32AH CHALLENGE PROBLEM 4

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1. Partial Differential Equations

2. The Laplace Equation

Result 2.1. Show that the following functions are harmonic (a-b), and come up with 2 other examples of harmonic functions (c-d).

a. $f(x, y) = e^{x}cos(y)$: $R^2 \to R$

Show that f is a harmonic function given the Laplace equation of a function of two variables,

$$\Delta f = 0 \text{ where } \Delta f = f_{xx} + f_{yy}.$$

Find f_{xx} defined by

$$f_{xx} = \frac{d}{dx}\left(\frac{df}{dx}\right).$$

Solve for the first partial derivative in the equation,

$$f_{xx} = \frac{d}{dx}\left(e^{x}cos(y)\right).$$

Then,

$$f_{xx} = e^{x}cos(y).$$

Find f_{yy} defined by

$$f_{yy} = \frac{d}{dy}\left(\frac{df}{dy}\right).$$

Solve for the first partial derivative in the equation,

$$f_{yy} = \frac{d}{dy}\left(-e^{x}sin(y)\right).$$

Then,

$$f_{yy} =- e^{x}cos(y).$$

Prove that $\Delta f = 0$ given f_{xx} and f_{yy},

$$\Delta f = e^{x}cos(y) - e^{x}cos(y)$$
$$\Delta f = 0.$$

Thus, we have shown that f is a harmonic function.

b. $g(x, y) = ln\left(x^2 + y^2\right)$: $R^2 \to R$

Show that g is a harmonic function given the Laplace equation of a function of two variables,

$$\Delta g = 0 \text{ where } \Delta g = g_{xx} + g_{yy}.$$

Find g_{xx} defined by

$$g_{xx} = \frac{d}{dx}\left(\frac{dg}{dx}\right).$$

Solve for the first partial derivative in the equation,

$$g_{xx} = \frac{d}{dx}\left(\frac{2x}{x^2+y^2}\right).$$

Then,

$$g_{xx} = \frac{2\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2}.$$

Find g_{yy} defined by

$$g_{yy} = \frac{d}{dy}\left(\frac{dg}{dy}\right).$$

Solve for the first partial derivative in the equation,

$$g_{yy} = \frac{d}{dy}\left(\frac{2y}{x^2+y^2}\right).$$

Then,

$$g_{yy} = \frac{2\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}.$$

Prove that $\Delta g = 0$ given g_{xx} and g_{yy},

$$\Delta g = \frac{2\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2} + \frac{2\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}$$

$$\Delta g = \frac{2\left(y^2-x^2+x^2-y^2\right)}{\left(x^2+y^2\right)^2}$$

$$\Delta g = \frac{2(0)}{\left(x^2+y^2\right)^2}$$

$$\Delta g = 0.$$

Thus, we have shown that g is a harmonic function.

c. Show that $u(x, y) = e^x sin(y)\colon R^2 \to R$ is an example of a harmonic function given the Laplace equation of a function of two variables,

$$\Delta u = 0 \text{ where } \Delta u = u_{xx} + u_{yy}.$$

Find u_{xx} defined by

$$u_{xx} = \frac{d}{dx}\left(\frac{du}{dx}\right).$$

Solve for the first-order partial derivative in the equation,

$$u_{xx} = \frac{d}{dx}\left(e^x sin(y)\right).$$

Then,

$$u_{xx} = e^x sin(y).$$

Find u_{yy} defined by

$$u_{yy} = \frac{d}{dy}\left(\frac{du}{dy}\right).$$

Solve for the first partial derivative in the equation,

$$u_{yy} = \frac{d}{dy}\left(e^{x}cos(y)\right).$$

Then,

$$u_{yy} =- e^{x}sin(y).$$

Prove that $\Delta u = 0$ given u_{xx} and u_{yy},

$$\Delta u = e^{x}sin(y) - e^{x}sin(y)$$
$$\Delta u = 0.$$

Thus, we have shown that u is a harmonic function.

