32AH CHALLENGE PROBLEM 3

1. Calculus of Vector-Valued Functions

2. The Frenet Frame

Result 2.1. Prove that the derivative rule for dot products of vector-valued functions $r,\ s\colon R \to R^n$ is

$$\frac{d}{dt}[r(t)\ \cdot\ s(t)] = r'(t)\ \cdot\ s(t)\ +\ r(t)\ \cdot\ s'(t).$$

By *Definition 1.4.*, the derivative of a vector-valued function $r(t)$ is defined as

$$r'(t)\ =\ \frac{d}{dt}r(t)\ =\ \lim_{h\to 0}\ \frac{r(t+h)-r(t)}{h}.$$

Then,

$$\frac{d}{dt}[r(t)\ \cdot\ s(t)] = \lim_{h\to 0}\ \frac{r(t+h)\cdot s(t+h)-r(t)\cdot s(t)}{h}.$$

Adding 0,

$$\frac{d}{dt}[r(t)\ \cdot\ s(t)] = \lim_{h\to 0}\ \frac{r(t+h)\cdot s(t+h)-r(t+h)\cdot s(t)+r(t+h)\cdot s(t)-r(t)\cdot s(t)}{h}.$$

Rearrange,

$$\frac{d}{dt}[r(t)\ \cdot\ s(t)] =\ \lim_{h\to 0}\ \frac{[r(t+h)\cdot s(t+h)-s(t)]+[r(t+h)-r(t)\cdot s(t)]}{h}$$

$$\frac{d}{dt}[r(t)\ \cdot\ s(t)] =\ \lim_{h\to 0}\ \frac{r(t+h)-r(t)}{h}\ \cdot\ s(t)\ +\ \lim_{h\to 0}\ \frac{s(t+h)-s(t)}{h}\ \cdot\ r(t\ +\ h).$$

We know that $r(t\ +\ h)$ evaluates to $r(t)$ when taking the limit, so

$$\frac{d}{dt}[r(t)\ \cdot\ s(t)] =\ \lim_{h\to 0}\ \frac{r(t+h)-r(t)}{h}\ \cdot\ s(t)\ +\ \lim_{h\to 0}\ \frac{s(t+h)-s(t)}{h}\ \cdot\ r(t).$$

We have proven that the derivative rule for dot products of vector-valued functions is

$$\frac{d}{dt}[r(t)\ \cdot\ s(t)] = r'(t)\ \cdot\ s(t)\ +\ r(t)\ \cdot\ s'(t).$$

Result 2.2. Prove that the Frenet frame is an orthonormal basis. Let a vector-valued function $r\colon\ R \to R^3$ be $r(t)\ =\ \langle x(t), y(t), z(t)\rangle$ The Frenet frame is an orthonormal basis if it satisfies all 4 properties:

1. All vectors in the set have unit length 1.
2. All vectors in the set are orthogonal to each other.
3. The set of vectors is linearly independent.
4. The basis spans R^3.

We know $r(t)\ =\ \langle x(t), y(t), z(t)\rangle$ and $r'(t)\ =\ \langle x'(t), y'(t), z'(t)\rangle$.

Prove that all vectors in the set are normalized with unit length 1.

T has unit length 1 if $||T(t)|| =\ 1$

$$T(t) = \frac{1}{||r'(t)||} r'(t) \text{ (\textit{Definition 2.1.})}$$

$$T(t) = \frac{\langle x'(t), y'(t), z'(t) \rangle}{\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}}$$

$$||T(t)|| = \sqrt{\left(\frac{x'(t)}{\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}}\right)^2 + \left(\frac{y'(t)}{\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}}\right)^2 + \left(\frac{z'(t)}{\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}}\right)^2}$$

