MATH 131A: HOMEWORK 1

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Problem 1. *Solution.*
$\boxed{\text{Ross 1.1}}$ Our nth proposition is

$$P_n : \text{“}1^2 + 2^2 + ... + n^2 = \frac{1}{6}n(n+1)(2n+1) \text{ for all positive integers } n.\text{”}$$

The base case asserts $1 = \frac{1}{6}(1+1)(2+1)$. For the induction step, suppose P_n is true. To prove P_{n+1}, express it as

$$\begin{aligned}
1^2 + 2^2 + ... + n^2 + (n+1)^2 &= \frac{1}{6}n(n+1)(2n+1) + (n+1)^2 \\
&= \frac{1}{6}(2n^3 + 3n^2 + n) + (n^2 + 2n + 1) \\
&= \frac{1}{6}(2n^3 + 9n^2 + 13n + 6) \\
&= \frac{1}{6}(n+1)(2n^2 + 7n + 6) \\
&= \frac{1}{6}(n+1)(n+2)(2n+3) \\
&= \frac{1}{6}(n+1)((n+1)+1)(2(n+1)+1).
\end{aligned}$$

Thus, P_{n+1} holds if P_n holds. By principle of mathematical induction, we conclude P_n is true for all n.
$\boxed{\text{Ross 1.4}}$ The sum for $n = 1, 2, 3, 4$ is as follows:

$$\begin{aligned}
&P_1 = 1 \\
&P_2 = 1 + (2 \cdot 2 - 1) = 4 \\
&P_3 = 1 + 3 + (2 \cdot 3 - 1) = 9 \\
&P_4 = 1 + 3 + 5 + (2 \cdot 4 - 1) = 16.
\end{aligned}$$

Our nth proposition is

$$P_n : \text{“}1 + 3 + ... + (2n-1) = n^2 \text{ for all positive integers } n.\text{”}$$

We have already proven the base case $1 = 1^2$. For the induction step, suppose P_n is true. To verify P_{n+1}, the trick is to write

$$\begin{aligned}
1 + 3 + ... + (2n-1) + (2(n+1)-1) &= n^2 + (2(n+1)-1) \\
&= n^2 + 2n + 1 \\
&= (n+1)^2.
\end{aligned}$$

Thus P_{n+1} holds if P_n holds. By principle of mathematical induction, we conclude P_n is true for all n.
$\boxed{\text{Ross 1.9}}$

(a) We are given the inequality $2^n > n^2$ and observe the following:

$$
\begin{aligned}
&n < 0 : \frac{1}{2^n} > n^2 && \textit{false} \\
&n = 0 : 1 > 0 && \textit{true} \\
&n = 1 : 2 > 1 && \textit{true} \\
&n = 2 : 4 > 4 && \textit{false} \\
&n = 3 : 8 > 9 && \textit{false} \\
&n = 4 : 16 > 16 && \textit{false} \\
&n = 5 : 32 > 25 && \textit{true} \\
&\qquad \vdots && \vdots
\end{aligned}
$$

We propose that $2^n > n^2$ if and only if $n = 0, 1$ or $n \geq 5$.

(b) We have already checked the cases $n = 0, 1, 2, 3, 4$. To prove $2^n > n^2$ is false for all $n < 0$, we simply notice that $2^n < 1$ and $n^2 \geq 1$ for all $n < 0$.

Now we prove $2^n > n^2$ is true for all $n \geq 5$. The base case $P_5 = 32 > 25$ is true. Suppose the statement is true for some $n \geq 5$. We now prove the statement for $n + 1$. We have

$$2^{n+1} = 2 \cdot 2^n > 2n^2,$$

by the induction hypothesis. Observe that $n^2 + n^2 > n^2 + 2n + 1$ for all $n \geq 5$ because the roots of $x^2 - 2x - 1$ are less than 5. Thus,

$$2^{n+1} > 2n^2 > (n+1)^2$$

is true whenever the nth case is true. By principle of mathematical induction, we conclude $2^n > n^2$ is true for all $n \geq 5$.

