EECS 376: Foundations of Computer Science

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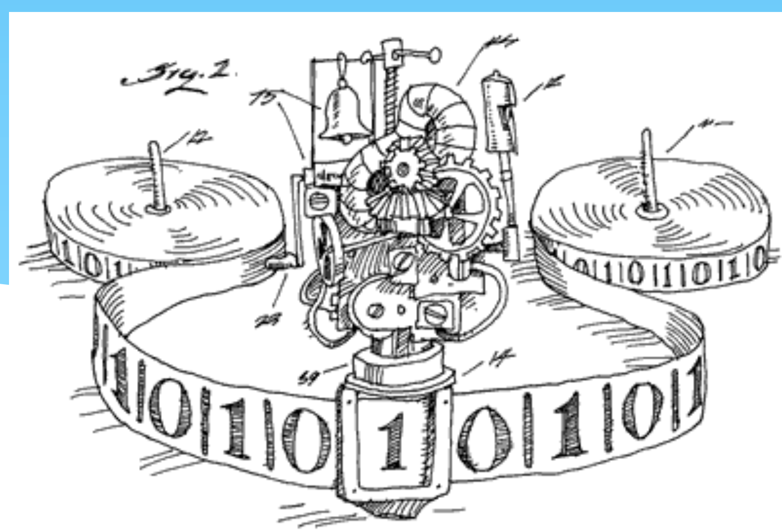
Lecture 7



Today's Agenda

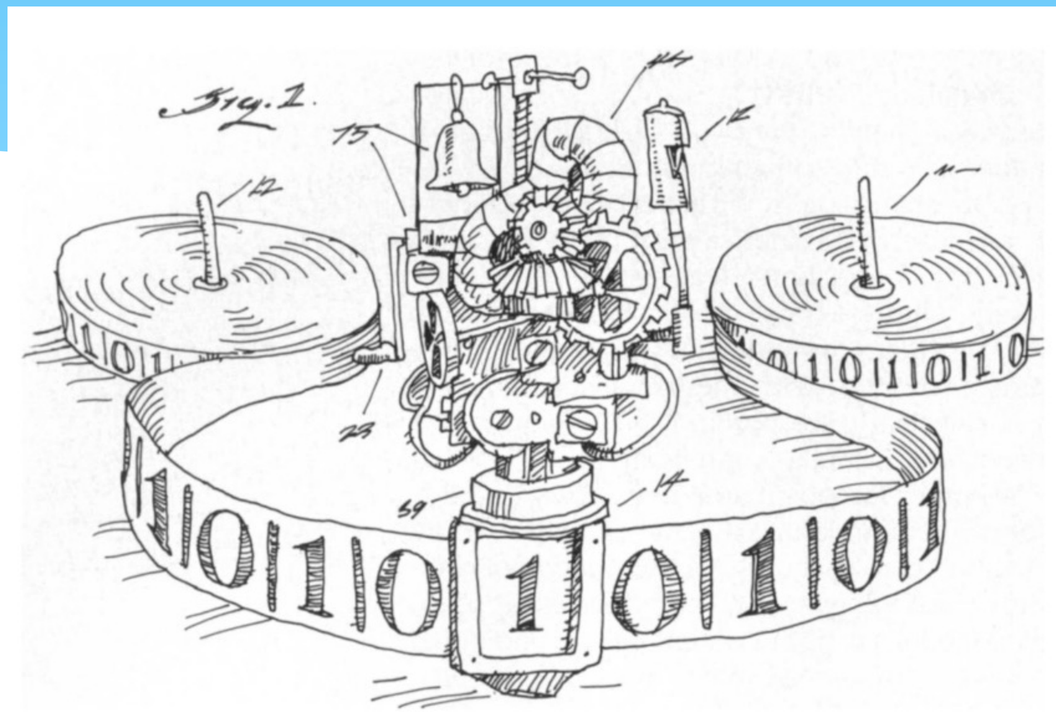
- Overview of **computability**
- **Decision** ("membership") problems
- Deterministic Finite Automata (DFA)
  - A weak class of computing machines
- A limit on the power of DFAs

Overview of Computability

- **Q:** Is every "problem" solvable on a "computer"? If so, how? If not, why not?
- **Modeling computation**
  - Formalize notion of "problem" and "computer"
- **The Halting problem and reductions**
  - Prove there are problems that cannot be solved on *any* computer *that could ever exist* (or so we believe)
- **Rice's Theorem**
  - "Any non-trivial program analysis is *impossible*"

"The question of whether a computer can think is no more interesting than the question of whether a submarine can swim." - Edsger Dijkstra.

