Forecast congruence, accuracy, and smoothing parameter shrinkage

Kandrika Pritularga

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Marketing Analytics and Forecasting

Lancaster University Management School

Forecasting and decisions

Managers use forecasts to make decisions:

- Producing forecasts;
- Making decisions based on forecasts.

It is often difficult to merge these activities into a single activity.

The literature argues that accurate forecasts lead to desirable decisions.

However, this might not always be the case.

Forecasting and decisions

Pritularga and Kourentzes (2024) propose a new measure to connect forecasting with inventory decisions – **forecast congruence**

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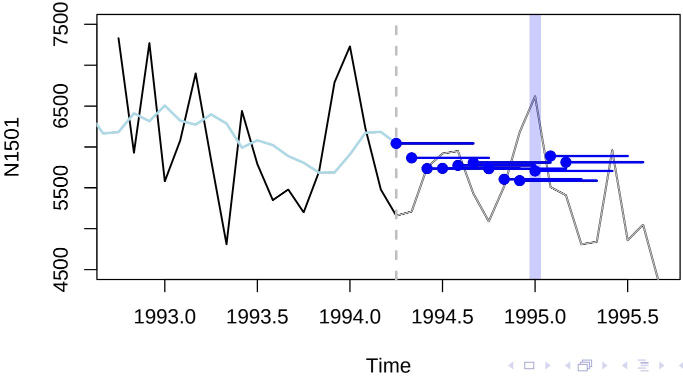
Let's forecast sales of double espresso tomorrow.

We produce many forecasts from many origins, e.g., from the last week to yesterday; from different step ahead forecasts.

The paper argues that we need to get congruent forecasts that do not harm accuracy.

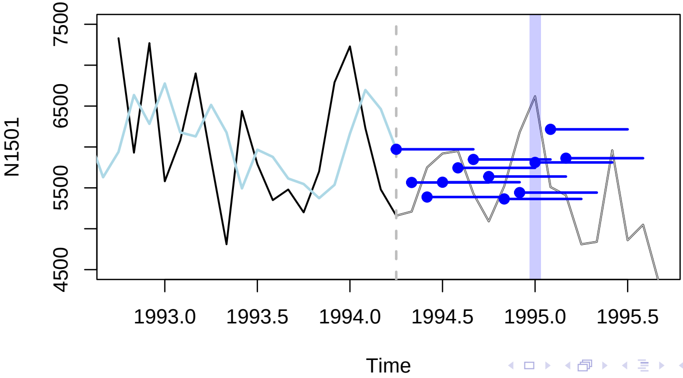
Forecasting and decisions

ETS(A,N,N) with $\alpha = 0.2$



Forecasting and decisions

ETS(A,N,N) with $\alpha = 0.5$



Forecasting and decisions

How can we achieve this?

Forecasting and decisions

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- Do not re-estimate the forecasting models across origins (ad-hoc approach);
- Multiple temporal aggregation approach (e.g., MAPA, THieF, (Kourentzes et al., 2014; Athanasopoulos et al., 2017));
- Exponential smoothing (ETS) with shrinkage estimators.
- Machine learning approach (Van Belle et al., 2023).

Changing the loss function to produce 'more stable' or congruent forecasts (Van Belle et al., 2023; Godahewa et al., 2023).

ETS and decisions

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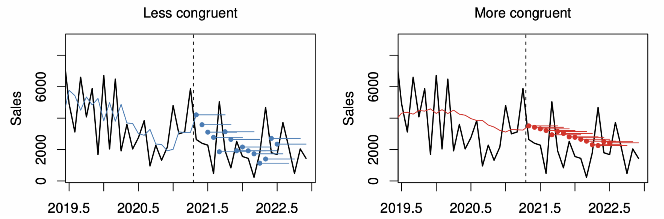
This then makes the states (level, trend, or seasons) tend to be less stochastic.

Then, the forecasts tend to be more congruent.



ETS and decisions

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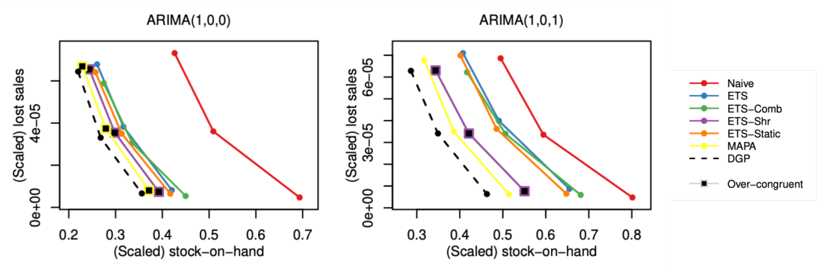
This leads to less desirable inventory decision metrics such as lost sales and service levels.



