UNIFORM TILINGS OF THE HYPERBOLIC PLANE

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Abstract. A uniform tiling of the hyperbolic plane is a tessellation by regular geodesic polygons with the property that each vertex has the same *vertex-type*, which is a cyclic tuple of integers that determine the number of sides of the polygons surrounding the vertex. We determine combinatorial criteria for the existence, and uniqueness, of a uniform tiling with a given vertex type, and pose some open questions.

1. Introduction

A *tiling* of the hyperbolic plane is a partition into regular geodesic polygons (the *tiles*) which are non-overlapping (interiors are disjoint) and such that tiles which touch, do so either at exactly one vertex, or along exactly one edge. The *vertex-type* of a vertex v is a cyclic tuple of integers $[k_1, k_2, \ldots, k_d]$ where d is the degree (or valence) of v, and each k_i (for $1 \leq i \leq d$) is the number of sides of the i-th of the d polygons in *counter-clockwise order* around v. A *uniform* tiling is one in which the vertex-type is identical for each vertex (see Figure 1). A pair of tilings are identical if they are combinatorially isomorphic, that is, there is an orientation-preserving homeomorphism of the plane to itself that takes vertices, edges and tiles of one tiling to those of the other.

Uniform tilings of the *Euclidean* plane have been studied from antiquity, and it is known that there are exactly eleven such tilings, up to scaling (see [DM], and [GS77] for an informative survey). Also classical is the fact that there are 14 such uniform tilings of the round sphere, along with two infinite families (the prisms and antiprisms). Thirteen of these tilings have an automorphism group that is vertex-transitive, and are the famed *Archimedean solids*.

For a tiling of the hyperbolic plane, it is easy to verify that the vertex-type $\mathsf{k} = [k_1, k_2, \ldots, k_d]$ of any vertex must satisfy

$$\alpha(\mathsf{k}) = \sum_{i=1}^{d} \frac{k_i - 2}{k_i} > 2 \tag{1}$$

since the sum of the interior angles of a regular hyperbolic polygon is strictly *less* than those of its Euclidean counterpart. We shall call $\alpha(\mathsf{k})$ the *angle-sum* of the cyclic tuple k.

It is well-known that there are infinitely many examples of uniform tilings with a vertex-type k satisfying the above geometric condition. Indeed, the Fuchsian triangle groups $G(p, q)$ generated by reflections on the sides of a hyperbolic triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{p}$ and $\frac{\pi}{q}$ generate a uniform tiling with vertex-type $[p^q] = [\underbrace{p, p, \ldots, p}_{\text{q times}}]$ whenever $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$ (see [EEK82]).

A basic question then is:

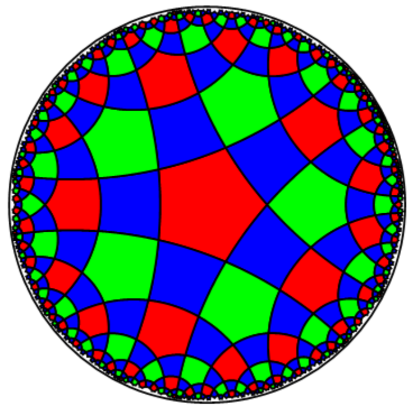

FIGURE 1. A uniform tiling of the hyperbolic plane with vertex-type $\mathsf{k} = [4,5,4,5]$, a tuple that satisfies conditions (A) and (B). Theorem 1.2 asserts that in fact, this is the unique uniform tiling with this vertex-type.

Question 1.1. *Which cyclic tuples of integers* $\mathsf{k} = [k_1, k_2, \ldots, k_d]$ *satisfying* $\alpha(\mathsf{k}) > 2$ *is the vertex-type for a uniform tiling of the hyperbolic plane? Which vertex-types have a unique such tiling?*

Apart from the condition concerning the angle-sum, there is an immediate necessary *combinatorial* condition for the vertex-type of a uniform tiling, that arises from our requirement that the polygons corresponding to the integers in k appear in *counter-clockwise* order around each vertex. Namely,

(A) *if an integer x follows an integer y in this cyclic tuple* $[k_1, k_2, \ldots, k_d]$, *that is, if xy appears, then so does yx.*

Indeed, if two polygons P_1 and P_2, with numbers of edges x and y respectively, share an edge, then they appear in different cyclic orders at the two endpoints of the common edge and contribute the two pairs xy and yx to the vertex-type (see Figure 2).

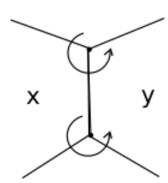

FIGURE 2. Adjacent polygons with x and y sides appear in different orders around the endpoints of the common edge.

The case when the degree $d = 3$, that is, when there are exactly three tiles around each vertex, is handled completely in Theorem 1.5 below. For the case when degree $d \geq 4$, we introduce another simple combinatorial condition:

(B) *if xy and yz appear in the cyclic tuple* $[k_1, k_2, \ldots, k_d]$, *then so does xyz.*

(We say uvw "appears" in a cyclic tuple k if the integers u, v, w appear in consecutive order in k, and similarly for tuples of other lengths.)

We prove the following *sufficient* criterion for the existence of "triangle-free" tilings:

Theorem 1.2 (Existence criteria - I)**.** *Consider a cyclic tuple* $\mathsf{k} = [k_1, k_2, ..., k_d]$ *such that*

- *the angle-sum* $\alpha(\mathsf{k}) > 2$*,*
- *conditions (A) and (B) are satisfied,*
- $d \geq 4$*, and each* $k_i \geq 4$*.*

Then there exists a uniform tiling of the hyperbolic plane with vertex-type k*. Moreover, this tiling is unique (up to an isomorphism) if any two consecutive elements of the cyclic tuple uniquely determine the rest of the tuple.*

Remark. It is a folklore result that for a general vertex-type, uniqueness does not hold. Indeed, in §4, we shall show that there are *uncountably many* pairwise distinct uniform tilings of the hyperbolic plane with the same vertex type $\mathsf{k} = [4, 4, 4, 6]$. Note that such a cyclic tuple does *not* satisfy the criterion for uniqueness.