Modeling Computation



What is a “problem”?

* We consider ***decision problems***, where the goal is to ***decide*** (say “yes” or “no”) if a given input object has a certain property

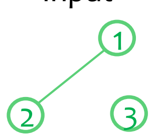
Input
376
Output
No

Is the given integer prime?

Input
abba
Output
Yes

Is the given string a palindrome?

Input



Output
No

Is the given graph connected?

…The list goes on!
We need to *unify* under a common framework

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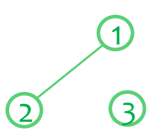
Languages and Their Decision Problems

* *Any* *finite* object (integer, graph, …) can be ***encoded*** as a *finite* ***string*** of characters from a *finite* ***alphabet*** (binary, ASCII, …).
* A property corresponds to a *set of strings: a.k.a., a* ***language***
* **Q:** What are the languages for the prior decision problems?

Input



Output

No

Is the given graph connected?

Input

010
100
000

Output

No

Is the given *string* in the *language* L_{CONN}?

$L_{CONN} = \{\langle G\rangle \mid G \text{ is a connected graph}\}$

The ***decision problem*** for a language L:

Given a string x, ***decide*** if $x \in L$ (say Y/N, acc/rej, etc).

Alphabets, Strings, Languages

- An ***alphabet*** is a finite set of characters, often denoted Σ
  - Often implicit, e.g., $\Sigma =$ASCII characters or $\Sigma = \{0,1\}$
- A (Σ-)***string*** is a finite sequence of characters from Σ
  - The ***length*** of a string x (# chars) is denoted $|x|$
  - The ***empty string*** is denoted ε; it has length $|\varepsilon| = 0$
- A (Σ-)***language*** is a (possibly infinite) set of (Σ-)strings
  - The language of *all* strings is denoted Σ^*
- **Example:** $\Sigma = \{0,1\}, \Sigma^* = \{\varepsilon, 0,1,00, \dots\}, |010| = 3$

What is a "computer"?

A "computer" used to mean a ***person*** who performed arithmetic calculations

* Alan Turing defined a formal model, called ***Turing machine***, which is *widely believed* to encompass what *any* real computing process could possibly do.
* **Beautiful Idea:** Abstracts the process of a *person/device* working on a (decision) problem, with access to an *unlimited amount of "scratch" paper/memory*.
* ***Church-Turing thesis:*** Any "computer" can be simulated by a Turing machine.
* **Warm-up:** DFAs (≡ person w/o paper): deterministic finite automata

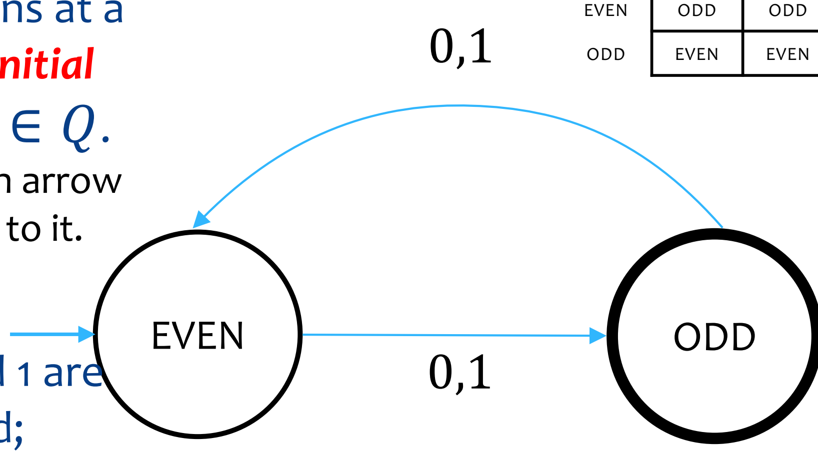
DFA Example (1/4)

Example: DFA that decides $L = \{x \in \{0,1\}^* : |x| \textit{ is odd}\}$, the set of binary strings whose lengths are *odd;* alphabet $\Sigma = \{0,1\}$.

(2) It begins at a special ***initial*** state $q_0 \in Q$.
Drawn with arrow pointing to it.

Strings 0 and 1 are accepted;
ε, 01, 0110 aren't

	0	1
EVEN	ODD	ODD
ODD	EVEN	EVEN



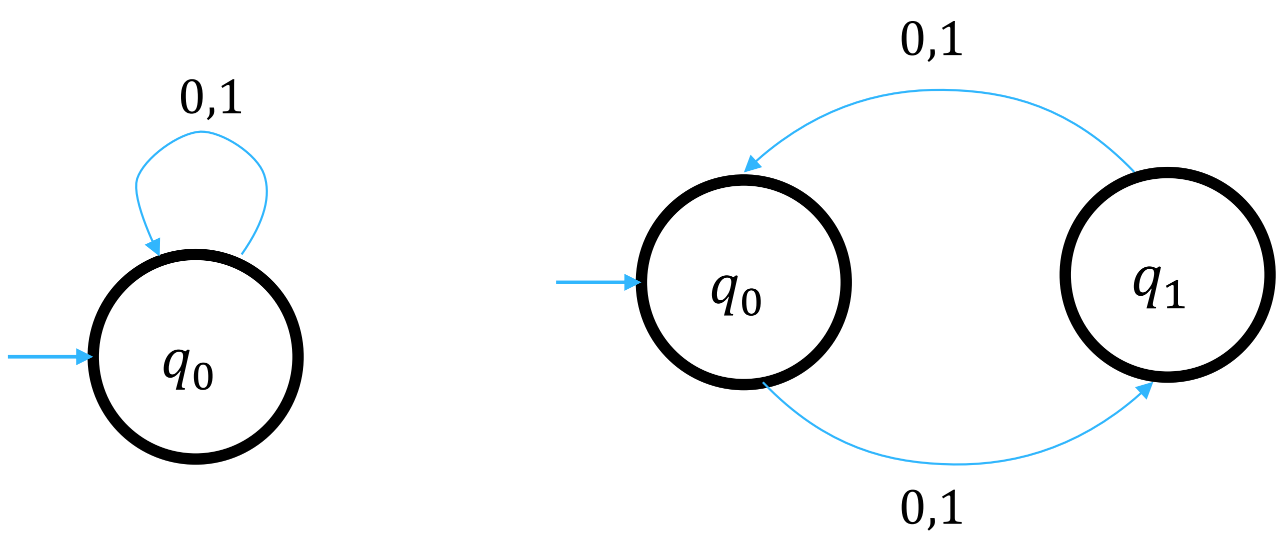
(1) A DFA has a <u>finite</u> set Q of ***states*** ('memory').
Each state represented by a distinct vertex.

(3) Given an input string x, DFA reads <u>one character</u> of x at a time and changes state according to the ***transition function*** $\delta: Q \times \Sigma \to Q$ ('program').
An edge $q \to q'$ labeled with $a \in \Sigma$ represents that $\delta(q, a) = q'$, i.e., at state q, go to state q' upon reading a.

(4) It ***accepts*** the input x if it stops at an ***accepting*** state after reading all of x.
Drawn as thick/outlined vertices.

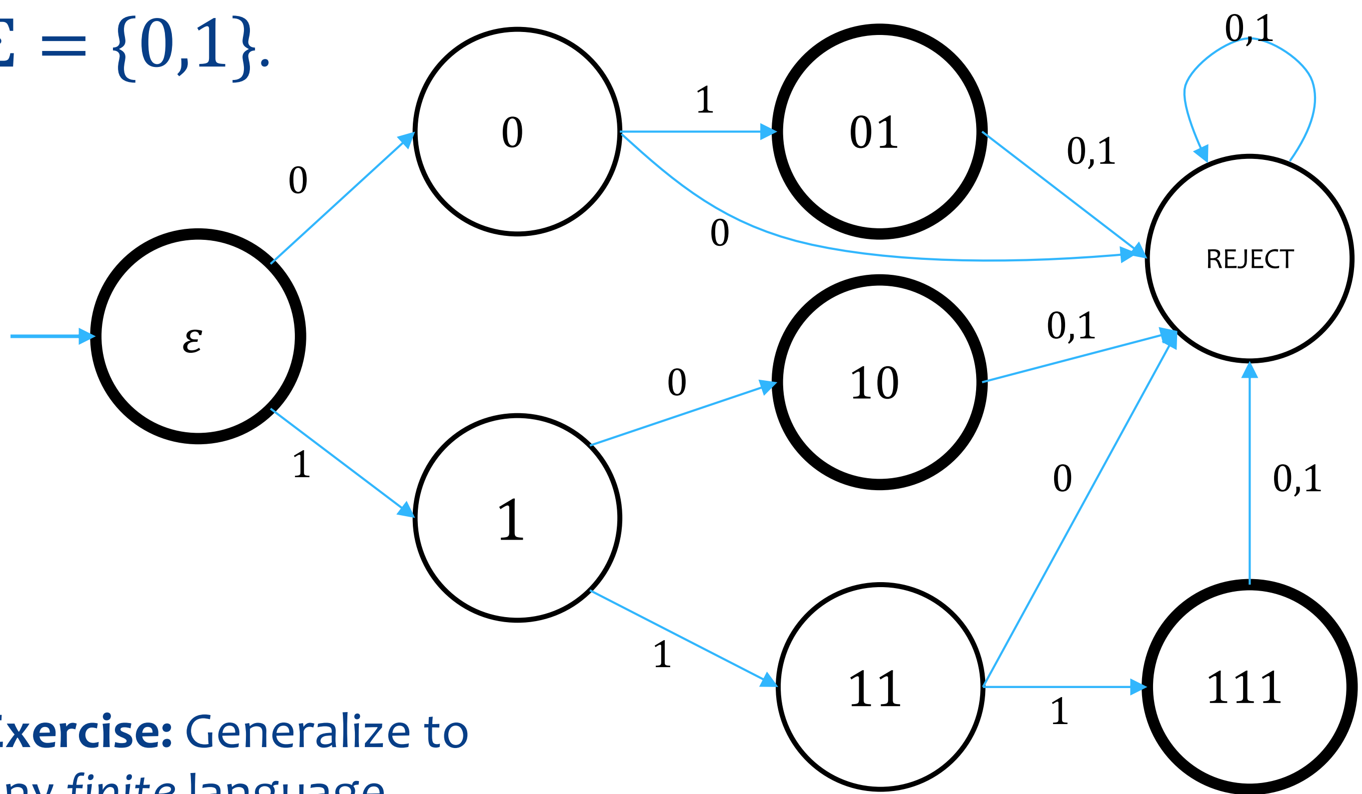
DFA Example (2/4)

Example: DFAs that decide $L = \{0,1\}^*$, the set of all binary strings, for alphabet $\Sigma = \{0,1\}$.



DFA Example (3/4)

Example: DFA that decides $L = \{\varepsilon, 01, 10, 111\}$, for alphabet $\Sigma = \{0,1\}$.



An inescapable "reject" state is often quite useful!

Exercise: Generalize to any *finite* language.

More Examples

L = {w : strings of length at most 5}
L = {w : all strings except the empty string}
L = {w : w contains exactly two 0's}
L = {w : every odd position of w is a 1}
L = {w : w contains the string 01}

L = {w : w does not contain the string 01}

#1 tip when making DFAs: Think about maintaining *state invariants*: properties that must hold whenever certain states are reached.

