Math 131A Homework 8
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Problem 1.
$\boxed{\text{Ross 18.6}}$ Define $f(x) = \cos x$. Then f is continuous on $I = (0, \frac{\pi}{2})$. Let $g(x) = f(x) - x$. Then g is continuous on I. Next invoke IVT on g: for $0 < \frac{\pi}{2}$ and any y between $g(0) = 1$ and $g(\frac{\pi}{2}) = -\frac{\pi}{2}$, there exists some $x_0 \in (0, \frac{\pi}{2})$ such that $g(x_0) = y$. Choose $y = 0$. Then $g(x_0) = 0$ implies $x_0 = f(x_0)$ and hence $x_0 = \cos x_0$. Therefore we have proven there exists some $x_0 \in (0, \frac{\pi}{2})$ satisfying $x_0 = \cos x_0$.

$\boxed{\text{Ross 18.8}}$ Suppose f is some real-valued continuous function on $\mathbb{R}$ and $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$. We want to prove there exists x between a and b such that $f(x) = 0$. Invoke IVT on f: for $a < b$ and any y between $f(a)$ and $f(b)$, there exists some $x \in (a, b)$ such that $f(x) = y$. With no loss of generality, assume $f(a) < 0$ and $f(b) > 0$ so that $f(a)f(b) < 0$ holds. Therefore choose $y = 0$. Then there exists $x \in (a, b)$ such that $f(x) = 0$.

$\boxed{\text{Ross 18.10}}$ Suppose f is continuous on $[0, 2]$ and $f(0) = f(2)$. We want to prove there exists x, y in $[0, 2]$ such that $|y - x| = 1$ and $f(x) = f(y)$. Consider $g(x) = f(x + 1) - f(x)$. Then g is continuous on $[0, 1]$. Next invoke IVT on g: for $0 < 1$ and any y between $g(0) = f(1) - f(0)$ and $g(1) = f(2) - f(1)$, there exists some $x_0 \in [0, 1]$ such that $g(x_0) = y$. With no loss of generality, assume $f(1) > f(0) = f(2)$. This gives us $g(0) > 0$ and $g(1) < 0$. Therefore choose $y = 0$. Then $g(x_0) = 0$ implies $f(x_0 + 1) = f(x_0)$, hence we have proven there exists $x_0, x_0 + 1$ in $[0, 2]$ such that $f(x_0 + 1) = f(x_0)$ and $|x_0 + 1 - x_0| = 1$.

Problem 2.
$\boxed{\text{Ross 19.2(b)}}$ Consider $f(x) = x^2$ on $[0, 3]$. We want to prove that f is uniformly continuous on the $[0, 3]$ by directly verifying the $\epsilon - \delta$ property in Definition 19.1. It suffices to show $\forall \epsilon > 0$, $\exists \delta > 0$ such that $x, y \in [0, 3]$ and $|x - y| < \delta$ imply $|f(x) - f(y)| < \epsilon$. First note that

$$|f(x) - f(y)| < |x^2 - y^2| = |x - y| \cdot |x + y| \leq |x - y| \cdot (|x| + |y|) \leq 6|x - y|.$$

Now choose any $\epsilon > 0$ and let $\delta = \frac{\epsilon}{6}$. Then whenever $x, y \in [0, 3]$ and $|x - y| < \frac{\epsilon}{6}$, it follows that $|f(x) - f(y)| \leq 6|x - y| < 6 \cdot \frac{\epsilon}{6} = \epsilon$. Therefore we say f is uniformly continuous on $[0, 3]$.

$\boxed{\text{Ross 19.4(a)}}$ We want to prove that if f is uniformly continuous on a bounded set S, then f is a bounded function on S. Suppose f is unbounded on S. That is, for any $n \in \mathbb{N}$ there exists $x_n \in S$ such that $|f(x_n)| > n$. Therefore we can construct a sequence (x_n) such that $|f(x_n)| \to +\infty$. Since S is bounded, (x_n) is a bounded sequence and therefore has a convergent subsequence (x_{n_k}). Since f is uniformly continuous on S and (x_{n_k}) is a Cauchy sequence, $(f(x_{n_k}))$ is also a Cauchy sequence. However, $|f(x_n)| \to +\infty$ implies $|f(x_{n_k})| \to +\infty$ and this is a contradiction. Therefore it must be the case that f is a bounded function on S.

Problem 3. Let f and g be uniformly continuous on $\mathbb{R}$. Then for any $\epsilon > 0$, $\exists \delta_1 > 0$ such that $x, y \in \mathbb{R}$ and $|x - y| < \delta_1$ implies $|g(x) - g(y)| < \epsilon$. For the given δ_1, there also exists $\delta > 0$ such that $x, y \in \mathbb{R}$ and $|x - y| < \delta$ implies $|f(x) - f(y)| < \delta_1$. Consequently, $|g(f(x)) - g(f(y))| = |(g \circ f)(x) - (g \circ f)(y)| < \epsilon$. Therefore we say $g \circ f$ is uniformly continuous on $\mathbb{R}$.