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...but also congruent enough so that the decision makers can benefit from the forecasts.

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From here forecast congruence is our proxy for decisions

ETS and forecast congruence

Let's take a simple example of ETS(ANN)

$$y_t = l_{t-1} + \varepsilon_t$$
$$l_t = l_{t-1} + \alpha\varepsilon_t$$

where α is the smoothing parameter and it controls the stochasticity of the local mean. l_0 is the level initial value and it initialises the time series.

We need to estimate the parameters of α and l_0 with any estimators (MSE, MAE, MLE, etc.).

ETS and forecast congruence

The available shrinkage estimators implement 'LASSO' and 'RIDGE'-like estimators

$$\text{LASSO} = (1-\lambda)\text{MSE}_{t+1|t} + \lambda \sum_{j=1}^{K} |g_i| \tag{1}$$

$$\text{RIDGE} = (1-\lambda)\text{MSE}_{t+1|t} + \lambda \sum_{j=1}^{K} g_i^2 \tag{2}$$

where $g_i \in \boldsymbol{g}$, λ is a hyperparameter and λ is between 0 and 1.

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where $g_i \in \boldsymbol{g}$, λ is a hyperparameter and λ is between 0 and 1.

This means that we shrink the smoothing parameters (g_i) to zero and putting the importance of the initial values in ETS. Note: the dampening parameter is shrunk to 1.

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The current implementation relies on the one-step ahead MSE in the validation set:

$$\mathcal{L}(\lambda, \hat{\boldsymbol{g}}) = \min_{\lambda} \frac{1}{L} \sum_{l=1}^{L} \mathsf{MSE}(\hat{y}_{t+1} | \mathcal{I}_{t,l}, \hat{\boldsymbol{g}})$$

where L is the number of forecast origins and $\mathcal{I}_{t,l}$ is the information at time t and origin l.

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where L is the number of forecast origins and $\mathcal{I}_{t,l}$ is the information at time t and origin l. What does it mean?

ETS and forecast congruence

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This does not align well with the nature of forecast congruence – **multistep forecast errors**.

$$
\begin{array}{cc}
\begin{pmatrix}
e_{1+l|l} & e_{1+l|l-1} & \cdots & e_{1+l|1+l-(l-1)} & e_{1+l|1} \\
e_{2+l|1+l} & e_{2+l|l} & \cdots & e_{2+l|2+l-(l-1)} & e_{1+l|2} \\
\vdots & \ddots & & & \vdots \\
 & & e_{t|t-h} & & \\
\vdots & & & \ddots & \vdots \\
e_{n-1|n-2} & e_{n-1|n-3} & \cdots & e_{n-1|n-1-(l-1)} & e_{n-1|n-1-l} \\
e_{n|n-1} & e_{n|n-2} & \cdots & e_{n|n-(l-1)} & e_{n|n-l}
\end{pmatrix}
&
\begin{array}{l}
\rightarrow \tau^2_{l+1} \\
\rightarrow \tau^2_{l+2} \\
\\
\quad \vdots \\
\\
\rightarrow \tau^2_{n-1} \\
\rightarrow \tau^2_{n}
\end{array}
\\
\begin{array}{ccccc}
\downarrow & \downarrow & & \downarrow & \downarrow \\
\sigma^2_1 & \sigma^2_2 & \cdots & \sigma^2_{l-1} & \sigma^2_l
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Steps in answering this question:

1. exploring the relationship between accuracy (MSE) and congruence (τ), conditional to different circumstances;
2. using the exploration findings to propose a new loss function for finding the hyperparameter.

Forecast congruence

Let's define forecast congruence. We are interested in collecting a set of forecasts from different origins and h for a specific t.

