MATH 131A: HOMEWORK 2

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Problem 1. *Solution.*

$\boxed{\text{Ross 3.6}}$

(a) Apply the triangle inequality:

$$|(a+b)+c| \leq |a+b| + |c|.$$

Note the base case $|a+b| \leq |a| + |b|$ $\forall a, b \in \mathbb{R}$. Let $c \in \mathbb{R}$. Add $|c|$ to both sides and we get $|a+b| + |c| \leq |a| + |b| + |c|$. Combine inequalities:

$$|(a+b)+c| \leq |a+b| + |c| \leq |a| + |b| + |c|.$$

By transitivity, we have proven $|a+b+c| \leq |a| + |b| + |c|$ $\forall a, b, c \in \mathbb{R}$.

(b) The base cases $n = 1, 2$ are obvious, they are $|a_1| \leq |a_1|$ and the triangle inequality. For the induction step, first assume the inductive hypothesis is true: $|a_1 + a_2 + \ldots + a_n| \leq |a_1| + |a_2| + \ldots + |a_n|$ is true for n numbers $a_1, a_2, \ldots, a_n$. To prove the $n+1$ case, start by applying the triangle inequality:

$$|(a_1 + a_2 + \ldots + a_n) + a_{n+1}| \leq |a_1 + a_2 + \ldots + a_n| + |a_{n+1}|.$$

Note if we add $|a_{n+1}|$ to both sides of the nth case, we get:

$$|a_1 + a_2 + \ldots + a_n| + |a_{n+1}| \leq |a_1| + |a_2| + \ldots + |a_n| + |a_{n+1}|.$$

Combine inequalities:

$$|(a_1 + a_2 + \ldots + a_n) + a_{n+1}| \leq |a_1 + a_2 + \ldots + a_n| + |a_{n+1}| \leq |a_1| + |a_2| + \ldots + |a_n| + |a_{n+1}|.$$

By transitivity, we have shown

$$|a_1 + a_2 + \ldots + a_n + a_{n+1}| \leq |a_1| + |a_2| + \ldots + |a_n| + |a_{n+1}|$$

is true whenever the nth case is true. By principle of mathematical induction, $|a_1 + a_2 + \ldots + a_n| \leq |a_1| + |a_2| + \ldots + |a_n|$ is true for n numbers $a_1, a_2, \ldots, a_n$.

$\boxed{\text{Ross 3.8}}$

Let $a, b \in \mathbb{R}$. We want to show that if $a \leq b_1$ for every $b_1 > b$ and $a \leq b$.

We will prove this statement by contradiction. Let $a \leq b_1$ for every $b_1 > b$ and $a > b$.

Given that $a > b$, we propose that there exists a real value r such that $b < r < a$. Let $r = b + \frac{a-b}{2}$, where we know $\frac{a-b}{2} > 0$. We claim that $b < b + \frac{a-b}{2} < a$.

We know $\frac{a-b}{2} > 0$, so $b < b + \frac{a-b}{2}$ follows. $b + \frac{a-b}{2} < a$ is also true because it is the same as writing $2b + a - b < 2a$ and this is just $b < a$. This was given in our assumption.

We have proven our claim that there exists a real value r such that $b < r < a$. $b < r$ satisfies the definition of $b_1 > b$. Our assumption states that $a \leq r$ in this case. However, we have established that $r < a$. We have formulated a contradiction, thus concluding that if $a \leq b_1$ for every $b_1 > b$, then $a \leq b$. $\square$

Problem 2. *Solution.*
$\boxed{\text{Ross 4.6}}$ Let S be a nonempty bounded subset of $\mathbb{R}$. By completeness, $sup(S)$ and $inf(S)$ exist and are real numbers.

(a) $sup(S)$ is the least upper bound of S such that $s \leq sup(S)\ \forall s \in S$.

Similarly, $inf(S)$ is the greatest upper bound of S such that $inf(S) \leq s\ \forall s \in S$.

Combining inequalities, we get $inf(S) \leq s \leq sup(S)$.

By transitivity, we have proven $inf(S) \leq sup(S)$.

(b) From part a, we know $inf(S) \leq s \leq sup(S)\ \forall s \in S$.

Let $inf(S) = sup(S)$.

Then, $inf(S) = s$ and $sup(S) = s$ is true $\forall s \in S$ if and only if S contains only one element.

$\boxed{\text{Ross 4.7}}$ Let S and T be nonempty bounded subsets of $\mathbb{R}$. Then, $sup(S)$, $inf(S)$, $sup(T)$, and $inf(T)$ exist and are real numbers.

(a) By definition of supremum, $\forall t \in T$, $t \leq sup(T)$.

Let $S \subseteq T$ such that $\forall s \in S$, $s \in T$.

By choosing t to be an arbitrary element $s \in S$, we see $s \leq sup(T)$ holds $\forall s \in S$. From this, we claim that $sup(T)$ is an upper bound of S. Since $sup(S)$ is the least upper bound of S, $sup(S) \leq sup(T)$.

Similarly, $\forall t \in T$, $inf(T) \leq t$ by definition of infimum. Let t be an arbitrary element $s \in S$, then $inf(T) \leq s$ is true $\forall s \in S$. We have shown $inf(T)$ is a lower bound of S. $inf(S)$ is the greatest lower bound of S, so $inf(T) \leq inf(S)$.

We also know $inf(S) \leq sup(S)$ by Ross 4.6.

Combining inequalities, we have proven $inf(T) \leq inf(S) \leq sup(S) \leq sup(T)$ to be true if $S \subseteq T$.

(b) The set $S \cup T$ can be written as $\{u \in S \cup T : u \in S \text{ or } u \in T\}$.

By definition of supremum, $\forall s \in S$, $s \leq sup(S)$.

We want to claim $sup(S) \leq max(sup(S), sup(T))$ by considering the two cases:

i. If $sup(S) \leq sup(T)$, then $max(sup(S), sup(T)) = sup(T)$.
As a result, $sup(S) \leq max(sup(S), sup(T)) \equiv sup(S) \leq sup(T)$ is true.

ii. If $sup(S) > sup(T)$, then $max(sup(S), sup(T)) = sup(S)$.
Thus, $sup(S) \leq max(sup(S), sup(T)) \equiv sup(S) \leq sup(S)$ is true.

Therefore, we have shown $sup(S) \leq max(sup(S), sup(T))$ holds. In other words, $max(sup(S), sup(T))$ is considered to be an upper bound for S.

Similarly, $\forall t \in T$, $t \leq sup(T)$ by definition of supremum.

We will also show that $sup(T) \leq max(sup(S), sup(T))$ is true:

i. If $sup(T) \leq sup(S)$, then $max(sup(S), sup(T)) = sup(S)$ and $sup(T) \leq max(sup(S), sup(T)) \equiv sup(T) \leq sup(S)$ is true.

ii. If $sup(T) > sup(S)$, then $max(sup(S), sup(T)) = sup(T)$ and $sup(T) \leq max(sup(S), sup(T)) \equiv sup(T) \leq sup(T)$ is true.

Thus, we can say $sup(T) \leq max(sup(S), sup(T))$, or $max(sup(S), sup(T))$ is an upper bound for T.

Now let $u \in S \cup T$. Pick u to be an arbitrary element of S such that the following is true $\forall s \in S$: $s \leq sup(S)$. Then, $u \leq sup(S) \leq max(sup(S), sup(T))$.

Similarly, $\forall t \in T$, we know $t \leq sup(T)$. If we choose u as an arbitrary element of T, we can write $u \leq sup(T) \leq max(sup(S), sup(T))$.

Therefore, we have shown $max(sup(S), sup(T))$ is an upper bound for any element $u \in S \cup T$ where $u \in S$ or $u \in T$. Equivalently, this is $sup(S \cup T) \leq max(sup(S), sup(T))$.

Next, observe $\forall u \in S \cup T$, $u \leq sup(S \cup T)$ by definition of supremum.

Note $\forall s \in S, s \in S \cup T$ by definition of union. Then, we can substitute u to be an element of S, say $s \in S$, and write it as $s \leq sup(S \cup T)$. Since s is arbitrary, this holds $\forall s \in S$.

We have shown $sup(S \cup T)$ is an upper bound for S.

Likewise, $\forall t \in T, t \in S \cup T$. Let u be an element of T, say $t \in T$, and write it as $t \leq sup(S \cup T)$. t is arbitrary, so this holds $\forall t \in T$.

Thus, we have shown $sup(S \cup T)$ is also an upper bound for T.

$sup(S \cup T)$ is an upper bound for S and T, hence $sup(S) \leq sup(S \cup T)$ and $sup(T) \leq sup(S \cup T)$.

The following must also be true: $max(sup(S), sup(T)) \leq sup(S \cup T)$.

i. This is because if $sup(S) \leq sup(T)$, then $max(sup(S), sup(T)) = sup(T)$ and we know $sup(T) \leq sup(S \cup T)$.
ii. Likewise, if $sup(S) > sup(T)$, then $max(sup(S), sup(T)) = sup(S)$ and $sup(S) \leq sup(S \cup T)$ holds.

Therefore, we have proven $max(sup(S), sup(T)) \leq sup(S \cup T)$ and $sup(S \cup T) \leq max(sup(S), sup(T))$ to be true. This proves the equality $sup(S \cup T) = max(sup(S), sup(T))$ is true.