$\square$

Problem 2. $\boxed{\text{Ross 2.3}}$ For the sake of contradiction, let $a = \sqrt{2 + \sqrt{2}}$ be a rational number. Then,

$$
\begin{aligned}
a^2 &= 2 + \sqrt{2} \\
a^2 - 2 &= \sqrt{2}
\end{aligned}
$$

Since a is a rational number, so is a^2. Consequently, $a^2 - 2 = \sqrt{2}$ is also a rational number. Contradiction!

$\boxed{\text{Ross 2.7}}$ Let $a = \sqrt{4 + 2\sqrt{3}} - \sqrt{3}$. Then,

$$
\begin{aligned}
a^2 &= \left(\sqrt{4 + 2\sqrt{3}} - \sqrt{3}\right)^2 \\
&= \left(\sqrt{4 + 2\sqrt{3}}\right)^2 - 2\left(\sqrt{4 + 2\sqrt{3}}\right)\left(\sqrt{3}\right) + \left(\sqrt{3}\right)^2 \\
&= 4 + 2\sqrt{3} - 2\left(\sqrt{3 + 2\sqrt{3} + 1}\right)\left(\sqrt{3}\right) + 3 \\
&= 7 + 2\sqrt{3} - 2\left(\sqrt{\left(\sqrt{3} + 1\right)^2}\right)\left(\sqrt{3}\right) \\
&= 7 + 2\sqrt{3} - 2\left(\sqrt{3} + 1\right)\left(\sqrt{3}\right) \\
&= 7 + 2\sqrt{3} - 2\left(3 + \sqrt{3}\right) \\
&= 7 + 2\sqrt{3} - 6 - 2\sqrt{3} \\
&= 1
\end{aligned}
$$

using perfect squares. $a^2 = 1 \Rightarrow a = \pm 1$. $\pm 1 \in \mathbb{Z}$ and $\mathbb{Z} \subset \mathbb{Q}$, so a is a rational number.

Problem 3. *Solution.* We are given $\sqrt{2}$ is not a rational number. Let $a = \sqrt{1+\sqrt{1+\sqrt{2}}}$ be a rational number. Then,

$$\begin{aligned} a^2 &= 1+\sqrt{1+\sqrt{2}} \\ (a^2-1)^2 &= 1+\sqrt{2} \\ a^4-2a^2+1 &= 1+\sqrt{2} \\ a^4-2a^2 &= \sqrt{2} \end{aligned}$$

$\sqrt{2}$ is written in terms of the rational number a, which contradicts the first part of the proof. Therefore, a is not a rational number given that $\sqrt{2}$ is not a rational number. $\square$

Problem 4. *Solution.* We are given the sequence $\{x_n\}_{n=1}^{\infty}$ defined by $x_1 = \frac{1}{6}$ and

$$x_{n+1} = \frac{n+1}{n+3}\left(x_n + \frac{1}{2}\right).$$

To find x_{2024}, begin by writing it from left to right:

$$x_{2024} = \frac{2024}{2026}\left(\frac{2023}{2025}\left(\frac{2022}{2024}\left((\dots)+\frac{1}{2}\right)+\frac{1}{2}\right)+\frac{1}{2}\right). \tag{*}$$

Similarly, write it from right to left:

$$x_{2024} = \dots\left(\frac{4}{6}\left(\frac{3}{5}\left(\frac{2}{4}\left(x_1+\frac{1}{2}\right)+\frac{1}{2}\right)+\frac{1}{2}\right)+\frac{1}{2}\right)\dots. \tag{**}$$

Note as we recursively call the sequence, most terms eventually get cancelled out until we are left with

$$x_{2024} = \frac{2\cdot 3}{2026\cdot 2025}x_1 + A$$

where A accounts for the $\frac{1}{2}$ terms in the original sequence. Let us define A. First, expand x_{2024} from (*):

$$\begin{aligned} x_{2024} &= \frac{2024}{2026}\left(\frac{2023}{2025}\left(\frac{2022}{2024}(\dots)+\frac{1}{2}\cdot\frac{2022}{2024}+\frac{1}{2}\right)+\frac{1}{2}\right) \\ &= \frac{2024}{2026}\left(\frac{2023}{2025}\frac{2022}{2024}(\dots)+\frac{1}{2}\cdot\frac{2023}{2025}\frac{2022}{2024}+\frac{1}{2}\cdot\frac{2023}{2025}+\frac{1}{2}\right) \\ &= \frac{2024}{2026}\frac{2023}{2025}\frac{2022}{2024}(\dots)+\frac{1}{2}\cdot\frac{2024}{2026}\frac{2023}{2025}\frac{2022}{2024}+\frac{1}{2}\cdot\frac{2024}{2026}\frac{2022}{2025}+\frac{1}{2}\cdot\frac{2024}{2026}. \end{aligned}$$