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$$\tau = \frac{1}{n}\sum_{t=1}^{n}\left(\sqrt{\mathrm{var}_h(e_{t|t-h})}\right)$$

τ measures the variance of $e_{t|t-h}$ over h

Accuracy & congruence: analytical approach

Random walk

$$\tau_t = \sqrt{\mathsf{E}\left[\sum_{i=1}^{h}(h-i+1)\varepsilon^2_{t-i+1}\right] - \mathsf{E}\left[\sum_{i=1}^{h}(h-1+1)\varepsilon_{t-i+1}\right]^2},$$

AR(1)

$$\tau_t = \sqrt{\mathsf{E}\left[\sum_{i=1}^{h}(h-i+1)\alpha^{2i}\varepsilon^2_{t-i+1}\right] - \mathsf{E}\left[\sum_{i=1}^{h}(h-i+1)\alpha^{i}\varepsilon_{t-i+1}\right]^2},$$

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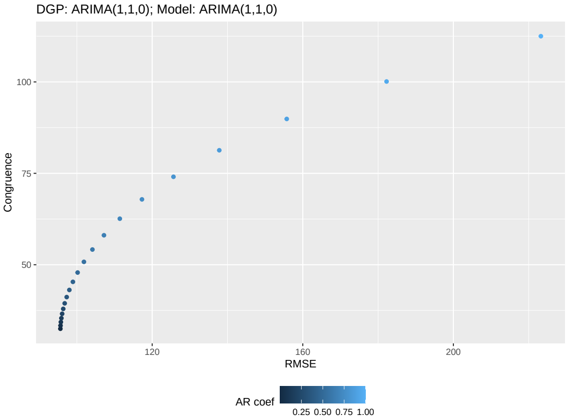
The positive relationship between τ and σ is not only mediated by the horizon but also the model parameters.

Accuracy & congruence: simulation approach

Correct models, 1000 repetitions



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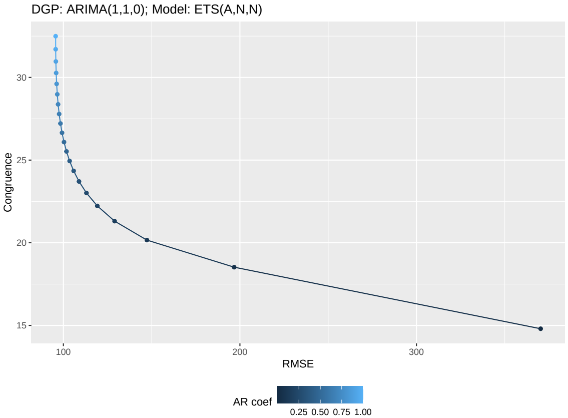

Multistep estimators

We need a metric that takes into account the 'multi-step' nature – **multistep estimator**

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Literature has studied the efficacy of multistep loss function in forecasting (Svetunkov et al., 2024, and the reference therein).

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GMSE_h is the geometric mean of $\log \text{MSE}_j$ for $j \in h$ - with a slight modification.

$$\mathsf{GMSE_h} = \frac{1}{h}\sum_{j=1}^{h}\log\frac{1}{T-h}\sum_{t=1}^{T-h}e^2_{t+j|t} = \frac{1}{h}\sum_{j=1}^{h}\log \mathsf{MSE_j}$$

A composite loss: multistep accuracy + congruence

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How do we find the 'optimal' λ? Note: $0 \leq \lambda < 1$

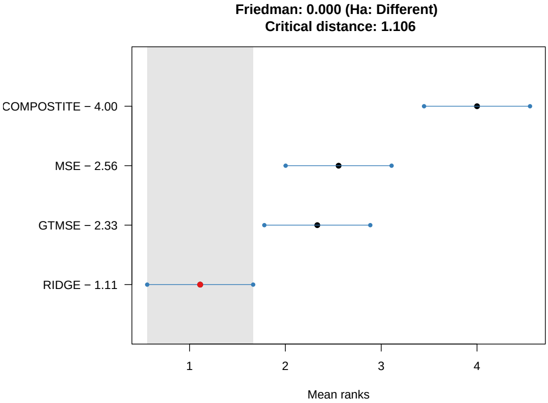
1. Split the dataset into (a) training and (b) validation set.
2. Estimate ETS parameters with a composite loss function for each λ.
3. Measure the composite loss in the validation set.
4. Find λ that results in the lowest composite loss.

Empirical analysis

- Dataset: M3 dataset for monthly only.
- Split the training set into an in-sample and a validation set (6 origins).
- The model structure is set to be incorrect: ETS(M,A,M)
- Forecast horizon: 18.
- MSE as the benchmark.
- Evaluated the point forecast accuracy for 1-h step ahead in the test set using RMSSE and scaled AME.
- Include τ that is calculated in the validation set.

