MATH 131A: HOMEWORK 4

Lana Lim 105817312

May 11, 2025

Problem 1. *Solution.*
$\boxed{\text{Ross 9.4}}$

(a) $s_1 = 1, s_2 = \sqrt{2}, s_3 = \sqrt{\sqrt{2}+1}, s_4 = \sqrt{\sqrt{\sqrt{2}+1}+1}$.

(b) Let $s_1 = 1$ and for $n \geq 1$ let $s_{n+1} = \sqrt{s_n + 1}$.

Lemma 1 (**Monotonicity of** (s_n))**.** *We will prove that (s_n) is increasing by induction. Let*

$$P_n : \text{“}s_n \leq s_{n+1} \textit{ for all } n \geq 1.\text{''} \tag{1}$$

Base Case: *$s_1 \leq s_2$ implies $1 \leq \sqrt{2}$ and this is true.*
Inductive Step: *assume P_n holds for some $n \geq 1$. We will then show that this is true for P_{n+1}. To show $s_{n+1} \leq s_{n+2}$, we need*

$$\begin{aligned} &\sqrt{s_n + 1} \leq \sqrt{s_{n+1} + 1} \\ &\implies s_n + 1 \leq s_{n+1} + 1 \\ &\implies s_n \leq s_{n+1}. \end{aligned}$$

$s_n \leq s_{n+1}$ is given by our hypothesis so (1) holds for $n+1$ whenever (1) holds for n. Hence (1) holds for all $n \geq 1$ by induction. Thus we have shown that (s_n) is increasing.

Assume (s_n) converges. Let $\lim s_n = s$ for some $s \in \mathbb{R}$. We want to prove $s = \frac{1}{2}(1 + \sqrt{5})$. We know $\lim s_n = \lim s_{n+1} = s$. Then consider the following

$$\begin{aligned} &\lim s_n = \lim s_{n+1} \\ &\implies \lim s_n = \lim \sqrt{s_n + 1} \\ &\implies (\lim s_n)^2 = (\lim \sqrt{s_n + 1})^2 \\ &\implies s^2 = \lim(s_n + 1) && (*) \\ &\implies s^2 = \lim s_n + \lim 1 && (**) \\ &\implies s^2 = s + 1 \\ &\implies s^2 - s - 1 = 0. \end{aligned}$$

Proof of (*): *Given that $(\sqrt{s_n + 1})$ converges to s, we can write*

$$(\lim \sqrt{s_n + 1})^2 = (\lim \sqrt{s_n + 1})(\lim \sqrt{s_n + 1}) = \lim(\sqrt{s_n + 1}\sqrt{s_n + 1}) = \lim(\sqrt{s_n + 1})^2 = \lim(s_n + 1). \tag{Theorem 9.4}$$

Proof of (**): *Given that (s_n) converges to s and constant sequences converge to their constant, we can write*

$$\lim(s_n + 1) = \lim s_n + \lim 1. \tag{Theorem 9.3}$$

Possible solutions to $s^2 - s - 1 = 0$ are $\frac{1}{2}(1 \pm \sqrt{5})$. (s_n) is increasing which implies $1 \leq s_n$ for all $n \geq 1$. Therefore $s = \frac{1}{2}(1 + \sqrt{5})$. Thus we have proven $\lim s_n = \frac{1}{2}(1 + \sqrt{5})$.

$\boxed{\text{Ross 9.10}}$

(a) Let $M > 0$ and $k > 0$. Since $\lim s_n = +\infty$ and $\frac{M}{k} > 0$, there exists $N \in \mathbb{N}$ such that $n > N$ implies $s_n > \frac{M}{k}$, hence $ks_n > M$. So there exists N such that $n > N$ implies $ks_n > M$ for all $M > 0$. Thus we have shown $\lim ks_n = +\infty$ given $\lim s_n = +\infty$ and $k > 0$.

(b) Suppose $\lim s_n = +\infty$. Let $M < 0$. Then there exists $N \in \mathbb{N}$ such that $n > N$ implies $s_n > -M$, hence $-s_n < M$. For sequence $(-s_n)$, we write $\lim -s_n = -\infty$ since we have shown for all $M < 0$ there exists N such that $n > N$ implies $-s_n < M$. Thus $\lim(-s_n) = -\infty$ given $\lim s_n = +\infty$.

Suppose $\lim(-s_n) = -\infty$. Let $M > 0$. Then there exists $N \in \mathbb{N}$ such that $n > N$ implies $-s_n < -M$, hence $s_n > M$. For sequence (s_n), we write $\lim s_n = +\infty$ since we have shown for all $M > 0$ there exists N such that $n > N$ implies $s_n > M$. Thus $\lim s_n = +\infty$ given $\lim(-s_n) = -\infty$.

