# A description of several coordinate systems for hyperbolic spaces

Sandro S. e Costa

Instituto Astronômico e Geofísico da Universidade de São Paulo (IAG-USP)

Av. Miguel Stéfano, 4200 - CEP 04301-904 - São Paulo - SP - Brazil

Electronic address: sancosta@iagusp.usp.br

February 7, 2008

#### Abstract

This article simply presents several coordinate systems for 2 and 3-dimensional hyperbolic spaces, describing the general solutions of Helmholtz equation in each one of these systems.

PACS numbers: 02.40.-k

# 1 Introduction

Although the most recent cosmological observations suggest we live in a flat universe, the possibility that such observations are indicating a non-flat space with almost vanishing curvature can not yet be excluded, since it is not easy to distinguish a truly flat universe, where the curvature is null, from a negatively or positively curved universe inflated until the point of having almost no curvature.

Such indeterminacy, somewhat fundamental, is linked to another one about the compactness or not of the universe, *i.e.*, of its topology, and in any case one can ask if it would be possible, in principle, to achieve a higher degree of certainty.

The line of research confronting such problem can be summarized by the question "can one see the shape of the universe?" [1], which is related to the

1 Introduction 2

problem of "hearing the shape of a drum" [2, 3]. Unfortunately, the answers obtained until now from the cosmological observations are unconclusive or, at most, not much restrictive. More complete studies would, for example, involve as a first step the solution of the eigenvalue problem of the Laplace-Beltrami operator, *i.e.*, the Helmholtz equation, in several spaces of different curvatures, compact and not compact, and this is yet an open field of research [4, 5].

For flat, Euclidean geometry, it is easy to find in textbooks all the systems of coordinates where the Helmholtz equation is solvable<sup>a</sup>: 4 systems for the 2-dimensional Euclidean space  $E^2$  and 11 systems for the 3-dimensional Euclidean space  $E^3$  [6, 8]. In contrast to this, one can see that almost all physically motivated studies of hyperbolic spaces present in literature use only the analogous of the spherical polar coordinates of flat geometry, probably because such coordinate system is obtained as a trivial generalization for hyperbolic geometry, and also probably because this system of coordinates is the one one would expect to use, for instance, in cosmological observations.

However, in the same way as in flat geometry, there are also other groups of coordinates in hyperbolic geometry which can be adequate for several specific problems, or where the Helmholtz equation can be solved. One recent example found in literature is the use of a set of hyperbolic coordinates analogous to the Euclidean cylindrical ones to solve a problem of quantum cosmology [9].

With no other purpose than serving as one more tool for works involving 2 and 3-dimensional hyperbolic spaces, and with no intention of being complete, this paper simply presents a description of several sets of coordinates for these spaces, showing in the majority of cases the general solutions of Helmholtz equation for the sets presented -4 sets for the 2-dimensional space  $H^2$  and 12 for the 3-dimensional space  $H^3$ , numbers which probably do not cover all possibilities for the spaces  $H^2$  and  $H^3$ .

The paper is organized as follows: section 2 describes two types of projections that can be used for visualizing the hyperbolic space; sections 3 and 4 present, respectively, different coordinate systems for 2 and 3-dimensional hyperbolic spaces; and section 5 is a brief conclusion.

<sup>&</sup>lt;sup>a</sup> In the case of Euclidean space the interested reader can find a good description of how to obtain all the coordinate systems in which the Helmholtz equation is separable, involving the use of complex variables and linear algebra, in the textbook of Morse and Feshbach [6]; see also the work of Eisenhart [7]. Here, we simply *describe some* coordinate systems for hyperbolic spaces.

# 2 Projections of the hyperbolic space

The infinite hyperbolic space  $H^n$  can be described by the n-dimensional surface described by the constraint

$$x^{\mu}x_{\mu} \equiv g_{\mu\nu}x^{\mu}x^{\nu} = (x^{1})^{2} + (x^{2})^{2} + \dots - (x^{n+1})^{2} = -1 , \qquad (1)$$

where the last equality shows explicitly that  $g_{\mu\nu} = \text{diag}[1,...,1,-1]$  is the metric of a Minkovski space of dimension n+1, also represented by the element of line

$$ds^{2} = (dx^{1})^{2} + \dots + (dx^{n})^{2} - (dx^{n+1})^{2} .$$
 (2)

Notice that equation (1) is in perfect analogy with the constraint equation for a spherical surface,  $x^{\mu}x_{\mu} = 1$ , where the metric used is the Euclidean one.

In order to better visualize the infinite hyperbolic surface one can use several types of geometric projections, in almost the same way one does to visualize the spherical surface of the Earth in one flat sheet of paper. Two kinds of projections, known as Klein and Poincaré projections, are such that the entire n-dimensional hyperbolic space is 'confined' into a circular unitary 'ball' of dimension n. The coordinates for these projections are defined, respectively, as

$$x_K^{\mu} \equiv \frac{x^{\mu}}{x^{n+1}} \tag{3}$$

and

$$x_P^{\mu} \equiv \frac{x^{\mu}}{1 + x^{n+1}} ,$$
 (4)

where  $\mu = 1, 2, ..., n$ . In the Klein projection the geodesics of the hyperbolic space – which are hyperbolas – appear as straight lines, while in the Poincaré projection they appear as arcs of circles *or* straight lines.

An artistic example of the use of these projections can be seen in some works of the Dutch painter M.C. Escher, who used them to present fillings of the entire infinite hyperbolic plane  $H^2$  with repetitive patterns, a procedure analogous to the covering of a wall with regular tiles<sup>b</sup>.

In the next section both the Klein and Poincaré projections for a 2-dimensional hyperbolic space will be used to show for each system of coordinates a representation of the curves produced when one of the coordinates is kept constant.

<sup>&</sup>lt;sup>b</sup>Similar examples are some works by Peter Raedschelders which appear in the book 'Surfing through the hyperspace', by Clifford A. Pickover [10].

# 3 2-dimensional hyperbolic space

### 3.1 Symmetries and coordinates

As said in the previous section, the 2-dimensional hyperbolic space can be seen as formed by all points that satisfy the constraint

$$(x^{1})^{2} + (x^{2})^{2} - (x^{3})^{2} = -1 . (5)$$

Notice that the 'central point' (0,0,1) obeys this constraint. Notice also that any change of coordinates that keeps this equation still valid are allowed, producing then parametric equations which will define a coordinate system.

In fact, any coordinate transformation that keeps equation (5) invariant is a representation of the internal symmetries of hyperbolic space  $H^2$ . One such symmetry is represented by Lorentz boosts in the  $x^1$  and  $x^2$  directions, and other is given by rotations in the plane  $x^1x^2$ . This symmetries are represented by the matrices

$$R(\varphi) \equiv \begin{pmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6}$$

and

$$\Lambda(\varphi, a) = R(\varphi) \times \begin{pmatrix} \cosh a & 0 & \sinh a \\ 0 & 1 & 0 \\ \sinh a & 0 & \cosh a \end{pmatrix} \times R^{-1}(\varphi) \qquad , \qquad (7)$$

which represent, respectively, a rotation of angle  $\varphi$  in the plane and a boost of rapidity a in the direction given by the angle  $\varphi$ .