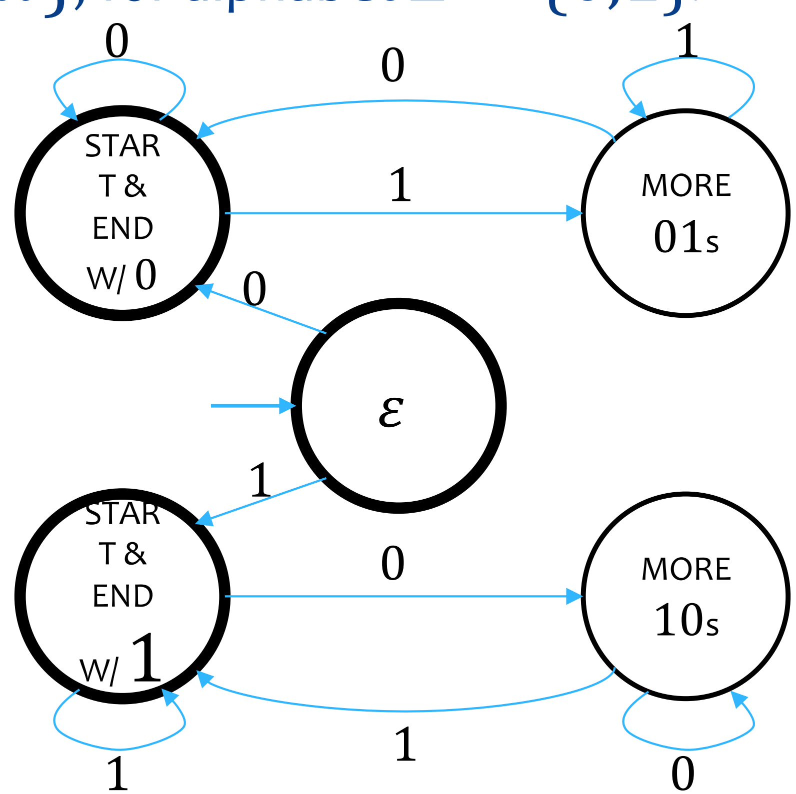
L = {w : w has equal number of 0's and 1's} (no DFA can decide this! (Later.))

Theorems:

If L is decidable by a DFA, then so is its complement $\overline{L}$.
If L_1 and L_2 are decidable by DFAs, then so are $L_1 \cup L_2$, and $L_1 \cap L_2$.

DFA Example (4/4)

Example: DFA that decides $L = \{x \in \{0,1\}^* \mid$ # of 01s in x = # of 10s in $x\}$, for alphabet $\Sigma = \{0,1\}$.



#1 tip when making DFAs: Think about maintaining *state invariants*: properties that must hold whenever certain states are reached.

<u>D</u>eterministic <u>F</u>inite <u>A</u>utomaton: Formal Definition

* Formally, a DFA M is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where:
  * Q is the <u>finite</u> set of ***states***
  * Σ is the ***input alphabet***
  * $\delta: Q \times \Sigma \to Q$ is the ***transition function***
  * $q_0 \in Q$ is the ***initial*** state
  * $F \subseteq Q$ is the subset of ***accepting*** states
* **Takeaway:** DFAs are a simple & weak, but well defined, kind of "computer."

The Language of a DFA

- We say that a DFA
  - ***accepts*** a string x if it ends at an *accepting* state, given x
  - ***decides*** a language L if it *accepts* every string $x \in L$ **and** *does not accept* ('rejects') every string $x \notin L$
- A language is said to be ***regular*** if some DFA decides it.
- **Q:** Is *every* language *regular*?
- *In other words:* is every *decision problem* "solvable" on this simple, weak model of a "computer"?
  - **No!** Intuitively, bottlenecked by DFA's finite memory