To handle the case of tilings with triangular tiles, we need to consider degree $d \geq 6$, and an additional combinatorial condition:

(C) *if the triples* $x3y$ *and* $3yz$ *appear in the cyclic tuple* $[k_1, k_2, \ldots, k_d]$*, then so does the* 4*-tuple* $x3yz$*.*

We shall prove:

Theorem 1.3 (Existence criteria - II)**.** *Consider a cyclic tuple* $\mathsf{k} = [k_1, k_2, ..., k_d]$ *such that*

- *the angle-sum* $\alpha(\mathsf{k}) > 2$*,*
- *conditions (A), (B) and (C) are satisfied, and*
- $d \geq 6$*.*

Then there exists a uniform tiling of the hyperbolic plane with vertex-type k*. Moreover, this tiling is unique (up to an isomorphism) if any two consecutive elements of the cyclic tuple uniquely determine the rest of the tuple.*

As a corollary, we obtain (see the end of §3) the uniqueness of the uniform tilings generated by the Fuchsian triangle-groups mentioned above:

Corollary 1.4. *A uniform tiling of the hyperbolic plane with vertex type* $[p^q]$ *(where* $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$*) is unique, that is, any pair of such tilings are related by an orientation-preserving isometry of the hyperbolic plane, that takes vertices and edges of one to those of the other.*

In §5, we provide the following necessary and sufficient conditions for the existence of uniform tilings with degree $d = 3$:

Theorem 1.5. *A cyclic tuple* $\mathsf{k} = [k_1, k_2, k_3]$ *is the vertex-type of a uniform tiling of the hyperbolic plane if and only if one of the following hold:*

- $\mathsf{k} = [p, p, p]$ *where* $p \geq 7$*, or*
- $\mathsf{k} = [2n, 2n, q]$ *where* $2n \neq q$*, and* $\frac{1}{n} + \frac{1}{q} < \frac{1}{2}$*.*

We now mention some questions that are still open.

First, our construction in Theorem 1.2 yields tilings that can have different symmetries, that is, could be invariant under different Fuchsian groups (or none at all) some of which have *compact quotients*.

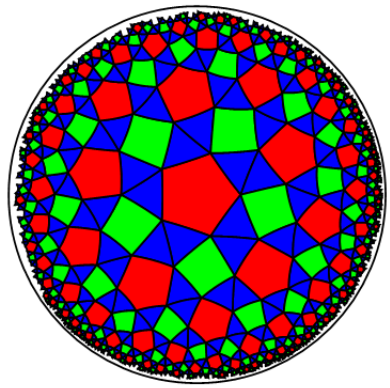

FIGURE 3. A uniform tiling with vertex-type [5,3,4,3,3]. This cyclic tuple does not satisfy condition (B).

Thus, we can ask:

Question 1.6. *Given a tuple* $\mathsf{k} = [k_1, k_2, \ldots, k_d]$*, which compact oriented hyperbolic surfaces have a uniform tiling with vertex-type* k*? How many such tilings does such a surface have?*

Remark. For the vertex-type $[p^q]$, the question above was answered in [EEK82], where it was shown that such a tiling exists whenever the appropriate Euler characteristic count holds. The case when the surface is a torus or Klein bottle, and the vertex-type is that of a Euclidean tiling, was dealt with in [DM17]. The work in [KN12] enumerates uniform tilings of surfaces of low genera which have a vertex-transitive automorphism group.

Second, the existence criterion (B) in Theorem 1.2 is not *necessary* – see, for example, Figure 3. It should be possible, albeit tedious, to enumerate all possible vertex-types for uniform tilings of degrees $d = 4$ or 5 (see [Mit]). For general degree d, not *all* tuples k that satisfy the angle-sum condition (1) can be realized by a uniform tiling – an example of this, that can be checked easily, is the tuple $\mathsf{k} = [3, 3, 4, 4, 3, 3, 4, 4]$. Note that this k also does not satisfy condition (B).

However, a set of necessary and sufficient conditions akin to conditions (A), (B) and (C) seems elusive. Although the work in [Ren08] develops algorithms for some related problems, we do not know if the answer to the following question is known:

Question 1.7. *Is there a finite-state automaton to test if a given tuple* k *is the vertex-type of a uniform tiling of the hyperbolic plane?*

2. A TILING CONSTRUCTION

In this section we prove Theorem 1.2, by describing a constructive procedure to tile the hyperbolic plane uniformly. As we shall see, our method shall work provided the cyclic tuple k satisfies the hypotheses of Theorem 1.2, including Conditions (A) and (B) mentioned in the introduction. Moreover, the algorithm will be free of choices under an additional hypothesis, proving the uniqueness statement in that case.

2.1. **Initial step: a fan.** We shall begin with region X_0 of the hyperbolic plane tiled by exactly d polygons of sides $k_1, k_2, \ldots, k_d$ around an initial vertex V. In what follows, we shall call such a configuration of tiles around a vertex a *fan*.

A standard continuity argument implies:

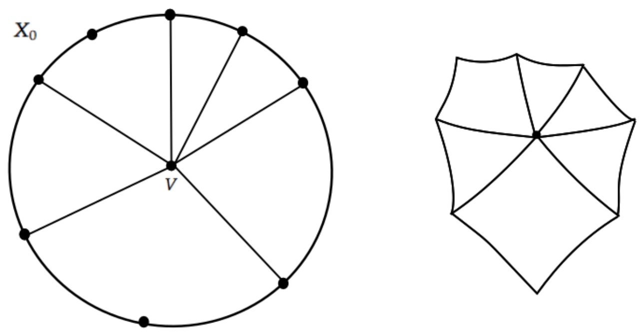

FIGURE 4. A representation of a fan for the vertex type [4,3,3,3,4,3]. The hyperbolic realization of this fan is shown on the right.

Lemma 2.1. *There is a unique choice of a side-length $l_0 > 0$ for the polygons in X_0 such that the total angle around the vertex V is exactly 2π.*

Proof. For sufficiently small $l > 0$, a regular hyperbolic polygon of k_i sides and side length l will be approximately Euclidean, and each interior angle will be close to $\pi(k_i - 2)/k_i$. This makes the total angle $\alpha(l)$ at vertex V close to $\pi \sum_{i=1}^{d} \frac{k_i - 2}{k_i} > 2\pi$, since the vertex-type satisfies the angle-sum condition to be a hyperbolic tiling. On the other hand, for large $l \gg 0$, as the vertices of the regular polygons tend to the ideal boundary, each interior angle will be close to 0, since for any ideal polygon adjacent sides bound cusps. The total angle $\alpha(l)$ is then close to 0. In fact, elementary hyperbolic trigonometry shows that α is a strictly monotonic function of l. Hence, there is a unique intermediate value $l_0 \in \mathbb{R}^+$ for which $\alpha(l) = 2\pi$. □

Remarks. (i) Throughout, we shall use polygons with side-lengths equal to the l_0 obtained in previous lemma. Moreover, we shall represent a fan as a disk with an appropriate division into wedges, together with vertices added to resulting boundary arcs to add more sides – see Figure 4.

