

Cohomology theories for Algebraic Varieties

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$$C^0(X) \xrightarrow{\delta^0} C^1(X) \xrightarrow{\delta^1} \dots$$

$$H_{\text{sing}}^i(X, R) = \ker(\delta^i)/\text{im}(\delta^{i-1}) = \text{singular cohomology groups.}$$

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Let's sketch Fixed Point Theorem: If $f: X \rightarrow X$ with isolated fixed points, then

$$\Lambda_f = \sum_{i=0}^{2\dim(X)} (-1)^i \text{Tr}(f^*|_{H_{\text{sing}}^i(X, \mathbb{Q})})$$

counts the fixed points of f with multiplicity.

Schemes and varieties

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Ex: P_A^1 the projective line. $P_A^1(B) = \{L \subset B \times B \text{ direct factor, } h \text{ free rank one}\}$

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Ex: X/\mathbb{Z} smooth and proper induces varieties: $X_{\mathbb{Q}}/\mathbb{Q}$ and $X_{\mathbb{F}_p}/\mathbb{F}_p$
via $\mathbb{Z} \rightarrow \mathbb{Q}$ and $\mathbb{Z} \rightarrow \mathbb{F}_p$.

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N.B. For any variety X/k with k of characteristic zero we get a cohomology theory. It **does not work** for varieties over fields of characteristic p .

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$$Z_{\mathbb{P}_{\mathbb{F}_p}^1}(t) = \exp\left(-\sum_{n=1}^{\infty} \frac{p^n + 1}{n} \cdot t^n\right) = \frac{1}{(1-t)(1-pt)}$$

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Key idea: Refine topology of X and use sheaf cohomology

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1) $H^i_{dR}(X/A)$ are A -modules

2) Naturally filtered via the "stupid" filtration on de Rham complex.

Recap

Recap

Var(h)

