MATH 131A: HOMEWORK 3

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Problem 1. *Solution.*
Ross 7.2

(b) Converges. For sufficiently large n, the n term dominates $\frac{3n+1}{4n-1}$ such that it is approximately equal to $\frac{3n}{4n} = \frac{3}{4}$. Therefore, $b_n \to \frac{3}{4}$.

(d) Does not converge. $sin(\frac{n\pi}{4})$ oscillates between values $[-1, 1]$. For values $n = 2, 10, 18 \ldots$, $sin(\frac{n\pi}{4}) = 1$ and for values $n = 6, 14, 22 \ldots$, $sin(\frac{n\pi}{4}) = -1$. Therefore, the sequence does not have a limit.

Ross 7.4

(a) Consider the sequence $\{s_n\}_{n\in\mathbb{N}}$ where $s_n = \frac{\sqrt{2}}{n}$. This is a sequence of irrational numbers having a rational limit where $\lim s_n = 0$.

First, we want to show $\frac{\sqrt{2}}{n}$ is irrational for any $n \in \mathbb{N}$. We will prove this by contradiction. Assume $\frac{\sqrt{2}}{n}$ is equal to some rational number r. Then, $r = \frac{\sqrt{2}}{n}$, hence $nr = \sqrt{2}$. $n \in \mathbb{N}$ shows that nr is an integer multiple of r, therefore $nr \in \mathbb{Q}$. However, we know $\sqrt{2}$ is irrational, so we have a contradiction. Thus, $\{s_n\}$ is a sequence of irrational numbers.

Second, we claim that the sequence has a rational limit $\lim s_n = 0$. Let $\epsilon > 0$. Let $N = \frac{\sqrt{2}}{\epsilon}$. Then $n > N$ implies $n > \frac{\sqrt{2}}{\epsilon}$ which implies $n\epsilon > \sqrt{2}$ and hence $\epsilon > \frac{\sqrt{2}}{n}$. Thus $n > N$ implies $|\frac{\sqrt{2}}{n} - 0| < \epsilon$. This proves $\lim s_n = 0$.

(b) Consider the sequence $\{r_n\}_{n\in\mathbb{N}} = \{1, 1.4, 1.41, 1.414, \ldots\}$ where $r_n = \{\text{first n digits of } \sqrt{2}\}$. This is a sequence of rational numbers having an irrational limit where $\lim r_n = \sqrt{2}$.

First, notice that every term in the sequence can be written as a rational of the form $\frac{p}{q}$ where p, q are integers and $q \neq 0$. For any $r_n \in \{r_n\}$, there are $n-1$ digits (not counting trailing 0s) to the right of the decimal place. Then, we can write $r_n = \frac{p}{q}$ where $p = r_n \cdot 10^{n-1}$ and $q = 10^{n-1}$. $p, q \in \mathbb{Z}$ and $q \neq 0$, so we have shown that $\{r_n\}$ is a sequence of rational numbers.

Second, we want to show that the sequence has an irrational limit $\lim r_n = \sqrt{2}$. To prove this, we must find an $N \in \mathbb{N}$ such that $n > N$ implies $|r_n - \sqrt{2}| < \epsilon$ for any $\epsilon > 0$. Take ϵ to be very small so that $\epsilon = 0.00\ldots$ Let $d =$ the number of leading 0s after the decimal place in ϵ. Then choose $N = d + 2$ so that r_n is a decimal place more precise than ϵ. Then $n > N$ implies $|r_n - \sqrt{2}| < \epsilon$ for all $\epsilon > 0$. Thus, we have shown $\lim r_n = \sqrt{2}$.

□

Problem 2. *Solution.*
Ross 8.6

(a) Suppose $\lim s_n = 0$. We will prove that this implies $\lim |s_n| = 0$. We know $\forall\epsilon > 0, \exists N \in \mathbb{N}$ such that $n > N \Rightarrow |s_n - 0| = |s_n| < \epsilon$. Observe $||s_n|| = |s_n|$. Then we could write $||s_n|| < \epsilon$ which implies $||s_n| - 0| < \epsilon$. Given that $\lim s_n = 0$, we have shown $\forall\epsilon > 0, \exists N \in \mathbb{N}$ such that $n > N \Rightarrow ||s_n| - 0| < \epsilon$.