□

Problem 2. *Solution.*
$\boxed{\text{Ross 10.6}}$

(a) Let $\epsilon > 0$. We want to find a N such that $m, n > N$ implies $|s_m - s_n| < \epsilon$. Without loss of generality, assume $m \geq n$. We can write

$$\begin{aligned} |s_m - s_n| &= |(s_m - s_{m-1}) + (s_{m-1} - s_{m-2}) + \ldots + (s_{n+1} - s_n)| \\ &\leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \ldots + |s_{n+1} - s_n| && \text{(Triangle inequality)} \\ &< \frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + \ldots + \frac{1}{2^n} \\ &< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \ldots \\ &= \frac{1}{2^n}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right) && \text{(Geometric sequence } \textstyle\sum_{n=0}^{\infty} \frac{1}{2^n}) \\ &= \frac{1}{2^n} \cdot 2 && (S = \tfrac{a}{1-r} \text{ where } a = 1 \text{ and } r = \tfrac{1}{2}) \\ &= \frac{1}{2^{n-1}}. \end{aligned}$$

Then let $N > 1 + log_2(\frac{1}{\epsilon})$ such that $m, n \geq N$ implies $|s_m - s_n| < \frac{1}{2^{n-1}} < \frac{1}{2^{N-1}} < \epsilon$ hence $2^{N-1} > \frac{1}{\epsilon}$ hence $N - 1 > log_2(\frac{1}{\epsilon})$ hence $N > 1 + log_2(\frac{1}{\epsilon})$.

Therefore we have proven that (s_n) is a Cauchy sequence, thus a convergent sequence.

(b) The result in (a) is false if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$. We will prove this by a counterexample. Let (s_n) be a sequence such that $s_{n+1} = s_n + \frac{1}{2^n}$ where $s_1 = 0$. This implies $s_{n+1} - s_n = \frac{1}{2^n}$ hence

$$|s_{n+1} - s_n| = \frac{1}{2^n} < \frac{1}{n}. \tag{*}$$

Proof of (*): *We will prove* $\frac{1}{2^n} < \frac{1}{n}$ *for all* $n \in \mathbb{N}$ *by induction. The base case* $\frac{1}{2} < 1$ *is true. Assume* $\frac{1}{2^n} < \frac{1}{n}$ *is true for some* n. *We want to show that it is also true for* $n + 1$. *Do this by writing*

$$\frac{1}{2^{n+1}} < \frac{1}{n+1} \implies \frac{1}{2}\frac{1}{2^n} < \frac{1}{n+1}.$$

We know $\frac{1}{2}\frac{1}{2^n} < \frac{1}{2}\frac{1}{n}$ *by our hypothesis, so we can write*

$$\frac{1}{2}\frac{1}{2^n} < \frac{1}{2}\frac{1}{n} = \frac{1}{n+n} < \frac{1}{n+1}.$$

Therefore we have shown $\frac{1}{2^{n+1}} < \frac{1}{n+1}$. *It follows by induction* $\frac{1}{2^n} < \frac{1}{n}$ *for all* $n \in \mathbb{N}$.
We will prove that (s_n) is divergent, therefore not Cauchy. Define (s_n) as

$$s_n = \frac{1}{2}\sum_{i=1}^{n-1}\frac{1}{i}$$

where $s_1 = 0, s_2 = \frac{1}{2}(1), s_3 = \frac{1}{2}\left(1 + \frac{1}{2}\right)$, and so on. Thus we have shown (s_n) diverges given that $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. Therefore we have shown there exists (s_n) satisfying $|s_{n+1} - s_n| < \frac{1}{n}$ that diverges and hence not Cauchy. Thus the result in (a) is false if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$.

□

Problem 3. *Solution.*

(a) We will prove $a_n < 2$ for all $n \in \mathbb{N}$ by induction. Let

$$P_n : \text{“}a_n < 2 \text{ for all } n \in \mathbb{N}.\text{''} \tag{2}$$

Base Case: $a_1 < 2 \Rightarrow 1 < 2$ is true.
Inductive Step: we assume P_n holds for some $n \geq 1$. To show that P_{n+1} holds, consider the following

$$\begin{aligned} &a_{n+1} < 2 \\ &\implies \sqrt{a_n + 2} < 2 \\ &\implies a_n + 2 < 4 \\ &\implies a_n < 2 \end{aligned}$$

$a_n < 2$ is given by our hypothesis so (2) holds for $n + 1$ whenever (2) holds for n. Hence (2) holds for all $n \in \mathbb{N}$ by induction. Thus we have shown that $a_n < 2$ for all $n \in \mathbb{N}$. That is, (a_n) is bounded above.