d. Show that $u(x, y) = x^{3} - 3xy^{2}$ is an example of a harmonic function given the Laplace equation of a function of two variables,

$$\Delta u = 0 \text{ where } \Delta u = u_{xx} + u_{yy}$$

Find u_{xx} defined by

$$u_{xx} = \frac{d}{dx}\left(\frac{du}{dx}\right).$$

Solve for the first partial derivative in the equation,

$$u_{xx} = \frac{d}{dx}\left(3x^{2} - 3y^{2}\right).$$

Then,

$$u_{xx} = 6x.$$

Find u_{yy} defined by

$$u_{yy} = \frac{d}{dy}\left(\frac{du}{dy}\right).$$

Solve for the first partial derivative in the equation,

$$u_{yy} = \frac{d}{dy}(- 6xy).$$

Then,

$$u_{yy} =- 6x.$$

Prove that $\Delta u = 0$ given u_{xx} and u_{yy},

$$\Delta u = 6x - 6x$$
$$\Delta u = 0.$$

Thus, we have shown that u is a harmonic function.

Result 2.2. Prove that if $u(x, y)\colon R^{2} \to R$ is harmonic, then both its partial derivatives u_{x} and u_{y} are also harmonic functions.

We know that the harmonic function $u(x, y)\colon R^{2} \to R$ satisfies

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0.$$

Let’s take a look at one of its partial derivatives, $f = \frac{du}{dx}$. f is a harmonic function if it satisfies

$$\frac{d^2f}{dx^2} + \frac{d^2f}{dy^2} = 0.$$

Then,

$$\frac{d^2f}{dx^2} + \frac{d^2f}{dy^2} = \frac{d^3u}{dx^2dx} + \frac{d^3u}{dy^2dx}$$
$$= \frac{d}{dx}\left(\frac{d^2u}{dx^2}\right) + \frac{d}{dx}\left(\frac{d^2u}{dy^2}\right)$$
$$= \frac{d}{dx}\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}\right).$$

We know that $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$, so

$$\frac{d}{dx}(0) = 0$$
$$\frac{d^2f}{dx^2} + \frac{d^2f}{dy^2} = 0.$$

We have proven that u_x is a harmonic function. Similarly, suppose that $g = \frac{du}{dy}$ where g is the other partial derivative of u. g is a harmonic function if it satisfies

$$\frac{d^2g}{dx^2} + \frac{d^2g}{dy^2} = 0.$$

Then,

$$\frac{d^2g}{dx^2} + \frac{d^2g}{dy^2} = \frac{d^3u}{dx^2dy} + \frac{d^3u}{dy^2dy}$$
$$= \frac{d}{dy}\left(\frac{d^2u}{dx^2}\right) + \frac{d}{dy}\left(\frac{d^2u}{dy^2}\right)$$
$$= \frac{d}{dy}\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}\right).$$

We know that $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$, so

$$\frac{d}{dy}(0) = 0$$
$$\frac{d^2g}{dx^2} + \frac{d^2g}{dy^2} = 0.$$

Thus, we have shown that the partial derivatives u_x and u_y are also harmonic functions for $u(x, y)$: $R^2 \to R$. Please note that this proof requires that Clairaut's theorem is true for the original function. That is, the order does not matter when taking partial derivatives because mixed partials are the same i.e.

$$\frac{d^3u}{dx^2dy} = \frac{d^3u}{dydx^2}.$$

Result 2.3. Newton’s universal law of gravitation is given by

$$g = -\frac{GM}{||r||^2} e_r.$$

a. Consider Newton's universal law of gravitation being written as a function of three variables

$$\phi(x, y, z) = -\frac{GM}{\sqrt{x^2+y^2+z^2}}.$$

where G and M are constants. Then, we can express g as the gradient

$$g = -\nabla\phi.$$

The gradient of ϕ, $\nabla\phi$: $R \to R^3$, is defined by

$$\nabla\phi := \begin{bmatrix} D_1\Phi(x) \\ D_2\Phi(x) \\ D_3\Phi(x) \end{bmatrix}.$$

Which can be rewritten as

$$\nabla\phi := \begin{bmatrix} \Phi_x \\ \Phi_y \\ \Phi_z \end{bmatrix}.$$

Find ϕ_x. Take out constants,

$$\phi_x = -GM\frac{d\Phi}{dx}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right).$$

