Lecture 8: Spatial-Temporal Models

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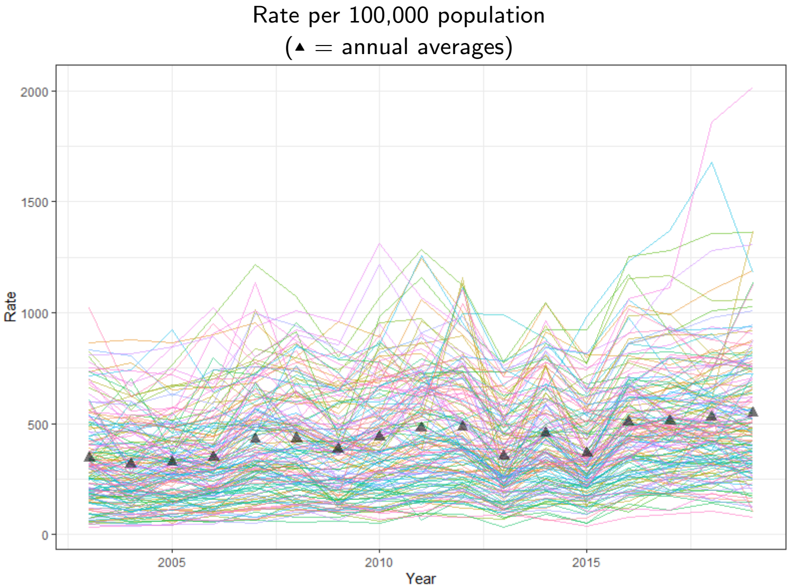
Annual Chlamydia Rate in Georgia

County-specific Temporal Trends



Additive Models

Let

y_{st} = cases of chlamybia in county s in year t.
P_{st} = population in county s in year t.
x_t = year since 2002 (per 10 year),

$$y_{st} \sim \text{Pois}\ (\ P_{st} \times \mu_{st}\ )$$

$$\log(\mu_{st}) = \beta_0 + \beta_1 x_t + \theta_{st}$$

In an additive model, we break the space-time residuals into a purely spatial component and a purely temporal component:

$$\theta_{st} = \theta_s + \theta_t\ .$$

- θ_s = overall county-specific residual log rate ratio .
- θ_t = overall year-specific residual log rate ratio.
- θ_s and θ_t are independent.

Additive Models

We will start with conventional random intercept models that ignore spatial/temporal dependence.

Model	θ_s	θ_t	WAIC
1	$N(0, \sigma_1^2)$	-	45,615
2	$N(0, \sigma_1^2)$	$N(0, \sigma_2^2)$	41,862

Model	β_0	β_1 (Time Trend)	σ_1^2	σ_2^2
1	-5.86 (-5.95, -5.78)	0.28 (0.28, 0.29)	0.30	-
2	-5.86 (-6.00, -5.71)	0.27 (0.16, 0.37)	0.29	0.015

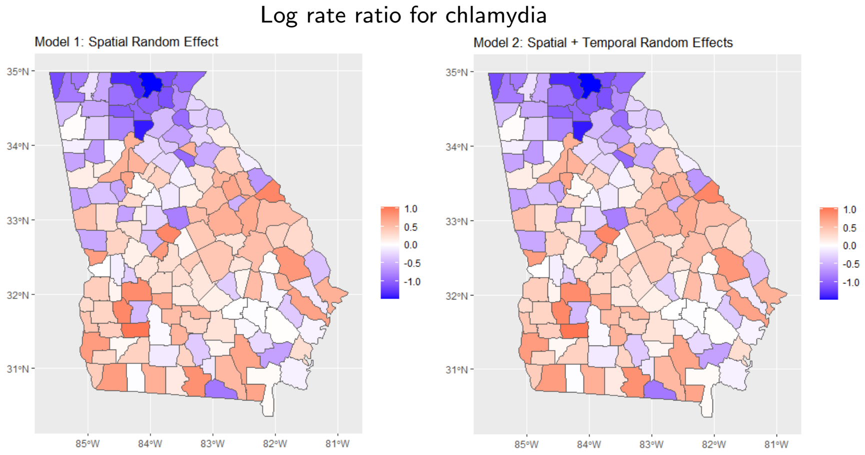
- Accounting for temporal clustering leads to larger interval estimates.
- Residual between-county variation is more important than between-year variation after removing a linear time trend.

