Homework 6

We may grade a **subset of the assigned questions**, to be determined after the deadline, so that we can provide better feedback on the graded questions.
Unless otherwise stated, each question requires sufficient justification to convince the reader of the correctness of your answer.
For bonus questions, we will not provide any insight during office hours or Piazza, and we do not guarantee anything about the difficulty of these questions.
We strongly encourage you to typeset your solutions in LaTeX.
If you collaborated with someone, you must state their name(s). You must write your own solution for all problems and may not look at any other student's write-up.

1. Determine whether the following languages are always, sometimes or never decidable and provide justification. If they are always or never decidable, provide a proof. If they are sometimes decidable, provide an example for when they are decidable and when they are not.

   (a) $L = (L_1 \setminus L_2) \cap L_3$, where L_1, L_2 and L_3 are all **decidable** languages.

   Note: If A and B are sets, then $A \setminus B = \{x \in A : x \notin B\}$ (set difference of A and B).

   Solution:

   (b) $L = L_1 \cap L_2$, where L_1 and L_2 are both **undecidable**.

   Solution:

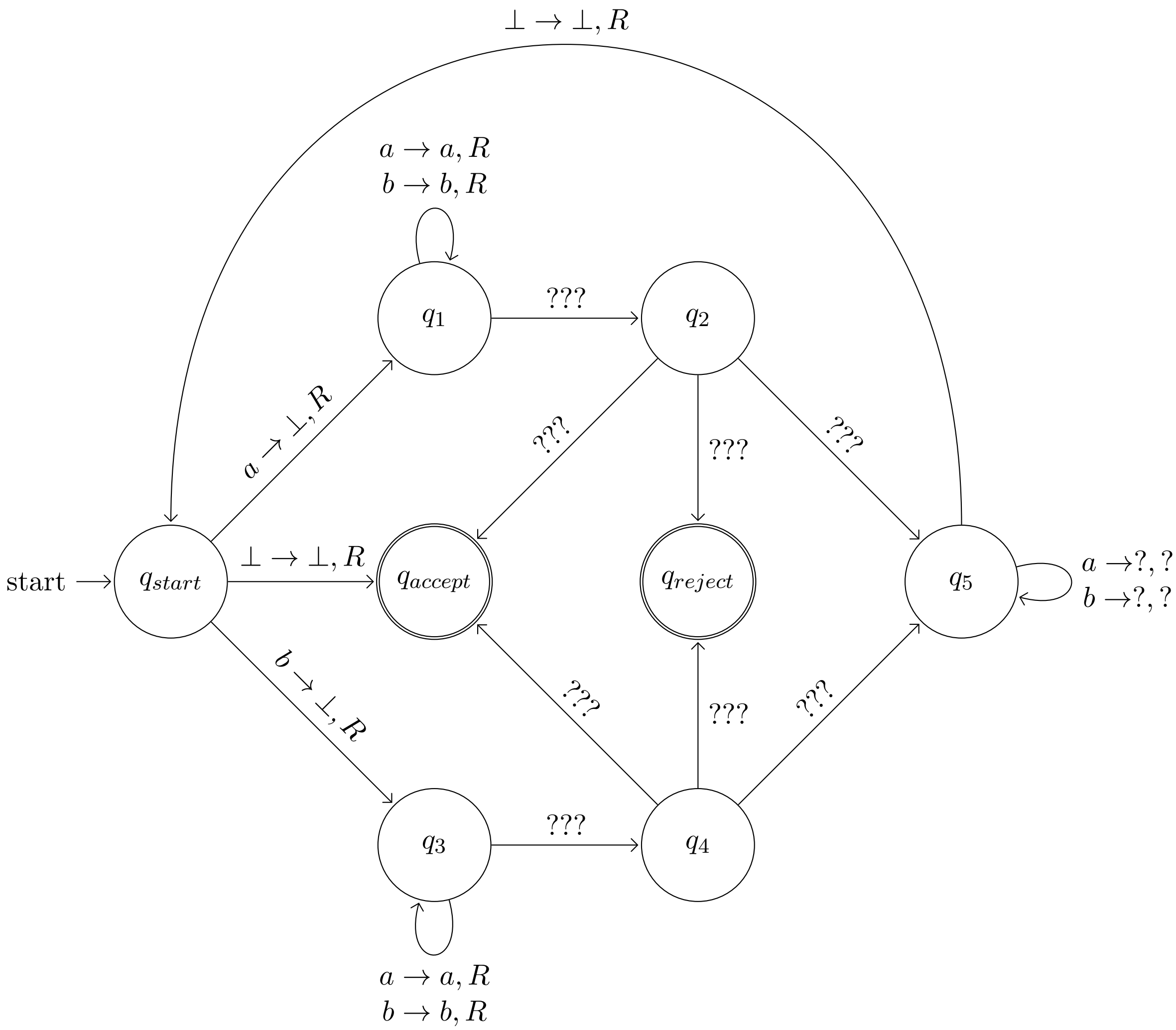
   (c) $L = L_1 \cup L_2$, where L_1 is decidable and L_2 is undecidable.