(ii) Question 1.1 is equivalent to asking:

Question 2.2. *When can a hyperbolic fan be extended to a uniform tiling of the hyperbolic plane?*

It is easy to see that the following two properties hold for the tiled region X_0:

Property 1. *All boundary vertices have valence 2 or 3. Moreover, there is at least one boundary vertex of valence 2.*

(Note that the second statement follows from the property that not all tiles are triangles, which is weaker than our assumption of a triangle-free tiling.)

Property 2. *The tiled region is homeomorphic to a disk.*

2.2. **Inductive step.** The tiling is constructed layer by layer, namely, we shall find a sequence of tiled regions

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_i \subset X_{i+1} \subset \cdots \tag{2}$$

such that their union is the entire hyperbolic plane, the interior vertex of each X_i has vertex type $[k_1, k_2, \ldots, k_d]$.

In the following construction, we shall describe how X_{i+1} is obtained from X_i by adding tiles around each boundary vertex of X_i (that is, completing a fan) , such that each boundary vertex of X_i becomes an interior vertex of X_{i+1}. Informally speaking, the tiles added to construct X_{i+1} form a "layer" around X_i; the tiling of the hyperbolic plane is thus built by successively adding concentric layers.

We shall now describe the inductive step of the construction, namely, how to add tiles to expand from X_i to X_{i+1}.

In the construction, we shall assume that Properties 1 and 2 (see §2.1) hold for X_i. As we saw, these were true for $i = 0$, namely for the tiled region X_0, and shall verify it for X_{i+1} when we complete the construction.

As a consequence of Property 2, the boundary ∂X_i is a topological circle. Let the boundary vertices be $v_0, v_1, \ldots, v_n$ in a counter-clockwise order. Note that the number n of vertices is certainly dependent on i, and in fact grows exponentially with i, but we shall suppress this dependence for the ease of notation.

Moreover, we shall choose this cyclic ordering such that v_0 is a vertex of valence 3, and v_n is a vertex of valence 2. (This is possible because Property 1 holds.)

Completing a fan at v_0. We begin by adding tiles to complete the fan F_0 at v_0.

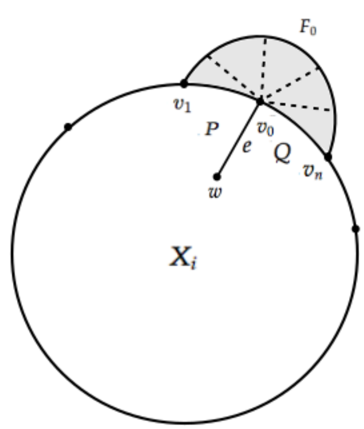

FIGURE 5. The fan F_0 is completed at the boundary-vertex v_0 of X_i. The added wedge is shown shaded.

Recall that v_0 has valence 3. Hence, v_0 is the common vertex of two polygons P and Q in X_i. The topological operation of adding the fan can be viewed as follows: consider a semi-circular arc centered at v_0 in the exterior of X_i and with endpoints at v_1 and v_n. We add $d-2$ "spokes" to the resulting "wedge" containing v_0: this results in a fan around v_0 comprising the initial polygon P, and exactly $d-2$ triangles. Finally, we add more valence 2 vertices to the sub-arcs of the boundary of the wedge, if need be, in order to obtain d polygons with the desired number of sides and cyclic order prescribed by the vertex-type.

The fact that we can do so requires Condition (A): Namely, let e be the common edge between P and Q, one of whose endpoints is v_0, and the other w. (See Figure 5.) Note that if the number of sides of P and Q are x and y respectively, then yx appears in that order for vertex-type for the vertex w. Condition (A) then ensures that xy appears in the cyclic

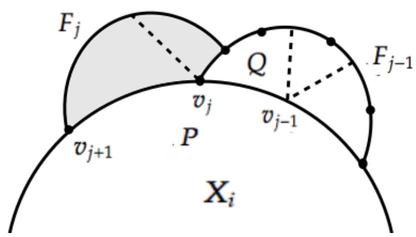

FIGURE 6. The fan F_j added at v_j, that has valence 3 after F_{j-1} was added.

vector $[k_1, k_2, \ldots, k_d]$. Hence, there is indeed a choice of polygons that completes the fan around v_0.

Completing a fan at $v_1, v_2, \ldots, v_{n-1}$**.** We then successively complete fans F_j at v_j for $1 \leq j \leq n-1$ as follows. Assume we have completed fans at $v_0, v_1, \ldots, v_{j-1}$. At the j-th stage, there are two cases:

Case I. The vertex v_j has valence 3. This implies that v_j had valence 2 in X_i; the additional edge incident to v_j comes from the fan F_{j-1} added at v_{j-1}. Let P be the polygon in X_i that has v_j as a vertex, and let Q be the polygon in the fan F_{j-1} that has v_j as a vertex. Note that the edge between v_{j-1} and v_j is the common edge of P and Q. To describe the fan F_j topologically, draw an arc in the exterior of $X_i \cup F_0 \cup F_1 \cup \cdots \cup F_{j-1}$, between v_{j+1} and the vertex in ∂F_{j-1} adjacent to v_j. (See Figure 6.) Divide this wedge region into $d-3$ triangles by adding spokes, and as before, add an appropriate number of vertices to the resulting circular arcs to have polygons with more than three sides. Note that if the number of sides of P and Q are x and y respectively, then yx appears in the vertex type of v_{j-1}, and Condition (A) ensures that xy appears in the cyclic vector $[k_1, k_2, \ldots, k_d]$. Hence there is indeed a choice of such polygons that completes the fan F_j.

Case II. The vertex v_j has valence 4. Then v_j has valence 3 in X_i. Let P and Q be the polygons in X_i sharing the vertex v_j, and having x and y sides respectively. Then xy appears in the vertex type of v_j. Let R be the polygon in the fan F_{j-1} that also has v_j as a vertex, and let z be the number of its sides. (See Figure 7.) Note that by the assumption that the degree $d \geq 4$, the polygons P, Q, R cannot be the only polygons around v_j in the final tiling; our task is to show that we can add more to complete the fan at v_j.