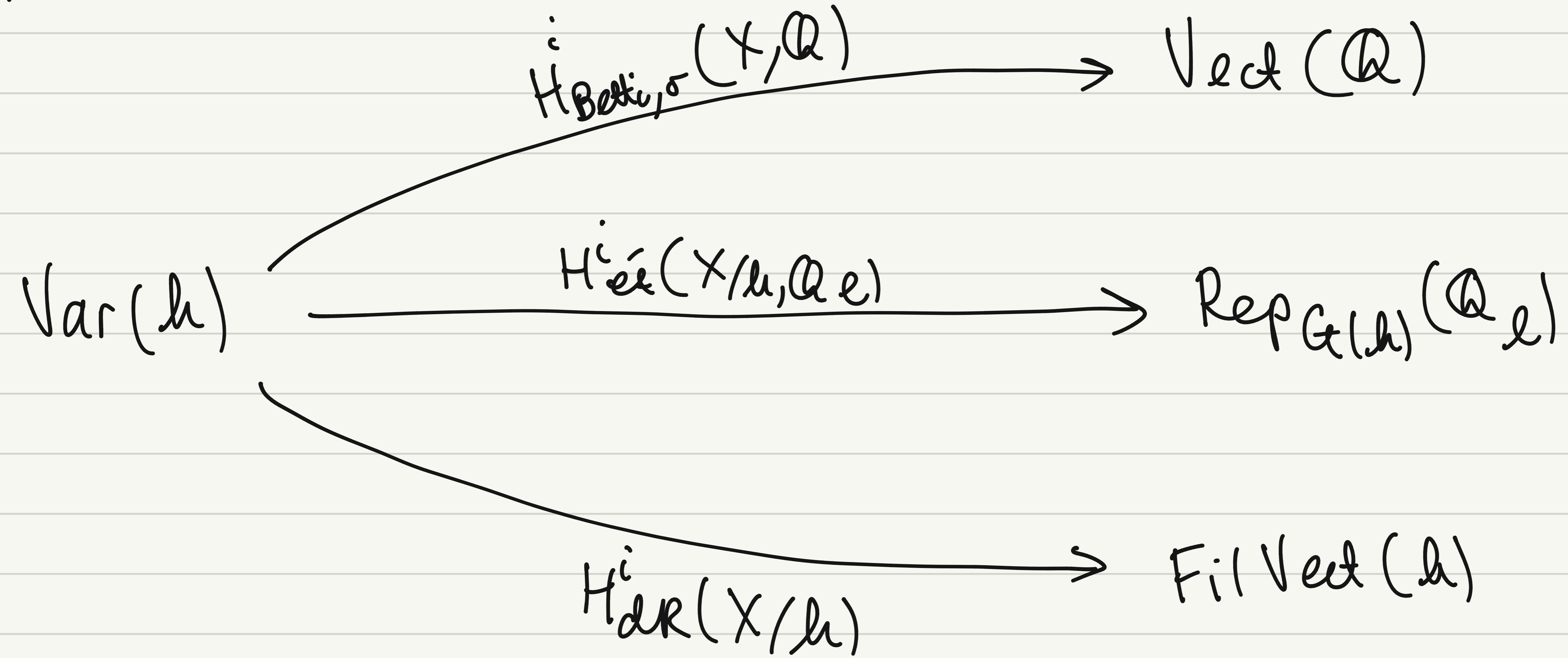
Recap

$$\text{Var}(h) \xrightarrow{i_* H_{\text{Betti}, 0}^{\wedge}(X, \mathbb{Q})} \text{Vect}(\mathbb{Q})$$

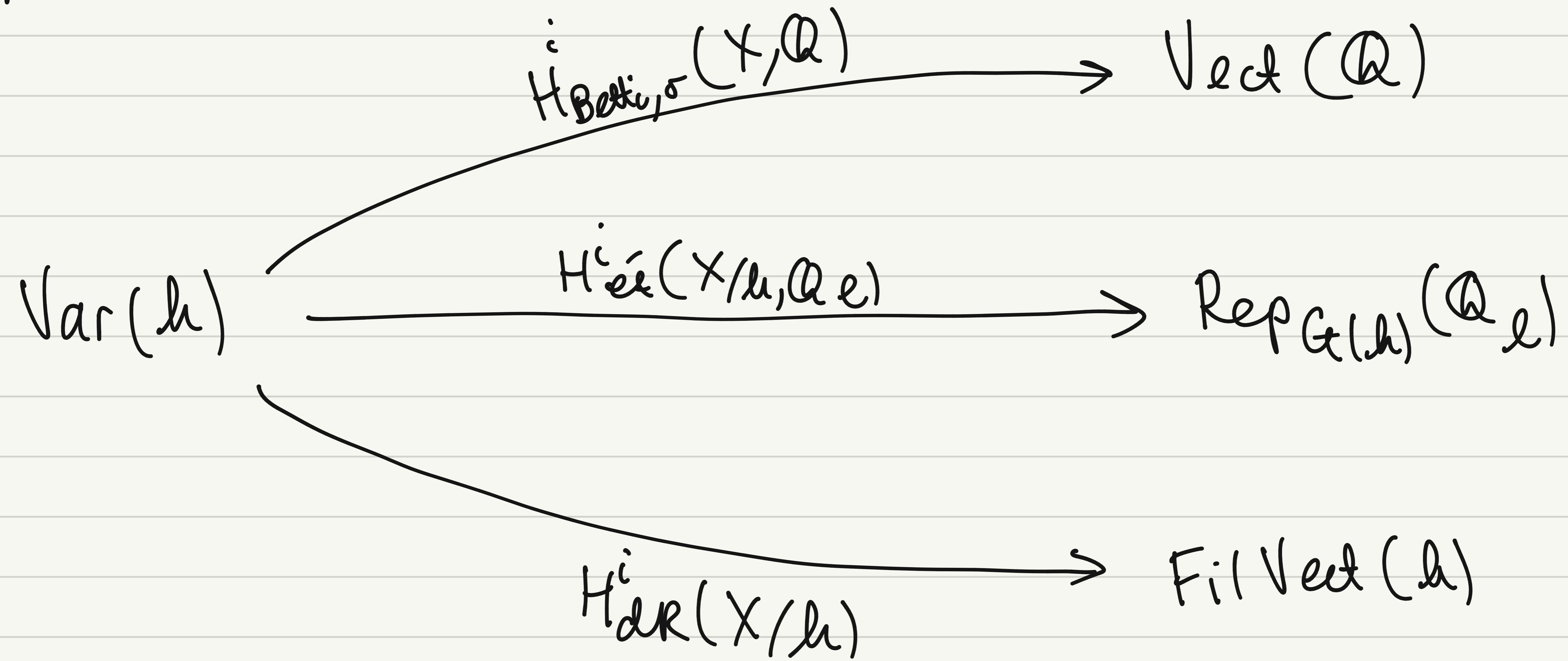
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$$\text{Var}(h) \xrightarrow{\text{H}_{\text{Betti}, \sigma}^i(X, \mathbb{Q})} \text{Vect}(\mathbb{Q})$$
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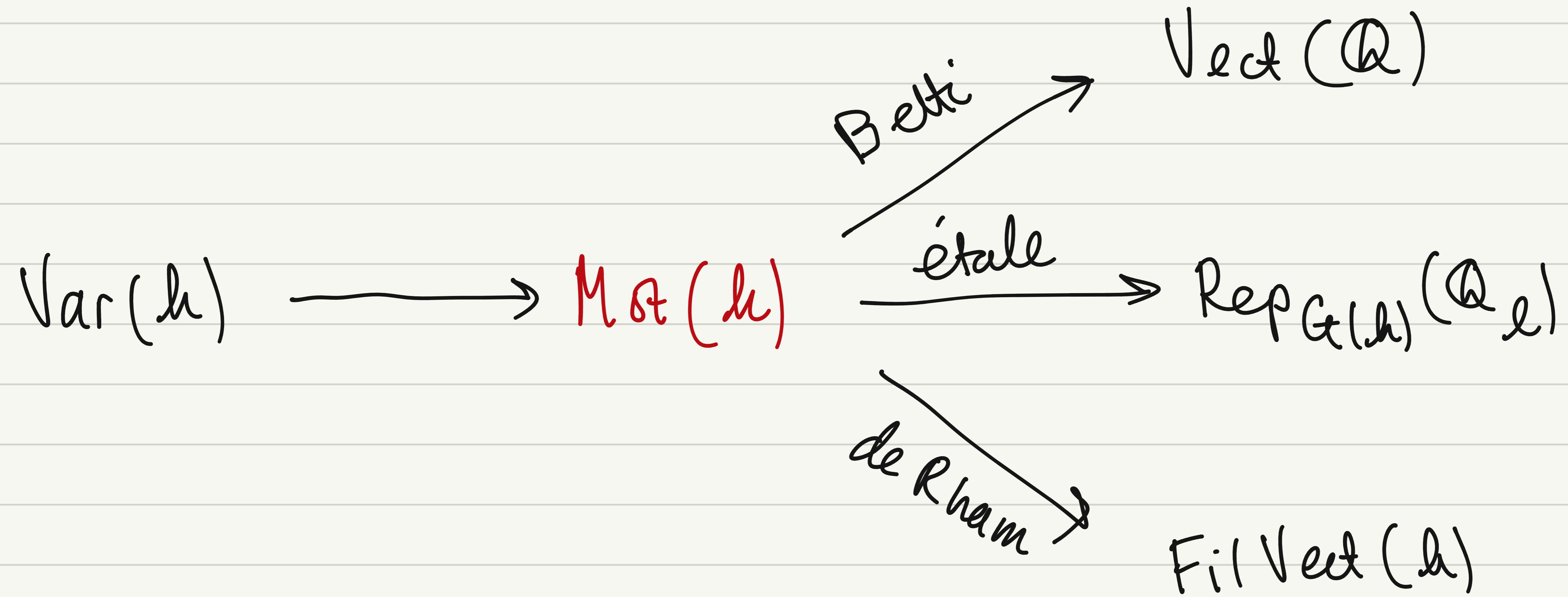


