Math 131A Homework 6
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Problem 1.

$\boxed{\text{Ross 14.2}}$

(a) $\sum \frac{n-1}{n^2}$ diverges by the Comparison Test. Note $\frac{n-\frac{n}{2}}{n^2} = \frac{n}{2n^2} = \frac{1}{2n} \leq \frac{n-1}{n^2}$ for $n > 1$. We know $\sum \frac{1}{n}$ diverges, therefore $\sum \frac{1}{2n}$ diverges. It follows that $\sum \frac{n-1}{n^2}$ diverges by comparison.

(b) $\sum(-1)^n$ diverges by the Divergence Test. $\lim_{n\to\infty}(-1)^n$ diverges therefore $\sum(-1)^n$ diverges.

(f) $\sum \frac{1}{n^n}$ converges by the Root test. Take $\lim |\frac{1}{n^n}|^{\frac{1}{n}} = \lim |n^{-n}|^{\frac{1}{n}} = \lim \frac{1}{n} = 0$. Therefore $\sum \frac{1}{n^n}$ converges given that $\lim \frac{1}{n} < 1$.

$\boxed{\text{Ross 14.4}}$

(b) Consider the series $\sum \sqrt{n+1} - \sqrt{n}$. The partial sum is

$$S_n = \sqrt{2} - \sqrt{1} + \sqrt{3} - \sqrt{2} + \sqrt{4} - \sqrt{3} + \ldots + \sqrt{n+1} - \sqrt{n}.$$

Notice that all terms on the right hand side cancel out except for $\sqrt{n+1}$ and -1. Therefore we say $S_n = \sqrt{n+1} - 1$. $S_n \to \infty$ as $n \to \infty$, therefore the series is divergent.

(c) ***Note:*** *Let* $s_n = (1 + \frac{1}{n})^n$. *Recall from Calculus that* $\lim s_n = e$. *Then* $\lim \frac{1}{s_n} = \frac{1}{e}$ *(*$s_n \neq 0$ *for all* $n \in \mathbb{N}$*) (Lemma 9.5).*

Now consider the series $\sum \frac{n!}{n^n}$ and define $a_n = \frac{n!}{n^n}$. Then

$$\begin{aligned}
\left|\frac{a^{n+1}}{a^n}\right| &= \left|\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}\right| && \text{(Positive)} \\
&= \left|\frac{(n+1)n!}{(n+1)n!} \cdot \frac{n^n}{(n+1)^n}\right| \\
&= \left|\left(\frac{n}{n+1}\right)^n\right| \\
&= \left|\left(\frac{n+1}{n}\right)^{-n}\right| \\
&= \left|\left(1+\frac{1}{n}\right)^{-n}\right| \\
&= \left|\frac{1}{(1+\frac{1}{n})^n}\right| \\
&= \frac{1}{(1+\frac{1}{n})^n}.
\end{aligned}$$

Applying the Ratio Test we get $\lim \left|\frac{a^{n+1}}{a^n}\right| = \lim \frac{1}{(1+\frac{1}{n})^n} = \lim \frac{1}{s_n} = \frac{1}{e}$. And $\frac{1}{e} < 1$, so we deduce that the series is absolutely convergent by the Ratio Test, therefore convergent.

$\boxed{\text{Ross 14.14}}$ We will prove $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the Comparison Test. Let $b_n = \frac{1}{n}$ and (a_n) be the sequence

$$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \dots\right)$$

First notice that

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \underbrace{\left(\frac{1}{8} + \dots + \frac{1}{8}\right)}_{\text{4 terms}} + \underbrace{\left(\frac{1}{16} + \dots + \frac{1}{16}\right)}_{\text{8 terms}} + \dots \\ &= \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \frac{8}{16} + \dots \\ &= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots .\end{aligned}$$

Thus $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2}$ and this series diverges. Next we claim that $b_n \geq a_n$ for all $n \in \mathbb{N}$. If we group the terms similar to the one above we get:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> \frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right)}_{> \frac{1}{2}} + \dots$$

Notice that each group of terms is greater than $\frac{1}{2}$. Therefore we conclude that $b_n \geq a_n$ for all $n \in \mathbb{N}$. Thus $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the Comparison Test.