One way of combining all these informations at once is to apply, in sequence, two different orthogonal boosts and a rotation to the point (0,0,1), obtaining then a general point P,

$$P = R(\varphi) \times \Lambda\left(\frac{\pi}{2}, a\right) \times \Lambda(0, b) \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cos\varphi \sinh b - \sin\varphi \sinh a \cosh b \\ \sin\varphi \sinh b + \cos\varphi \sinh a \cosh b \\ \cosh a \cosh b \end{pmatrix} . \tag{8}$$

If a = 0 or b = 0 this equation gives the polar parametrization of the space  $H^2$ ,

$$\begin{cases} x^1 = \sinh \chi \cos \varphi \\ x^2 = \sinh \chi \sin \varphi \\ x^3 = \cosh \chi \end{cases}$$
 (9)

The element of line obtained for these coordinates is<sup>c</sup>

$$ds^{2} \equiv (dx^{1})^{2} + (dx^{2})^{2} - (dx^{3})^{2} = d\chi^{2} + \sinh^{2}\chi d\varphi^{2} . \tag{10}$$

Putting  $\varphi = 0$  in (8) one obtains another parametrization,

$$\begin{cases} x^1 = \sinh \rho \\ x^2 = \cosh \rho \sinh \omega \\ x^3 = \cosh \rho \cosh \omega \end{cases} , \tag{11}$$

with the element of line<sup>d</sup>

$$ds^2 = d\rho^2 + \cosh^2 \rho d\omega^2 \quad . \tag{12}$$

Another distinct parametrization of the space  $H^2$  can be obtained from the central point (0,0,1) by the product

$$\begin{pmatrix} 1 & -\mu & \mu \\ \mu & 1 - \frac{\mu^2}{2} & \frac{\mu^2}{2} \\ \mu & -\frac{\mu^2}{2} & 1 + \frac{\mu^2}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \sigma & \sinh \sigma \\ 0 & \sinh \sigma & \cosh \sigma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{13}$$

revealing then that the matrix

$$M(\mu) \equiv \begin{pmatrix} 1 & -\mu & \mu \\ \mu & 1 - \frac{\mu^2}{2} & \frac{\mu^2}{2} \\ \mu & -\frac{\mu^2}{2} & 1 + \frac{\mu^2}{2} \end{pmatrix} , \qquad (14)$$

with the property  $M(\mu) M(\nu) = M(\mu + \nu)$ , inverse  $M^{-1}(\mu) = M(-\mu)$ , and with det [M] = 1, also represents a symmetry of  $H^2$ .

<sup>&</sup>lt;sup>c</sup>Here one can use the substitution  $\sinh \chi = \tan \theta$ , where  $-\pi/2 < \theta < \pi/2$  is the angle known as Gudermannian or hyperbolic amplitude [11], to obtain the element of line  $ds^2 = \sec^2 \theta \left(d\theta^2 + \sin^2 \theta d\varphi^2\right)$ , an equality which shows that these coordinates produce a metric conformal to the one of the sphere.

<sup>&</sup>lt;sup>d</sup>Here the use of the substitution  $\sinh \rho = \tan \theta$ , where  $-\pi/2 < \theta < \pi/2$  is again the Gudermannian, gives the element of line  $ds^2 = \sec^2 \theta \left( d\theta^2 + d\omega^2 \right)$ .

The parametrization thus obtained,

$$\begin{cases} x^1 = e^{-\sigma}\mu \\ x^2 = \sinh \sigma + e^{-\sigma}\frac{\mu^2}{2} \\ x^3 = \cosh \sigma + e^{-\sigma}\frac{\mu^2}{2} \end{cases} , \tag{15}$$

gives the element of line<sup>e</sup>

$$ds^2 = d\sigma^2 + e^{-2\sigma}d\mu^2 \quad . \tag{16}$$

Such parametrization appears interestingly related to an aplication in microwave engineering [13], producing a graph known as 'Smith chart'.

The three parametrizations shown until now are not symmetric in the coordinates  $x^1$  and  $x^2$ . A fourth parametrization with complete equivalence between these coordinates is

$$\begin{cases} x^{1} = \sqrt{2} \cosh u \sinh v \\ x^{2} = \sqrt{2} \cosh v \sinh u \\ x^{3} = \sqrt{\cosh 2u \cosh 2v} \end{cases}, \tag{17}$$

with element of line

$$ds^{2} = \left(\cosh 2u + \cosh 2v\right) \left(\frac{du^{2}}{\cosh 2u} + \frac{dv^{2}}{\cosh 2v}\right) . \tag{18}$$

Table I presents a summary of the coordinates shown here for the space  $H^2$ , with arbitrary names given for each system, while Figures 1 to 4 show, in Klein and Poincaré projections, the lines obtained for each set of coordinates, when one of the coordinates is kept constant.

# 3.2 Helmholtz equation

The Helmholtz equation [14]

$$\nabla^2 \Psi + k^2 \Psi \equiv \frac{1}{\sqrt{g}} \partial_\mu \left[ \sqrt{g} g^{\mu\nu} \partial_\nu \right] \Psi + k^2 \Psi = 0 \quad , \tag{19}$$

is separable in all four coordinate systems presented here for the space  $H^2$ .

<sup>&</sup>lt;sup>e</sup>It is also interesting to use, in this parametrization, the substitution  $z = e^{\sigma}$ , with  $0 < z \le \infty$ , what gives  $ds^2 = \left(d\mu^2 + dz^2\right)/z^2$ . Such element of line represents the upper half-space model of the space  $H^2$  [12].

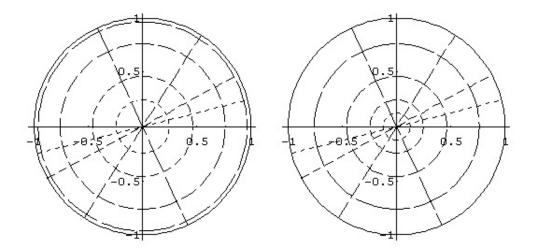


Figure 1: Visualization in the projections of Klein (left) and Poincaré (right) of the radial coordinates  $(\chi, \varphi)$ : constant  $\chi$  produces circles while constant  $\varphi$  produces straight lines.

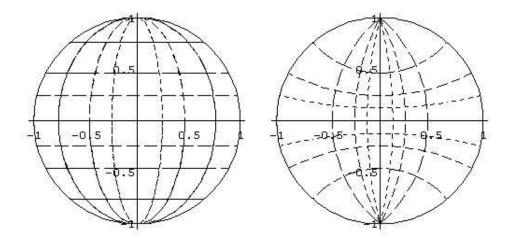


Figure 2: Visualization in the projections of Klein (left) and Poincaré (right) of the hyperbolic coordinates  $(\rho, \omega)$ .