In other words, $\lim |s_n| = 0$.

Now we want to prove that if $\lim |s_n| = 0$, then $\lim s_n = 0$. $|s_n|$ converges to 0 provided that $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n > N \Rightarrow ||s_n| - 0| = ||s_n|| = |s_n| < \epsilon$. We have shown $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n > N \Rightarrow |s_n| = |s_n - 0| < \epsilon$. Therefore, we can say $\lim s_n = 0$ given $\lim |s_n| = 0$.

We have proven $\lim s_n = 0$ if and only if $\lim |s_n| = 0$.

(b) Let $s_n = (-1)^n$. We will show that $\lim s_n$ does not exist and $\lim |s_n|$ exists. First, we claim that $\lim s_n$ does not exist. Proof by contradiction: assume $\lim s_n = s$ for some $s \in \mathbb{R}$. Letting $\epsilon = 1$ in the definition of the limit, we see that there exists $N \in \mathbb{N}$ such that $n > N \Rightarrow |s_n - s| < 1$. Consider the 2 cases:

- n is even. Then $|(-1)^n - s| = |1 - s| < 1$.
- n is odd. Then $|(-1)^n - s| = |-1 - s| = |-(1 + s)| = |1 + s| < 1$.

Apply the triangle inequality:

$$2 = |(1 - s) + (1 + s)| \leq |1 - s| + |1 + s| < 1 + 1 = 2$$

$$2 < 2.$$

$2 < 2$ shows our assumption $\lim s_n = s$ must be wrong. Because s is arbitrary, the sequence $(-1)^n$ does not converge. Therefore, we can conclude that the limit $\lim s_n$ does not exist.

Now consider the sequence $|s_n|$. We claim that its limit exists and $\lim |s_n| = 1$. Note $|s_n|$ is a constant sequence since $|(-1)^n| = 1 \; \forall n \in \mathbb{N}$. By definition of the limit, $||s_n| - s| = |1 - 1| = 0 < \epsilon$ and this completes the proof. The limit of $|s_n|$ exists where $\lim |s_n| = 1$.

Ross 8.8

(b) Prove $\lim[\sqrt{n^2 + n} - n] = \frac{1}{2}$.
$\forall \epsilon > 0$, we need to find some $N \in \mathbb{N}$ such that $n > N$ implies $\left|\sqrt{n^2 + n} - n - \frac{1}{2}\right| < \epsilon$. Consider the following:

$$\begin{aligned}
\left|\sqrt{n^2+n} - n - \frac{1}{2}\right| &= \left|\sqrt{n^2+n} - (n + \frac{1}{2})\right| \\
&= \left|\sqrt{n^2+n} - \sqrt{\left(n + \frac{1}{2}\right)^2}\right| \\
&= \left|\frac{(n^2+n) - \left(n + \frac{1}{2}\right)^2}{\sqrt{n^2+n} + \sqrt{\left(n+\frac{1}{2}\right)^2}}\right| \\
&= \left|\frac{n^2 + n - n^2 - n - \frac{1}{4}}{\sqrt{n^2+n} + \sqrt{n^2+n+\frac{1}{4}}}\right| \\
&= \left|-\frac{1/4}{\sqrt{n^2+n} + \sqrt{n^2+n+\frac{1}{4}}}\right| \\
&= \frac{1/4}{\sqrt{n^2+n} + \sqrt{n^2+n+\frac{1}{4}}}. && \text{(Positive)}
\end{aligned}$$

From this we conclude that $\left|\sqrt{n^2+n}-n-\frac{1}{2}\right| = \frac{1/4}{\sqrt{n^2+n}+\sqrt{n^2+n+\frac{1}{4}}} < \epsilon$. Note that we do not need to find the least N such that this inequality is true. So we will simplify matters by making estimates. The idea is that we want to bound $\frac{1/4}{\sqrt{n^2+n}+\sqrt{n^2+n+\frac{1}{4}}}$ for sufficiently large n. To find such a bound it is enough to find a lower bound for the denominator. To make the denominator smaller, we note $\sqrt{n^2+n}+\sqrt{n^2+n+\frac{1}{4}} \geq \sqrt{n^2+n}+\sqrt{n^2+n}$. We can bound it to be even smaller: $\sqrt{n^2+n}+\sqrt{n^2+n} \geq \sqrt{n^2}+\sqrt{n^2} = n+n = 2n$ given n is large. In summary,

$$\frac{1/4}{\sqrt{n^2+n}+\sqrt{n^2+n+\frac{1}{4}}} \leq \frac{1/4}{\sqrt{n^2+n}+\sqrt{n^2+n}} \leq \frac{1/4}{\sqrt{n^2}+\sqrt{n^2}} = \frac{1/4}{n+n} = \frac{1/4}{2n} = \frac{1}{8n}.$$