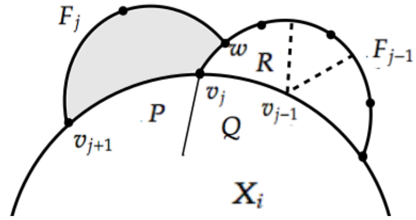

FIGURE 7. The fan F_j added at a v_j of valence 4: here we need Condition (B).

Claim 1. The triple xyz appears in the cyclic tuple $\mathsf{k} = [k_1, k_2, \ldots, k_d]$.

The edge between v_{j-1} and v_j is common between Q and R, and the vertex-type at v_{j-1} includes zy. By condition (A), the tuple k will have yz also. We have already seen above

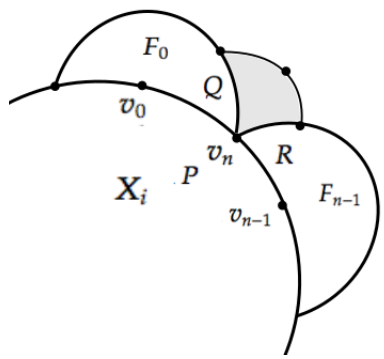

FIGURE 8. The final fan F_n involves adding a wedge between F_0 and F_{n-1}.

that k contains xy. Hence, by condition (B), the triple xyz appears in k. This proves Claim 1.

Hence, we can choose numbers of sides of successive polygons to follow P, Q and R around v_j, to complete a fan F_j. This we do by adding and subdividing a wedge, just as in Case I.

Completing a fan at v_n. Finally, we need to complete the final fan F_n around v_n. Note that we had chosen v_n to have valence 2 in X_i; after adding the fans F_j for $0 \leq j \leq n-1$, v_n has valence 4 in $X_i \cup \left(\bigcup_{j=0}^{n-1} F_j\right)$, where the two additional edges belong to F_0 and F_{n-1}. Consider the three polygons Q, P and R in counter-clockwise order around v_n, where Q is a polygon in F_0, P is a polygon in X_i, and R is a polygon in F_n, each having v_n as a vertex. Let P, Q, R have x, y, z sides respectively. (See Figure 8.)

Claim 2. The triple yxz appears in the cyclic tuple $\mathsf{k} = [k_1, k_2, \ldots, k_d]$.

Since Q and P share the edge between v_n and v_0, and since they are successive polygons in counter-clockwise order in the fan of v_0, xy appears in the vertex-type $[k_1, k_2, \ldots, k_d]$. Similarly, R and Q share the edge between v_{n-1} and v_n, and are successive polygons in the fan of v_{n-1}, and so zx appears in the vertex-type. Applying condition (A), both yx and xz appears in the vertex-type, and hence by condition (B), so does yxz. This completes the proof of Claim 2.

Hence by adding a wedge based at v_n between the fans F_0 and F_{n-1}, and subdividing into polygons, there is a choice of numbers of sides such that the successive polygons Q, P, R are completed to a fan F_n. Note that once again, we have implicitly used the assumption that $d \geq 4$, as in the process we are adding at least four polygons around v_n.

Verifying Properties 1 and 2. We can now define

$$X_{i+1} := X_i \cup F_0 \cup F_1 \cup \cdots \cup F_n, \tag{3}$$

where by construction, all the boundary vertices $v_0, v_1, \ldots, v_n$ are in the interior, and every interior vertex has vertex-type k.

By the inductive hypothesis, X_i is topologically a disk, and by construction, at each step of adding a fan at a boundary vertex, one is adjoining a simply-connected wedge to an arc of the boundary. Hence $X_i \cup F_0 \cup F_1 \cup \cdots \cup F_j$ is topologically a disk for each $j = 0, 1, 2, \ldots, n$, establishing Property 2 for X_{i+1}.

To check Property 1 for X_{i+1}, notice that the new boundary vertices are the vertices of the fans $F_0, F_1, \dots, F_n$ that lie on the boundary arcs of the wedges that we added. Vertices that lie in the *interior* of such a boundary arc have valence 2 or 3, like for an isolated fan X_0. There must be one such vertex in the boundary of the wedge that is added, since there cannot be a triangular tile. Such a vertex will be of valence 2, verifying the second statement of Property 1.

Now a vertex w that lies at the intersection of two adjacent wedges, say for F_j and F_{j-1}, is the endpoint of the edge from v_j to w that is common to F_j and F_{j-1}. (See Figure 7.) Note that there cannot be an edge from w to v_{j+1}, since then $v_{j+1}wv_j$ will form a triangular tile, contradicting our assumption that our tiling is triangle-free. Similarly, there cannot be an edge from w to v_{j-1}. Hence this boundary vertex w is of valence 3.

Thus, all boundary vertices of X_{i+1} have valence 2 or 3, verifying the first statement of Property 1 also.

2.3. **The endgame.** It only remains to show:

Lemma 2.3. *The union of the tiled regions X_i for $i \geq 0$ is the entire hyperbolic plane.*

Proof. Clearly, the initial fan X_0 contains disk of hyperbolic radius $r_0 > 0$ centered at the central vertex V.

Claim 3. There is an $r > 0$ such that any point on the boundary of X_{i+1} is at least a distance r from X_i, for each $i \geq 0$.

It is enough to verify this for the portion α of the boundary of the fan F_j that we completed at the vertex v_j, that is disjoint from the adjacent fans F_{j-1} or F_{j+1} (where the indices are taken modulo n). For a fixed vertex-type k, there are only finitely many configurations of hyperbolic polygons to check, and hence there is a minimum such distance of α from the vertex v_j. This proves Claim 3.

Thus, for each $i \geq 0$, the distance of the boundary of X_i from V is at least $r_0 + i \cdot r$, and hence the region X_i includes a disk of hyperbolic radius $r_0 + i \cdot r$ centered at V. As $i \to \infty$, the radius tends to infinity, and hence we cover the entire hyperbolic plane. This proves the lemma. ◻

This completes the proof of the existence statement of Theorem 1.2.

The uniqueness statement of Theorem 1.2 follows from the fact that if the vertex-type $\mathsf{k} = [k_1, k_2, \dots, k_d]$ has the stated property, then there is a unique way of completing the fan F_j for each $0 \leq j \leq n$ in our construction. This is because at any such step, the partial fan that was already at v_j, had (at least) two polygons P and Q already in place around it. If x and y are the number of sides of P and Q respectively, then this determines two successive elements x and y in k. Then by the assumed property of k, there is a unique sequence of polygons that can follow P and Q in the final fan. This determines a unique way of choosing the subdivision of the added wedge to determine these polygons. Hence the i-th stage of the construction (expanding from X_i to X_{i+1}) is determined uniquely for each $i \geq 0$, and thus the final tiling is determined uniquely.

