Math 131A Homework 7
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Problem 1.

$\boxed{\text{Ross 17.8}}$ Let f and g be real-valued functions.

(a) Show $\min(f,g) = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$.
Suppose $f > g$. We want $\min(f,g) = g$. Consider

$$\begin{aligned}
\min(f,g) &= \frac{1}{2}(f+g) - \frac{1}{2}|f-g| \\
&= \frac{1}{2}(f+g) - \frac{1}{2}(f-g) && (f > g \text{ implies } |f-g| = f-g) \\
&= g.
\end{aligned}$$

Suppose $f \le g$. We want $\min(f,g) = f$. Consider

$$\begin{aligned}
\min(f,g) &= \frac{1}{2}(f+g) - \frac{1}{2}|f-g| \\
&= \frac{1}{2}(f+g) - \frac{1}{2}|g-f| && (\text{By def. of } ||,\ |f-g| = |g-f|) \\
&= \frac{1}{2}(f+g) - \frac{1}{2}(g-f) && (f \le g \text{ implies } |g-f| = g-f) \\
&= f.
\end{aligned}$$

Therefore we have shown $\min(f,g) = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$.

(b) Show $\min(f,g) = -\max(-f,-g)$. First show $\max(f,g) = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$.
Suppose $f > g$. We want $\max(f,g) = f$. Consider

$$\begin{aligned}
\max(f,g) &= \frac{1}{2}(f+g) + \frac{1}{2}|f-g| \\
&= \frac{1}{2}(f+g) + \frac{1}{2}(f-g) && (f > g \text{ implies } |f-g| = f-g) \\
&= f.
\end{aligned}$$

Suppose $f \le g$. We want $\max(f,g) = g$. Consider

$$\begin{aligned}
\max(f,g) &= \frac{1}{2}(f+g) + \frac{1}{2}|f-g| \\
&= \frac{1}{2}(f+g) + \frac{1}{2}|g-f| && (\text{By def. of } ||,\ |f-g| = |g-f|) \\
&= \frac{1}{2}(f+g) + \frac{1}{2}(g-f) && (f \le g \text{ implies } |g-f| = g-f) \\
&= g.
\end{aligned}$$

Therefore we have shown $\max(f,g) = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$. Now consider

$$\begin{aligned}
-\max(-f,-g) &= -\left(\frac{1}{2}(-f-g) + \frac{1}{2}|-f+g|\right) \\
&= -\left(-\frac{1}{2}(f+g) + \frac{1}{2}|f-g|\right) \quad \text{(By def. of } ||, |-f+g| = |g-f| = |f-g|) \\
&= \frac{1}{2}(f+g) - \frac{1}{2}|f-g| \\
&= \min(f,g).
\end{aligned}$$

Therefore we have shown $\min(f,g) = -\max(-f,-g)$.

(c) Let f and g be continuous at x_0. By Theorem 17.4(i), $f+g$ and $f-g$ are continuous at x_0. Hence $|f-g|$ is continuous at x_0 by Theorem 17.3. Then $\frac{1}{2}(f+g)$ and $\frac{1}{2}|f+g|$ are continuous at x_0 by Theorem 17.3. Finally, another application of Theorem 17.4(i) shows $\min(f,g) = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$ is continuous at x_0.

$\boxed{\text{Ross 17.10(b)}}$ Define $g(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. We will prove that g is discontinuous at $x_0 = 0$. Assume g is continuous at x_0. Then we apply the sequential definition of continuity. That is, for all $\{x_n\}_{n=1}^{\infty}$ such that $x_n \to x_0$, it necessarily holds that $g(x_n) \to g(x_0)$. Note that $\sin(2\pi n + \frac{\pi}{2}) = 1$ for all $n \in \mathbb{N}$. Then let $x_n = \frac{1}{2\pi n + \frac{\pi}{2}}$. Clearly $x_n \to x_0$. By our original assumption this also means $g(x_n) \to g(x_0)$. However, $g(x_n) = \sin(2\pi n + \frac{\pi}{2}) = 1$ for all $n \in \mathbb{N}$. So it is actually $g(x_n) \to 1$ and this is a contradiction. Therefore for all $\{x_n\}_{n=1}^{\infty}$ such that $x_n \to x_0$, it does not necessarily hold that $g(x_n) \to g(x_0)$. Hence g is discontinuous at $x_0 = 0$.

