MATH 131A: HOMEWORK 5

Lana Lim 105817312

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Problem 1. *Solution.*
$\boxed{\text{Ross 11.2}}$ Let $a_n = (-1)^n$ and $d_n = \frac{6n+4}{7n-3}$

(a) $\{1, 1, 1, 1, 1\}$, $\{\frac{10}{4}, \frac{16}{11}, \frac{22}{18}, \frac{28}{25}, \frac{34}{32}\}$.

(b)
- $s_n = a_n$, $S = \{1, -1\}$.
- $s_n = d_n$, $S = \{\frac{6}{7}\}$.

(c)
- $\limsup a_n = \max\{1, -1\} = 1, \liminf a_n = \min\{1, -1\} = -1$.
- $\limsup d_n = \liminf d_n = \frac{6}{7}$.

$\boxed{\text{Ross 11.8}}$ Use Definition 10.6 and Exercise 5.4 to prove $\liminf s_n = -\limsup(-s_n)$ for every sequence (s_n).

Recall in Exercise 5.4 that if S is a nonempty subset of $\mathbb{R}$ and $-S = \{-s : s \in S\}$, then $\inf S = -\sup(-S)$. Let $V_N = \inf\{s_n : n > N\}$ and $U_N = \sup\{-s_n : n > N\}$. Note $\{s_n : n > N\}$ and $\{-s_n : n > N\}$ are nonempty subsets of $\mathbb{R}$, therefore we say

$$V_N = -U_N. \tag{1}$$

We are only interested in the case where (s_n) is bounded. This means V_N is a real number for all $N \in \mathbb{N}$. Then (1) implies U_N is a real number for all $N \in \mathbb{N}$. By definition of limit supremum/infimum, the limit is always defined (as a finite real number in this case). So we can send $N \to \infty$ and apply Theorem 9.2:

$$\begin{aligned} \lim_{N\to\infty} V_N &= \lim_{N\to\infty} -U_N \\ &= -\lim_{N\to\infty} U_N && \text{(Theorem 9.2)} \end{aligned}$$

$$\implies \lim_{N\to\infty} \inf\{s_n : n > N\} = -\lim_{N\to\infty} \sup\{-s_n : n > N\}.$$

□

Problem 2. *Solution.*

(a) Let n be arbitrary.

$$\begin{aligned} &|s_{n+1} - s_n| \le r|s_n - s_{n-1}| \\ &\implies |s_{n+1} - s_n| \le r|s_n - s_{n-1}| \le r^2|s_{n-1} - s_{n-2}| \le \ldots \le r^{n-1}|s_2 - s_1| && (*) \\ &\implies |s_{n+1} - s_n| \le r^{n-1}|s_2 - s_1|. \end{aligned}$$

Note (*): *We write n terms starting from $|s_{n+1} - s_n|$ to reach our final inequality. Given that $|s_{n+1} - s_n| \le r|s_n - s_{n-1}|$, we eventually get $|s_{n+1} - s_n| \le r^{n-1}|s_2 - s_1|$.*

Thus we have proven $|s_{n+1} - s_n| \le r^{n-1}|s_2 - s_1|$ for all $n \in \mathbb{N}$.

(b) Let $\epsilon > 0$. We want to find a N such that $m, n \geq N$ implies $|s_m - s_n| < \epsilon$. Without loss of generality, assume $m > n$. We can write

$$\begin{aligned} |s_m - s_n| &= |(s_m - s_{m-1}) + (s_{m-1} - s_{m-2}) + \ldots + (s_{n+1} - s_n)| \\ &\leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \ldots + |s_{n+1} - s_n| \\ &\leq r^{m-2}|s_2 - s_1| + r^{m-3}|s_2 - s_1| + \ldots + r^{n-1}|s_2 - s_1| \end{aligned}$$

$$\begin{aligned} \implies |s_m - s_n| &\leq (r^{m-2} + r^{m-3} + \ldots + r^{n-1})|s_2 - s_1| \\ &< (r^{n-1} + r^n + \ldots)|s_2 - s_1| \qquad (*) \\ &= \frac{r^{n-1}}{1-r}|s_2 - s_1|. \qquad (**) \end{aligned}$$

Note (*): *Given that* $r, |s_2 - s_1| > 0$ *and* $m > n$*, the sum of an infinite series* $\sum_{k=n-1}^{\infty} r^k |s_2 - s_1|$ *is larger than the sum of a finite series* $\sum_{k=n-1}^{m-2} r^k |s_2 - s_1|$.
Note (**): *The formula for the sum of an infinite geometric series is denoted* $S = \frac{a}{1-r}$ *where* $a = r^{n-1}$ *and* $r = r$.