3. Handling triangular tiles: Proof of Theorem 1.3

Suppose we now have a vertex-type $\mathsf{k} = [k_1, k_2, \dots, k_d]$ that satisfies the hypotheses of Theorem 1.3. This time, $d \geq 6$, but we could have $k_i = 3$ for some (or all) $i \in \{1, 2, \dots, d\}$.

Constructing the exhaustion. The construction is the inductive procedure same as before: we start with a fan X_0 around a single vertex V, and proceed to build a sequence of tiled regions

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_i \subset X_{i+1} \subset \cdots$$

which exhausts the hyperbolic plane, and such that each interior vertex of X_i has vertex-type k, for each $i \geq 0$.

In what follows we shall point out some of the additional considerations that we handle in this new case, that differs from the proof of Theorem 1.2 in §3.

The key difference is this time a vertex of valence 4 may appear on the boundary of a tiled region X_{i+1} after completing the fans for the boundary vertices of X_i. (See Figure 11.)

Each region X_i shall satisfy Property 2 as before, but the following different analogue of Property 1:

Property $1'$**.** *The following properties hold for X_i:*

(i) *All boundary vertices in ∂X_i have valence $2, 3$ or 4 in X_i.*
(ii) *Any boundary vertex $v \in \partial X_i$ of valence 4 is the vertex of a triangular tile in X_i that intersects the boundary ∂X_i only at v.*
(iii) *Finally, either there is a boundary vertex of valence 2, or there is a boundary edge that belongs to a triangular tile.*

Is it easy to see that X_0 satisfies (i) and (ii) above. Also, (iii) is vacuously true as ∂X_0 does not have any vertex of valence 4. As mentioned, valence 4 vertices may arise on the boundary of X_i for $i \geq 1$ because of the presence of triangular tiles.

As before, the boundary ∂X_i is a topological circle because of Property 2, and we denote the boundary vertices of ∂X_i by $v_0, v_1, \ldots, v_n$ in counter-clockwise order. We also require that:

(a) v_0 has valence 3 or valence 4.
(b) One of the two hold:
  - Either v_0v_n is a boundary edge that belongs to a triangular tile, or
  - v_n has valence 2 in X_i.

This is possible for X_0 since if $\mathsf{k} = [3^d]$, then each boundary vertex has valence 3, and the first condition of (b) holds. Otherwise, as in §3, we can in fact choose v_0 to have valence 3 and v_n to have valence 2, that is, satisfying the second condition of (b). For X_i where $i \geq 1$, we shall verify that such a choice of v_0 and v_n is possible at the end of the inductive step.

In what follows we shall complete fans F_j around v_j for each $0 \leq j \leq n$ as before, and define

$$X_{i+1} := X_i \cup F_0 \cup F_1 \cup \cdots \cup F_n \tag{4}$$

for each $i \geq 0$.

Completing the fan at v_0**.** When the valence of v_0 is 3, we complete the fan F_0 around v_0 exactly as in the construction for Theorem 1.2. When the valence of v_0 equals 4, then the three polygons of X_i around v_0 have number of sides $x, 3$ and y because of part (ii) of Property $1'$.

We need to ensure that we can continue placing polygons around v_0 to complete a fan, that is, we need to prove:

Claim 4. $x3y$ is a consecutive triple in the cyclic tuple $\mathsf{k} = [k_1, k_2, \ldots, k_d]$.

Let e and f be the two edges in the interior of X_i that are incident on v_0, and let w and w' be their other endpoints. Note that part (iii) of Property 1′ implies that e and f are in fact, two sides of a triangular tile. Suppose P and Q are the other polygons in X_i with v_0 as a vertex, having x and y sides respectively. Then considering the vertices w and w' that lie in the interior of X_i (and consequently have vertex-type k) we see that the pairs $3x$ and $y3$ must appear in the tuple k. By Condition (A), this implies that $x3$ and $3y$ appear in k, and by Condition (B), so does $x3y$. This completes the proof of Claim 4.

Then, as before, we can add a wedge at v_0 in the exterior of X_i, and subdivide into polygons by adding spokes and vertices on the resulting boundary arcs of the wedge, having the numbers of sides that determine the rest of the tuple k following $x, 3$ and y. This completes the fan F_0 at v_0.

Completing the fan at v_j**.** Now suppose we have completed fans around $v_0, v_1, \ldots, v_{j-1}$ for $1 \leq j \leq n-1$, and we need to complete the fan F_j at v_j.

As before, our analysis divides into cases depending on the valency of the vertex v_j in X_i. When v_j has valence 3 or 4 in the already-tiled region $X_i \cup F_0 \cup F_1 \cup \cdots \cup F_{j-1}$, the completion of the fan F_j proceeds exactly as in the corresponding step in the proof of Theorem 1.2. The new case is when v_j has valence 5, that is, when it had valence 4 in X_i (before the other fans were completed). In this case, there are three polygons P, T, and Q in X_i which shares a vertex v_j, where T is a triangular tile, and there is another polygon R in the fan F_{j-1} that has v_j as a vertex. (See Figure 9.)

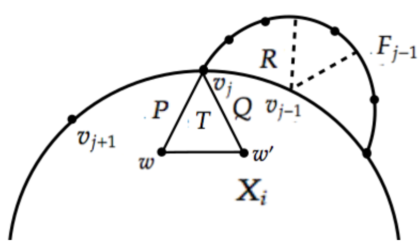

FIGURE 9. Completing a fan F_j at a vertex v_j of valence 5 .

Thus, around v_j, there are polygons P, T, Q, R, in that counter-clockwise order. If the corresponding numbers of sides are $x, 3, y$ and z, in order to be able to complete a fan at v_j with vertex-type k, we need to show:

Claim 5. The 4*-tuple* $x3yz$ *appears as consecutive elements in* k*.*

This is where we shall use Condition (C). Let w and w' be other endpoints of the tirangular tile T that has v_j as a vertex. Note that w, w' both have vertex-type k by the inductive hypothesis, since they lie in the interior of X_i. Then, since the polygons T and P appear in counter-clockwise order around w, the pair $3x$ appears in k. Similarly, considering the polygons around w', we see that the pair $y3$ appears in k. Using Conditions (A) and (B), we deduce, exactly as in a previous claim, that the triple $x3y$ appears in k. Now the vertex v_{j-1} has the polygons R and Q in counter-clockwise order around it, so the pair zy also appears in k. Applying the sam argument involving Conditions (A) and (B), we conclude that $3yz$ appears in k. Finally, since the triples $x3y$ and $3yz$ are in k, an application of Condition (C) proves the claim. This completes the proof of Claim 5.