Model Parameter Interpretations

From Model 2:

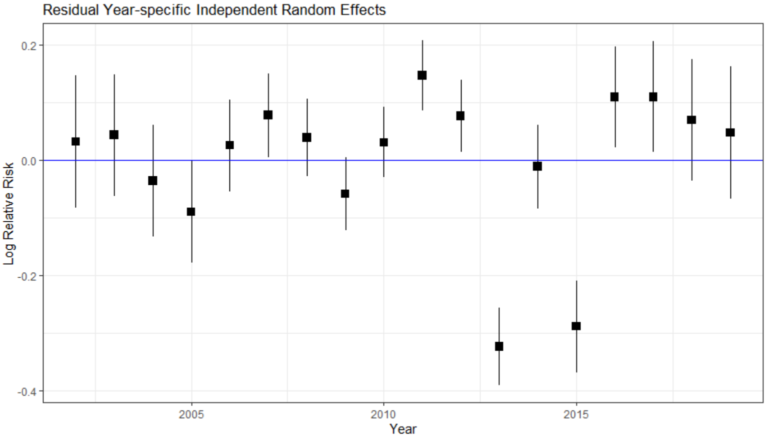
- $e^{\beta_0} = 0.0028$ is the estimated rate of chlamydia in year 2002.
- $e^{\beta_1} = 1.32$ describes a 32% increase in rate of chlamydia for every 10 years.
- $\sigma_1 = 0.54$ is the between-county standard deviation of log rate
- $\sigma_2 = 0.12$ is the between-year standard deviation of log rate

County-specific Residual Random Effects



Evidence of spatial dependence?

Temporal Residual Random Effects



Evidence of temporal dependence?

Modeling Temporal Dependence

One common approach is to use a random-walk specification for equally-spaced covariate values.

Let $\Delta_t = f(x_t) - f(x_{t-1})$ for $t = 1, \ldots, n-1$. If we assume that the differences $\Delta_t \sim N(0, \sigma_x^2)$, the conditional distribution of $f(x)$ is

$$f(x_t)|f(x_{-t}) \sim N\left(\frac{1}{2}[f(x_{t-1}) + f(x_{t+1})],\ \frac{\sigma^2}{2}\right)$$

.

- This is the same model as an iCAR where the neighbors are defined as the first-order in time unit.
- The function $f(t)$ will be smoother if we assume a 2nd-order structure.

Additive Models with ST Dependence

Let's now consider spatial/temporally-dependent random effects

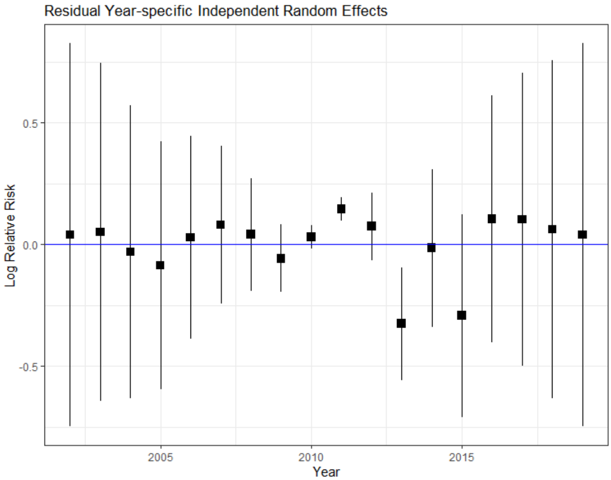
Model	θ_1	θ_2	β_1	WAIC
3	iCAR (τ_1^2)	RW-1 (τ_2^2)	0.28 (-0.69, 1.25)	41,861
4	Conv (τ_1^2, σ_1^2)	RW-1 (τ_2^2)	0.28 (-0.60, 1.16)	41,861
5	iCAR (τ_1^2)	RW-2 (τ_2^2)	0.09 (-51.0, 51.0)	41,862
6	Conv (τ_1^2, σ_1^2)	RW-2 (τ_2^2)	0.09 (-51.0, 51.0)	41.862

Conv (τ_1^2, σ_1^2) denotes the convolution (BYM) model: $\theta_s = \theta_{1,s} + \theta_{2,s}$ with $\theta_{1,s} \sim$ iCAR(θ_1^2) and $\theta_{2,s} \sim N(0, \sigma_1^2)$.

All the models have similar WAIC. We shall go with the simplest Model (i.e. Model 3).

What happened to our time trend β_1 estimate?

Temporal Residual Random Effects



The annual estimates are very noisy, especially at the ends. Too much smoothing?

Non-additive Models

Additive models are quite restrictive because they may not capture how the spatial map evolves over time.

Let's consider the following model

$$y_{st} \sim \text{Pois}\ \left(P_{st} \times \mu_{st} \right)$$

$$\log(\mu_{st}) = \beta_0 + \beta_1 x_t + \theta_s + \theta_t + \theta_{st}$$

$$\theta_s \sim \text{iCAR}(\sigma_1^2) \qquad \theta_t \sim N(0, \sigma_2^2) \qquad \theta_{st} \sim N(0, \nu^2)\ .$$

We can think of θ_{st} as space-time interaction that is independent across space and time.

The improvement in WAIC is huge!

Model	θ_s	θ_t	β_1	WAIC
7	iCAR	iid	0.33 (0.23, 0.35)	23,021

$$\sigma_1^2 = 0.91 \qquad \sigma_2^2 = 0.04 \qquad \nu^2 = 0.11$$

Space-time Interaction Model

There are several ways we can build in spatially-correlated residual errors. One common approach is to model time-specific random maps using separable covariance functions.

Let Σ_S be the covariance matrix between counties defined by an iCAR model in any year. And let Σ_T be the correlation matrix between time for any county.

If we stack all the space-time random effects by time, then space:

$$\boldsymbol{\theta} = (\boldsymbol{\theta}_{t=1}, \boldsymbol{\theta}_{t=2}, \ldots, \boldsymbol{\theta}_T)' \ ,$$

a separable model entails a covariance for the entire vector as

$$\mathsf{Cov}(\boldsymbol{\theta}) = \Sigma_T \otimes \Sigma_S \ .$$

where $\otimes$ is the Kronecker product.

An Example

Assume we have 2 time points $T = 2$ and 3 locations ($S = 3$). Assume the stacked random effects and two covariance matrices are:

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_{11} \\ \theta_{12} \\ \theta_{13} \\ \theta_{21} \\ \theta_{22} \\ \theta_{23} \end{bmatrix} \qquad \Sigma_T = \begin{bmatrix} 1 & \gamma \\ \gamma & 1 \end{bmatrix} \qquad \Sigma_S = \begin{bmatrix} \sigma^2 & \rho_1 & \rho_2 \\ \rho_1 & \sigma^2 & \rho_3 \\ \rho_2 & \rho_3 & \sigma^2 \end{bmatrix}$$

$$\text{Cov}(\boldsymbol{\theta}) = \Sigma_T \otimes \Sigma_S = \begin{bmatrix} \Sigma_S & \gamma\Sigma_S \\ \gamma\Sigma_S & \Sigma_S \end{bmatrix} = \begin{bmatrix} \sigma^2 & \rho_1 & \rho_2 & \gamma\sigma^2 & \gamma\rho_1 & \gamma\rho_2 \\ \rho_1 & \sigma^2 & \rho_3 & \gamma\rho_1 & \gamma\sigma^2 & \gamma\rho_3 \\ \rho_2 & \rho_3 & \sigma^2 & \gamma\rho_2 & \gamma\rho_3 & \gamma\sigma^2 \\ \gamma\sigma^2 & \gamma\rho_1 & \gamma\rho_2 & \sigma^2 & \rho_1 & \rho_2 \\ \gamma\rho_1 & \gamma\sigma^2 & \gamma\rho_3 & \rho_1 & \sigma^2 & \rho_3 \\ \gamma\rho_2 & \gamma\rho_3 & \gamma\sigma^2 & \rho_2 & \rho_3 & \sigma^2 \end{bmatrix}$$

Note the correlation of the same location i but at different time points:

$$\text{Cor}(\theta_{i1}, \theta_{i2}) = \gamma$$

Separable Space-Time Interaction Model

Here are a few commonly used models.

A. Independent.
When $\Sigma_T = \mathbf{I}$, we the spatial random effects are independent maps across time

$$\boldsymbol{\theta}_t \overset{iid}{\sim} iCAR\ .$$

B. Exchangeable.
When $\Sigma_T = \rho\, \mathbf{1}^T \mathbf{1} + (1-\rho)\mathbf{I}$ for $\rho < 1$, random effects in the same county by at any two points in time are correlated by ρ:

$$\mathrm{Cov}(\theta_{st}, \theta_{s',t'}) = \rho\, \Sigma_{T,(s,s')}\ .$$

C. Dynanmic (Auto-regressive).
Assume $\boldsymbol{\theta}_t$ evolves in time depending on the previous value (AR-1).

$$\boldsymbol{\theta}_t = \alpha \boldsymbol{\theta}_{t-1} + \boldsymbol{\epsilon}_t \qquad \boldsymbol{\epsilon}_t \sim \mathrm{iCAR}\ .$$

Separable Space-Time Interaction Model

Model	β_0	β_1	ρ/α	WAIC
8 (Ind)	-5.9 (-6.0, -5.8)	0.33 (0.22, 0.43)		23,736
9 (Exch)	-5.9 (-6.0, -5.9)	0.33 (0.23, 0.44)	0.86 (0.83, 0.86)	23,274
10 (Dynamic)	-5.9 (-6.0, -5.8)	0.33 (0.23, 0.43)	0.96 (0.95, 0.97)	22,961

- Very robust β_0 and β_1 estimates.
- ρ and α being close to 1 indicate very strong temporal similarity in residual maps.
- WAIC prefers the dynamic model and suggests the residual has a temporal (ordered) dependency.

Case Study: Estimating Unrecognized COVID-19 Deaths in the US

- COVID-19 deaths reported from death certificates is likely an underestimate.
- It is important to estimate the full burden of infectious disease.
- One approach is to use infection activity proxies (e.g., weekly percent test positive) to estimate counterfactual baseline mortality rates.
- Need to handle spatial-temporal correlation, sparse data situations and over-dispersion in mortality counts.
- This analysis is only for ages $\geqslant 85$ from March 2020 to April 2021.

Zhang Y, Chang HH, Iuliano AD, Reed C (2022). Application of Bayesian spatial-temporal models for estimating unrecognized COVID-19 deaths in the United States. *Spatial Statistics* 50:100584.

Spatial-Temporal Binomial Model for Positive Tests

Let N_{st} = number of positive SARS-CoV-2 test results for state s during week t.

$$N_{st} \sim \text{Binom}\left(D_{st}, p_{st}\right)$$

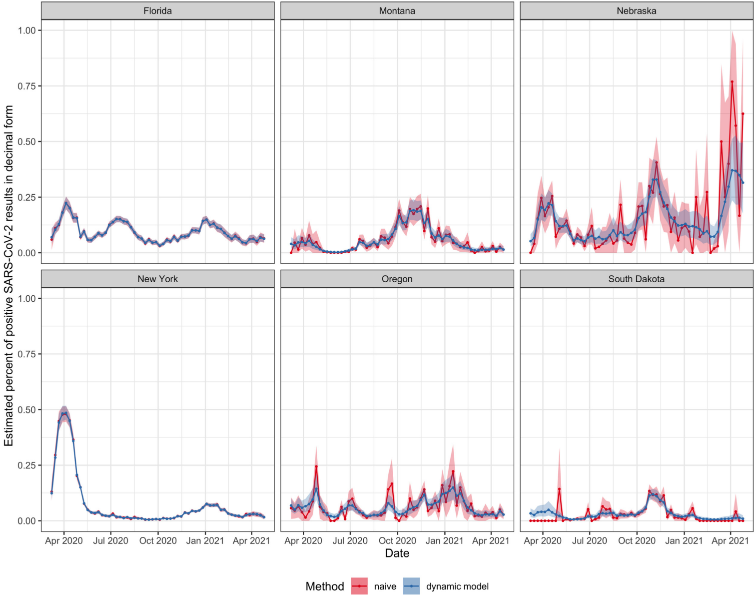
$$\log\left(\frac{p_{st}}{1-p_{st}}\right) = \alpha_0 + \theta_{s,t}$$

- D_{st} = number of tests conducted
- p_{st} percent positive test results

For θ_{st},

- Exchangeable: $\theta_s \sim N(0, \sigma^2)$ and $\theta_t \sim$ RW-1.
- Spatial: $\theta_s \sim iCAR$ and $\theta_t \sim$ RW-1.
- Dynamic (AR-1)

Estimated Percent Positivity (2020-03 to 2021-04)



Spatial-Temporal Negative-Binomial Model for Mortality

Let Y_{st} denote the number of all-cause deaths with COVID-19 coded deaths removed.

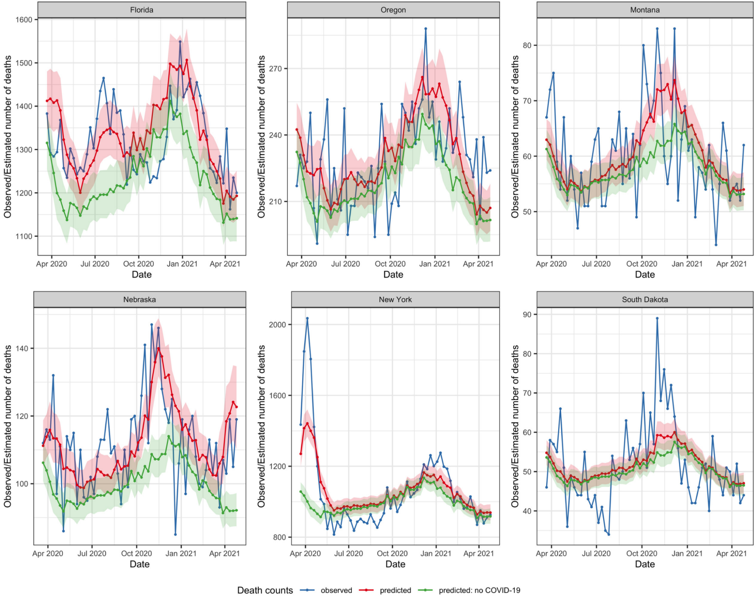
$$Y_{st} \sim \text{Pois}(\lambda_{st}) \qquad \lambda_{st} \sim \text{Gamma}\left(\xi, e^{\eta_{st}}\right)$$

$$\log \eta_{st} = \log \text{Pop}_{st} + \beta_0 + \beta_1 p_{s,t-1} + \beta_2 p_{s,t-2} + \delta_{s,t} \ .$$

- Similar space-time options for δ_{st}
- Unrecognized deaths are estimated as

$$\xi \left(e^{\eta_{st}} - e^{\log \text{Pop}_{st} + \beta_0 + \delta_{s,t}} \right)$$

Weekly Mortality Counts (2020-03 to 2021-04)



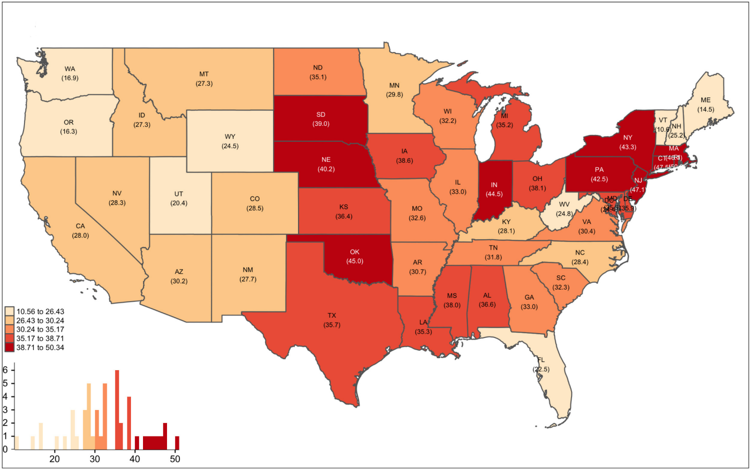
Model Selection

Model	Exchangeable	Spatial	Dynamic
Binomial (% Positive)	59,699	59,619	18,271
Neg-Binom (Mortality)	26,015	26,040	26,943