Then we have

$$|s_m - s_n| < \frac{r^{n-1}}{1-r}|s_2 - s_1|.$$

$m, n \geq N$ implies

$$|s_m - s_n| < \frac{r^{n-1}}{1-r}|s_2 - s_1| < \frac{r^{N-1}}{1-r}|s_2 - s_1|.$$

We want to choose N such that $\frac{r^{N-1}}{1-r}|s_2 - s_1| < \epsilon$ holds. Let

$$\begin{aligned} &N < log_r\left(\epsilon\frac{1-r}{|s_2 - s_1|}\right) + 1 \\ &\implies N - 1 < log_r\left(\epsilon\frac{1-r}{|s_2 - s_1|}\right) \\ &\implies r^{N-1} < \epsilon\frac{1-r}{|s_2 - s_1|} \\ &\implies \frac{r^{N-1}}{1-r}|s_2 - s_1| < \epsilon. \end{aligned}$$

Thus choosing $N < log_r(\epsilon\frac{1-r}{|s_2 - s_1|}) + 1$ implies $|s_m - s_n| < \epsilon$ (by transitivity) for any $\epsilon > 0$. Therefore we deduce that the sequence is Cauchy.

$\square$

Problem 3. *Solution.*

(a) We will prove that if $v_M = +\infty$ for some $M \in \mathbb{N}$, then $v_N = +\infty$ for all $N \geq M$. Observe that the sequence is not bounded from above given $v_M = +\infty$.

- The base case $N = M$ is already given.
- Assume $v_N = +\infty$ is true for some $N > M$, we want to show that it is also true for $N + 1$.
- We know
  $v_N = \sup\{s_n : n > N\} = \sup\{s_N, s_{N+1}, s_{N+2}, \ldots\}$
  $v_{N+1} = \sup\{s_n : n > N + 1\} = \sup\{s_{N+1}, s_{N+2}, \ldots\}$.

- That is, we obtain v_{N+1} after removing the term s_N from the sequence $\{s_N, s_{N+1}, s_{N+2}, \ldots\}$.
- Let us assume $v_N \neq v_{N+1}$. This implies that removing the term s_N from the sequence is nontrivial, therefore we assume that s_N is an upper bound for $\{s_N, s_{N+1}, s_{N+2}, \ldots\}$. s_N is a real number, therefore this contradicts our first assumption that the sequence is not bounded from above. This is because by definition, there should exist a $s_n > M$ for all $M \in \mathbb{R}$ where $M = s_N$.
- Therefore we have shown $v_{N+1} = +\infty$ is true whenever $v_N = +\infty$ is true, thus we have proven if $v_M = +\infty$ for some $M \in \mathbb{N}$, then $v_N = +\infty$ is true for all $N \geq M$.

$\square$

Problem 4. *Solution.*

(a) We are only interested in the case where the 2 sequences are bounded. By the Completeness Axiom, (s_n) and (t_n) have inf's and are real numbers. That is, for all $n > N$

$$s_n \geq \inf\{s_n : n > N\}$$
$$t_n \geq \inf\{t_n : n > N\}$$

$$\implies s_n + t_n \geq \inf\{s_n : n > N\} + \inf\{t_n : n > N\}.$$

$\inf\{s_n : n > N\}$ and $\inf\{t_n : n > N\}$ are finite and real, therefore $\inf\{s_n : n > N\} + \inf\{t_n : n > N\}$ is a lower bound for $\{s_n + t_n : n > N\}$. Then there exists a greatest lower bound such that

$$\inf\{s_n + t_n : n > N\} \geq \inf\{s_n : n > N\} + \inf\{t_n : n > N\}.$$

Let $a_N = \inf\{s_n + t_n : n > N\}$ and $b_N = \inf\{s_n : n > N\} + \inf\{t_n : n > N\}$. By definition of limit infimum, the limit is always defined. We want to prove $\lim_{N\to\infty} a_N$ and $\lim_{N\to\infty} b_N$ are finite and real by using MCT:

- a_N and b_N are increasing in N (inf increases as you remove more terms from the sequence).
- a_N is bounded from above given that (s_n) and (t_n) are bounded from above by some $s, t \in \mathbb{R}$ for all $n \in \mathbb{N}$. Therefore this implies $a_N = \inf\{s_n + t_n : n > N\} \leq s_n + t_n \leq s + t$.
- Similarly, b_N is bounded from above given that $b_N = \inf\{s_n : n > N\} + \inf\{t_n : n > N\} \leq s_n + t_n \leq s + t$.

By MCT, $a_N \to a$ and $b_N \to b$ where $a, b \in \mathbb{R}$. Then $a_N \geq b_N$ for all $N \in \mathbb{N}$ implies $a \geq b$ (Homework 3 Problem 4). Therefore

$$\lim_{N\to\infty} \inf\{s_n + t_n : n > N\} \geq \lim_{N\to\infty} (\inf\{s_n : n > N\} + \inf\{t_n : n > N\})$$

$$\implies \lim_{N\to\infty} \inf\{s_n + t_n : n > N\} \geq \lim_{N\to\infty} \inf\{s_n : n > N\} + \lim_{N\to\infty} \inf\{t_n : n > N\}.$$

(b)
- Let $s_n = (-1)^n$ and $t_n = (-1)^{n+1}$.
- $(s_n + t_n)$ is a constant sequence $\{0, 0, \ldots\}$
- $lim_{N\to\infty} \inf\{s_n : n > N\} = lim_{N\to\infty} \inf\{t_n : n > N\} = \min\{-1, 1\} = -1$.
- Then $\lim_{N\to\infty} \inf\{s_n + t_n : n > N\} \geq \lim_{N\to\infty} \inf\{s_n : n > N\} + \lim_{N\to\infty} \inf\{t_n : n > N\}$ implies $0 > -2$.

$\square$

Problem 5. *Solution.*

(a) Let $\epsilon > 0$. We want to find a N such that $m, n \geq N$ implies $|s_m - s_n| < \epsilon$. Without loss of generality, assume $m > n$. We can write

$$|s_m - s_n| \leq \frac{1}{mn} < \frac{1}{n} < \frac{1}{N}.$$

Let $N < \frac{1}{\epsilon}$. Then $\frac{1}{N} < \epsilon$ implies $|s_m - s_n| < \epsilon$. Thus choosing $N < \frac{1}{\epsilon}$ implies $|s_m - s_n| < \epsilon$ for any $\epsilon > 0$.

(b) Given that the sequence is Cauchy, for any $\epsilon > 0$ there exists a N such that $m, n \geq N$ implies $|s_m - s_n| < \epsilon$. Fix $n = N_1$. Then for all $m \geq N$ we have $|s_m - s_{N_1}| < \epsilon$. Let $\epsilon = \frac{1}{mN_1}$. For large values of m, s_m is getting infinitesimally close to equal s_{N_1}.

Now fix $n = N_2$ such that $s_{N_1} \neq s_{N_2}$. Then for all $m \geq N$ we have $|s_m - s_{N_2}| < \epsilon$. Let $\epsilon = \frac{1}{mN_2}$. We also see that for large values of m, s_m is getting infinitesimally close to equal s_{N_2}. We arrive at a contradiction–convergent sequences cannot be getting arbitrarily close to different values for large m. Therefore this implies $s_{N_1} = s_{N_2}$. Since s_{N_1} and s_{N_2} are arbitrary, it follows that $s_m = s_n$ for all $m, n \in \mathbb{N}$.

$\square$