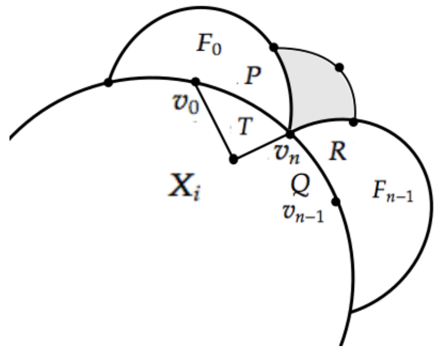

FIGURE 10. Completing the final fan in the case when v_0v_n is the edge of a triangular tile T.

This allows a wedge to be added at v_j, and divided into polygons, so that the polygons P, T, Q and R are part of a fan F_j of vertex-type k that is thus completed around v_j.

The final fan. To complete the fan F_n at the remaining boundary-vertex v_n, we would need to add a wedge that goes between the fans F_0 and F_{n-1}, subdivide into polygons.

The case when v_n had valence 2 in X_i is exactly as in the case of completing the final fan in the proof of Theorem 1.2.

The remaining case is when v_0v_n is an edge of a triangular tile T: in this case the polygons in $X_i \cup F_0 \cup F_1 \cup \cdots \cup F_{n-1}$ that share a vertex with v_n are P (which is part of F_0), T and Q (which are part of X_i) and R (which is part of F_{n-1}), where P, T, Q and R are in counter-clockwise order around v_n. (See Figure 10.) Suppose the numbers of vertices of P, Q and R are x, y and z respectively. Then, by exactly the same argument as in Claim 5, we have that $x3yz$ belongs to the vertex-type k, and hence there is indeed a completion of these four polygons to a fan F_n at v_n.

This completes the new tiled region X_{i+1}.

Verifying Property $1'$ and completing the proof. By construction, all interior vertices of X_{i+1} have vertex-type k, and it is easy to see that Property 2 holds, namely, X_{i+1} is homeomorphic to a disk.Thus, it only remains to verify Property $1'$, which is where the degree condition $d \geq 6$ is used.

The key observation is that when the fans F_j (for $0 \leq j \leq n$) are added to X_i, then the following holds:

Claim 6. A portion of the wedge added while completing F_j lies on the boundary of X_{i+1}. In other words, $F_j \setminus (F_{j-1} \cup F_{j+1})$ is not empty.

For example, when a fan F_j is added to a boundary vertex $v_j \in \partial X_i$ having valence 4 in X_i, then there are already four polygons around v_j in $X_i \cup F_0 \cup F_1 \cup \cdots \cup F_{j-1}$, and the added wedge (to complete the fan F_j) needs to havr at least one spoke, since the total number of polygons need to be at least 6. If q is the endpoint of the first spoke (in counter-clockwise order around v_j), then $F_{j+1} \cap F_j$ cannot contain the portion of the wedge boundary that lies between q and F_{j-1}. Hence this portion of the boundary of F_j is on the boundary of X_{i+1}.

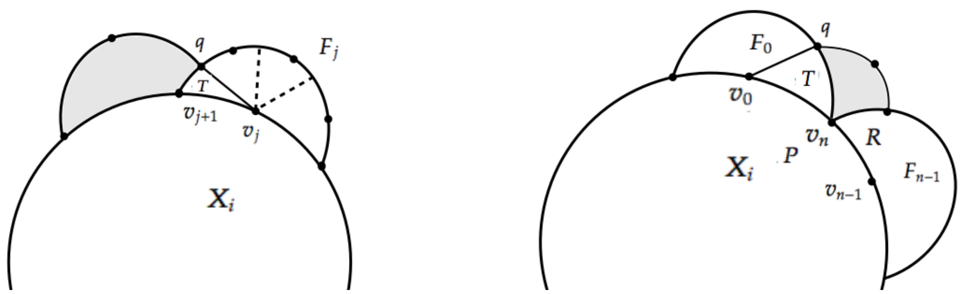

FIGURE 11. Ways that a valence 4 vertex q can appear in the boundary of X_{i+1} while completing the fans. Note that a triangular tile T is always involved.

The same holds for the other cases (when v_j has valencies 2 or 3); note that then the added wedge needs to be divided with even more spokes, to have a final valence at least 6. This proves Claim 6.

Recall now that Property 1′ had three parts.

Proof of (i) and (ii). A valence 4 vertex is created in the boundary of X_{i+1} when the fan F_j (for $0 \leq j \leq n-1$) has a triangular tile T in the subdivided wedge, one of whose edges is $v_j v_{j+1}$. In that case, if q is the other vertex of T, then q lies in the boundary of X_{i+1}, and is disjoint from F_{j-1}, by the claim above. Moreover, it has valence 4 on X_{i+1}, since the edge qv_{j+1} will be shared by a polygon in the wedge added at v_{j+1} to complete the next fan F_{j+1}. (See Figure 11.) The only other case when a valence 4 vertex appears in the boundary of X_{i+1} is when the fan F_0 has a triangular tile in the added wedge that has side $v_0 v_n$. Then, the other vertex q of T lies in the boundary of X_{i+1}, and has valence 4 in X_{i+1} when the wedge (for F_n) is added at v_n. In both these cases, the triangular tile T lies in the interior of X_{i+1}, and (ii) is satisfied.

In all other cases, when completing a fan, the extreme points of the boundary of any added wedge has valence 3. Note that if a portion α of the boundary of an added wedge lies in the boundary of X_{i+1}, then α is also a portion of the boundary of a fan, with no other edges from it to other parts of X_{i+1}, and hence all vertices that lie in α have valence either 2 or 3. This proves (i).

Proof of (iii). Finally, recall that the subdivision of the wedge into polygons involves adding spokes, and then, in the case of non-triangular tiles, adding valence 2 vertices to the resulting boundary arcs to achieve the desired number of sides. We claim above implies that there is a portion α of an added wedge that lies in the boundary of X_{i+1}. This boundary arc α is either the boundary of (one or more) triangular tiles tiling the wedge, or there is some polygon in the added wedge having a number of sides greater than 3. In the latter case, the subdivision procedure implies that there is a vertex in α, and consequently in the boundary of X_{i+1}, that has valence 2. In the former case, there is a boundary edge of X_{i+1} that belongs to a triangular tile. This proves (iii).

