +\dots c_n v_n, v_i\rangle = c_1\langle v_1, v_i\rangle +\dots c_n\langle v_n, v_i\rangle= \sum_{j=1}^{n} c_j\langle v_j, v_i\rangle$$

By *Definition 1.2.*, an orthonormal basis is an ordered set of normalized and orthogonal vectors. The inner product of two non-zero vectors v_i and v_j in an orthonormal set can be represented by the Kronecker delta $\langle v_i, v_j \rangle = \delta_{ij}$ in *Proposition 1.3.*.

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$\langle v_i, v_j \rangle = 1$ if $i = j$ because the dot product of a vector with itself is the square of its magnitude, and the magnitude of all vectors in an orthonormal basis is length 1. Similarly, $\langle v_i, v_j \rangle = 0$ for all $i \neq j$, because the dot product of v_i with a different vector orthogonal to v_i is 0. This is supported by *Result 1.1.* where $\boldsymbol{u} \cdot \boldsymbol{v} = 0$ when the angle between vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is orthogonal. With this in mind,

$$\sum c_j \langle v_j, v_i \rangle = c_1(0) +\dots c_j(1) = 0 +\dots c_j = c_j$$

The sum of $\sum c_j \langle v_j, v_i \rangle = c_j$ for all c_j because there is only one instance in the linear combination where v_j is dotted with itself, when $i = j$, and the inner product is 1. In the rest of the equation, v_j is dotted with different orthogonal vectors and the inner product is 0. The only term left in the sum is c_j. Therefore,

$$v = c_1 v_1 +\dots c_n v_n$$

is the same as

$$v = \langle v, v_1 \rangle v_1 +\dots \langle v, v_n \rangle v_n$$

for all c_j because $\langle v, v_i \rangle =$ a scalar quantity c_j when $i = j$.

2. Gram-Schmidt Orthogonalization

Result 2.3. Show that $|u|^2 = |u_{||v}|^2 + |u_{\perp v}|^2$. Let us define the components in the expression.

By Gram-Schmidt orthogonalization, let there be two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in R^n. The sum of $\boldsymbol{u}$ can be broken down into 2 components: a vector parallel to $\boldsymbol{v}$ and a vector perpendicular or orthogonal to $\boldsymbol{v}$. This can be drawn as

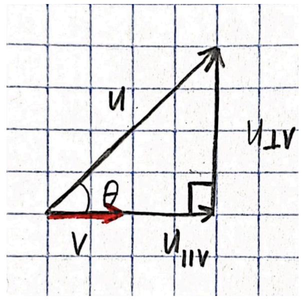

Figure 2.3.

We can identify similarities between $|u|^2 = \left|u_{||v}\right|^2 + \left|u_{\perp v}\right|^2$ and the pythagorean theorem $a^2 + b^2 = c^2$ where ***u*** is the hypotenuse and $u_{||v}$ and $u_{\perp v}$ are the triangle's legs--but let's prove this for an abstract vector space.

Recall in *Result 1.1.* that the angle between two non-zero vectors ***u*** and ***v*** is orthogonal when ***u***·***v*** = 0.

$$0 = u_{||v} \cdot u_{\perp v} = \langle u_{||v}, u_{\perp v}\rangle$$

By *Definition 2.1.* and *Definition 2.2.*,

$$u_{||v} = \left(\frac{\langle u, v\rangle}{\langle v, v\rangle}\right)v \text{ and } u_{\perp v} = u - u_{||v}$$

Then, $\langle u_{||v}, u_{\perp v}\rangle$ can be written as

$$\langle\left(\frac{\langle u, v\rangle}{\langle v, v\rangle}\right)v,\ u - u_{||v}\rangle$$
$$= \langle\left(\frac{\langle u, v\rangle}{\langle v, v\rangle}\right)v,\ u - \left(\frac{\langle u, v\rangle}{\langle v, v\rangle}\right)v\rangle$$

By definition of bilinearity of the inner product, it can be expressed as

$$\langle\left(\frac{\langle u, v\rangle}{\langle v, v\rangle}\right)v,\ u\rangle - \langle\left(\frac{\langle u, v\rangle}{\langle v, v\rangle}\right)v,\ \left(\frac{\langle u, v\rangle}{\langle v, v\rangle}\right)v\rangle = 0$$
$$\left(\frac{\langle u, v\rangle}{\langle v, v\rangle}\right)\langle v,\ u\rangle - \left(\frac{\langle u, v\rangle}{\langle v, v\rangle}\right)^2\langle v,\ v\rangle = 0$$
$$\langle v,\ u\rangle - \left(\frac{\langle u, v\rangle}{\langle v, v\rangle}\right)\langle v,\ v\rangle = 0$$
$$\langle v,\ u\rangle - \langle v,\ u\rangle = 0$$

Observe that $u_{||v}$ and $u_{\perp v}$ are orthogonal to each other and form a right angle. By Gram-Schmidt orthogonalization,

$$u = u_{||v} + u_{\perp v}$$

By definition of the dot product, a vector dotted with itself is the square of its magnitude.

$$|u|^2 = u \cdot u$$
$$= \langle u_{||v} + u_{\perp v},\ u_{||v} + u_{\perp v}\rangle$$
$$= \langle u_{||v},\ u_{||v}\rangle + \langle u_{\perp v},\ u_{\perp v}\rangle$$

Thus proving,

$$|u|^2 = |u_{||v}|^2 + |u_{\perp v}|^2$$

Preliminary 2.4. Applying the Gram-Schmidt algorithm to the following set of vectors in R^3:

$$x_1 = \langle 1, 0, 0\rangle,\ x_2 = \langle 1, 1, 0\rangle,\ x_3 = \langle 1, 1, 1\rangle$$

$$u_1 = x_1$$

$$u_1 = \langle 1, 0, 0\rangle$$

$$u_2 = x_2 - \frac{x_2 \cdot u_1}{u_1 \cdot u_1} u_1$$

$$u_2 = \langle 1, 1, 0\rangle - \frac{1+0+0}{1+0+0}\langle 1, 0, 0\rangle$$

$$u_2 = \langle 0, 1, 0\rangle$$

$$u_3 = x_3 - \frac{x_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{x_3 \cdot u_2}{u_2 \cdot u_2} u_2$$

$$u_3 = \langle 1, 1, 1\rangle - \frac{1+0+0}{1+0+0}\langle 1, 0, 0\rangle - \frac{0+1+0}{1+0+0}\langle 0, 1, 0\rangle$$

$$u_3 = \langle 0, 0, 1\rangle$$

Result 2.4.

$$u_1 = \langle 1, 0, 0\rangle, u_2 = \langle 0, 1, 0\rangle, u_3 = \langle 0, 0, 1\rangle$$

Result 2.5. Let $B = \{v_1, v_2, ..., v_n\}$ with *n* elements be a basis of vector space *V*, and $B' = \{u_1, u_2, ..., u_n\} \subseteq V$ be the output of the Gram-Schmidt algorithm. Show that B' is an orthogonal basis of V.

By *Theorem 2.4.*, any set of vectors $\{v_1, v_2, ..., v_n\}$ with *n* elements that are linearly independent or span V also forms a basis for V. We can prove B' is a basis of V by linear independence.

By *Definition 2.3.*, we know that the cardinality of *B* is the same as the cardinality of *B'*. This means *B'* has the same number of elements as *B*. If *B* is a basis of *V* with *n* elements and *B'* has the same number of elements as *B*, showing *B'* is also linearly independent proves *B'* is a basis of V.

By definition of linear dependence, a set of vectors $\{v_1, v_2, ... , v_k\}$ is said to be linearly dependent if there exist scalars c_i, not all zero, such that

$$c_1 v_1 + ... c_n v_n = 0$$

Otherwise, it is said to be linearly independent. Recall in *Definition 2.1.* and *Definition 2.2.*, the parallel and orthogonal projection of ***u*** along ***v*** are non-zero quantities and vectors ***u*** and ***v*** are also non-zero in the Gram-Schmidt orthogonalization. As a result, the set of orthogonal vectors $B' = \{u_1, u_2, ..., u_n\}$are non-zero vectors, so

$$0 = c_1 u_1 +... c_n u_n$$

can only be true when $c_i = 0$ for all c_i. Therefore B' is linearly independent, which means B' is an orthogonal basis of V.