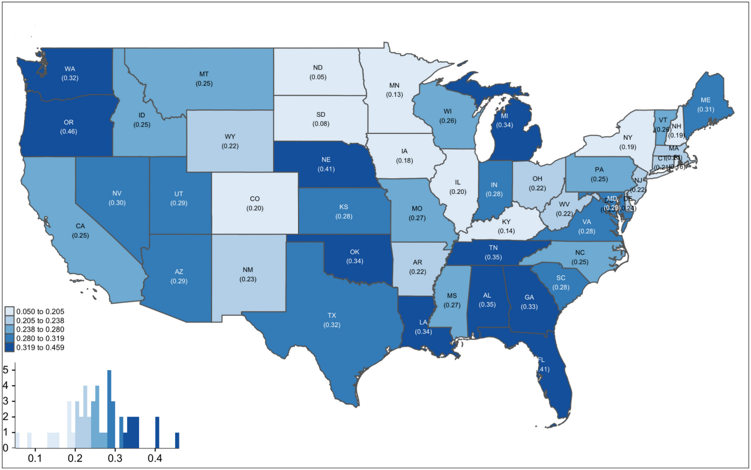
- The dynamic spatial model is favored for modeling percent positives.
- Across imputations the exchangeable and spatial models perform similarly for mortality counts.

Total COVID-19 Deaths (2020-03 to 2021-04)

Total Reported = 162,085 versus
Total Estimated = 220,285 (95% PI: 213,385 226,985)



Percent Unrecognized COVID-19 Deaths (2020-03 to 2021-04)



Time-varying Associations?

Extend analysis from March 8, 2020 to July 3, 2022.

Let Y_{st} denote the number of all-cause deaths with COVID-19 coded deaths removed.

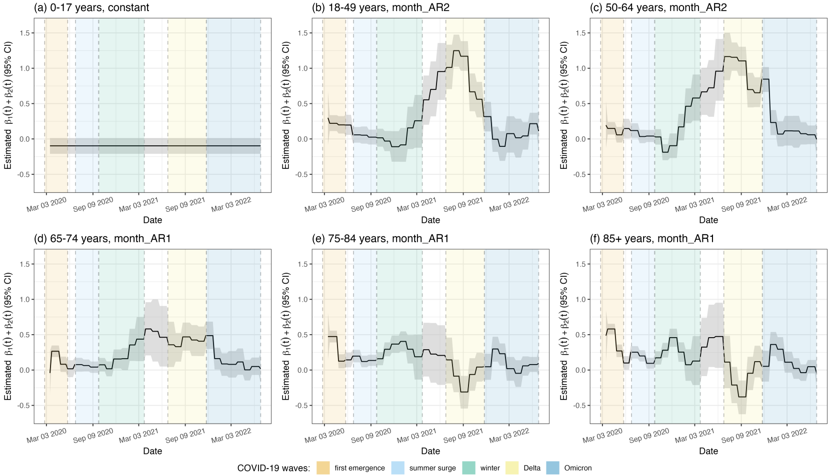
$$Y_{st} \sim \text{Pois}(\lambda_{st}) \qquad \lambda_{st} \sim \text{Gamma}\left(\xi, e^{\eta_{st}}\right)$$

$$\log \eta_{st} = \log \text{Pop}_{st} + \beta_0 + \beta_1(t) p_{s,t-1} + \beta_2(t) p_{s,t-2} + \delta_s + \gamma_t \ .$$

CAR models for $\beta_1(t)$, $\beta_2(t)$, δ_s and γ_t.

Zhang Y, Chang HH, Iuliano AD, Reed C. A Bayesian spatial-temporal varying coefficients model for estimating excess deaths associated with respiratory infections. J R Stat Soc Ser A Stat Soc. 2024 188(3):843-858. PMCID: PMC12256124

Time-varying Associations $\beta_1(t) + \beta_2(t)$



Time-varying Unrecognized Deaths (2020-03-08 to 2022-07-03)