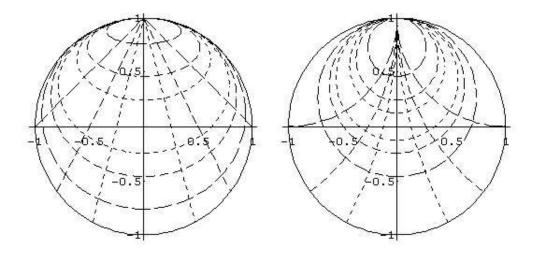


Figure 3: Visualization in the projections of Klein (left) and Poincaré (right) of the coordinates  $(\sigma, \mu)$ : constant  $\sigma$  produces ellipses or circles while constant  $\mu$  produces convergent lines. The Poincaré projection of these coordinates serves as basis for a graph called "Smith Chart", of interest in microwave engineering [13].

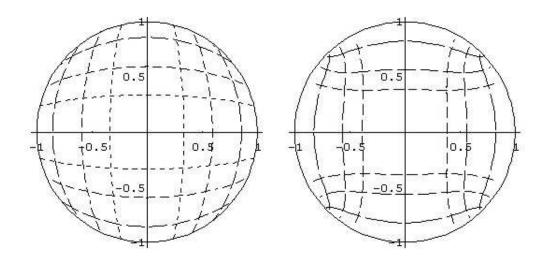


Figure 4: Visualization in the projections of Klein (left) and Poincaré (right) of the hyperbolic symmetric coordinates (u, v).

Coordinate systems for the hyperbolic 2-D space			
system	coords.	element of line	
polar	$\chi, \varphi$	$d\chi^2 + \sinh^2\chi d\varphi^2$	
hyperbolic	$\rho, \omega$	$d\rho^2 + \cosh \rho^2 d\omega^2$	
exponential	$\sigma, \mu$	$d\sigma^2 + e^{-2\sigma}d\mu^2$	
symmetric	u, v	$(\cosh 2u + \cosh 2v) \left(\frac{du^2}{\cosh 2u} + \frac{dv^2}{\cosh 2v}\right)$	

Table I: Summary of systems for space  $H^2$ .

In polar coordinates  $(\chi, \varphi)$  one has

$$\frac{1}{\sinh \chi} \left[ \frac{\partial}{\partial \chi} \left( \sinh \chi \frac{\partial \Psi}{\partial \chi} \right) + \frac{\partial}{\partial \varphi} \left( \frac{1}{\sinh \chi} \frac{\partial \Psi}{\partial \varphi} \right) \right] = -k^2 \Psi , \qquad (20)$$

with solutions of the type  $\Psi\left(\chi,\varphi\right)=X\left(\chi\right)\Phi\left(\varphi\right),$  with

$$\Phi = a_1 \cos \lambda \varphi + a_2 \sin \lambda \varphi \tag{21}$$

and

$$X = b_1 P_{-\frac{1}{2} + \sqrt{\frac{1}{4} - k^2}}^{\lambda} (\cosh \chi) + b_2 Q_{-\frac{1}{2} + \sqrt{\frac{1}{4} - k^2}}^{\lambda} (\cosh \chi) , \qquad (22)$$

or

$$X = \left(\cosh \chi\right)^{-\frac{1}{2}} \left[ b_1' P_{-\frac{1}{2} + \lambda}^{\sqrt{\frac{1}{4} - k^2}} \left(\tanh \chi\right) + b_2' Q_{-\frac{1}{2} + \lambda}^{\sqrt{\frac{1}{4} - k^2}} \left(\tanh \chi\right) \right] , \qquad (23)$$

where  $P^{\mu}_{\nu}(z)$  and  $Q^{\mu}_{\nu}(z)$  are associated Legendre functions, and where  $\lambda$  is just a separation constant.

For the hyperbolic coordinates  $(\rho, \omega)$  one has

$$\frac{1}{\cosh \rho} \left[ \frac{\partial}{\partial \rho} \left( \cosh \rho \frac{\partial \Psi}{\partial \rho} \right) + \frac{\partial}{\partial \omega} \left( \frac{1}{\cosh \rho} \frac{\partial \Psi}{\partial \omega} \right) \right] = -k^2 \Psi , \qquad (24)$$

an equation solved by the function  $\Psi(\rho,\omega) = R(\rho)W(\omega)$ , with

$$W = a_3 \cos \lambda \omega + a_4 \sin \lambda \omega \tag{25}$$

and

$$R = (\cosh \rho)^{-\frac{1}{2}} \left[ b_3 P_{-\frac{1}{2} + i\lambda}^{\sqrt{\frac{1}{4} - k^2}} \left( \tanh \rho \right) + b_4 Q_{-\frac{1}{2} + i\lambda}^{\sqrt{\frac{1}{4} - k^2}} \left( \tanh \rho \right) \right] , \qquad (26)$$

or

$$R = b_3' P_{-\frac{1}{2} + \sqrt{\frac{1}{4} - k^2}}^{i\lambda} (i \sinh \rho) + b_4' Q_{-\frac{1}{2} + \sqrt{\frac{1}{4} - k^2}}^{i\lambda} (i \sinh \rho) , \qquad (27)$$

with  $\lambda$  being again a separation constant. For completeness, it is important to say that one can also find solutions for  $R(\rho)$  in terms of Gegenbauer or hypergeometric functions by use of the substitution  $\zeta' \equiv \tanh^2 \rho$ .

The exponential coordinate system,  $(\sigma, \mu)$ , gives the equation

$$\frac{\partial^2 \Psi}{\partial \sigma^2} - \frac{\partial \Psi}{\partial \sigma} + e^{2\sigma} \frac{\partial^2 \Psi}{\partial \mu^2} = -k^2 \Psi . \tag{28}$$

In this case the solutions are  $\Psi(\sigma, \mu) = \Sigma(\sigma) \Theta(\mu)$ , where

$$\begin{cases}
\Theta(\mu) = a_{+}e^{i\lambda\mu} + a_{-}e^{i\lambda\mu} \\
\Sigma(\sigma) = e^{\sigma/2} \left[ bI_{\sqrt{\frac{1}{4} - k^{2}}} (\lambda e^{\sigma}) + cK_{\sqrt{\frac{1}{4} - k^{2}}} (\lambda e^{\sigma}) \right]
\end{cases},$$
(29)

with  $I_{\nu}$  and  $K_{\nu}$  being modified Bessel functions (Bessel functions of imaginary argument).  $\lambda$  is again a separation constant.

Finally, the symmetric set of coordinates (u, v) produces the equation

$$\cosh 2u \frac{\partial^2 \Psi}{\partial u^2} + \sinh 2u \frac{\partial \Psi}{\partial u} + \cosh 2v \frac{\partial^2 \Psi}{\partial v^2} + \sinh 2v \frac{\partial \Psi}{\partial v} + k^2 \left(\cosh 2u + \cosh 2v\right) \Psi = 0,$$
(30)

which the function  $\Psi(u, v) = U(u) V(v)$  divides into

$$\frac{d^2U}{du^2} + \tanh 2u \frac{dU}{du} + \left(k^2 - \frac{\lambda^2}{\cosh 2u}\right)U = 0 \tag{31}$$

and

$$\frac{d^2V}{dv^2} + \tanh 2v \frac{dV}{dv} + \left(k^2 + \frac{\lambda^2}{\cosh 2v}\right)V = 0.$$
 (32)

Unfortunately, it is not easy to find simple analytical solutions for these two equations, except for the case  $\lambda = 0$ , when the solutions are

$$U(u) = (\cosh 2u)^{-1/4} \left[ a P_{-\frac{1}{4}}^{\frac{1}{2}\sqrt{\frac{1}{4}-k^2}} \left(\tanh 2u\right) + b Q_{-\frac{1}{4}}^{\frac{1}{2}\sqrt{\frac{1}{4}-k^2}} \left(\tanh 2u\right) \right]$$
(33)

and

$$V(v) = (\cosh 2v)^{-1/4} \left[ a' P_{-\frac{1}{4}}^{\frac{1}{2}\sqrt{\frac{1}{4}-k^2}} \left(\tanh 2v\right) + b' Q_{-\frac{1}{4}}^{\frac{1}{2}\sqrt{\frac{1}{4}-k^2}} \left(\tanh 2v\right) \right] . (34)$$

# 4 3-dimensional hyperbolic space

### 4.1 Coordinate systems

The 3-dimensional hyperbolic space can be seen as formed by the set of points following the 4-dimensional version of the constraint (5),

$$(x^{1})^{2} + (x^{2})^{2} + (x^{3})^{2} - (x^{4})^{2} = -1 . (35)$$

As discussed in the introduction, for the flat Euclidean 3-dimensional space  $E^3$  there are eleven coordinate systems in which the Helmholtz equation is separable [8]. In this section almost the same number of systems is presented for the hyperbolic  $H^3$  space. The presentation follows a classification based on generalizations of the four basic types of coordinates found for the 2-dimensional hyperbolic space  $H^2$ .

#### 4.1.1 Polar coordinates

The most known coordinate system for the  $H^3$  space is the hyperbolic equivalent of a sphere,

$$\begin{cases}
 x^1 &= \sinh \chi \cos \varphi \sin \theta \\
 x^2 &= \sinh \chi \sin \varphi \sin \theta \\
 x^3 &= \sinh \chi \cos \theta \\
 x^4 &= \cosh \chi
\end{cases} ,$$
(36)

which produces the element of line

$$ds^{2} = d\chi^{2} + \sinh^{2}\chi \left(d\theta^{2} + \sin^{2}\theta d\varphi^{2}\right) . \tag{37}$$

Notice that this element of line has a part conformal to the element of line of a spherical surface.

However, a positively curved surface allows also a parametrization distinct from the one of the sphere. Such parametrization, of a semi-sphere, is totally symmetric:

$$x^{1} = \sqrt{2}\cos\theta\sin\phi, \quad x^{2} = \sqrt{2}\sin\theta\cos\phi, \quad x^{3} = \sqrt{\cos 2\theta\cos 2\phi},$$
 (38)

where  $-\pi/4 \leq \theta, \phi \leq \pi/4$ . So, changing the spherical part of the polar

hyperbolic coordinates by this one one has

$$\begin{cases} x^{1} = \sqrt{2} \sinh \chi \cos \theta \sin \phi \\ x^{2} = \sqrt{2} \sinh \chi \sin \theta \cos \phi \\ x^{3} = \sinh \chi \sqrt{\cos 2\theta \cos 2\phi} \\ x^{4} = \cosh \chi \end{cases}$$
(39)

what produces the element of line

$$ds^{2} = d\chi^{2} + \sinh^{2}\chi \left(\cos 2\theta + \cos 2\phi\right) \left(\frac{d\theta^{2}}{\cos 2\theta} + \frac{d\phi^{2}}{\cos 2\phi}\right) , \qquad (40)$$

where yet  $-\pi/4 \le \theta, \phi \le \pi/4$ .

#### 4.1.2 Coordinates related to the hyperbolic parametrization

There are two immediate distinct ways of generalizing the 2-dimensional set of hyperbolic coordinates  $(\rho, \omega)$ . The first one is

$$\begin{cases} x^1 = \sinh \rho \cos \varphi \\ x^2 = \sinh \rho \sin \varphi \\ x^3 = \cosh \rho \sinh \omega \end{cases}, \tag{41}$$

$$x^4 = \cosh \rho \cosh \omega$$

with element of line

$$ds^{2} = d\rho^{2} + \cosh^{2}\rho d\omega^{2} + \sinh^{2}\rho d\varphi^{2} . \tag{42}$$

The second one is

$$\begin{cases} x^1 &= \sinh \rho \\ x^2 &= \cosh \rho \sinh \gamma \\ x^3 &= \cosh \rho \cosh \gamma \sinh \omega \\ x^4 &= \cosh \rho \cosh \gamma \cosh \omega \end{cases} , \tag{43}$$

with element of line

$$ds^{2} = d\rho^{2} + \cosh^{2}\rho \left(d\gamma^{2} + \cosh^{2}\gamma d\omega^{2}\right) . \tag{44}$$

By noticing that this second element of line has a part conformally equivalent to an hyperbolic 2-dimensional space one can write down other three

parametrizations by simply changing this part by any one of the set of coordinates described in the previous section.

So, one can write

$$\begin{cases} x^{1} = \cosh \rho \sinh \varsigma \cos \varphi \\ x^{2} = \cosh \rho \sinh \varsigma \sin \varphi \\ x^{3} = \sinh \rho \\ x^{4} = \cosh \rho \cosh \varsigma \end{cases}$$
(45)

obtaining then the element of line

$$ds^{2} = d\rho^{2} + \cosh^{2}\rho \left(d\varsigma^{2} + \sinh^{2}\varsigma d\varphi^{2}\right) . \tag{46}$$

In the same way one can have

$$\begin{cases} x^{1} = e^{-\sigma}\mu \cosh \rho \\ x^{2} = \sinh \rho \\ x^{3} = \left(\sinh \sigma + \frac{1}{2}e^{-\sigma}\mu^{2}\right)\cosh \rho \end{cases}, \tag{47}$$
$$x^{4} = \left(\cosh \sigma + \frac{1}{2}e^{-\sigma}\mu^{2}\right)\cosh \rho$$

a parametrization with element of line

$$ds^{2} = d\rho^{2} + \cosh^{2}\rho \left(d\sigma^{2} + e^{-2\sigma}d\mu^{2}\right) . \tag{48}$$

Finally, one can write

$$\begin{cases} x^{1} = \sqrt{2}\cosh\rho\sinh u\cosh v \\ x^{2} = \sqrt{2}\cosh\rho\sinh v\cosh u \\ x^{3} = \sinh\rho \\ x^{4} = \cosh\rho\sqrt{\cosh 2u\cosh 2v} \end{cases}, \tag{49}$$

obtaining then the element of line

$$ds^{2} = d\rho^{2} + \cosh^{2}\rho \left(\cosh 2u + \cosh 2v\right) \left(\frac{du^{2}}{\cosh 2u} + \frac{dv^{2}}{\cosh 2v}\right) . \tag{50}$$

#### 4.1.3 Coordinates related to the exponential parametrization

The exponential 2-dimensional coordinates  $(\sigma, \mu)$  can be generalized for the space  $H^3$  by the parametric equations

$$\begin{cases} x^{1} = e^{-\sigma}\mu \\ x^{2} = e^{-\sigma}\nu \\ x^{3} = \sinh \sigma + e^{-\sigma} (\mu^{2} + \nu^{2})/2 \end{cases},$$

$$x^{4} = \cosh \sigma + e^{-\sigma} (\mu^{2} + \nu^{2})/2$$
(51)

with the subsequent element of line

$$ds^{2} = d\sigma^{2} + e^{-2\sigma} \left( d\mu^{2} + d\nu^{2} \right) . \tag{52}$$

Since the last part of this element of line is conformal to a flat 2-dimensional space<sup>f</sup>, one can change it for any of the parametrizations of the flat space and still obtains a coordinate system where the Helmholtz equation is separable.

The first of these sets of coordinates is

$$\begin{cases}
x^{1} = e^{-\sigma}\rho\cos\varphi \\
x^{2} = e^{-\sigma}\rho\sin\varphi \\
x^{3} = \sinh\sigma + \frac{1}{2}e^{-\sigma}\rho^{2} \\
x^{4} = \cosh\sigma + \frac{1}{2}e^{-\sigma}\rho^{2}
\end{cases} , (53)$$

with the subsequent element of line

$$ds^2 = d\sigma^2 + e^{-2\sigma} \left( d\rho^2 + \rho^2 d\varphi^2 \right) . \tag{54}$$

Another one is

$$\begin{cases}
 x^1 = ae^{-\sigma}\cosh u \cos v \\
 x^2 = ae^{-\sigma}\sinh u \sin v \\
 x^3 = \sinh \sigma + \frac{1}{2}a^2e^{-\sigma}\left(\cosh^2 u - \sin^2 v\right) \\
 x^4 = \cosh \sigma + \frac{1}{2}a^2e^{-\sigma}\left(\cosh^2 u - \sin^2 v\right)
\end{cases} , \tag{55}$$

with element of line

$$ds^{2} = d\sigma^{2} + a^{2}e^{-2\sigma} \left(\sinh^{2} u + \sin^{2} v\right) \left(du^{2} + dv^{2}\right) . \tag{56}$$

<sup>&</sup>lt;sup>f</sup>In the same way of the 2-dimensional space  $H^2$  here the substitution  $z = e^{\sigma}$ , with  $0 < z \le \infty$ , gives  $ds^2 = \left(d\mu^2 + d\nu^2 + dz^2\right)/z^2$ , the element of line which represents the upper half-space model of the space  $H^3$  [12].

A third system of coordinates following this line<sup>g</sup> is

$$\begin{cases} x^{1} = e^{-\sigma} \xi \eta \\ x^{2} = \frac{1}{2} e^{-\sigma} (\eta^{2} - \xi^{2}) \\ x^{3} = \sinh \sigma + \frac{1}{8} e^{-\sigma} (\eta^{2} + \xi^{2})^{2} \\ x^{4} = \cosh \sigma + \frac{1}{8} e^{-\sigma} (\eta^{2} + \xi^{2})^{2} \end{cases},$$
(57)

with the subsequent element of line

$$ds^{2} = d\sigma^{2} + e^{-2\sigma} \left( \eta^{2} + \xi^{2} \right) \left( d\eta^{2} + d\xi^{2} \right) . \tag{58}$$

#### 4.1.4 Coordinates related to the symmetric parametrization

It is not easy to build a symmetric parametrization for the space  $H^3$ . However, it is very simple to find one almost symmetric generalization of the last coordinate system presented in the section 3. This generalization is

$$\begin{cases} x^{1} = \sqrt{2} \cosh u \sinh v \\ x^{2} = \sqrt{2} \cosh v \sinh u \\ x^{3} = \sqrt{\cosh 2u \cosh 2v} \sinh \chi \\ x^{4} = \sqrt{\cosh 2u \cosh 2v} \cosh \chi \end{cases}$$
(59)

with element of line

$$ds^{2} = \cosh 2u \cosh 2v d\chi^{2} + (\cosh 2u + \cosh 2v) \left(\frac{du^{2}}{\cosh 2u} + \frac{dv^{2}}{\cosh 2v}\right) . (60)$$

Table II presents a summary of all coordinate systems shown in this section for the space  $H^3$ . The names used there for each system are arbitrary and will be used in the next subsection.

$$\begin{cases} x^{1} &= ae^{-\sigma} \sinh \eta \left(\cosh \eta - \cos \xi\right)^{-1} \\ x^{2} &= ae^{-\sigma} \sin \xi \left(\cosh \eta - \cos \xi\right)^{-1} \\ x^{3} &= \sinh \sigma + a^{2}e^{-\sigma} \left(\cosh \eta + \cos \xi\right) \left(\cosh \eta - \cos \xi\right)^{-1} \\ x^{4} &= \cosh \sigma + a^{2}e^{-\sigma} \left(\cosh \eta + \cos \xi\right) \left(\cosh \eta - \cos \xi\right)^{-1} \end{cases}$$

with element of line

$$ds^2 = d\sigma^2 + a^2 e^{-2\sigma} \left(\cosh \eta - \cos \xi\right)^{-2} \left(d\eta^2 + d\xi^2\right)$$
.

 $<sup>^{\</sup>rm g}$ Also following this line there is an interesting example of a system of coordinates where the Helmholtz equation  $is\ not$  separable:

Coordinate systems for the hyperbolic 3-D space				
system	coords.	element of line		
spherical polar	$\chi, \theta, \varphi$	$d\chi^2 + \sinh^2\chi \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$		
semi-spherical polar	$\chi, \theta, \phi$	$d\chi^2$		
		$+\sinh^2\chi\left(1+\cos 2\phi\sec 2\theta\right)d\theta^2$		
		$+\sinh^2\chi \left(1+\cos 2\theta\sec 2\phi\right)d\phi^2$		
hyperbolic	$\rho, \omega, \varphi$	$d\rho^2 + \cosh^2 \rho d\omega^2 + \sinh^2 \rho d\varphi^2$		
bi-hyperbolic	$ ho, \omega, \gamma$	$d\rho^2 + \cosh^2\rho \left(d\gamma^2 + \cosh^2\gamma d\omega^2\right)$		
polar hyperbolic	$\rho, \varsigma, \varphi$	$d\rho^2 + \cosh^2\rho \left(d\varsigma^2 + \sinh^2\varsigma d\varphi^2\right)$		
exponential hyperbolic	$\rho, \sigma, \mu$	$d\rho^2 + \cosh^2\rho \left(d\sigma^2 + e^{-2\sigma}d\mu^2\right)$		
symmetric hyperbolic	$\rho, u, v$	$d\rho^2$		
		$+\cosh^2\rho\left(1+\cosh 2u/\cosh 2v\right)dv^2$		
		$+\cosh^2\rho\left(1+\cosh 2v/\cosh 2u\right)du^2$		
exponential	$\sigma, \mu, \nu$	$d\sigma^2 + e^{-2\sigma} \left( d\mu^2 + d\nu^2 \right)$		
polar exponential	$\sigma, \rho, \varphi$	$d\sigma^2 + e^{-2\sigma} \left( d\rho^2 + \rho^2 d\varphi^2 \right)$		
elliptic exponential	$\sigma, u, v$	$d\sigma^2$		
		$+a^2e^{-2\sigma}\left(\sinh^2u+\sin^2v\right)\left(du^2+dv^2\right)$		
parabolic exponential	$\sigma, \xi, \eta$	$d\sigma^{2} + e^{-2\sigma} (\eta^{2} + \xi^{2}) (d\eta^{2} + d\xi^{2})$		
symmetric	$\chi, u, v$	$\cosh 2u \cosh 2v d\chi^2$		
		$+ (1 + \cosh 2u/\cosh 2v) dv^2$		
		$+ (1 + \cosh 2v/\cosh 2u) du^2$		
bipolar exponential	$\sigma, \xi, \eta$	$d\sigma^2$		
		$+a^{2}e^{-2\sigma}\left(\cosh\eta-\cos\xi\right)^{-2}\left(d\eta^{2}+d\xi^{2}\right)$		

Table II: Summary of systems for space  $H^3$ . Notice that the last one of these systems, named bipolar exponential, does not allow complete separation of the Helmholtz equation.

### 4.2 Helmholtz equation

All the twelve coordinate systems presented in the previous subsection allow to separate the Helmholtz equation, still given by the formula in equation (19), in a system of three differential equations, and these equations are shown here. However, not all the differential equations obtained are easy to solve analytically, and, therefore, in this paper some of the solutions are not presented. In fact, one such case, concerning the 2-dimensional symmetric coordinates, already appeared at the end of section 3.

#### 4.2.1 Polar coordinates

The polar coordinates presented in equations (36) and (37) produce the most known solutions of Helmholtz equation for the space  $H^3$ . Such solutions are obtained by use of the tentative function  $\Psi(\chi, \theta, \varphi) = X(\chi) Y(\theta, \varphi)$ , yielding then the system of equations

$$\frac{\partial^2 Y}{\partial \theta^2} + \frac{1}{\tan \theta} \frac{\partial Y}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \varphi^2} = -\ell (\ell + 1) Y$$
 (61)

and

$$\frac{d^2X}{d\chi^2} + \frac{2}{\tanh\chi} \frac{dX}{d\chi} + \left[k^2 - \frac{\ell(\ell+1)}{\sinh^2\chi}\right] X = 0.$$
 (62)

While the first, angular, equation admits as solutions the functions known as spherical harmonics,  $Y_{\ell m}(\theta, \varphi)$ , the last equation, radial, also known as hyperspherical Bessel equation [15], produces

$$X = \left(1 - \cosh^2 \chi\right)^{-1/4} \left[ a_5 P_{-\frac{1}{2} + \sqrt{1 - k^2}}^{\ell + \frac{1}{2}} \left(\cosh \chi\right) + a_6 Q_{-\frac{1}{2} + \sqrt{1 - k^2}}^{\ell + \frac{1}{2}} \left(\cosh \chi\right) \right] ,$$
(63)

or

$$X = \left(1 - \coth^2 \chi\right)^{1/2} \left[ a_5' P_\ell^{\sqrt{1-k^2}} \left(\coth \chi\right) + a_6' Q_\ell^{\sqrt{1-k^2}} \left(\coth \chi\right) \right] . \tag{64}$$

Another possible solution for the radial equation is found for  $\kappa_1 = \cosh \chi$  by use of the 'Leibniz's theorem',

$$\frac{d^m}{dx^m} \left[ f\left( x \right) g\left( x \right) \right] = \sum_{n=0}^m \left( \begin{array}{c} m \\ n \end{array} \right) \frac{d^{m-n}}{dx^{m-n}} f\left( x \right) \frac{d^n}{dx^n} g\left( x \right) \tag{65}$$

in the equation

$$\left(\kappa_1^2 - 1\right) \frac{d^2 Z}{d\kappa_1^2} + \kappa_1 \frac{dZ}{d\kappa_1} - \left(1 - k^2\right) Z = 0 \tag{66}$$

where

$$Z = \frac{1}{2} \left[ \left( \kappa_1 + \sqrt{\kappa_1^2 - 1} \right)^{1 - k^2} + \left( \kappa_1 - \sqrt{\kappa_1^2 - 1} \right)^{1 - k^2} \right] . \tag{67}$$

The coefficients obtained for each term must be compared to the ones in the equation obtained from (62) by the tentative function

$$X = \left(1 - \kappa_1^2\right)^a Y \ . \tag{68}$$

Such procedure shows that [5, 16, 17]

$$X = a_{\sqrt{1-k^2}}^{\ell} \sinh^{\ell} \chi \frac{d^{\ell+1} \cosh \sqrt{(1-k^2) \chi^2}}{d \left(\cosh \chi\right)^{\ell+1}}$$
 (69)

where  $a_{\sqrt{1-k^2}}^{\ell}$  is a constant. Just for completeness, it must be cited that a solution of equation (62) in terms of Gegenbauer functions is also possible.

The coordinate system based in the parametrization of a semi-sphere produces three separate differential equations by use of the function  $\Psi(\chi, \theta, \phi) = X(\chi) \Theta(\theta) \Phi(\phi)$ . These equations are equation (62), and the pair

$$\frac{d^{2}\Theta}{d\theta^{2}} - \tan 2\theta \frac{d\Theta}{d\theta} + \left[\ell(\ell+1) + \frac{\lambda^{2}}{\cos 2\theta}\right]\Theta = 0$$
 (70)

and

$$\frac{d^2\Phi}{d\phi^2} - \tan 2\phi \frac{d\Phi}{d\phi} + \left[\ell \left(\ell + 1\right) - \frac{\lambda^2}{\cos 2\phi}\right] \Phi = 0.$$
 (71)

Unfortunately, it is not easy to find analytical solutions for these two last equations, except for the case  $\lambda = 0$ , when the solutions are

$$\Theta = (\cos \theta)^{\frac{1}{4}} \left[ a P_{\frac{1}{4}(2\ell-1)}^{\frac{1}{4}} (\sin \theta) + b Q_{\frac{1}{4}(2\ell-1)}^{\frac{1}{4}} (\sin \theta) \right] , \qquad (72)$$

and

$$\Phi = (\cos \phi)^{\frac{1}{4}} \left[ a' P_{\frac{1}{4}(2\ell-1)}^{\frac{1}{4}} (\sin \phi) + b' Q_{\frac{1}{4}(2\ell-1)}^{\frac{1}{4}} (\sin \phi) \right] . \tag{73}$$

#### 4.2.2 Group of the hyperbolic coordinates

Using a solution of the form

$$\Psi\left(\rho,\omega,\varphi\right) = R\left(\rho\right)\Omega\left(\omega\right)\Phi\left(\varphi\right) \tag{74}$$

one obtains, for the first set of coordinates derivated from the 2-dimensional hyperbolic coordinates, the differential equations

$$\frac{d^2\Omega}{d\omega^2} = -\lambda^2\Omega \; , \quad \frac{d^2\Phi}{d\omega^2} = -\ell^2\Phi \; , \tag{75}$$

two equations whose solutions are given by linear combinations of a sine and a cosine, and

$$\frac{d^2R}{d\rho^2} + \frac{\sinh^2\rho + \cosh^2\rho}{\sinh\rho\cosh\rho} \frac{dR}{d\rho} - \left[ \frac{\lambda^2}{\cosh^2\rho} + \frac{\ell^2}{\sinh^2\rho} - k^2 \right] R = 0 . \tag{76}$$

This last differential equation can be solved by the transformation of coordinates

$$\tau = \cosh^{-2} \rho \Rightarrow \tau^{-2} d\tau = -2 \sinh \rho \cosh \rho d\rho \tag{77}$$

which yields solutions of the kind

$$R = (1 - \tau)^p \tau^q S \tag{78}$$

with

$$p = \frac{\ell}{2} , \quad q = \frac{1}{2} \left[ 1 + \sqrt{1 - k^2} \right]$$
 (79)

and

$$S = aF \left[ p + q + i\frac{\lambda}{2}, p + q - i\frac{\lambda}{2}; 2q; \tau \right]$$

$$+ b\tau^{1-2q}F \left[ p + (1-q) + i\frac{\lambda}{2}, p + (1-q) - i\frac{\lambda}{2}; 2(1-q); \tau \right] , (80)$$

where  $F(\alpha, \beta; \gamma; z)$  is an hypergeometric function. Another equivalent solution, linked to this one by relations between hypergeometric functions, is obtained from the substitution  $\tau' = \tanh^2 \rho$ .

For the second set of this group, named bi-hyperbolic, a function of the form

$$\Psi = \Gamma(\gamma) R(\rho) \Omega(\omega) \tag{81}$$

also divides the problem in three differential equations,

$$\frac{d^2\Omega}{d\omega^2} = -\ell^2\Omega \quad , \tag{82}$$

$$\frac{d^2\Gamma}{d\gamma^2} + \tanh\gamma \frac{d\Gamma}{d\gamma} - \frac{\ell^2}{\cosh^2\gamma} \Gamma = -\lambda^2\Gamma$$
 (83)

and

$$\frac{d^2R}{d\rho^2} + 2\tanh\rho \frac{dR}{d\rho} - \left[\frac{\lambda^2}{\cosh^2\rho} - k^2\right]R = 0.$$
 (84)

The two first equations are equivalent to the equations obtained from (24), and therefore have solutions equivalent to (25) and (26), while the last one produces

$$R = (\cosh \rho)^{-\frac{1}{2}} \left[ a P_{-\frac{1}{2} + \sqrt{1 - k^2}}^{\sqrt{\frac{1}{4} - \lambda^2}} (i \sinh \rho) + b Q_{-\frac{1}{2} + \sqrt{1 - k^2}}^{\sqrt{\frac{1}{4} - \lambda^2}} (i \sinh \rho) \right]$$
(85)

or

$$R = \sqrt{\cosh \rho} \left[ a' P_{-\frac{1}{2} + \sqrt{\frac{1}{4} - \lambda^2}}^{\sqrt{1 - k^2}} (\tanh \rho) + b' Q_{-\frac{1}{2} + \sqrt{\frac{1}{4} - \lambda^2}}^{\sqrt{1 - k^2}} (\tanh \rho) \right] . \tag{86}$$

For the third set of coordinates,  $(\rho, \varsigma, \varphi)$ , named polar hyperbolic, one can use a tentative function of the type  $\Psi(\sigma, \varsigma, \varphi) = R(\rho) \Xi(\varsigma) \Phi(\varphi)$ , and thus obtain equations similar to (82) and (84) for the functions  $\Phi(\varphi)$  and  $R(\rho)$ , respectively, and the equation

$$\frac{d^2\Xi}{d\varsigma^2} + \coth\varsigma \frac{d\Xi}{d\varsigma} - \left(\frac{\ell^2}{\sinh^2\varsigma} - \lambda^2\right)\Xi = 0 , \qquad (87)$$

whose solution is equivalent to the one given in (22).

The fourth set of coordinates of this group, named exponential hyperbolic, allows the use of the function  $\Psi(\sigma, \mu, \rho) = \Sigma(\sigma) M(\mu) R(\rho)$  to produce, beyond eq. (84), the equations

$$\frac{d^2M}{d\mu^2} + \ell^2 M = 0 , (88)$$

equivalent to (82), and

$$\frac{d^2\Sigma}{d\sigma^2} - \frac{d\Sigma}{d\sigma} - \left(e^{2\sigma}\ell^2 - \lambda^2\right)\Sigma = 0 , \qquad (89)$$

whose solution appears in (29).

Finally, the function  $\Psi(\rho, u, v) = R(\rho) U(u) V(v)$  produces for the symmetric hyperbolic coordinates the equations (84) and the pair

$$\frac{d^2U}{du^2} + \tanh 2u \frac{dU}{du} + \left(\lambda^2 - \frac{\ell^2}{\cosh 2u}\right)U = 0 \tag{90}$$

and

$$\frac{d^2V}{dv^2} + \tanh 2v \frac{dV}{dv} + \left(\lambda^2 + \frac{\ell^2}{\cosh 2v}\right)V = 0, \qquad (91)$$

equivalent to (31) e (32).

#### 4.2.3 Group of the exponential coordinates

The metric present in the element of line given by equation (52), of the exponential coordinates, when in combination with the tentative function  $\Psi(\sigma, \mu, \nu) = \Sigma(\sigma) M(\mu) N(\nu)$ , produces from the Helmholtz equation three equations,

$$\frac{d^2M}{d\mu^2} + \ell^2 M = 0 , \quad \frac{d^2N}{d\nu^2} + \left(\lambda^2 - \ell^2\right) N = 0 , \tag{92}$$

solved in terms of sines and cosines, and

$$\frac{d^2\Sigma}{d\sigma^2} - 2\frac{d\Sigma}{d\sigma} - \left(e^{2\sigma}\lambda^2 - k^2\right)\Sigma = 0, \qquad (93)$$

whose solution is

$$\Sigma(\sigma) = e^{\sigma} \left[ bI_{\sqrt{1-k^2}}(\lambda e^{\sigma}) + cK_{\sqrt{1-k^2}}(\lambda e^{\sigma}) \right] . \tag{94}$$

The parametrization with coordinates  $(\sigma, \rho, \varphi)$ , named here polar exponential, produces the equations

$$\frac{d^2\Phi}{d\varphi^2} + \ell^2\Phi = 0 , \qquad (95)$$

solved in terms of sines and cosines,

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(\lambda^2 - \frac{\ell^2}{\rho^2}\right)R = 0, \qquad (96)$$

which is Bessel's equation for the variable  $\lambda \rho$ , and eq. (93).

With the coordinates present in the third parametrization derivated of the 2-dimensional exponential coordinates, named elliptic exponential, one can built a tentative function  $\Psi(\sigma, \mu, \nu) = \Sigma(\sigma) M(\mu) N(\nu)$ , and thus obtain equation (93) and the pair

$$\frac{d^2U}{du^2} - \left(a^2\lambda^2\sinh^2u + \ell^2\right)U = 0\tag{97}$$

and

$$\frac{d^2V}{dv^2} - \left(a^2\lambda^2 \sin^2 v - \ell^2\right)V = 0.$$
 (98)

These two equations are Mathieu's modified equation and Mathieu's equation, respectively, whose solutions are given in terms of Mathieu functions [11].