$\boxed{\text{Ross 4.14}}$ Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let $A + B$ be the set of all sums $a + b$ where $a \in A$ and $b \in B$.

(a) By definition of supremum, $sup(A+B)$ is the least upper bound for $A+B$ such that $a+b \leq sup(A+B)$ $\forall a \in A$ and $\forall b \in B$.

Subtracting b from both sides, we get $a \leq sup(A + B) - b$.

Since a is arbitrary and this holds $\forall b \in B$, we have shown $sup(A + B) - b$ is an upper bound for A. Equivalently, $sup(A) \leq sup(A + B) - b$.

Rearranging the inequality, we get $b \leq sup(A + B) - sup(A)$.

We previously considered b to be arbitrary, so this holds $\forall b \in B$.

We have shown $sup(A + B) - sup(A)$ is an upper bound for B, or $sup(B) \leq sup(A + B) - sup(A)$.

This is the same as $sup(A) + sup(B) \leq sup(A + B)$.

Next, pick an arbitrary $c \in A + B$ such that $c = a + b$ where $a \in A$ and $b \in B$.

Recognize $a \leq sup(A)$ $\forall a \in A$ and $b \leq sup(B)$ $\forall b \in B$ by definition of supremum.

Combining inequalities, we get $a + b \leq sup(A) + sup(B)$.

This holds $\forall c \in A + B$, so we have shown $sup(A) + sup(B)$ is an upper bound for $A + B$.

Hence, $sup(A + B) \leq sup(A) + sup(B)$.

The inequalities $sup(A) + sup(B) \leq sup(A+B)$ and $sup(A+B) \leq sup(A) + sup(B)$ prove the equality $sup(A + B) = sup(A) + sup(B)$.

(b) We will follow the same reasoning as part (a).

By definition of infimum, $inf(A+B)$ is the greatest lower bound for $A+B$ such that $inf(A+B) \leq a+b$ $\forall a \in A$ and $\forall b \in B$.

We can rewrite it as $inf(A + B) - b \leq a$.

Since a is arbitrary and this holds $\forall b \in B$, we have shown $inf(A + B) - b$ is a lower bound for A.

Therefore, this is the same as writing $inf(A + B) - b \leq inf(A)$.

We can also get a lower bound for B: $inf(A + B) - inf(A) \leq b$ holds $\forall b \in B$, since we previously considered b to be arbitrary. So, $inf(A + B) - inf(A) \leq inf(B)$.

This is the same as $inf(A+B) \leq inf(A) + inf(B)$.

Next pick an arbitrary $c \in A+B$ such that $c = a+b$ where $a \in A$ and $b \in B$.

Note $inf(A) \leq a \; \forall a \in A$ and $inf(B) \leq b \; \forall b \in B$ by definition of infimum.

Adding the inequalities, we have $inf(A) + inf(B) \leq a + b$.

This holds $\forall c \in A+B$, therefore $inf(A) + inf(B)$ is a lower bound for $A+B$ and can be written as such: $inf(A) + inf(B) \leq inf(A+B)$.

Thus, the inequalities $inf(A+B) \leq inf(A) + inf(B)$ and $inf(A) + inf(B) \leq inf(A+B)$ prove the equality $inf(A+B) = inf(A) + inf(B)$.

$\square$

Problem 4. *Solution.* Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences satisfying $a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots \leq M$ and $b_1 \leq b_2 \leq \ldots \leq b_n \leq \ldots \leq M$.

Let $\{a_n + b_n\}_{n=1}^{\infty}$ be the sequence satisfying $(a_1 + b_1) \leq (a_2 + b_2) \leq \ldots \leq (a_n + b_n) \leq \ldots \leq (M + M)$.

We see that $\{a_n\}$ and $\{b_n\}$ are bounded above by M and $\{a_n + b_n\}$ has an upper bound $2M$. By completeness, $sup(a_n : n \in \mathbb{N})$, $sup(b_n : n \in \mathbb{N})$, and $sup(a_n + b_n : n \in \mathbb{N})$ exist and represent real numbers.

First, we want to claim $sup(a_n + b_n : n \in \mathbb{N}) \leq sup(a_n : n \in \mathbb{N}) + sup(b_n : n \in \mathbb{N})$.

Consider $x \in \{a_n + b_n\}$, so that $x = a_k + b_k$ for some $a_k \in \{a_n\}$ and $b_k \in \{b_n\}$ where $k \in [[1, n]]$.

By definition of supremum, we know $a_k \leq sup(a_n : n \in \mathbb{N}) \; \forall a_k \in \{a_n\}$ and $b_k \leq sup(b_n : n \in \mathbb{N}) \; \forall b_k \in \{b_n\}$ where $k \in [[1, n]]$.