(b) We will prove that the sequence (a_n) is increasing by induction. Let

$$P_n : \text{“}a_n \leq a_{n+1} \text{ for all } n \in \mathbb{N}.\text{''} \tag{3}$$

Base Case: $a_1 \leq a_2 \Rightarrow 1 \leq \sqrt{3}$ and this is true.
Inductive Step: we assume P_n holds for some $n \geq 1$. We will show P_{n+1} holds by the following

$$\begin{aligned} &a_{n+1} \leq a_{n+2} \\ &\implies \sqrt{a_n + 2} \leq \sqrt{a_{n+1} + 2} \\ &\implies a_n + 2 \leq a_{n+1} + 2 \\ &\implies a_n \leq a_{n+1} \end{aligned}$$

$a_n \leq a_{n+1}$ is given by our hypothesis so (3) holds for $n + 1$ whenever (3) holds for n. Hence (3) holds for all n by induction. Thus we have shown that $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$. That is, (a_n) is increasing.

(c) We have shown that (a_n) is increasing and bounded above. Therefore (a_n) converges. Let $\lim a_n = a$ for some $a \in \mathbb{R}$. We will prove that $a = 2$. We know $\lim a_n = \lim a_{n+1}$. Then consider the following

$$\begin{aligned} &\lim a_n = \lim a_{n+1} \\ &\implies \lim a_n = \lim \sqrt{a_n + 2} \\ &\implies (\lim a_n)^2 = (\lim \sqrt{a_n + 2})^2 \\ &\implies a^2 = \lim(a_n + 2) && (*) \\ &\implies a^2 = \lim a_n + \lim 2 && (**) \\ &\implies a^2 = a + 2 \\ &\implies a^2 - a - 2 = 0. \end{aligned}$$

Proof of (*): *Given that* $(\sqrt{a_n+2})$ *converges to* a, *we can write*

$$(\lim \sqrt{a_n+2})^2 = (\lim \sqrt{a_n+2})(\lim \sqrt{a_n+2}) = \lim(\sqrt{a_n+2}\sqrt{a_n+2}) = \lim(\sqrt{a_n+2})^2 = \lim(a_n+2). \quad \text{(Theorem 9.4)}$$

Proof of (**): *Given that* (a_n) *converges to* a *and constant sequences converge to their constant, we can write*

$$\lim(a_n+2) = \lim a_n + \lim 2. \quad \text{(Theorem 9.3)}$$

Possible solutions to $a^2 - a - 2 = 0$ are 2 and -1. (a_n) is increasing which implies $1 \le a_n$ for all $n \ge 1$. Therefore $a = 2$. Thus we have proven $\lim a_n = 2$.

□

Problem 4. *Solution.*

Lemma 2 (**(a_n) and (b_n) are strictly positive**). *We will prove that* (a_n) *and* (b_n) *are strictly positive such that* $a_n > 0$ *and* $b_n > 0$ *for all* $n \in \mathbb{N}$. *The base case is proven given* $0 < a < b$. *Assume* $a_n > 0$ *and* $b_n > 0$ *for some* n. *To prove the* $n+1$ *case, consider*

$$\begin{aligned} & a_{n+1} > 0 & \qquad & b_{n+1} > 0 \\ \implies & \sqrt{a_n b_n} > 0 & \qquad \implies & \frac{a_n + b_n}{2} > 0 \\ \implies & a_n b_n > 0 & \qquad \implies & a_n + b_n > 0 \end{aligned}$$

$a_n b_n > 0$ *and* $a_n + b_n > 0$ *is true given our hypothesis. Thus we have shown* $a_{n+1} > 0$ *and* $b_{n+1} > 0$ *is true whenever* $a_n > 0$ *and* $b_n > 0$ *is true. Thus* $a_n > 0$ *and* $b_n > 0$ *is true for all* $n \in \mathbb{N}$ *by induction. That is,* (a_n) *and* (b_n) *are strictly positive sequences.*

Lemma 3 (**(a_n) and (b_n) are bounded monotonic sequences**). *We want to prove that* (a_n) *is increasing and bounded above by* b *and* (b_n) *is decreasing and bounded below by* a. *That is, we will prove* $b_n \ge b_{n+1} \ge a_{n+1} \ge a_n$ *is true for all* $n \in \mathbb{N}$ *by induction. Let*

$$P_n : \text{“}b_n \ge b_{n+1} \ge a_{n+1} \ge a_n \text{ for all } n \in \mathbb{N}.\text{”} \tag{4}$$

Base Case: $b_1 \ge b_2 \ge a_2 \ge a_1$ *implies* $b \ge \frac{a+b}{2} \ge \sqrt{ab} \ge a$. *There are three inequalities to prove:*

(1) $b \ge \frac{a+b}{2}$ *implies* $2b \ge a+b$ *which is* $b+b \ge a+b$ *and this is true given* $a < b$. *This establishes the first inequality.*

(2) $\frac{a+b}{2} \ge \sqrt{ab}$ *implies* $a+b \ge 2\sqrt{ab}$. *This is true given* $0 < a < b$. *Thus we obtain* $\frac{a+b}{2} \ge \sqrt{ab}$.