Finally, one can obtain for the fourth set of coordinates in this group, named parabolic exponential, again equation (93) and the pair

$$\frac{d^2N}{d\eta} - \left(\lambda^2 \eta^2 + \ell^2\right) N = 0 \tag{99}$$

and

$$\frac{d^2\Xi}{d\xi^2} - (\lambda^2 \xi^2 - \ell^2) \Xi = 0 , \qquad (100)$$

solved with the use of parabolic cylinder functions [11, 18].

#### 4.2.4 Symmetric coordinates

The almost symmetric coordinates  $(\chi, u, v)$  produce, by use of the tentative function  $\Psi(\chi, u, v) = X(\chi) U(u) V(v)$ , the equation

$$\frac{d^2X}{d\chi^2} + \lambda^2 X = 0 , \qquad (101)$$

whose general solution is a linear combination of a sine and a cosine, and the pair

$$\frac{d^2U}{du^2} + 2\tanh 2u\frac{dU}{du} + \left(k^2 - \frac{\lambda^2}{\cosh^2 2u} - \frac{\ell^2}{\cosh 2u}\right)U = 0$$
 (102)

and

$$\frac{d^2V}{dv^2} + 2\tanh 2v\frac{dV}{dv} + \left(k^2 - \frac{\lambda^2}{\cosh^2 2v} + \frac{\ell^2}{\cosh 2v}\right)V = 0.$$
 (103)

5 Conclusion 23

These two last equations have simple analytical solutions only when the separation constant  $\ell$  is null, a case in which one has

$$U(u) = (\cosh 2u)^{-\frac{1}{2}} \left[ a P_{-\frac{1}{2}(1-i\lambda)}^{\frac{1}{2}\sqrt{1-k^2}} \left( \tanh 2u \right) + b Q_{-\frac{1}{2}(1-i\lambda)}^{\frac{1}{2}\sqrt{1-k^2}} \left( \tanh 2u \right) \right]$$
(104)

and

$$V(v) = (\cosh 2v)^{-\frac{1}{2}} \left[ a' P_{-\frac{1}{2}(1-i\lambda)}^{\frac{1}{2}\sqrt{1-k^2}} \left( \tanh 2v \right) + b' Q_{-\frac{1}{2}(1-i\lambda)}^{\frac{1}{2}\sqrt{1-k^2}} \left( \tanh 2v \right) \right] . \quad (105)$$

## 5 Conclusion

Specific boundary conditions imposed by a certain problem are, in general, better dealt with by use of one specific general solution of the Helmholtz equation obtained in some particular parametrization. This idea is easily seen when one tries to find the adequate solutions to describe the movements of waves inside a box: of all the 11 coordinate systems suitable for the Euclidean 3-dimensional space only the Cartesian one is perfectly fit for this job.

An interesting problem following this idea consists in finding the modes allowed in a finite flat space known as flat 2-torus, a non-trivial, compact manifold represented by a rectangle with opposite sides identified. Several equal copies of such manifold can be put side by side to fill entirely the Euclidean plane  $E^2$ , in a regular tiling – or tessellation –, and therefore the adequate solutions of the Helmholtz equation in this manifold must be such that they are periodic. It is not hard to see that the adequate solutions are written as plane waves, *i.e.*, linear combinations of sines and cosines of the rectangular Cartesian coordinates x and y.

There are similar problems for the hyperbolic spaces. The representation of the simplest compact 2-dimensional surface of negative curvature, the torus of genus 2, topologically equivalent to a double-doughnut, is built by the identification of pairs of sides of a regular octagon [19]. In the Klein projection the octagon appears with sides formed by segments of straight lines, while in the Poincaré projection it appears as formed by arcs of circles. Another examples of manifolds of this type can be seen in the polyhedra that represent compact 3-dimensional hyperbolic spaces, drawn by the software SnapPea [20] in the Klein projection.

For the hyperbolic case, however, differently from the Euclidean one, the properties of compact -i.e., finite – spaces are far from being completely

5 Conclusion 24

known. Several studies with problems involving these spaces need numerical integrations which demand considerable computational time – see as examples refs. [21, 22, 23]. The use of adequate coordinates could facilitate such studies, as exemplified in a recent paper where a problem of quantum cosmology, previously dealt with by use of crude estimates, is reanalyzed with the use of an adequate coordinate system, in a procedure that allowed also to calculate the volume of non-trivial compact 3-dimensional hyperbolic spaces [9]. Following such line of research, this article, therefore, simply presents a table of basic results that could be of some help for those dealing with the study of hyperbolic spaces, in the hope of motivate new developments in this field, particularly in the unveiling of properties of hyperbolic compact spaces, which are of interest in cosmology.

### Acknowledgments

The author thanks the Brazilian agency FAPESP for the financial support (grant 00/13762-6).

## References

- [1] Lachièze-Rey, M.; Luminet, J.P. Phys. Rep. **254**, 135 (1995).
- [2] Kac, M. Amer. Math. Monthly **73**, 1 (1966).
- [3] Gutiérrez, G.; Yáñez, J.M. Am. J. Phys. 65, 739 (1997).
- [4] Inoue, K.T. Class. Quant. Grav. 16, 3071 (1999).
- [5] Cornish, N.J.; Spergel, D.N. -e-print math/DG 9906017 (1999).
- [6] Morse, P.M.; Feshbach, H. "Methods of Theoretical Physics", vol. I; McGraw-Hill, 1953.
- [7] Eisenhart, L.P. Phys. Rev. 45, 427 (1934).
- [8] Arfken, G. "Mathematical methods for physicists", 2<sup>nd</sup> ed.; Academic Press, 1970.
- [9] Costa, S.S. Phys. Rev. **D62** (4), 047303 (2000).
- [10] Pickover, C.A. "Surfing through hyperspace"; Oxford University Press, New York, 1999.
- [11] Gradshteyn, I.S.; Ryzhik, I.M. "Table of integrals, series, and products", 5<sup>th</sup> ed.; Academic Press, 1994.
- [12] Thurston, W.P. "Three-dimensional geometry and topology", vol. 1, Princeton University Press, 1997.
- [13] Terras, A. "Harmonic analysis on symmetric spaces and applications", vol. I; Springer-Verlag, 1985.
- [14] Frankel, T. "The geometry of physics: an introduction", Cambridge University Press, 1997.
- [15] Kosowsky, A. *e-print* astro-ph 9805173 (1998).
- [16] Cornish, N.J.; Spergel, D.N.; Starkman, G. Phys. Rev. D57, 5982 (1998).
- [17] Hu, W. -e-print astro-ph 9508126 (1995).

- [18] Erdélyi, A. et al. "Higher Transcendental Functions", vol. II; McGraw Hill, New York, 1953.
- [19] Balazs, N.L.; Voros, A. Phys. Rep. 143, 109 (1986).
- [20] SnapPea, free software obtained from www.northnet.org/weeks.
- [21] De Lorenci, V.A. et al. Phys. Rev. **D56** (2), 3329 (1997).
- [22] Cornish, N.J.; Turok, N.G. Class. Quantum Grav. 15, 2669 (1998).
- [23] Inoue, K.T. -e-print astro-ph 0103158 (2001).