Cohomology theories for Algebraic Varieties

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$$C^0(X) \xrightarrow{\delta^0} C^1(X) \xrightarrow{\delta^1} \cdots$$

$H^i_{sing}(X, R) = \ker(\delta^i)/\mathrm{im}(\delta^{i+1})$ = singular cohomology groups.

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Lefschetz Fixed Point Theorem: If $f: X \to X$ with isolated fixed points, then

$$\Lambda_f = \sum_{i=0}^{2\dim(X)} (-1)^i \operatorname{Tr}(f^* \mid H^i_{sing}(X, \mathbb{Q}))$$

counts the fixed points of f with multiplicity.

Schemes and varieties

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Ex: $\mathbb{P}^1_A$ the projective line. $\mathbb{P}^1_A(B) = \left\{ \begin{array}{l} L \subset B \times B \text{ direct factor,} \\ \text{loc free rank one} \end{array} \right\}$

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Ex: $X/\mathbb{Z}$ smooth and proper induces varieties: $X_{\mathbb{Q}}/\mathbb{Q}$ and $X_{\mathbb{F}_p}/\mathbb{F}_p$ via $\mathbb{Z} \to \mathbb{Q}$ and $\mathbb{Z} \to \mathbb{F}_p$.

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N.B. For any variety X/k with k of characteristic zero we get a cohomology theory. It doesn't work for varieties over fields of characteristic p

Varieties over $\mathbb{F}_p$ and Weil

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$$Z_{\mathbb{P}^1_{\mathbb{F}_p}}(t) = \exp\left(-\sum_{n=1}^{\infty} \frac{p^n+1}{n} \cdot t^n\right) = \frac{1}{(1-t)(1-pt)}$$

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Key idea:

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Key idea: Refine topology of X and use sheaf cohomology

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$H^i_{dR}(X) = H^i(\Omega^{\bullet,sm}(X)) =$ de Rham cohomology.

X/A a smooth scheme the algebraic de Rham complex of algebraic diff. forms

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1) $H^i_{dR}(X/A)$ are A-modules

2) Naturally filtered via the "stupid" filtration on de Rham complex.

Recap

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$\mathrm{Var}(h)$

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$$\mathrm{Var}(k) \xrightarrow{H^i_{\mathrm{Betti},\sigma}(X,\mathbb{Q})} \mathrm{Vect}(\mathbb{Q})$$

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Existence relies on the "Standard Conjectures"

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3) Due largely to Fontaine.

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3) Prismatic cohomology unifies de Rham and p-adic étale cohom.

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3) Canonically determines the de Rham cohomology of $X_{\mathbb{Q}}/\mathbb{Q}$ with its filtration.

Thank You!