Solve for the first-order partial derivative,

$$\phi_x = -GM\frac{d\Phi}{dx}\left(x^2 + y^2 + z^2\right)^{-1/2}$$

$$\phi_x = -GM \cdot -\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2} \cdot 2x.$$

Simplify,

$$\phi_x = \frac{-GM2x}{-2\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\phi_x = \frac{GMx}{\left(x^2+y^2+z^2\right)^{3/2}}.$$

Similarly, find ϕ_y and ϕ_z.

$$\phi_y = -GM\frac{d\Phi}{dy}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\phi_y = \frac{GMy}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\phi_z = -\ GM\frac{d\Phi}{dz}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\phi_z = \frac{GMz}{\left(x^2+y^2+z^2\right)^{3/2}}.$$

Solve for $\nabla\phi$ given ϕ_x, ϕ_y, ϕ_z.

$$\nabla\phi = \begin{bmatrix} \frac{GMx}{\sqrt{x^2+y^2+z^2}} \\ \frac{GMy}{\sqrt{x^2+y^2+z^2}} \\ \frac{GMz}{\sqrt{x^2+y^2+z^2}} \end{bmatrix}.$$

Then, g can be expressed as the gradient

$$g = -\begin{bmatrix} \frac{GMx}{\sqrt{x^2+y^2+z^2}} \\ \frac{GMy}{\sqrt{x^2+y^2+z^2}} \\ \frac{GMz}{\sqrt{x^2+y^2+z^2}} \end{bmatrix}$$

$$g = \begin{bmatrix} -\frac{GMx}{\sqrt{x^2+y^2+z^2}} \\ -\frac{GMy}{\sqrt{x^2+y^2+z^2}} \\ -\frac{GMz}{\sqrt{x^2+y^2+z^2}} \end{bmatrix}.$$

b. Show that ϕ is a harmonic function given the Laplace equation of a function of three variables defined by

$$\Delta\Phi = 0 \text{ where } \Delta\Phi = \phi_{xx} + \phi_{yy} + \phi_{zz}.$$

Find the second partial derivatives given the first partials calculated in part (a),

$$\frac{d\phi}{dx} = \phi_x = \frac{GMx}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\frac{d\phi}{dy} = \phi_y = \frac{GMy}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\frac{d\phi}{dz} = \phi_z = \frac{GMz}{\left(x^2+y^2+z^2\right)^{3/2}}.$$

Find ϕ_{xx} given by

$$\phi_{xx} = \frac{d}{dx}\left(\frac{d\phi}{dx}\right).$$

Plug in first-order partial derivative,

$$\phi_{xx} = \frac{d}{dx}\left(\frac{GMx}{\left(x^2+y^2+z^2\right)^{3/2}}\right).$$

Take out constants,

$$\phi_{xx} = GM\frac{d}{dx}\left(\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}}\right).$$

Apply the quotient rule,

$$\phi_{xx} = GM\frac{1\cdot\left(x^2+y^2+z^2\right)^{3/2}-x\cdot\frac{3}{2}\left(x^2+y^2+z^2\right)^{1/2}2x}{\left(\left(x^2+y^2+z^2\right)^{3/2}\right)^2}.$$

Simplify,

$$\phi_{xx} = GM\frac{\left(x^2+y^2+z^2\right)^{3/2}-3x^2\left(x^2+y^2+z^2\right)^{1/2}}{\left(x^2+y^2+z^2\right)^3}$$

$$\phi_{xx} = GM\frac{x^2+y^2+z^2-3x^2}{\left(x^2+y^2+z^2\right)^{5/2}}$$

$$\phi_{xx} = GM\frac{-2x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^{5/2}}.$$

Similarly, find ϕ_{yy}

$$\phi_{yy} = \frac{d}{dy}\left(\frac{d\phi}{dy}\right)$$

$$\phi_{yy} = GM\frac{d}{dy}\left(\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}}\right)$$

$$\phi_{yy} = GM\frac{-2y^2+x^2+z^2}{\left(x^2+y^2+z^2\right)^{5/2}}.$$

Find ϕ_{zz}

$$\phi_{zz} = \frac{d}{dz}\left(\frac{d\phi}{dz}\right)$$

$$\phi_{zz} = GM\frac{d}{dz}\left(\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}}\right)$$

$$\phi_{zz} = GM\frac{-2z^2+x^2+y^2}{\left(x^2+y^2+z^2\right)^{5/2}}.$$

Prove that $\Delta\phi = 0$ given $\Delta\phi = \phi_{xx} + \phi_{yy} + \phi_{zz}$.