Problem 2.
$\boxed{\text{Ross 15.2(a)}}$ Consider the series $\sum [\sin(\frac{n\pi}{6})]^n$. Observe that the terms $\{(1/2)^n, (\sqrt{3}/2)^n, (-1/2)^n, (-\sqrt{3}/2)^n\} \to 0$ as $n \to \infty$. However the series oscillates between the terms $\{-1, 1\}$, thus the sequence of terms does not converge to 0, therefore the series is divergent.

Problem 3. We will prove that the series $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ is convergent. Let $a_n = \frac{1}{(\ln n)^{\ln n}}$.

Lemma 1 **((a_n) is a decreasing sequence and $a_n \geq 0$ for all $n \geq 2$).** *$\ln n > 0$ implies $(\ln n)^{\ln n} > 0$ and hence $\frac{1}{(\ln n)^{\ln n}} > 0$ for all $n \geq 2$. Next we will prove (a_n) is a decreasing sequence by induction.*

Base Case: *$\frac{1}{(\ln 2)^{\ln 2}} \geq \frac{1}{(\ln 3)^{\ln 3}}$ holds.*
Inductive Hypothesis: *Assume that $a_n \geq a_{n+1}$ for some $n \geq 2$.*
Inductive Step: *We will show that $a_{n+1} \geq a_{n+2}$ given the Inductive Hypothesis.*
Note $n+1 < n+2$ implies $\ln(n+1) < \ln(n+2)$ which implies $(\ln(n+1))^{\ln(n+1)} < (\ln(n+2))^{\ln(n+2)}$ and hence $\frac{1}{(\ln(n+1))^{\ln(n+1)}} \geq \frac{1}{(\ln(n+2))^{\ln(n+2)}}$. Therefore

$$\frac{1}{(\ln(n+1))^{\ln(n+1)}} \geq \frac{1}{(\ln(n+2))^{\ln(n+2)}} \implies a_{n+1} \geq a_{n+2}.$$

We have shown $n+1$ holds whenever the nth case is true, therefore we say (a_n) is decreasing for all $n \geq 2$.

We have shown that $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ satisfies the requirements for the Dyadic Criterion (Lemma 1). Therefore we will apply the Dyadic Criterion to prove that it is convergent. Let the dyadic series $\sum_{k=1}^{\infty} 2^k a_{2^k} = \sum_{k=1}^{\infty} \frac{2^k}{(\ln 2^k)^{\ln 2^k}}$. We can rewrite the series as:

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{2^k}{(\ln 2^k)^{\ln 2^k}} &= \sum_{k=1}^{\infty} \frac{2^k}{(k \ln 2)^{k \ln 2}} \\
&= \sum_{k=1}^{\infty} \frac{2^k}{(k^{\ln 2})^k (\ln 2^{\ln 2})^k} \\
&= \sum_{k=1}^{\infty} \left(\frac{2/c}{k^{\ln 2}} \right)^k . && (c = \ln 2^{\ln 2})
\end{aligned}$$

Applying the Root Test we get $\limsup \left| \left(\frac{2/c}{k^{\ln 2}} \right)^k \right|^{\frac{1}{k}} = \limsup \left| \frac{2/c}{k^{\ln 2}} \right| = 0$. And $0 < 1$ implies the dyadic series is absolutely convergent, thus convergent. Therefore by the Dyadic Criterion, we have proven that $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ is convergent.

Problem 4.

(a) Let (a_n) be a decreasing sequence. Suppose $\sum_{n=1}^{\infty} a_n$ converges. Assume for the sake of contradiction that there exists some index k such that $a_k < 0$. Since (a_n) is decreasing, $a_k \geq a_n$ for all $n \geq k$. This implies $|a_k| \leq |a_n|$ for all $n \geq k$. Now take the formal definition of the limit, notice that $\sum_{n=1}^{\infty} a_n$ converges implies $\lim a_n = 0$. This means for all $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n| < \epsilon$. Choose $\epsilon = |a_k|$. Then $n \geq k$ implies $|a_n| < |a_k|$. However, we have established that $|a_k| \leq |a_n|$. We arrive at a contradiction, therefore it must be true that (a_n) is strictly non-negative.