Thus X_{i+1} satisfies Property 1′, and this completes the inductive step.

Finally, we verify that we can choose an ordering of the new boundary vertices $v'_0, v'_1, \ldots v'_n$ of X_{i+1} such that v'_0 and v'_n satisfy (a) and (b) stated after Property 1′. In fact, these successive vertices v'_n, v'_0 can be chosen to lie along the boundary of the final wedge added while completing F_n. Indeed, an extreme point w (in counter-clockwise order) in the boundary of such a wedge, that also belongs to F_0, has valence 3 or 4. This satisfies (a). The vertex v in the wedge boundary that precedes w either has valence 2, in case the edge wv belongs to a polygon in the wedge having more than three sides, or else wv is an edge of a triangular tile in the added wedge. This satisfies (b), and thus, the vertices w and v can be taken to the first and last vertices (v_0' and v_n') respectively, in our new counter-clockwise ordering of the boundary vertices of X_{i+1}.

Thus, we get a sequence of nested tiled regions

$$X_0 \subset X_1 \subset \cdots \subset X_i \subset X_{i+1} \subset \cdots$$

such that any interior vertex of X_i has vertex-type k. Lemma 2.3 still applies (its proof did not use the hypotheses of Theorem 1.2), and this sequence of regions exhausts the hyperbolic plane, defining the desired uniform tiling.

If k satisfies the property that any pair of consecutive elements of k determines the rest of the cyclic tuple uniquely, then as before, the completions of the fans F_j for $0 \leq j \leq n$ is uniquely determined. This is because at least two polygons are in place around the boundary vertex, and therefore allows a unique way of adding polygons to complete the fan. The construction of the uniform tiling is thus determined uniquely.

Proof of Corollary 1.4. We first show that two uniform tilings T and T' with the same vertex-type $\mathsf{k} = [p^q]$ where $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$ are combinatorially isomorphic. Note that the above inequality arises from the angle-sum condition (1).

First, it is easy to see that the property of k that implies uniqueness holds in the case $\mathsf{k} = [p^q]$.

If $p = 3$, then the inequality arising from the angle-sum condition implies that $q \geq 7$. Hence, in this case, the hypotheses of Theorem 1.3 are satisfied by $\mathsf{k} = [p^q]$, and we deduce that the two tilings are isomorphic.

If both $p, q \geq 4$ the the hypotheses of Theorem 1.2 are satisfied by k, and we similarly deduce that the two tilings are isomorphic.

Finally, if $q = 3$, then $p \geq 7$, and the tilings are the duals to uniform tilings with vertex-type $[3^p]$. The latter tilings are isomorphic, as noted above, and hence so are their duals.

Since T and T' are ismomorphic, there is an orientation-preserving homeomorphism h of the hyperbolic plane to itself, that maps vertices and edges of T to those of T'. By Lemma 2.1, there is a unique choice of a hyperbolic length of the edges for a uniform tiling with vertex-type k. Thus, h can be taken to be length-preserving on each edge, and this can be extended to be an isometry on each tile. Thus, we in fact have an orientation-preserving isometry of the hyperbolic plane to itself, that realizes the isomorphism between T and T'. □

4. Examples of non-uniqueness

In this section we give examples of distinct tilings with the same vertex-type.

Uncountably many distinct tilings. Consider the vetex type $\mathsf{k} = [4, 4, 4, 6]$. In this case, there is a uniform tiling T of vertex type k such that there is an action of Γ, a free subgroup of $\mathrm{PSL}_2(\mathbb{R})$ on 3 generators, that acts transitively on the hexagonal tiles. (See the tiling on the right in Figure 12.)

Notice that it has a Γ-invariant collection $\mathcal{R}$ of bi-infinite rows of squares $\{R_\gamma | \gamma \in \Gamma\}$ (see the rows of blue squares in Figure 12). Any such row R_γ has adjacent layers L_+ and L_- that comprise alternating hexagons and squares.

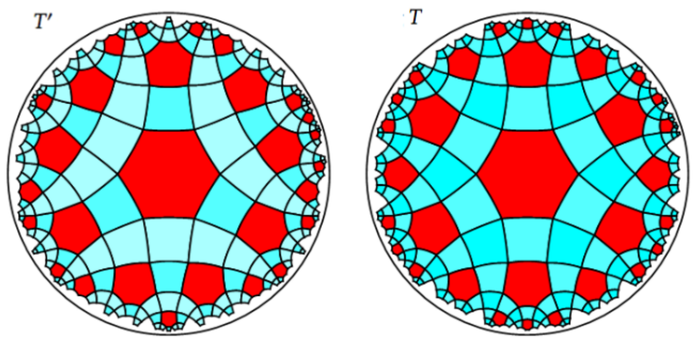

FIGURE 12. Spot the difference: these are distinct uniform tilings with identical vertex-type [4,4,4,6].

Then, a tiling T' that is *distinct* from T is obtained by shifting one side of each R_γ relative to the other. For example, performing this shift for three such bi-infinite rows adjacent to alternating sides (left, right, and bottom) of the central red hexagon in T produces a new tiling (on the left in Figure 12).

The same technique works for the vertex type $[4,4,4,n]$ for $n > 4$. There is a uniform tiling with this vertex-type which has an infinite collection $\mathcal{R}$ of bi-infinite rows of squares. The relative shift as above can in fact be performed at any subset of the collection of rows $\mathcal{R}$: for each row, the change in the tiling is shown below (Figure 13.) Since there are uncountably many such subsets of $\mathcal{R}$, we obtain uncountably many distinct tilings.

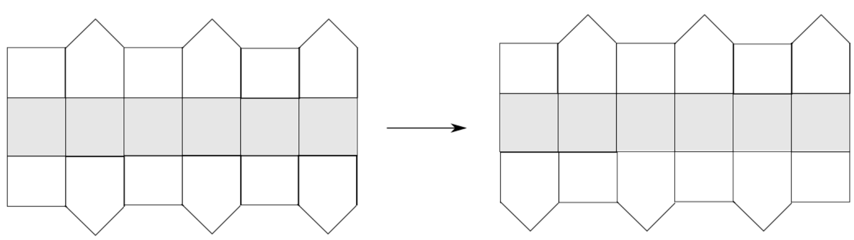

FIGURE 13. A relative shift of the tiles on either of the row R_γ (shown shaded) produces a different tiling with the same vertex-type $[4,4,4,5]$.