$$\Delta\phi = GM\frac{-2x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^{5/2}} + GM\frac{-2y^2+x^2+z^2}{\left(x^2+y^2+z^2\right)^{5/2}} + GM\frac{-2z^2+x^2+y^2}{\left(x^2+y^2+z^2\right)^{5/2}}.$$

$$\Delta\phi = GM\left(\frac{-2x^2+y^2+z^2-2y^2+x^2+z^2-2z^2+x^2+y^2}{\left(x^2+y^2+z^2\right)^{5/2}}\right)$$
$$\Delta\phi = GM\left(\frac{2x^2-2x^2+2y^2-2y^2+2z^2-2z^2}{\left(x^2+y^2+z^2\right)^{5/2}}\right)$$
$$\Delta\phi = GM\left(\frac{0}{\left(x^2+y^2+z^2\right)^{5/2}}\right)$$
$$\Delta\phi = GM(0)$$
$$\Delta\phi = 0.$$

Thus, we have shown that ϕ is a harmonic function.

3. Change of Variables

Result 3.1. Suppose that $u = f(x, y)$: $R^2 \to R$ is a harmonic function, then the Laplace's equation in R^2 is

$$\Delta u = 0 \text{ where } \Delta u = u_{xx} + u_{yy}.$$

However, suppose we change to thinking in terms of polar coordinates in R^2 by using the transformations

$$x = rcos\theta,\ y = rsin\theta.$$

In this exercise, you will prove that *u* satisfies the polar Laplace equation

$$r^2u_{rr} + ru_r + u_{\theta\theta} = 0.$$

Note: *u* can be written as a function of *r* and θ, say $u = g(r, \theta)$, where *g* is some new function. We will use the chain rule to calculate the partial derivatives in (a-e) since *u* is a function of (x, y) and (x, y) are functions of (r, θ).

a. Calculate the first and second partial derivatives of u with respect to x and y. In other words, find u_x, u_y, u_{xx}, and u_{yy}.

   First and second partial derivatives with respect to x,

$$u_x = \frac{du}{dx}$$
$$u_{xx} = \frac{d^2u}{dx^2}.$$

   First and second partial derivatives with respect to y,

$$u_y = \frac{du}{dy}$$
$$u_{yy} = \frac{d^2u}{dy^2}.$$

b. Calculate the first and second partial derivatives of *x* and *y* with respect to *r* and θ. In other words, find x_r, y_r, x_{rr}, y_{rr}, x_θ, y_θ, $x_{\theta\theta}$, and $y_{\theta\theta}$.

   First partial derivatives with respect to *r*,

$$x_r = \frac{dx}{dr} = cos\theta$$

$$y_r = \frac{dy}{dr} = sin\theta.$$

Second partial derivatives with respect to *r*,

$$x_{rr} = \frac{d^2x}{dr^2} = 0$$

$$y_{rr} = \frac{d^2y}{dr^2} = 0.$$

First partial derivatives with respect to θ,

$$x_\theta = \frac{dx}{d\theta} =- rsin\theta$$

$$y_\theta = \frac{dy}{d\theta} = rcos\theta.$$

Second partial derivatives with respect to θ,

$$x_{\theta\theta} = \frac{d^2x}{d\theta^2} =- rcos\theta$$

$$y_{\theta\theta} = \frac{d^2y}{d\theta^2} =- rsin\theta.$$

c. Use the chain rule to calculate u_r. Use this formula to find an expression for ru_r purely in terms of $x,\ y,\ u_x$, and u_y.

Solve for u_r using the chain rule,

$$u_r = \frac{du}{dx}\frac{dx}{dr} + \frac{du}{dy}\frac{dy}{dr}$$

$$u_r = u_x x_r + u_y y_r$$

$$u_r = cos\theta u_x + sin\theta u_y.$$

Then solve for ru_r,

$$ru_r = rcos\theta u_x + rsin\theta u_y.$$

Recall that $x = rcos\theta$ and $y = rsin\theta$, so

$$ru_r = xu_x + yu_y.$$

d. Use the chain rule to calculate u_{rr}. Use this formula to find an expression for r^2u_{rr} purely in terms of $x,\ y,\ u_{xx},\ u_{xy}$, and u_{yy}.