(b) Consider the sequence (na_n). We will prove that it converges to 0 by the Cauchy Criterion. That is, we will show that the partial sum (s_n) is a Cauchy sequence. Namely, for any $\epsilon > 0$ we need to find a $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have $|ns_n - ms_m| < \epsilon$. Without loss of generality, assume $n > m$. Then

$$\begin{aligned}
|ns_n - ms_m| &= \left| \sum_{k=1}^{n} na_k - \sum_{k=1}^{m} ma_k \right| \\
&= |(na_1 + na_2 + \ldots + na_m + na_{m+1} + \ldots + na_n) - (ma_1 + ma_2 + \ldots ma_m)| \\
&= |(n-m)(a_1 + a_2 + \ldots + a_m) + n(a_{m+1} + a_{m+2} + \ldots + a_n)| \\
&= (n-m)(a_1 + a_2 + \ldots + a_m) + n(a_{m+1} + a_{m+2} + \ldots + a_n) && \text{(Non-negative)} \\
&\leq m(n-m)(a_1) + n(n-m)(a_1) && \text{(Decreasing)} \\
&\leq 2n(n-m)(a_1) && (n > m) \\
&\leq 2n^2(a_1).
\end{aligned}$$

Therefore choose $\epsilon = 2n^2(a_1)$. Then we have found ϵ such that for all $m, n \geq N$ we have $|ns_n - ms_m| < \epsilon$. Thus (na_n) satisfies the Cauchy Criterion, hence $\sum_{n=1}^{\infty} na_n$ is convergent, hence $\lim na_n = 0$.

Problem 5. We will prove that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for all $p > 1$ using a similar trick in Ross 14.14. Let (a_n) be the sequence

$$\left(\frac{1}{2^p}, \frac{1}{2^p}, \frac{1}{4^p}, \frac{1}{4^p}, \frac{1}{4^p}, \frac{1}{4^p}, \frac{1}{8^p}, \frac{1}{8^p}, \frac{1}{8^p}, \frac{1}{8^p}, \frac{1}{8^p}, \frac{1}{8^p}, \frac{1}{8^p}, \ldots\right)$$

Express $\sum_{n=1}^{\infty} a_n$ as

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n &= \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \underbrace{\left(\frac{1}{4^p} + \ldots + \frac{1}{4^p}\right)}_{\text{4 terms}} + \underbrace{\left(\frac{1}{8^p} + \ldots + \frac{1}{8^p}\right)}_{\text{8 terms}} + \ldots \\
&= 2 \cdot \left(\frac{1}{2}\right)^p + 4 \cdot \left(\frac{1}{4}\right)^p + 8 \cdot \left(\frac{1}{8}\right)^p + \ldots \\
&= \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \ldots.
\end{aligned}$$

Notice that this is a geometric series with a calculable sum. The common ratio is $\frac{1}{2^{p-1}}$, therefore $S = \frac{\frac{1}{4^{p-1}}}{\frac{1}{2^{p-1}}} = \left(\frac{1}{2}\right)^{p-1}$. Next we claim that this series is greater than $\sum_{n=1}^{\infty} \frac{1}{n^p}$. Observe

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{n^p} &= \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots + \frac{1}{7^p} + \frac{1}{8^p} + \ldots + \frac{1}{15^p} + \ldots \\
\sum_{n=1}^{\infty} a_n &= \frac{1}{2^p} + \frac{1}{2^p} + \underbrace{\frac{1}{4^p} + \ldots + \frac{1}{4^p}}_{\text{4 terms}} + \underbrace{\frac{1}{8^p} + \ldots + \frac{1}{8^p}}_{\text{8 terms}} + \ldots
\end{aligned}$$

where a_n is greater than all terms in the sequence $\left(\frac{1}{n^p}\right)$ for $n > 2$. Note that we omitted the first term when comparing the two series. Pay no mind to this, recall that a series is convergent if its partial sum is convergent. Therefore removing/adding finitely many terms does not affect convergence/divergence of the partial sum. $\sum_{n=1}^{\infty} a_n$ converges, therefore by the Comparison Test $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.