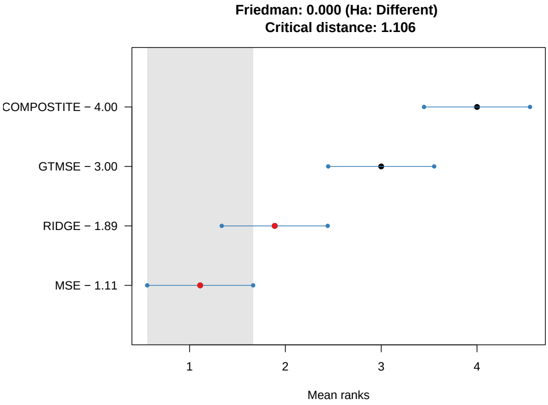
Empirical analysis

Accuracy – MSE (across hi



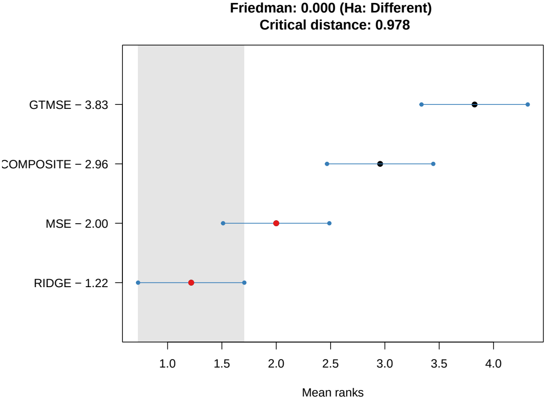
Empirical analysis

Bias – AME



Empirical analysis

Congruence – τ



Discussion & conclusion

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The experiment design could have been improved by providing the case when the model structure is estimated via an information criterion.

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The data-driven approach to find λ is not suitable in this particular situation.

Discussion & conclusion

Congruence offers a theoretical understanding of the ‘good’ forecasts, where forecasts agree with each other.

Forecasting models/ methods that have the smoothing feature on them:

- MAPA;
- ETS shrinkage;
- Forecast combination.

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Forecasting models/ methods that have the smoothing feature on them:

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- Forecast combination.

However, more questions than answers...

Discussion & conclusion

The congruence measures the balance between the column-wise and the row-wise of the forecast error matrix.

- Can we create a ‘seemingly-like ’covariance matrix of the multistep errors that contains accuracy and congruence?
  - a multistep error matrix has a dimension of $n \times h$, where n is the number of the origins and h is the forecast horizon
  - the covariance matrix of the multistep error will have a dimension of $n \times n$
- But then τ can have a matrix of $h \times h$ – does τ have another version of the covariance matrix?

Is λ too restrictive? Should we have a different way of formulating the hyperparameter loss function?

Why not forecast stability?

The concept of stability has been well known in time series analysis – ARIMA concept!

The stability feature is attached to a statistical forecasting model, which explains the long-term behaviour of a model.

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Nonetheless, we consider a few alternatives until we find a ‘free’ word.

Congruence here means ‘agreement’ between forecasts, due to updating information (re-estimate each model on each origin).

Thank you for your attention!

Kandrika Pritularga

k.pritularga@lancaster.ac.uk

Linkedin: Kandrika Pritularga

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Forecast congruence

$e_{t|t-h}$ contains the innovation at time t (ε_t) and the 'other' errors $\eta_{t|t-h}$ due to incorrect model structures and parameters.

$$\begin{aligned}
\tau &= \frac{1}{n}\sum_{t=1}^{n}\left(\sqrt{\mathsf{var}_h(\hat{y}_{t|t-h})}\right) \\
&= \frac{1}{n}\sum_{t=1}^{n}\left(\sqrt{\mathsf{var}_h(y_t - e_{t|t-h})}\right) \\
&= \frac{1}{n}\sum_{t=1}^{n}\left(\sqrt{\mathsf{var}_h(\eta_{t|t-h} + \varepsilon_t)}\right) \\
&= \frac{1}{n}\sum_{t=1}^{n}\left(\sqrt{\mathsf{var}_h(\eta_{t|t-h})}\right)
\end{aligned}$$

τ eventually measures the variance of misspesification errors