   Solution:

2. A string x is a *palindrome* if $x = x^R$, where x^R denotes the reversal of x (i.e., the characters of x in reverse order). For example, `abba` and `ababa` are palindromes, and `abbab` and `aaba` are not palindromes.

   The state diagram below is for a Turing machine that decides the language of palindromes over the alphabet $\Sigma = \{a, b\}$. **Complete the diagram by supplying the appropriate transitions** of the form "$e \to e', L/R$" where e, e' are symbols from the tape alphabet $\Gamma = \{a, b, \perp\}$. As a hint, each of the eight states serves one of the following purposes:

   - The initial, accept, and reject states.
   - After seeing an a (respectively, b) at the left end of the (remaining) input string, move the head to the right end of the (remaining) string.
   - After seeing an a (respectively, b) at the left end, the head is at the right end.
   - Move the head to the left end of the (remaining) input string.


$\perp \to \perp, R$
$a \to a, R$
$b \to b, R$
q_1
???
q_2
$a \to \perp, R$
???
???
???
start
q_{start}
$\perp \to \perp, R$
q_{accept}
q_{reject}
q_5
$a \to ?, ?$
$b \to ?, ?$
$b \to \perp, R$
???
???
???
q_3
???
q_4
$a \to a, R$
$b \to b, R$

Solution:

3. (a) In discussion, we used diagonalization to show that the set $\mathbb{R}$ of real numbers is uncountably infinite and to construct an example of an undecidable language. Generalize the diagonalization method to prove that for a *countably* infinite set A, the power set $\mathcal{P}(A)$ is uncountably infinite.

   Note: The power set of a set A is the set of all subsets of A. For instance, $\mathcal{P}(\{0,1\}) = \{\varnothing, \{0\}, \{1\}, \{0,1\}\}$.

   Solution:

   (b) Use part (a) to prove that any countably infinite language L has an undecidable subset $L' \subseteq L$.

   Solution:

4. Determine, with proof, which of the following languages are undecidable. That is, for an undecidable language, present a proof via Turing reduction of its undecidability, and for a decidable language, exhibit a decider for that language.

   (a) $L_{\varepsilon\text{-ACC}} = \{\langle M \rangle : M \text{ accepts } \varepsilon\}$

   Solution:

   (b) $L = \{(\langle M_1 \rangle, \langle M_2 \rangle) \mid \varepsilon \in L(M_1) \cup L(M_2)\}$
   Hint: consider $L_{\varepsilon\text{-ACC}}$.

   Solution:

   (c) $L_{\text{376-POW-STEP}} = \{(\langle M \rangle, x) : M \text{ halts on } x \text{ within } |x|^{376} \text{ steps}\}$

   Solution:

   (d) $L_{\text{QUICK-HALT}} = \{\langle M \rangle : \text{There is an input } w \text{ such that } M \text{ halts on } w \text{ within } |w| \text{ steps}\}$

   Solution: