We seek the phonon dispersion curves for a crystal of Cu (which forms an fcc lattice) in the [110] direction in k-space. A general vector that points in that direction is given by:

$$\mathbf{k} = \frac{k}{\sqrt{2}}(\hat{i} + \hat{j}) \tag{1}$$

The expression for the force on the atom located at the origin due to it's motion and the motion of on of it's neighbors is given by:

$$\mathbf{F} = -\alpha \left[\hat{\mathbf{R}} \cdot \mathbf{u}(000) - \hat{\mathbf{R}} \cdot \mathbf{u}(\mathbf{R}) \right] \hat{\mathbf{R}} \tag{2}$$

Each cu atom has 12 nearest neighbors. They are located at:

$$\begin{aligned}
a(\frac{1}{2}\frac{1}{2}0) \qquad & a(\tfrac{1}{2}0\tfrac{1}{2}) \qquad a(0\frac{1}{2}\frac{1}{2}) && (3)\\
a(-\frac{1}{2}\frac{1}{2}0) \qquad & a(-\tfrac{1}{2}0\tfrac{1}{2}) \qquad a(0-\frac{1}{2}\frac{1}{2}) && (4)\\
a(\frac{1}{2}-\frac{1}{2}0) \qquad & a(\tfrac{1}{2}0-\tfrac{1}{2}) \qquad a(0\frac{1}{2}-\frac{1}{2}) && (5)\\
a(-\frac{1}{2}-\frac{1}{2}0) \quad & a(-\tfrac{1}{2}0-\tfrac{1}{2}) \quad a(0-\frac{1}{2}-\frac{1}{2}) && (6)\\
& && (7)
\end{aligned}$$

Let's attack them one at a time.
For $\mathbf{r}_n = a(\frac{1}{2}\frac{1}{2}0)$:

$$\hat{\mathbf{R}} = \frac{1}{\sqrt{2}}(\hat{i} + \hat{j}) \tag{8}$$

and assuming

$$\begin{aligned}
\mathbf{u}_n &= \mathbf{A}e^{i\mathbf{k}\cdot\mathbf{r}_n - i\omega t} && (9)\\
&= \mathbf{A}e^{i(\frac{ka}{2\sqrt{2}} + \frac{ka}{2\sqrt{2}}) - i\omega t} && (10)\\
&= \mathbf{A}e^{i(\frac{ka}{\sqrt{2}}) - i\omega t} && (11)\\
&= \mathbf{u}(000)e^{i(\frac{ka}{\sqrt{2}})} && (12)\\
& && (13)
\end{aligned}$$

$$
\begin{aligned}
\mathbf{F} &= -\alpha\left[\hat{\mathbf{R}}\cdot\mathbf{u}(000) - \hat{\mathbf{R}}\cdot\mathbf{u}(\mathbf{R})\right]\hat{\mathbf{R}} && (14)\\
&= -\alpha\left[\frac{1}{\sqrt{2}}(\hat{i}+\hat{j})\cdot\mathbf{u}(000) - \frac{1}{\sqrt{2}}(\hat{i}+\hat{j})\cdot\mathbf{u}(\frac{1}{2}\frac{1}{2}0)\right]\frac{1}{\sqrt{2}}(\hat{i}+\hat{j}) && (15)\\
&= -\frac{\alpha}{2}\left[(\hat{i}+\hat{j})\cdot\mathbf{u}(000) - (\hat{i}+\hat{j})\cdot\mathbf{u}(\frac{1}{2}\frac{1}{2}0)\right](\hat{i}+\hat{j}) && (16)\\
&= -\frac{\alpha}{2}\left[u_x(000) + u_y(000) - u_x(000)e^{i(\frac{ka}{\sqrt{2}})} - u_y(000)e^{i(\frac{ka}{\sqrt{2}})}\right](\hat{i}+\hat{j}) && (17)\\
&= -\frac{\alpha}{2}\left[u_x(000)\left(1-e^{i(\frac{ka}{\sqrt{2}})}\right) + u_y(000)\left(1-e^{i(\frac{ka}{\sqrt{2}})}\right)\right](\hat{i}+\hat{j}) && (18)\\
& && (19)
\end{aligned}
$$

For $\mathbf{r}_n = a(-\frac{1}{2}\frac{1}{2}0)$:

$$
\hat{\mathbf{R}} = \frac{1}{\sqrt{2}}(-\hat{i}+\hat{j}) \tag{20}
$$

and assuming

$$
\begin{aligned}
\mathbf{u}(-\frac{1}{2}\frac{1}{2}0) &= \mathbf{A}e^{i\mathbf{k}\cdot\mathbf{r}_n - i\omega t} && (21)\\
&= \mathbf{A}e^{i(\frac{-ka}{2\sqrt{2}}+\frac{ka}{2\sqrt{2}}) - i\omega t} && (22)\\
&= \mathbf{A}e^{i0 - i\omega t} && (23)\\
&= \mathbf{u}(000) && (24)\\
& && (25)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{F} &= -\alpha\left[\hat{\mathbf{R}}\cdot\mathbf{u}(000) - \hat{\mathbf{R}}\cdot\mathbf{u}(\mathbf{R})\right]\hat{\mathbf{R}} && (26)\\
&= -\alpha\left[\frac{1}{\sqrt{2}}(-\hat{i}+\hat{j})\cdot\mathbf{u}(000) - \frac{1}{\sqrt{2}}(-\hat{i}+\hat{j})\cdot\mathbf{u}(-\frac{1}{2}\frac{1}{2}0)\right]\frac{1}{\sqrt{2}}(-\hat{i}+\hat{j}) && (27)\\
&= -\frac{\alpha}{2}\left[(-\hat{i}+\hat{j})\cdot\mathbf{u}(000) - (-\hat{i}+\hat{j})\cdot\mathbf{u}(-\frac{1}{2}\frac{1}{2}0)\right](-\hat{i}+\hat{j}) && (28)\\
&= -\frac{\alpha}{2}\left[-u_x(000) + u_y(000) + u_x(000) - u_y(000)\right](-\hat{i}+\hat{j}) && (29)\\
&= 0 && (30)
\end{aligned}
$$

For $\mathbf{r}_n = a(\frac{1}{2}-\frac{1}{2}0)$:

$$
\hat{\mathbf{R}} = \frac{1}{\sqrt{2}}(\hat{i}-\hat{j}) \tag{31}
$$

and assuming

$$
\begin{aligned}
\mathbf{u}(\frac{1}{2} - \frac{1}{2}0) &= \mathbf{A}e^{i\mathbf{k}\cdot\mathbf{r}_n - i\omega t} && (32)\\
&= \mathbf{A}e^{i(\frac{ka}{2\sqrt{2}} - \frac{ka}{2\sqrt{2}}) - i\omega t} && (33)\\
&= \mathbf{A}e^{i0 - i\omega t} && (34)\\
&= \mathbf{u}(000) && (35)\\
& && (36)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{F} &= -\alpha \left[\hat{\mathbf{R}}\cdot\mathbf{u}(000) - \hat{\mathbf{R}}\cdot\mathbf{u}(\mathbf{R})\right]\hat{\mathbf{R}} && (37)\\
&= -\alpha\left[\frac{1}{\sqrt{2}}(\hat{i}-\hat{j})\cdot\mathbf{u}(000) - \frac{1}{\sqrt{2}}(\hat{i}-\hat{j})\cdot\mathbf{u}(\frac{1}{2}-\frac{1}{2}0)\right]\frac{1}{\sqrt{2}}(\hat{i}-\hat{j}) && (38)\\
&= -\frac{\alpha}{2}\left[(\hat{i}-\hat{j})\cdot\mathbf{u}(000) - (\hat{i}-\hat{j})\cdot\mathbf{u}(\frac{1}{2}-\frac{1}{2}0)\right](\hat{i}-\hat{j}) && (39)\\
&= -\frac{\alpha}{2}\left[u_x(000) - u_y(000) - u_x(000) + u_y(000)\right](\hat{i}-\hat{j}) && (40)\\
&= 0 && (41)\\
& && (42)
\end{aligned}
$$

Makes sense since these atoms lie on the same wavefront and so should always be in phase with the atom at the origin.