Solve for u_{rr} using the chain rule,

$$u_{rr} = \frac{d}{dr}\left(\frac{du}{dx}\frac{dx}{dr} + \frac{du}{dy}\frac{dy}{dr}\right)$$

$$u_{rr} = \frac{d}{dr}\frac{du}{dx}\frac{dx}{dr} + \frac{d}{dr}\frac{du}{dy}\frac{dy}{dr}$$

$$u_{rr} = cos\theta\frac{d}{dr}\frac{du}{dx} + sin\theta\frac{d}{dr}\frac{du}{dy}$$

$$u_{rr} = cos\theta\left(\frac{d}{dx}\frac{du}{dx}\frac{dx}{dr} + \frac{d}{dy}\frac{du}{dx}\frac{dy}{dr}\right) + sin\theta\left(\frac{d}{dx}\frac{du}{dy}\frac{dx}{dr} + \frac{d}{dy}\frac{du}{dy}\frac{dy}{dr}\right)$$

$$u_{rr} = \cos\theta\left(\frac{dx}{dr}\frac{d^2u}{dx^2} + \frac{dy}{dr}\frac{d^2u}{dxdy}\right) + \sin\theta\left(\frac{dx}{dr}\frac{d^2u}{dydx} + \frac{dy}{dr}\frac{d^2u}{dy^2}\right)$$
$$u_{rr} = \cos\theta\left(\cos\theta\frac{d^2u}{dx^2} + \sin\theta\frac{d^2u}{dxdy}\right) + \sin\theta\left(\cos\theta\frac{d^2u}{dydx} + \sin\theta\frac{d^2u}{dy^2}\right)$$
$$u_{rr} = \cos^2\theta\frac{d^2u}{dx^2} + \sin\theta\cos\theta\frac{d^2u}{dxdy} + \sin\theta\cos\theta\frac{d^2u}{dydx} + \sin^2\theta\frac{d^2u}{dy^2}.$$

Mixed partials are equal according to Clairaut's theorem, so the answer simplifies to

$$u_{rr} = \cos^2\theta\frac{d^2u}{dx^2} + 2\sin\theta\cos\theta\frac{d^2u}{dxdy} + \sin^2\theta\frac{d^2u}{dy^2}.$$

Solve for r^2u_{rr} given u_{rr},

$$r^2u_{rr} = r^2\left(\cos^2\theta\frac{d^2u}{dx^2} + 2\sin\theta\cos\theta\frac{d^2u}{dxdy} + \sin^2\theta\frac{d^2u}{dy^2}\right)$$
$$r^2u_{rr} = r^2\cos^2\theta\frac{d^2u}{dx^2} + 2r^2\sin\theta\cos\theta\frac{d^2u}{dxdy} + r^2\sin^2\theta\frac{d^2u}{dy^2}$$
$$r^2u_{rr} = x^2u_{xx} + 2xyu_{xy} + y^2u_{yy}.$$

e. Use the chain rule to calculate $u_{\theta\theta}$. Use this formula to find an expression for $u_{\theta\theta}$ purely in terms of x, y, u_x, u_y, u_{xy}, and u_{yy}.

Solve for the first partial derivative u_θ using the chain rule,

$$u_\theta = \frac{du}{dx}\frac{dx}{d\theta} + \frac{du}{dy}\frac{dy}{d\theta}$$
$$u_\theta = \frac{du}{dx}(-r\sin\theta) + \frac{du}{dy}(r\cos\theta)$$
$$u_\theta = -r\sin\theta\frac{du}{dx} + r\cos\theta\frac{du}{dy}$$
$$u_\theta = -r\sin\theta u_x + r\cos\theta u_y.$$