Problem 2. Let $f(x)$ be a function that is continuous at $x_0 \in U$. Suppose $f(x_0) > 0$. Then let $\epsilon = \frac{f(x_0)}{2}$. By the $\epsilon - \delta$ definition, there exists a $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \frac{f(x_0)}{2}$. First notice

$$|x - x_0| < \delta \Rightarrow x_0 - \delta < x < x_0 + \delta.$$

Therefore $x \in (x_0 - \delta, x_0 + \delta)$. Next observe

$$\begin{aligned}
|f(x) - f(x_0)| < \frac{f(x_0)}{2} &\Rightarrow f(x_0) - \frac{f(x_0)}{2} < f(x) < f(x_0) + \frac{f(x_0)}{2} \\
&\Rightarrow \frac{f(x_0)}{2} < f(x) < \frac{3f(x_0)}{2}. \qquad (*)
\end{aligned}$$

(*) implies $f(x) > 0$. Therefore we have shown there exists a $\delta > 0$ such that $f(x) > 0$ on $(x_0 - \delta, x_0 + \delta)$ by setting $\epsilon = \frac{f(x_0)}{2}$.

Problem 3.

(a) Suppose $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $h(r) = 0$ for every rational number $r \in \mathbb{Q}$. We will prove that $h(x) = 0$ for all $x \in \mathbb{R}$. First, we claim that h is continuous at some irrational number $q \in \mathbb{Q}^c$ and assume $h(q) \neq 0$. By definition of sequential continuity, for all $\{x_n\}_{n=1}^{\infty}$ such that $x_n \to q$, it necessarily holds that $h(x_n) \to h(q)$. Let us construct a sequence $\{x_n\}$ of strictly rationals converging to q. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a rational x_1 satisfying $|x_1 - q| < 1$. Similarly, choose a rational x_k such that it satisfies $|x_k - q| < \max(\frac{1}{2^k}, |x_{k-1} - q|)$. Then $x_n \to q$. By our original assumption this also means $h(x_n) \to h(q)$. But clearly $h(x_n) = 0$ for all $n \in \mathbb{N}$, so $h(x_n) \to 0$. However we first assumed that $h(q) \neq 0$. Therefore we arrive at a contradiction. Then it must be the case $h(q) = 0$ for every irrational number q, hence $h(x) = 0$ for all $x \in \mathbb{R}$.

(b) Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ such that $f(r) = g(r)$ for every rational number $r \in \mathbb{Q}$. We will prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$. Consider $h(x) = f(x) - g(x)$. The difference of two continuous functions is continuous, therefore $h(x)$ is continuous on $\mathbb{R}$. Observe that $h(r) = 0$ for every rational number $r \in \mathbb{Q}$. Then $h(x) = 0$ for all $x \in \mathbb{R}$ as proven in Problem 3(a). Thus we have shown that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose there exists a constant $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$. We will prove that f is continuous on $\mathbb{R}$ by proving f is continuous at some arbitrary $x_0 \in dom(f)$. Choose any $\epsilon > 0$ and set $\delta = \frac{\epsilon}{M}$. We argue that $x \in dom(f)$ and $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$. First notice

$$|x - x_0| < \frac{\epsilon}{M} \Rightarrow M|x - x_0| < \epsilon.$$

And we know $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$, therefore

$$|f(x) - f(x_0)| \leq M|x - x_0| < \epsilon.$$

Therefore we have shown that f is continuous at some arbitrary x_0, hence we say f is continuous on $\mathbb{R}$.