For $\mathbf{r}_n = a(-\frac{1}{2}, -\frac{1}{2}, 0)$:

$$
\hat{\mathbf{R}} = \frac{1}{\sqrt{2}}(-\hat{i}-\hat{j}) \tag{43}
$$

and assuming

$$
\begin{aligned}
\mathbf{u}(-\frac{1}{2}, -\frac{1}{2}, 0) &= \mathbf{A}e^{i\mathbf{k}\cdot\mathbf{r}_n - i\omega t} && (44)\\
&= \mathbf{A}e^{i(\frac{-ka}{2\sqrt{2}} - \frac{ka}{2\sqrt{2}}) - i\omega t} && (45)\\
&= \mathbf{A}e^{i\frac{-ka}{\sqrt{2}} - i\omega t} && (46)\\
&= \mathbf{u}(000)e^{i\frac{-ka}{\sqrt{2}}} && (47)\\
& && (48)
\end{aligned}
$$

$$\mathbf{F} = -\alpha\left[\hat{\mathbf{R}}\cdot\mathbf{u}(000) - \hat{\mathbf{R}}\cdot\mathbf{u}(\mathbf{R})\right]\hat{\mathbf{R}} \tag{49}$$

$$= -\alpha\left[\frac{1}{\sqrt{2}}(-\hat{i}-\hat{j})\cdot\mathbf{u}(000) - \frac{1}{\sqrt{2}}(-\hat{i}-\hat{j})\cdot\mathbf{u}(-\frac{1}{2},-\frac{1}{2},0)\right]\frac{1}{\sqrt{2}}(-\hat{i}-\hat{j}) \tag{50}$$

$$= -\frac{\alpha}{2}\left[(-\hat{i}-\hat{j})\cdot\mathbf{u}(000) - (-\hat{i}-\hat{j})\cdot\mathbf{u}(-\frac{1}{2},-\frac{1}{2},0)\right](-\hat{i}-\hat{j}) \tag{51}$$

$$= -\frac{\alpha}{2}\left[-u_x(000) - u_y(000) + u_x(000)e^{-i\frac{ka}{\sqrt{2}}} + u_y(000)e^{-i\frac{ka}{\sqrt{2}}}\right](-\hat{i}-\hat{j}) \tag{52}$$

$$= -\frac{\alpha}{2}\left[u_x(000)\left(-1+e^{-i(\frac{ka}{\sqrt{2}})}\right) + u_y(000)\left(-1+e^{-i(\frac{ka}{\sqrt{2}})}\right)\right](-\hat{i}-\hat{j}) \tag{53}$$

$$\tag{54}$$

For $\mathbf{r}_n = a(\frac{1}{2},0,\frac{1}{2})$:

$$\hat{\mathbf{R}} = \frac{1}{\sqrt{2}}(\hat{i}+\hat{k}) \tag{55}$$

and assuming

$$\mathbf{u}(\frac{1}{2},0,\frac{1}{2}) = \mathbf{A}e^{i\mathbf{k}\cdot\mathbf{r}_n - i\omega t} \tag{56}$$

$$= \mathbf{A}e^{i(\frac{ka}{2\sqrt{2}}+0)-i\omega t} \tag{57}$$

$$= \mathbf{A}e^{i\frac{ka}{2\sqrt{2}}-i\omega t} \tag{58}$$

$$= \mathbf{u}(000)e^{i\frac{ka}{2\sqrt{2}}} \tag{59}$$

$$\tag{60}$$

$$\mathbf{F} = -\alpha\left[\hat{\mathbf{R}}\cdot\mathbf{u}(000) - \hat{\mathbf{R}}\cdot\mathbf{u}(\mathbf{R})\right]\hat{\mathbf{R}} \tag{61}$$

$$= -\alpha\left[\frac{1}{\sqrt{2}}(\hat{i}+\hat{k})\cdot\mathbf{u}(000) - \frac{1}{\sqrt{2}}(\hat{i}+\hat{k})\cdot\mathbf{u}(\frac{1}{2},0,\frac{1}{2})\right]\frac{1}{\sqrt{2}}(\hat{i}+\hat{k}) \tag{62}$$

$$= -\frac{\alpha}{2}\left[(\hat{i}+\hat{k})\cdot\mathbf{u}(000) - (\hat{i}+\hat{k})\cdot\mathbf{u}(\frac{1}{2},0,\frac{1}{2})\right](\hat{i}+\hat{k}) \tag{63}$$

$$= -\frac{\alpha}{2}\left[u_x(000) + u_z(000) - u_x(000)e^{i\frac{ka}{2\sqrt{2}}} - u_z(000)e^{i\frac{ka}{2\sqrt{2}}}\right](\hat{i}+\hat{k}) \tag{64}$$

$$= -\frac{\alpha}{2}\left[u_x(000)\left(1-e^{i(\frac{ka}{2\sqrt{2}})}\right) + u_z(000)\left(1-e^{i(\frac{ka}{2\sqrt{2}})}\right)\right](\hat{i}+\hat{k}) \tag{65}$$

$$\tag{66}$$