(3) $\sqrt{ab} \ge a$ *implies* $ab \ge a^2$ *and this is true given* $a < b$. *Therefore we have verified the base case.*

Inductive Step: *assume* P_n *is true for some* $n \ge 1$. *We want to prove* P_{n+1} *such that* $b_{n+1} \ge b_{n+2} \ge a_{n+2} \ge a_{n+1}$. *There are three inequalities to prove:*

(1) $b_{n+1} \ge b_{n+2}$ *implies* $b_{n+1} \ge \frac{a_{n+1}+b_{n+1}}{2}$ *which is* $2b_{n+1} \ge a_{n+1} + b_{n+1}$. *From our hypothesis we know* $b_{n+1} \ge a_{n+1}$ *so we have shown that the first inequality holds.*

(2) $b_{n+2} \ge a_{n+2}$ *implies* $\frac{a_{n+1}+b_{n+1}}{2} \ge \sqrt{a_{n+1}b_{n+1}}$ *which is* $a_{n+1} + b_{n+1} \ge 2\sqrt{a_{n+1}b_{n+1}}$. *This is true since* $a_n > 0$ *and* $b_n > 0$ *for all* $n \in \mathbb{N}$ *(Lemma 2). Thus we obtain* $b_{n+2} \ge a_{n+2}$.

(3) $a_{n+2} \ge a_{n+1}$ *implies* $\sqrt{a_{n+1}b_{n+1}} \ge a_{n+1}$ *which is* $a_{n+1}b_{n+1} \ge (a_{n+1})^2$. *This is given by our hypothesis because we assume* $b_{n+1} \ge a_{n+1}$. *Thus we have shown* $a_{n+2} \ge a_{n+1}$ *holds.*

Therefore we have shown (4) holds for $n+1$ *whenever (4) holds for* n. *Hence (4) holds for all* $n \in \mathbb{N}$ *by induction. Therefore we state the following:*

- $b_n \geq b_{n+1}$ *implies that* (b_n) *is decreasing and* $a_{n+1} \geq a_n$ *implies* (a_n) *is increasing.*

- $a_{n+1} \geq a_n$ *implies* $a_n \geq a$*. Since* $a_n \leq b_n$ *for all* $n \in \mathbb{N}$*, we say* $a \leq b_n$ *is true for all* $n \in \mathbb{N}$*. Therefore we have shown* (b_n) *is bounded below by* a*.*

- $b_n \geq b_{n+1}$ *implies* $b \geq b_n$*. Since* $b_n \geq a_n$ *for all* $n \in \mathbb{N}$*, we say* $b \geq a_n$ *is true for all* $n \in \mathbb{N}$*. Therefore we have shown* (a_n) *is bounded above by* b*.*

It follows that (a_n) and (b_n) converge since we have proven that they are bounded monotonic sequences (Lemma 4). Now we want to show that (a_n) and (b_n) converge to the same limit. Let $\lim a_{n+1} = L_a$ and $\lim b_{n+1} = L_b$. Then we can write

$$\begin{aligned}
&\lim a_{n+1} = L_a \\
&\implies \lim \sqrt{a_n b_n} = L_a \\
&\implies (\lim \sqrt{a_n b_n})^2 = {L_a}^2 \\
&\implies \lim a_n b_n = {L_a}^2 && (*) \\
&\implies (\lim a_n)(\lim b_n) = {L_a}^2 && (**) \\
&\implies L_a L_b = {L_a}^2 \\
&\implies L_b = L_a.
\end{aligned}$$

Proof of (*): *Given that* (a_n) *converges to* L_a *and* (b_n) *converges to* L_b*, we can write*

$$(\lim \sqrt{a_n b_n})^2 = (\lim \sqrt{a_n b_n})(\lim \sqrt{a_n b_n}) = \lim(\sqrt{a_n b_n}\sqrt{a_n b_n}) = \lim(\sqrt{a_n b_n})^2 = \lim a_n b_n.$$

(Theorem 9.4)

Proof of (**): *Given that* (a_n) *converges to* L_a *and* (b_n) *converges to* L_b*,* $(a_n b_n)$ *converges to* $L_a L_b$*. That is,*

$$(\lim a_n)(\lim b_n) = \lim a_n b_n = L_a L_b. \qquad \text{(Theorem 9.4)}$$

Then $L_a = L_b$ implies $\lim a_{n+1} = \lim b_{n+1}$ which implies $\lim a_n = \lim b_n$. Thus we have shown (a_n) and (b_n) converge to the same limit.

$\square$