$$||T(t)|| = \sqrt{\frac{x'(t)^2}{x'(t)^2+y'(t)^2+z'(t)^2} + \frac{y'(t)^2}{x'(t)^2+y'(t)^2+z'(t)^2} + \frac{z'(t)^2}{x'(t)^2+y'(t)^2+z'(t)^2}}$$

$$||T(t)|| = \sqrt{\frac{x'(t)^2+y'(t)^2+z'(t)^2}{x'(t)^2+y'(t)^2+z'(t)^2}}$$

$$||T(t)|| = 1.$$

N has unit length 1 if $||N(t)|| = 1$

$$N(t) = \frac{1}{||T'(t)||} T'(t) \text{ (\textit{Definition 2.2.})}$$

$$T'(t) = \frac{\langle x''(t), y''(t), z''(t) \rangle}{\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}}$$

$$||T'(t)|| = \sqrt{\frac{x''(t)^2}{x'(t)^2+y'(t)^2+z'(t)^2} + \frac{y''(t)^2}{x'(t)^2+y'(t)^2+z'(t)^2} + \frac{z''(t)^2}{x'(t)^2+y'(t)^2+z'(t)^2}}$$

$$||T'(t)|| = \sqrt{\frac{x''(t)^2+y''(t)^2+z''(t)^2}{x'(t)^2+y'(t)^2+z'(t)^2}}$$

$$N(t) = \frac{\frac{\langle x''(t), y''(t), z''(t) \rangle}{\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}}}{\sqrt{\frac{x''(t)^2+y''(t)^2+z''(t)^2}{x'(t)^2+y'(t)^2+z'(t)^2}}}$$

$$N(t) = \frac{\langle x''(t), y''(t), z''(t) \rangle}{\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}} \cdot \frac{\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}}{\sqrt{x''(t)^2+y''(t)^2+z''(t)^2}}$$

$$N(t) = \frac{\langle x''(t), y''(t), z''(t) \rangle}{\sqrt{x''(t)^2+y''(t)^2+z''(t)^2}}$$

$$||N(t)|| = \sqrt{\left(\frac{x''(t)}{\sqrt{x''(t)^2+y''(t)^2+z''(t)^2}}\right)^2 + \left(\frac{y''(t)}{\sqrt{x''(t)^2+y''(t)^2+z''(t)^2}}\right)^2 + \left(\frac{z''(t)}{\sqrt{x''(t)^2+y''(t)^2+z''(t)^2}}\right)^2}$$

$$||N(t)|| = \sqrt{\frac{x''(t)^2}{x''(t)^2+y''(t)^2+z''(t)^2} + \frac{y''(t)^2}{x''(t)^2+y''(t)^2+z''(t)^2} + \frac{z''(t)^2}{x''(t)^2+y''(t)^2+z''(t)^2}}$$

$$||N(t)|| = \sqrt{\frac{x''(t)^2+y''(t)^2+z''(t)^2}{x''(t)^2+y''(t)^2+z''(t)^2}}$$

$$||N(t)|| = 1.$$

Show that *T* and *N* are orthogonal such that

$$T \cdot N = 0.$$

Assume $r(t)$ has constant length, then

$$r(t) \cdot r(t) = ||r(t)||^2 = c,\ c \text{ is some constant.}$$

Use the derivative rule for dot products of vector-valued functions proven in *Result 1.1*.

$$\frac{d}{dt}[r(t) \cdot r(t)] = r'(t) \cdot r(t) + r(t) \cdot r'(t) = \frac{d}{dt}c$$

$$\frac{d}{dt}[r(t) \cdot r(t)] = r'(t) \cdot r(t) = 0.$$

If $r(t)$ and $r'(t)$ are orthogonal, this implies *T* and *N* are orthogonal.
Now, *B* has unit length 1 if $||B|| = 1$

$$B = T \times N \text{ (Definition 2.3.)}$$

$$||B|| = ||T \times N|| = ||T||||N||sin(\theta).$$

Recall that $||T(t)|| = 1$, $||N(t)|| = 1$, and *T* and *N* are orthogonal such that $\theta = 90°$. Then,

$$||B|| = 1.$$

We have proven that the vectors in the Frenet frame have unit length 1. Next, prove that the vectors in the set are orthogonal to each other. From the first property of an orthonormal basis, we have proven that *T* and *N* are orthogonal. By definition of the cross product, the binormal vector is orthogonal to the osculating plane created by *T* and *N*, so *T, N,* and *B* are orthogonal to each other.

Now prove that the set of vectors form a basis in R^3 by showing that they are linearly independent and span R^3. A set of vectors $\{v_1, \ldots, v_k\} \in V$ is said to be linearly dependent if there exists scalars a_i, not all zero, such that

$$a_1v_1 + \ldots + a_kv_k = 0.$$

Otherwise, they are said to be linearly independent.
The set $\{T, N, B\}$ can be expressed as

$$a_1T + a_2N + a_3B = 0.$$

Where a_1, a_2, and a_3 are some scalars. We know that *T, N,* and *B* are non-zero vectors, so

$$a_1T + a_2N + a_3B = 0$$

if and only if $a_1 = a_2 = a_3 = 0$. Then we can say that the set $\{T, N, B\}$ is linearly independent.

The set span$\{T, N, B\}$ also consists of all the linear combinations of $T, N, B \in R^3$, so the set spans R^3. Therefore, we can conclude that the Frenet frame forms an orthonormal basis in R^3 because the set of vectors *T, N,* and *B* are of unit length 1, orthogonal to each other, linearly independent, and span R^3.

Result 2.3. Sketch and compute the Frenet frame for the vector-valued function $r\colon R \to R^3$ given by $r(t) = \langle ln(t), 1, t\rangle$. The Frenet frame is an orthonormal basis consisting of the unit tangent vector T, unit normal vector N, and binormal vector B to create: $\{T, N, B\}$ (*Definition 2.3.*).
First, solve for the unit tangent vector *T* given by

$$T(t) = \frac{1}{||r'(t)||}r'(t) \text{ (Definition 2.1.).}$$

From *Theorem 1.2*, we know that a limit of a vector-valued function can be computed componentwise, so a derivative of a vector valued function can also be found componentwise. For example, if $r\colon\ R \rightarrow R^3$ is given by $r(t) = \langle x(t), y(t), z(t)\rangle$, then

$$r'(t) = \langle x'(t), y'(t), z'(t)\rangle.$$

Solve for $r'(t)$

$$r'(t) = \langle \tfrac{d}{dt} ln(t), \tfrac{d}{dt} 1, \tfrac{d}{dt} t\rangle$$

$$r'(t) = \langle \tfrac{1}{t}, 0, 1\rangle.$$

Let u be a vector in R^n. The norm or magnitude of a vector can be calculated using the formula

$$||u|| = \sqrt{u_1^2 + u_2^2 + \ \dots \ + u_n^2}.$$

Solve for $||r'(t)||$

$$||r'(t)|| = \sqrt{\tfrac{1}{t^2} + 1}.$$

The unit tangent vector T is

$$T(t) = \tfrac{1}{||r'(t)||} r'(t)$$

$$T(t) = \frac{1}{\sqrt{\frac{1}{t^2}+1}}\langle \tfrac{1}{t}, 0, 1\rangle$$

$$T(t) = \langle \frac{1}{\sqrt{t^2+1}}, 0, \frac{1}{\sqrt{\frac{1}{t^2}+1}}\rangle.$$

Solve for the unit normal vector N given by

$$N(t) = \tfrac{1}{||T'(t)||} T'(t) \text{ (\textit{Definition 2.2.})}.$$

Solve for $T'(t)$

$$T'(t) = \langle \frac{d}{dt}\frac{1}{\sqrt{t^2+1}}, \frac{d}{dt}0, \frac{d}{dt}\frac{1}{\sqrt{\frac{1}{t^2}+1}}\rangle$$

$$T'(t) = \langle -\frac{t}{\left(t^2+1\right)^{3/2}}, 0, \frac{1}{\left(\frac{1}{t^2}+1\right)^{3/2} t^3}\rangle.$$