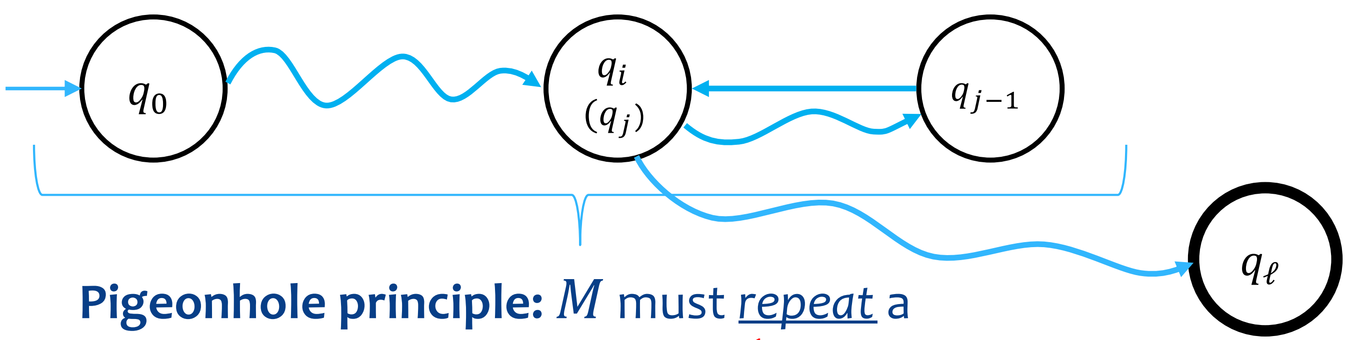
A Thought Experiment

- Imagine you're given a *huge* binary string x
  - $|x| \gg$ number of neurons in your brain
- You can read x as many times as you want, and in any order, but *you can't write anything down*
- **Q:** Can you decide if $x = 0^k 1^k$ for some k?
  - $0^k 1^k$ means k 0s followed by k 1s
- **Rabin and Scott [1959]:** 'Many-read' DFAs $\equiv$ DFAs.
  - DFAs can read the input only once, in sequence.
    This theorem says that more reads don't help.

Theorem: No DFA decides $\{0^k 1^k \mid k \geq 0\}$.

No DFA decides $\{0^k 1^k \mid k \geq 0\}$

- Suppose that some DFA M decides $\{0^k 1^k \mid k \geq 0\}$.
- Let n $=$# of states of M, and let $x = 0^{n+1}1^{n+1}$.
- **Claim:** We can write $x = uwv$ so that M is in the *same state* before and after reading substring $w \neq \varepsilon$.
- M must accept $uwwv \notin \{0^k 1^k \mid k \geq 0\}$. Contradiction!



Pigeonhole principle: M must *repeat* a state while reading 0^{n+1}

Regular Expressions

* Regular Expressions are a succinct way to describe simple languages. If R is a regular expression, $L(R) \subseteq \Sigma^*$ is its *language*.
  * If $R = a$ (a letter in Σ), $L(R) = \{a\}$.
  * $L(\epsilon) = \{\epsilon\}$ (the empty string).
  * $L(R_1R_2) = \{\mathrm{w}_1\mathrm{w}_2 \mid \mathrm{w}_1 \in L(R_1), w_2 \in L(R_2)\}$ (concatenation)
  * $L(R_1|R_2) = \mathrm{L}(\mathrm{R}_1) \cup L(R_2)$
  * $L(R^*) = \{\epsilon\} \cup L(R) \cup L(RR) \cup L(RRR) \cup \cdots$ (Kleene star)

* **Theorem.** Every RegExp language is decided by a DFA; the language of every DFA is a RegExp lang.

Regular Expression Exercises

- All strings over $\{a, b\}$ with an ***even*** number of as.
  - $b^*(b^*ab^*ab^*)^*$
- All strings over $\{a, b\}$ without 2 consecutive as.
  - $(b^*ab)^*\big(b^*(a|\epsilon)\big)$
- All strings over $\{0,1\}$ that begin and end with the same symbol.
  - $(0(0|1)^*0)|(1(0|1)^*1)$
- $N = (0|1|2|\cdots|9)$ $\qquad L = (A|B|\cdots|Z)$
  - Dates: $NN - LLL - NN(NN|\epsilon)$ (E.g., 16-Feb-2023 or 16-Feb-23)
  - Michigan License Plates: $LLL\ NNNN$

DFA Exercises

* Design a DFA to decide $\{x \in \{0,1\}^* \mid (unsigned\ int)\ x\ is\ divisible\ by\ 5\}$
* Design a DFA to decide $\{x \in \{0,1\}^* \mid (unsigned\ int)\ x^R\ is\ divisible\ by\ 5\}$
  * x^R is the *reversal* of x, i.e., least significant digits comes first.

* You need to keep track of the score in a tennis game between A and B. The sequence of points scored is represented by a string $x \in \{A, B\}^*$.
  * The first player to get ≥4 points AND be ahead by 2 wins.
  * For weird historical reasons, 0 pts is "love", 1 is "15", 2 is "30", 3 is "40".
* Design a DFA to decide $\left\{ \begin{matrix} x \in \{A,B\}^* \mid A \text{ has already won after seeing} \\ x \text{ points recorded.} \end{matrix} \right\}$

DFA Impossibility Exercises

* Prove $L_{\text{Palindrome}} = \{x \in \{0,1\}^* \mid x \text{ is a palindrome}\}$ cannot be decided by a DFA.

* Prove $L_{\text{Prime}} = \{x \in \{0,1\}^* \mid x = 1^p \text{ and } p \text{ is prime}\}$ cannot be decided by a DFA.