Adding the inequalities, we get $a_k + b_k \leq sup(a_n : n \in \mathbb{N}) + sup(b_n : n \in \mathbb{N})$.

a_k and b_k are arbitrary, so this holds $\forall x \in \{a_n + b_n\}$.

Therefore, we have shown $sup(a_n : n \in \mathbb{N}) + sup(b_n : n \in \mathbb{N})$ is an upper bound for the sequence $\{a_n + b_n\}$.

Given that $sup(a_n + b_n : n \in \mathbb{N})$ is the least upper bound for $\{a_n + b_n\}$, $sup(a_n + b_n : n \in \mathbb{N}) \leq sup(a_n : n \in \mathbb{N}) + sup(b_n : n \in \mathbb{N})$.

Next, we will prove $sup(a_n : n \in \mathbb{N}) + sup(b_n : n \in \mathbb{N}) \leq sup(a_n + b_n : n \in \mathbb{N})$.

By definition of supremum, $x \leq sup(a_n + b_n : n \in \mathbb{N}) \; \forall x \in \{a_n + b_n\}$.

Recall that we defined x as $x = a_k + b_k$, so this is the same as $a_k + b_k \leq sup(a_n + b_n : n \in \mathbb{N}) \; \forall a_k \in \{a_n\}$ and $\forall b_k \in \{b_n\}$ where $k \in [[1, n]]$.

We can rearrange this to get $a_k \leq sup(a_n + b_n : n \in \mathbb{N}) - b_k$.

Since a_k is arbitrary and this holds $\forall b_k \in \{b_n\}$, $sup(a_n + b_n : n \in \mathbb{N}) - b_k$ is an upper bound for $\{a_n\}$. In other words, $sup(a_n : n \in \mathbb{N}) \leq sup(a_n + b_n : n \in \mathbb{N}) - b_k$.

Rewrite it and we have $b_k \leq sup(a_n + b_n : n \in \mathbb{N}) - sup(a_n : n \in \mathbb{N})$. b_k is arbitrary, so we see that $sup(a_n + b_n : n \in \mathbb{N}) - sup(a_n : n \in \mathbb{N})$ is an upper bound for $\{b_n\}$, or $sup(b_n : n \in \mathbb{N}) \leq sup(a_n + b_n : n \in \mathbb{N}) - sup(a_n : n \in \mathbb{N})$.

This is equivalent to $sup(a_n : n \in \mathbb{N}) + sup(b_n : n \in \mathbb{N}) \leq sup(a_n + b_n : n \in \mathbb{N})$.

Finally, we have proven the inequalities $sup(a_n + b_n : n \in \mathbb{N}) \leq sup(a_n : n \in \mathbb{N}) + sup(b_n : n \in \mathbb{N})$ and $sup(a_n : n \in \mathbb{N}) + sup(b_n : n \in \mathbb{N}) \leq sup(a_n + b_n : n \in \mathbb{N})$ to be true.

As a result, it is also true that $sup(a_n + b_n : n \in \mathbb{N}) = sup(a_n : n \in \mathbb{N}) + sup(b_n : n \in \mathbb{N})$. $\square$

Problem 5. *Solution.* Let $S \subseteq \mathbb{R}$ be a non-empty subset that is bounded from above. Let $b > 0$ be a constant. Define $bS = \{bs : s \in S\}$.

By the Completeness Axiom, if S is bounded from above, then S has a least upper bound such that $s \leq sup(S)$ $\forall s \in S$.

Multiplying b to both sides, we get $bs \leq bsup(S)$. Since s is arbitrary, this holds $\forall x \in bS$ where $x = bs, s \in S$.

Therefore, we have shown bS is bounded from above with an upper bound $bsup(S)$. Then, bS must have a least upper bound $sup(bS)$ by completeness to get $sup(bS) \leq bsup(S)$.

The least upper bound for bS is denoted $x \leq sup(bS)$ $\forall x \in bS$ where $x = bs, s \in S$. This is the same as writing $bs \leq sup(bS)$ $\forall s \in S$.

Divide b on both sides and it's $s \leq \frac{sup(bS)}{b}$. s is arbitrary, so we have shown $\frac{sup(bS)}{b}$ is an upper bound for S, hence $sup(S) \leq \frac{sup(bS)}{b}$. Multiply b to both sides and we get $bsup(S) \leq sup(bS)$. Thus, we have shown $bsup(S) \leq sup(bS)$.

The inequalities $sup(bS) \leq bsup(S)$ and $bsup(S) \leq sup(bS)$ prove the equality $sup(bS) = bsup(S)$. $\square$