Solve for $u_{\theta\theta}$ using the chain rule,

$$u_{\theta\theta} = \frac{d}{d\theta}\left(-r\sin\theta\frac{du}{dx} + r\cos\theta\frac{du}{dy}\right)$$
$$u_{\theta\theta} = -r\cos\theta\frac{du}{dx} - r\sin\theta\frac{d}{d\theta}\frac{du}{dx} - r\sin\theta\frac{du}{dy} + r\cos\theta\frac{d}{d\theta}\frac{du}{dy}$$
$$u_{\theta\theta} = -r\sin\theta\frac{d}{d\theta}\frac{du}{dx} + r\cos\theta\frac{d}{d\theta}\frac{du}{dy} - r\cos\theta\frac{du}{dx} - r\sin\theta\frac{du}{dy}$$
$$u_{\theta\theta} = -r\sin\theta\left(\frac{d}{dx}\frac{du}{dx}\frac{dx}{d\theta} + \frac{d}{dy}\frac{du}{dx}\frac{dy}{d\theta}\right) + r\cos\theta\left(\frac{d}{dx}\frac{du}{dy}\frac{dx}{d\theta} + \frac{d}{dy}\frac{du}{dy}\frac{dy}{d\theta}\right) - r\cos\theta\frac{du}{dx} - r\sin\theta\frac{du}{dy}$$
$$u_{\theta\theta} = -r\sin\theta\left(\frac{dx}{d\theta}\frac{d^2u}{dx^2} + \frac{dy}{d\theta}\frac{d^2u}{dxdy}\right) + r\cos\theta\left(\frac{dx}{d\theta}\frac{d^2u}{dydx} + \frac{dy}{d\theta}\frac{d^2u}{dy^2}\right) - r\cos\theta\frac{du}{dx} - r\sin\theta\frac{du}{dy}$$
$$u_{\theta\theta} = -r\sin\theta\left(-r\sin\theta\frac{d^2u}{dx^2} + r\cos\theta\frac{d^2u}{dxdy}\right) + r\cos\theta\left(-r\sin\theta\frac{d^2u}{dxdy} + r\cos\theta\frac{d^2u}{dy^2}\right) - r\cos\theta\frac{du}{dx} - r\sin\theta\frac{du}{dy}$$
$$u_{\theta\theta} = r^2\sin^2\theta\frac{d^2u}{dx^2} - r^2\sin\theta\cos\theta\frac{d^2u}{dxdy} - r^2\sin\theta\cos\theta\frac{d^2u}{dydx} + r^2\cos^2\theta\frac{d^2u}{dy^2} - r\cos\theta\frac{du}{dx} - r\sin\theta\frac{du}{dy}$$

Mixed partials are equal according to Clairaut's theorem, so the answer simplifies to

$$u_{\theta\theta} = r^2 sin^2\theta \frac{d^2u}{dx^2} - 2r^2 sin\theta cos\theta \frac{d^2u}{dxdy} + r^2 cos^2\theta \frac{d^2u}{dy^2} - rcos\theta \frac{du}{dx} - rsin\theta \frac{du}{dy}$$

$$u_{\theta\theta} = r^2 sin^2\theta u_{xx} - 2r^2 sin\theta cos\theta u_{xy} + r^2 cos^2\theta u_{yy} - rcos\theta u_x - rsin\theta u_y$$

$$u_{\theta\theta} = y^2 u_{xx} - 2xyu_{xy} + x^2 u_{yy} - xu_x - yu_y.$$

f. Conclude that *u* satisfies the polar Laplace equation.

Prove that *u* satisfies the polar Laplace equation given

$$r^2 u_{rr} + ru_r + u_{\theta\theta} = 0.$$

We have already solved for $r^2 u_{rr}$, ru_r, and $u_{\theta\theta}$,

$$r^2 u_{rr} = x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy}$$

$$ru_r = xu_x + yu_y$$

$$u_{\theta\theta} = y^2 u_{xx} - 2xyu_{xy} + x^2 u_{yy} - xu_x - yu_y.$$

Then,

$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} + xu_x + yu_y + y^2 u_{xx} - 2xyu_{xy} + x^2 u_{yy} - xu_x - yu_y = 0.$$

Simplify,

$$x^2 u_{xx} + y^2 u_{yy} + y^2 u_{xx} + x^2 u_{yy} = 0$$

$$r^2 cos^2\theta u_{xx} + r^2 sin^2\theta u_{yy} + r^2 sin^2\theta u_{xx} + r^2 cos^2\theta u_{yy} = 0$$

$$x^2 (u_{xx} + u_{yy}) + y^2 (u_{xx} + u_{yy}) = 0.$$

Recall that *u* is a harmonic function where $u_{xx} + u_{yy} = 0$, so

$$x^2 (0) + y^2 (0) = 0$$

$$0 + 0 = 0.$$

Therefore we have shown that *u* satisfies the polar Laplace equation.