Note writing out the $(\dots)$ term would generate the rest of the $\frac{1}{2}$ terms. Now, we propose

$$A = \frac{1}{2}\sum_{i=1}^{2024}\prod_{h=1}^{i}\frac{2025-h}{2027-h}$$

where the first term is $\frac{1}{2}\cdot\frac{2024}{2026}$, the second term is $\frac{1}{2}\cdot\frac{2024}{2026}\frac{2023}{2025}$, the third term is $\frac{1}{2}\cdot\frac{2024}{2026}\frac{2023}{2025}\frac{2022}{2024}$, and so on. We also see that some terms will cancel, so we can rewrite A as:

$$A = \frac{1}{2}\sum_{i=2}^{2024}\frac{i(i+1)}{2026\cdot 2025}.$$

The index $i=2$ is clear when we expand x_{2024} from (**):

$$\begin{aligned} x_{2024} &= \dots\left(\frac{4}{6}\left(\frac{3}{5}\left(\frac{2}{4}x_1+\frac{1}{2}\cdot\frac{2}{4}+\frac{1}{2}\right)+\frac{1}{2}\right)+\frac{1}{2}\right)\dots \\ &= \dots\left(\frac{4}{6}\left(\frac{3}{5}\frac{2}{4}x_1+\frac{1}{2}\cdot\frac{3}{5}\frac{2}{4}+\frac{1}{2}\cdot\frac{3}{5}+\frac{1}{2}\right)+\frac{1}{2}\right)\dots \\ &= \dots\left(\frac{4}{6}\frac{3}{5}\frac{2}{4}x_1+\frac{1}{2}\cdot\frac{4}{6}\frac{3}{5}\frac{2}{4}+\frac{1}{2}\cdot\frac{4}{6}\frac{3}{5}+\frac{1}{2}\cdot\frac{4}{6}+\frac{1}{2}\right)\dots. \end{aligned}$$

Note the first term that is multiplied by $\frac{1}{2}$ is $\frac{2\cdot 3}{2026\cdot 2025}$ when all the terms cancel. Then, A becomes:

$$A = \frac{1}{2\cdot 2026\cdot 2025}\left(\sum_{i=2}^{2024} i^2 + \sum_{i=2}^{2024} i\right).$$

$1+2+\cdots+n = \frac{n(n+1)}{2}$ and $1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$ are known series proven by induction. They are the sum of squares and the sum of integers proven in Problem 1, Ross 1.1 and Chapter 1, Example 1 in the textbook respectively. Letting $n = 2024$ and accounting for the shift, we have

$$A = \frac{1}{2\cdot 2026\cdot 2025}\left(\frac{2024\cdot 2025}{2} - 1 + \frac{2024\cdot 2025\cdot 4049}{6} - 1^2\right).$$

Finally, x_{2024} can be calculated:

$$\begin{aligned} x_{2024} &= \frac{2\cdot 3}{2026\cdot 2025}\left(\frac{1}{6}\right) + \frac{1}{2\cdot 2026\cdot 2025}\left(\frac{2024\cdot 2025}{2} + \frac{2024\cdot 2025\cdot 4049}{6} - 2\right) \\ &\approx 337.33. \end{aligned}$$

□

Problem 6. *Solution.*

(a) I did not go to the store.

(b) $2+3 \le 6$.

(c) $2+3 \le 6$ and $2+3 \ge 0$.

(d) It is Monday and we don't have class.

(e) x^2 is odd and x is not odd.

□

Problem 7. *Solution.*

(a) If we cannot go, then the light is red.

(b) $x \notin \mathbb{Z} \Rightarrow x^2 \notin \mathbb{Z}$.

□