Other examples. Note our construction in §2 and §3 can be done starting with any initial tiled region X_0 that satisfies Property 1 (in the case that no tile is triangular and $d \geq 4$) or Property 1′ (in the case $d \geq 6$) and Property 2, together with the property that each interior vertex has the same vertex-type $\mathsf{k} = [k_1, k_2, \ldots, k_d]$.

If k satisfies the hypotheses of Theorem 1.3, but has a pair xy of consecutive elements which can be completed to the same cyclic tuple in two *different* ways, and these choices show up while completing the fans, then the final uniform tilings could be different. (See Figure 14).

5. DEGREE 3 TILINGS

In this final section we give the following necessary and sufficient conditions for the existence of tilings of degree $d = 3$:

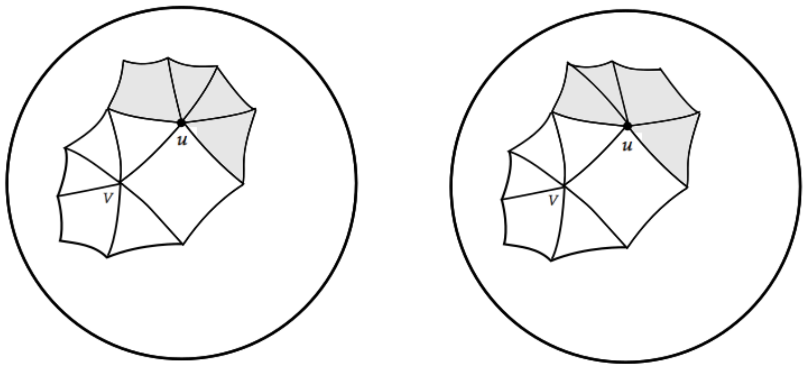

Figure 14. The unshaded tiles form a fan around V with vertex-type $\mathsf{k} = [4,3,3,3,4,3]$. The figures show two ways of completing another fan with the same vertex-type at a boundary vertex u. This choice, used judiciously in the tiling construction, gives rise to two distinct uniform tilings.

Proof of Theorem 1.5. Our proof divides into several cases which we handle separately.

Case 1: $\mathsf{k} = [p,p,p]$. Note that for the angle sum (1) to be satisfied, we have $p \geq 7$. Uniform tilings with this vertex-type k exist, as they are dual to the uniform tilings $[3^p]$ which exist by the Fuchsian triangle-group construction mentioned in the introduction.

Case 2: $\mathsf{k} = [p,q,r]$ *where* p,q,r *are distinct.* Such a triple does not satisfy our *necessary* condition (A), and hence a uniform tiling with vertex-type k cannot exist.

Case 3: $\mathsf{k} = [p,p,q]$ *where* $p \neq q$*, and* p *is odd.* In this case, suppose there is a uniform tiling with vertex-type k. Consider a p-gon in this tiling, with vertices $v_0, v_1, \ldots v_{p-1}$, and edges e_i between v_i and v_{i+1} for $0 \leq i \leq p-1$ (considered modulo p). Since each of these vertices have degree 3, the edges alternately share an edge with a p-gon and q-gon, respectively. However, if p is odd, then there is a vertex with three p-gons or three q-gons around it, which contradicts uniformity. (See Figure 15.) Thus, there can be no uniform tiling with vertex-type k. This argument also appears in [DM17], Lemma 2.2 (ii), in the context of maps on surfaces.

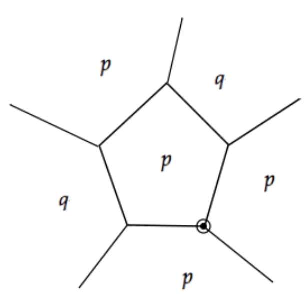

Figure 15. When p is odd, there cannot be a uniform tiling with vertex-type $[p,p,q]$: the offending vertex is circled.

Case 4: $\mathsf{k} = [p,p,q]$ *where* $p \neq q$*, and* p *is even.* Let $p = 2n$. Then the angle-sum in (1) can be easily seen to yield that $\frac{1}{n} + \frac{1}{q} < \frac{1}{2}$. Note that this is exactly the same condition that implies the existence of an $[n^q]$ tiling. We can in fact construct a uniform tiling with vertex-type k by modifying an $[n^q]$ tiling T_0 in the following way: replace each vertex in T_0 by a q-gon, which has vertices along the edges of T. The tiles of T_0 are now $2n$-gons, since each such tile acquires an extra edge from the q-gon added at each vertex, and there are n vertices. Although this construction is a *topological tiling*, we can replace them by regular hyperbolic polygons since the angle-sum condition is satisfied (*c.f.* Lemma 2.1). (See Figure 16.)

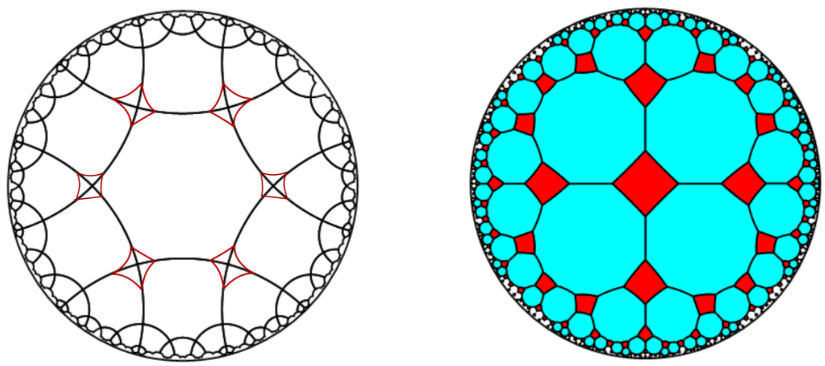

FIGURE 16. A uniform tiling with vertex type $\mathsf{k} = [12, 12, 4]$ (figure on the right) is obtained by introducing squares at each vertex of a uniform tiling with vertex-type $[6^4]$ (figure on the left).

This covers all possibilities for a triple k, and completes the proof. □

Acknowledgements. The first author was supported by SERB, DST (Grant No. MTR/2017/000410). The second author acknowledges the SERB , DST (Grant no. MT/2017/000706) and the Infosys Foundation for its support. Several figures in this article were made using the `L2Primitives` and `Tess` packages for Mathematica, available online at the Wolfram Library Archive.

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