Result 3.2. Suppose that $u = f(x, y)$ satisfies the equation

$$x^2 u_{xx} + y^2 u_{yy} + xu_x + yu_y = 0.$$

Show using the transformation

$$x = e^s,\ y = e^t.$$

that $u = g(s, t)$ is harmonic.

By using the transformation such that (x, y) are functions of (s, t), we can simplify the PDE in terms of *s* and *t* to prove that *u* is harmonic.

$u = g(s, t)$ is harmonic if it satisfies the Laplace equation

$$\Delta u = u_{ss} + u_{tt} = 0.$$

Calculate the partial derivatives of *x* and *y* with respect to *s* and *t*

$$\frac{dx}{ds} = e^s,\ \frac{dy}{ds} = 0,\ \frac{dx}{dt} = 0,\ \frac{dy}{dt} = e^t.$$

Note that $\frac{dx}{ds} = x$ and $\frac{dy}{dt} = y$.

Calculate the first partial derivative of u with respect to *s* using the chain rule,

$$u_s = \frac{du}{dx}\frac{dx}{ds} + \frac{du}{dy}\frac{dy}{ds}$$

$$u_s = x\frac{du}{dx} + 0$$

$$u_s = xu_x.$$

Calculate the second partial derivative of u with respect to *s* using the chain rule,

$$u_{ss} = \frac{d}{ds}\left(x\frac{du}{dx}\right)$$

$$u_{ss} = \left(\frac{d}{ds}x\right)\frac{du}{dx} + \left(\frac{d}{ds}\frac{du}{dx}\right)x$$

$$u_{ss} = \frac{d}{dx}x\frac{dx}{ds}\frac{du}{dx} + \frac{d}{dx}\frac{du}{dx}\frac{dx}{ds}x$$

$$u_{ss} = \frac{dx}{ds}\frac{du}{dx} + \frac{d^2u}{dx^2}\frac{dx}{ds}x$$

$$u_{ss} = xu_x + u_{xx}xx$$

$$u_{ss} = xu_x + x^2u_{xx}.$$

Calculate the first partial derivative of u with respect to *t* using the chain rule,

$$u_t = \frac{du}{dx}\frac{dx}{dt} + \frac{du}{dy}\frac{dy}{dt}$$

$$u_t = 0 + y\frac{du}{dy}$$

$$u_t = yu_y.$$

Calculate the second partial derivative of u with respect to *t* using the chain rule,

$$u_{tt} = \frac{d}{dt}\left(y\frac{du}{dy}\right)$$

$$u_{tt} = \left(\frac{d}{dt}y\right)\frac{du}{dy} + \left(\frac{d}{dt}\frac{du}{dy}\right)y$$

$$u_{tt} = \frac{d}{dy}y\frac{dy}{dt}\frac{du}{dy} + \frac{d}{dy}\frac{du}{dy}\frac{dy}{dt}y$$

$$u_{tt} = \frac{dy}{dt}\frac{du}{dy} + \frac{d^2y}{dy^2}\frac{dy}{dt}y$$

$$u_{tt} = yu_y + u_{yy}yy$$

$$u_{tt} = yu_y + y^2u_{yy}.$$

Solve for x^2u_{xx} and y^2u_{yy} given the calculations above,

$$x^2u_{xx} = u_{ss} - xu_x$$

$$y^2u_{yy} = u_{tt} - yu_y.$$

Substitute in the equation

$$x^2 u_{xx} + y^2 u_{yy} + xu_x + yu_y = 0$$

$$u_{ss} - xu_x + u_{tt} - yu_y + xu_x + yu_y = 0.$$

$$u_{ss} + u_{tt} = 0.$$

Thus proving that $u = g(s, t)$ is a harmonic function.