Solve for $||T'(t)||$

$$||T'(t)|| = \sqrt{\left(-\frac{t}{\left(t^2+1\right)^{3/2}}\right)^2 + \left(\frac{1}{\left(\frac{1}{t^2}+1\right)^{3/2} t^3}\right)^2}$$

$$||T'(t)|| = \sqrt{\frac{t^2}{\left(t^2+1\right)^3} + \frac{1}{\left(\frac{1}{t^2}+1\right)^3 t^6}}.$$

The unit normal vector N is

$$N(t) = \tfrac{1}{||T'(t)||} T'(t)$$

$$N(t) = \frac{1}{\sqrt{\frac{t^2}{(t^2+1)^3} + \frac{1}{\left(\frac{1}{t^2}+1\right)^3 t^6}}} \langle -\frac{t}{(t^2+1)^{3/2}}, 0, \frac{1}{\left(\frac{1}{t^2}+1\right)^{3/2} t^3} \rangle$$

$$N(t) = \langle -\frac{t}{\sqrt{t^2+1}}, 0, \frac{1}{\sqrt{t^2+1}} \rangle.$$

Last, solve for the binormal vector B given by

$$B = T \times N \text{ (*Definition 2.3.*)}.$$

$$B = \langle \frac{1}{\sqrt{t^2+1}}, 0, \frac{1}{\sqrt{\frac{1}{t^2}+1}} \rangle \times \langle -\frac{t}{\sqrt{t^2+1}}, 0, \frac{1}{\sqrt{t^2+1}} \rangle .$$

The matrix produced by the cross product, $T \times N$, is

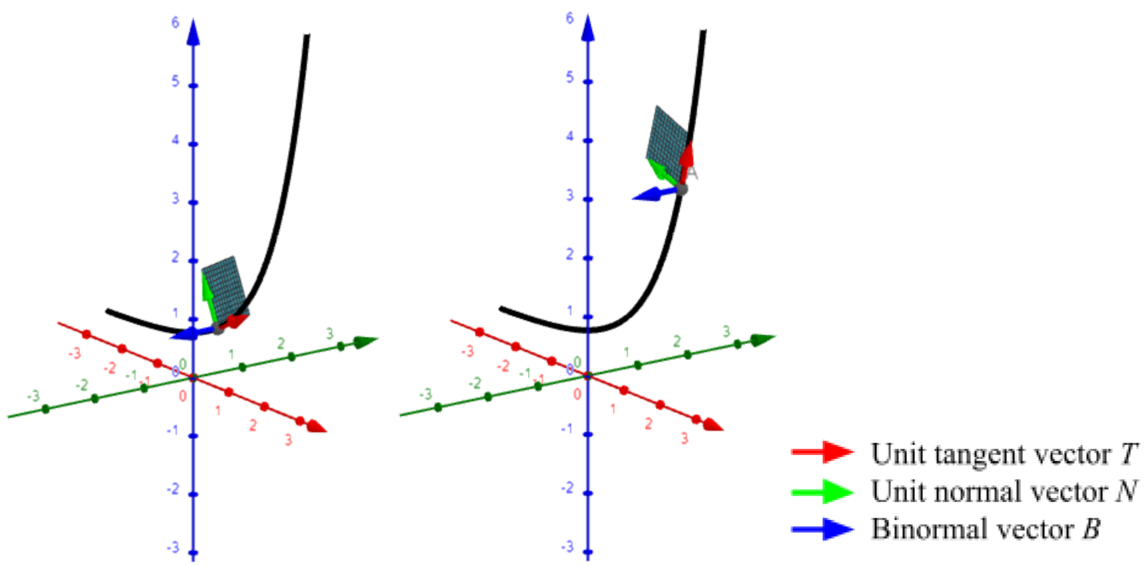
$$T \times N = \begin{bmatrix} i & j & k \\ \frac{1}{\sqrt{t^2+1}} & 0 & \frac{1}{\sqrt{\frac{1}{t^2}+1}} \\ -\frac{t}{\sqrt{t^2+1}} & 0 & \frac{1}{\sqrt{t^2+1}} \end{bmatrix} \begin{aligned} &= 0i - \left(\frac{1}{t^2+1} + \frac{t^2}{t^2+1}\right)j + 0k \\ &= 0i - \left(\frac{t^2+1}{t^2+1}\right)j + 0k \end{aligned}$$