We want to choose an N where $n > N$ implies $\frac{1}{8n} < \epsilon$ for all $\epsilon > 0$. From $\frac{1}{8n} < \epsilon$, we get $n > \frac{1}{8\epsilon}$, so it is suffice to set $N = \frac{1}{8\epsilon}$. Then, we can say that for the same N, $\frac{1/4}{\sqrt{n^2+n}+\sqrt{n^2+n+\frac{1}{4}}} < \epsilon$ holds by transitivity. We have shown $\forall\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $n > N$ implies $\left|\sqrt{n^2+n}-n-\frac{1}{2}\right| < \epsilon$. Thus, we have proven $\lim[\sqrt{n^2+n}-n] = \frac{1}{2}$.

□

Problem 3. *Solution.* Let $\{a_n\}_{n\in\mathbb{N}}$ be a convergent sequence that has a limit L. Then, $\forall\epsilon > 0$, there exists a number N such that $n > N$ implies $|a_n - L| < \epsilon$. We will prove that limits are unique. i.e., if the sequence has another limit $\tilde{L}$ which satisfies this statement, then $\tilde{L} = L$. In short, the values a_n cannot be getting arbitrarily close to different values for large n. To prove this, consider $\epsilon > 0$. By definition of the limit, there exists N_1 so that

$$n > N_1 \text{ implies } |a_n - L| < \frac{\epsilon}{2} \tag{1}$$

and there exists N_2 so that

$$n > N_2 \text{ implies } |a_n - \tilde{L}| < \frac{\epsilon}{2}. \tag{2}$$

For $n > max\{N_1, N_2\}$, the triangle inequality shows

$$|L-\tilde{L}| = |(L-a_n)+(a_n-\tilde{L})| \leq |L-a_n| + |a_n-\tilde{L}| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

This shows $|L-\tilde{L}| < \epsilon\ \forall\epsilon > 0$. It follows that $|L-\tilde{L}| = 0$, hence $L = \tilde{L}$. □

Problem 4. *Solution.* Proof by contradiction. Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be two sequences. Suppose that $a_n \to a$ and $b_n \to b$ for some $a, b \in \mathbb{R}$. Let $a_n \leq b_n\ \forall n \in \mathbb{N}$ and $a > b$. Let $\epsilon > 0$. Since $\lim a_n = a$, there exists N_1 such that $|a_n - a| < \epsilon$ for $n > N_1$. In particular,

$$n > N_1 \text{ implies } a - \epsilon < a_n.$$

Likewise there exists N_2 such that $|b_n - b| < \epsilon$ for $n > N_2$, so

$$n > N_2 \text{ implies } b_n < b + \epsilon.$$

Given that $a_n \leq b_n\ \forall n \in \mathbb{N}$, $n > max(N_1, N_2)$ implies

$$\begin{aligned} a - \epsilon < a_n \leq b_n &< b + \epsilon \\ a - \epsilon &< b + \epsilon \\ a &< b + 2\epsilon \end{aligned}$$

We assumed $a > b$, so $\frac{a-b}{4} > 0$. Choose $\epsilon = \frac{a-b}{4}$, then we get $a < b + \frac{a-b}{2}$ which is just $2a < 2b + a - b$, hence $a < b$ implies $a \leq b$. Our assumption states that $a > b$ if $a_n \leq b_n\ \forall n \in \mathbb{N}$. But we have shown $a \leq b$ in this case, thus forming a contradiction. Therefore, we have proven if $a_n \leq b_n\ \forall n \in \mathbb{N}$, then $a \leq b$ given $a_n \to a$ and $b_n \to b$ for some $a, b \in \mathbb{R}$. □