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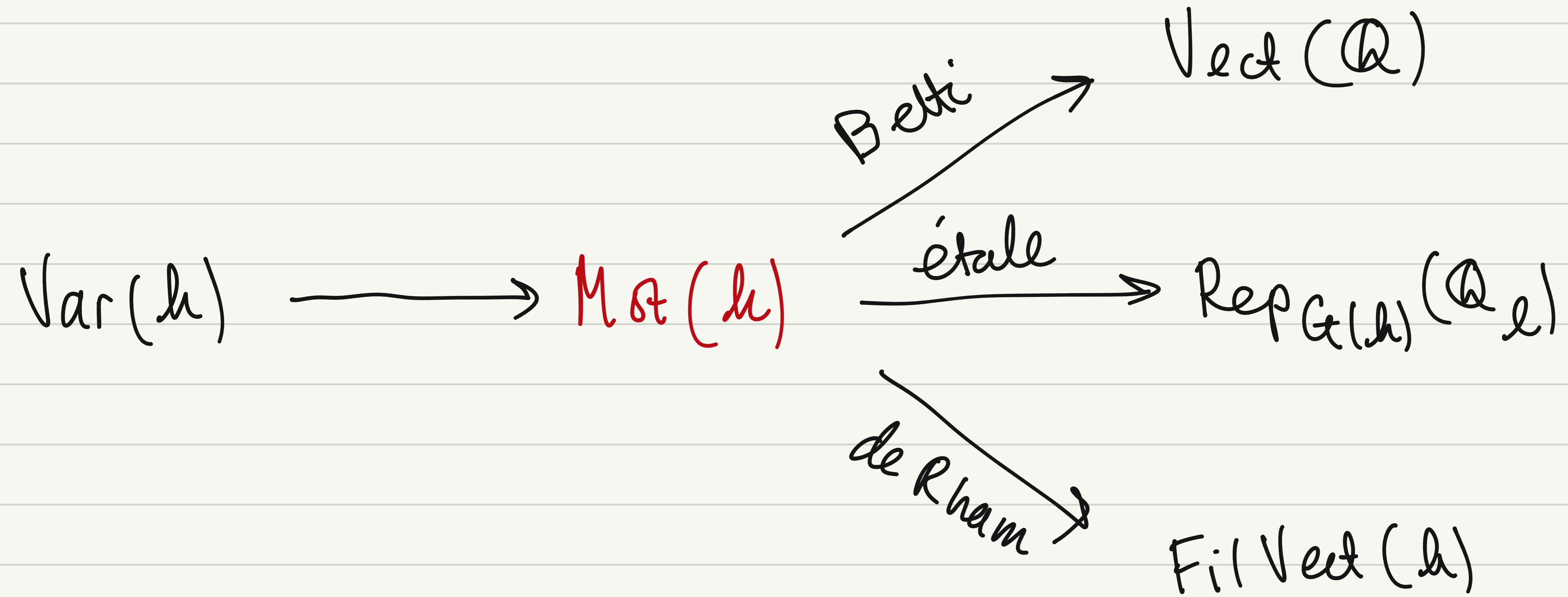
Grothendieck conjectured a **universal cohomology** - the theory of **motives**

Recap



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Recap



Grothendieck conjectured a **universal cohomology** - the theory of **motives**
Existence relies on the "Standard Conjectures"

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For X/\mathbb{Q} a variety we have two rational cohomology theories de Rham and Betti

$$H^i_{\text{dR}}(X/\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\int} H^i_{\text{Betti}}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

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$$\begin{aligned} H^i_{\text{dR}}(X/\mathbb{Q}) \otimes H_{2\dim(X)-i}(X, \mathbb{Q}) &\xrightarrow{\int} \mathbb{C} \\ \omega \otimes \gamma &\mapsto \int_{\gamma} \omega \end{aligned}$$

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- 3) Due largely to Fontaine.

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- 3) Prismatic cohomology unifies de Rham and p-adic étale cohom.

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3) Canonically determines the de Rham cohomology of $X_{\mathbb{Q}/\mathbb{Q}}$ with its filtration.

Thank You!