$$B = T \times N = \langle 0, -1, 0 \rangle.$$

Let the Frenet frame $F = \{T, N, B\}$ be an orthonormal basis of R^3. The Frenet frame for $r(t) = \langle ln(t), 1, t \rangle$ is

$$F = \left\{ \langle \frac{1}{\sqrt{t^2+1}}, 0, \frac{1}{\sqrt{\frac{1}{t^2}+1}} \rangle, \langle -\frac{t}{\sqrt{t^2+1}}, 0, \frac{1}{\sqrt{t^2+1}} \rangle, \langle 0, -1, 0 \rangle \right\}.$$

The Frenet frame can be represented in 3-D as:



Notice T, N, B are orthogonal to each other. The tangent and normal vector of the curve $r(t)$ form the osculating plane. The normal vector of the osculating plane, the binormal vector, is created by the cross product of T and N.

3. Curvature

Result 3.3. Prove that an alternative formula for curvature is:

$$k(t) = \frac{||r'(t) \times r''(t)||}{||r'(t)||^3}.$$

Observe that

$$r'(t) = T(t)||r'(t)||.$$

For simplicity, we will define the magnitude of $r'(t)$ as $\frac{ds}{dt}$. In other words, $||r'(t)||$ can be expressed as the derivative of arc length with respect to $r'(t)$ over t. Then,

$$r'(t) = T(t)||r'(t)|| = \frac{ds}{dt}T(t).$$

Solve for the second derivative

$$r''(t) = \frac{d}{dt}\left[\frac{ds}{dt}T(t)\right].$$

Apply the chain rule

$$r''(t) = \frac{d^2s}{dt}T(t) + \frac{ds}{dt}T'(t).$$

Solve for the cross product

$$r'(t) \times r''(t) = \frac{ds}{dt}T(t) \times \left[\frac{d^2s}{dt}T(t) + \frac{ds}{dt}T'(t)\right].$$

The cross product is distributive, so

$$r'(t) \times r''(t) = \left[\frac{ds}{dt}T(t) \times \frac{d^2s}{dt}T(t)\right] + \left[\frac{ds}{dt}T(t) \times \frac{ds}{dt}T'(t)\right]$$

$$r'(t) \times r''(t) = \frac{ds}{dt}\frac{d^2s}{dt}[T(t) \times T(t)] + \left(\frac{ds}{dt}\right)^2[T(t) \times T'(t)].$$

Notice $T(t) \times T(t)$ is the 0 vector because the cross product of two vectors in the same direction is 0

$$r'(t) \times r''(t) = \left(\frac{ds}{dt}\right)^2[T(t) \times T'(t)].$$

Solve for the absolute value of the cross product

$$||r'(t) \times r''(t)|| = \left|\left|\left(\frac{ds}{dt}\right)^2[T(t) \times T'(t)]\right|\right|$$

$$||r'(t) \times r''(t)|| = \left(\frac{ds}{dt}\right)^2||T(t) \times T'(t)||.$$

Prove $T(t)$ and $T'(t)$ are orthogonal such that

$$||r'(t) \times r''(t)|| = \left(\frac{ds}{dt}\right)^2||T'(t)||.$$

Recall that the tangent vector T is one of the unit vectors forming an orthonormal basis called the Frenet frame. An orthonormal basis is a collection of orthogonal and normalized vectors with length 1. Therefore, the length of $T(t)$ is 1.

$$T(t) \cdot T(t) = ||T(t)||^2 = 1.$$

Use the derivative rule for dot products of vector-valued functions proven in *Result 1.1*.

$$\frac{d}{dt}[T(t)\cdot T(t)] = T'(t)\cdot T(t) + T(t)\cdot T'(t) = \frac{d}{dt}1$$
$$\frac{d}{dt}[T(t)\cdot T(t)] = T'(t)\cdot T(t) = 0.$$
If the dot product between two non-zero vectors is 0, then they are orthogonal to each other. Now that we have proven $T(t)$ and $T'(t)$ are orthogonal, evaluate the cross product
$$||T(t)\times T'(t)|| = ||T(t)||||T'(t)||sin(90°)$$
$$||T(t)\times T'(t)|| = 1\cdot ||T'(t)||\cdot 1$$
$$||T(t)\times T'(t)|| = ||T'(t)||.$$
Revisit $||r'(t)\times r''(t)||$ so it turns into
$$||r'(t)\times r''(t)|| = \left(\frac{ds}{dt}\right)^2 ||T(t)\times T'(t)||$$
$$||r'(t)\times r''(t)|| = \left(\frac{ds}{dt}\right)^2 ||T'(t)||.$$
Express $||T'(t)||$ as
$$||T'(t)|| = \frac{||r'(t)\times r''(t)||}{\left(\frac{ds}{dt}\right)^2}.$$
Recall that $||r'(t)|| = \frac{ds}{dt}$, so
$$||T'(t)|| = \frac{||r'(t)\times r''(t)||}{||r'(t)||^2}.$$
The original formula used to compute the curvature of a vector valued function $r(t)$ in *Definition 3.1.* can be expressed as
$$k(t) = \frac{1}{||r'(t)||}||T'(t)||$$
$$k(t) = \frac{1}{||r'(t)||}\cdot\frac{||r'(t)\times r''(t)||}{||r'(t)||^2}$$
$$k(t) = \frac{||r'(t)\times r''(t)||}{||r'(t)||^3}.$$
We have proven that the alternative formula for curvature is
$$k(t) = \frac{||r'(t)\times r''(t)||}{||r'(t)||^3}.$$
The alternative formula is used for *Result 3.1* and *3.2.*.

Result 3.1. Compute the curvature k of the circle of radius R ($r(t) = \langle -Rcos(t), Rsin(t), 0\rangle$) using the alternative formula for curvature
$$k(t) = \frac{||r'(t)\times r''(t)||}{||r'(t)||^3}.$$
Find $r'(t)$
$$r'(t) = \langle\frac{d}{dt} - Rcos(t), \frac{d}{dt}Rsin(t), \frac{d}{dt}0\rangle$$
$$r'(t) = \langle Rsin(t), Rcos(t), 0\rangle.$$
Find $r''(t)$
$$r''(t) = \langle\frac{d}{dt}Rsin(t), \frac{d}{dt}Rcos(t), \frac{d}{dt}0\rangle$$
$$r''(t) = \langle Rcos(t), -Rsin(t), 0\rangle.$$

The matrix produced by $r'(t) \times r''(t)$ is

$$r'(t) \times r''(t) = \begin{bmatrix} i & j & k \\ R\sin(t) & R\cos(t) & 0 \\ R\cos(t) & -R\sin(t) & 0 \end{bmatrix} \begin{aligned} &= 0i - 0j + \left[-R^2 sin^2(t) - R^2 cos^2(t)\right]k \\ &= 0i - 0j + \left[-R^2(sin^2(t) + cos^2(t))\right]k \\ &= 0i - 0j - R^2 k \end{aligned}$$

$$r'(t) \times r''(t) = \langle 0, 0, -R^2 \rangle.$$

Solve for $||r'(t) \times r''(t)||$

$$||r'(t) \times r''(t)|| = \sqrt{\left(-R^2\right)^2} = R^2.$$

Solve for $||r'(t)||$. Recall $r'(t) = \langle Rsin(t), Rcos(t), 0\rangle$, so

$$||r'(t)|| = \sqrt{R^2 sin^2(t) + R^2 cos^2(t)}$$

$$||r'(t)|| = \sqrt{R^2(sin^2(t) + cos^2(t))}$$

$$||r'(t)|| = \sqrt{R^2}$$

$$||r'(t)|| = R.$$

Solve for $||r'(t)||^3$

$$||r'(t)||^3 = R^3.$$

Solve for $k(t)$

$$k(t) = \frac{||r'(t) \times r''(t)||}{||r'(t)||^3}$$

$$k(t) = \frac{R^2}{R^3}.$$

The curvature of the circle of radius R ($r(t) = \langle -Rcos(t), Rsin(t), 0\rangle$) is

$$k(t) = \frac{1}{R}.$$

Let us evaluate the equation for the curvature of the circle of radius R. As R increases, curvature decreases. Similarly, as R decreases, curvature increases. This captures the relationship between the radius of a circle and curvature described in *Definition 3.2.*. A circle with a large radius has a function that doesn't curve much, and a circle with a small radius has a function that makes a tight curve at point *P*.

Result 3.2. Compute the curvature for the vector valued function $r(t) = \langle ln(t), 1, t\rangle$ using the alternative formula for curvature

$$k(t) = \frac{||r'(t) \times r''(t)||}{||r'(t)||^3}.$$

Find $r'(t)$ and $r''(t)$

$$r'(t) = \langle \tfrac{1}{t}, 0, 1\rangle \text{ and } r''(t) = \langle -\tfrac{1}{t^2}, 0, 0\rangle.$$

The matrix produced by $r'(t) \times r''(t)$ is

$$r'(t) \times r''(t) = \begin{bmatrix} i & j & k \\ \frac{1}{t} & 0 & 1 \\ -\frac{1}{t^2} & 0 & 0 \end{bmatrix} = 0i - \left(0 + \frac{1}{t^2}\right)j + 0k$$

$$r'(t) \times r''(t) = \langle 0, -\frac{1}{t^2}, 0 \rangle.$$

Solve for $||r'(t) \times r''(t)||$

$$||r'(t) \times r''(t)|| = \sqrt{0 + \left(-\frac{1}{t^2}\right)^2 + 0}$$

$$||r'(t) \times r''(t)|| = \frac{1}{t^2}.$$

Solve for $||r'(t)||^3$. Recall $r'(t) = \langle \frac{1}{t}, 0, 1 \rangle$, so

$$||r'(t)|| = \sqrt{\left(\frac{1}{t}\right)^2 + 0 + 1}$$

$$||r'(t)|| = \sqrt{\frac{1}{t^2} + 1}.$$

$$||r'(t)||^3 = \left(\sqrt{\frac{1}{t^2} + 1}\right)^3.$$

Solve for $k(t)$

$$k(t) = \frac{||r'(t) \times r''(t)||}{||r'(t)||^3}$$

$$k(t) = \frac{1}{t^2} \cdot \frac{1}{\left(\sqrt{\frac{1}{t^2}+1}\right)^3} = \frac{1}{t^2} \cdot \frac{1}{\left(\frac{1}{t^2}+1\right)\sqrt{\frac{1}{t^2}+1}}.$$

The curvature for the vector-valued function $r(t) = \langle ln(t), 1, t \rangle$ is

$$k(t) = \frac{1}{\left(t^2+1\right)\sqrt{\frac{1}{t^2}+1}}.$$