We may grade a **subset of the assigned questions**, to be determined after the deadline, so that we can provide better feedback on the graded questions.
Unless otherwise stated, each question requires sufficient justification to convince the reader of the correctness of your answer.
For bonus questions, we will not provide any insight during office hours or Piazza, and we do not guarantee anything about the difficulty of these questions.
We strongly encourage you to typeset your solutions in LaTeX.
If you collaborated with someone, you must state their name(s). You must write your own solution for all problems and may not look at any other student's write-up.

0. If applicable, state the name(s) and uniqname(s) of your collaborator(s).

Solution:

1. The ideas behind the Karatsuba algorithm can be applied to multiply things other than integers. Let $A(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$ and $B(x) = b_0 + b_1x + \cdots + b_{n-1}x^{n-1}$ be two polynomials of degree at most $n-1$. For simplicity, assume that n is a power of 2, and that any arithmetic operations (e.g., addition, multiplication) between two coefficients can be done in constant time. Give a Divide-and-Conquer algorithm to compute the coefficients of the product polynomial $A(x) \cdot B(x)$ in $O(n^{\log_2 3})$ time, and prove its correctness and running time.

Solution:

2. Design an algorithm that given two positive integers x and y, computes x^y, using $O(\log y)$ integer multiplications. You can assume that multiplying any two integers takes 1 operation. For example, computing x^2 takes 1 operation.

   Hint: Consider the cases when y is even and y is odd separately.

Solution:

3. Consider the recurrence
$$T(n) = \sqrt{n} \cdot T(\sqrt{n}) + O(n).$$
   (a) Explain why the master theorem cannot be applied *directly* to give a closed form for $T(n)$.
   (b) Define $S(n) = T(n)/n$. Using substitution, write a recurrence for $S(n)$.
   (c) Let $n = 2^m$ and define $R(m) = S(2^m) = S(n)$. Using this substitution, write a recurrence for $R(m)$.
   (d) Use the Master Theorem and the above recurrence to get an asymptotic expression for $R(m)$, then use it to get asymptotic expressions for $S(n)$ and finally $T(n)$.

Hint: You may need to use that $\sqrt{n} = \sqrt{2^m} = 2^{m/2}$

Solution:

4. Rank the following recurrences by asymptotic growth. You must show all work.

   - $T(n) = 9T(n/4) + 1$
   - $U(n) = 6U(n/4) + n^{1.5} \log^2 n$
   - $V(n) = 8V(n/4) + n^{1.5}$

Solution:

5. The matrix H_t is a $n \times n$ matrix where $n = 2^t$. It is defined recursively, where $H_0 = (1)$, and in general,

$$H_{t+1} = \begin{pmatrix} H_t & H_t \\ H_t & -H_t \end{pmatrix}$$

For example,

$$H_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Given a vector $x \in \mathbb{Z}^n$, there is a trivial algorithm that uses $O(n^2)$ operations to compute the matrix-vector product $H_t x$, by first computing the matrix H_t and then computing its product with x. Describe an algorithm that computes $H_t x$ in $O(n \cdot t) = O(n \log n)$ operations.

Note: Recall how matrix vector multiplication works. If we have a $n \times n$ matrix A and a vector $x \in \mathbb{Z}^n$, their product is calculated by taking the dot product of x with each row in A. (That is, if the row is $[a_1, a_2, ..., a_n]$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then the dot product is $a_1 x_1 + a_2 x_2 + ... + a_n x_n$). For example, here we have a 2×2 matrix A and a vector $x \in \mathbb{Z}^2$.

$$Ax = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + cx_2 \end{bmatrix}$$

Moreover, instead of multiplying each individual row and column, we can also multiply matrices in blocks. For example, if we have a 4×4 matrix A and a vector $x \in \mathbb{Z}^4$, we can break A into 4 smaller matrices, say 2×2 matrices H, I, J, K. We can also break x into 2 vectors $\vec{x}_1$ and $\vec{x}_2$ each of size 2. This would give the following formulation of the product Ax.

$$Ax = \begin{bmatrix} H & I \\ J & K \end{bmatrix} \begin{bmatrix} \vec{x}_1 \\ \vec{x}_2 \end{bmatrix} = \begin{bmatrix} H\vec{x}_1 + I\vec{x}_2 \\ J\vec{x}_1 + K\vec{x}_2 \end{bmatrix}.$$

Solution: