

About a new splittings for the algorithmic study
of
the tilings $\{p, q\}$ of the hyperbolic plane
when q is odd

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Abstract

In this paper, we remind previous results about the tilings $\{p, q\}$ of the hyperbolic plane. We introduce two new ways to split the hyperbolic plane in order to algorithmically construct the tilings $\{p, q\}$ when q is odd.

Keywords: discrete hyperbolic geometry, combinatorial approach, tilings.

1 Introduction

As mentioned in the abstract, the goal of this paper is to introduce two ways within the algorithmic approach to the study of the tilings $\{p, q\}$ when q is odd. This approach is in the spirit of what has been done by the author starting from the basic paper [2].

In Section 2, we remind the reader the basic features of hyperbolic geometry, in particular what is needed to define and to study the tilings $\{p, q\}$. In Section 3, we remind the splitting of the hyperbolic plane leading to a spanning tree of the tiling $\{p, q\}$ when q is even. In Section 4, we consider the tiling $\{p, q\}$ when q is odd. In Subsection 4.1, we remind the construction performed in [6]. In Subsections 4.2 and 4.3, we present the two ways of splitting mentioned in the abstract. Both ways have nice properties except in one case, when $p = 4$ and $q = 5$. Subsection 4.3.1 is devoted to the study of this case.

2 Tessellations in the hyperbolic plane

In this section, we first remind Poincaré's disc model and a few features which will allow us to define tessellations in the hyperbolic plane which we shall study in Subsection 2.2

2.1 Hyperbolic geometry

Hyperbolic geometry appeared in the first half of the 19th century, proving the independence of the parallel axiom of Euclidean geometry. Models were devised in the second half of the 19th century and we shall use here one of the most popular ones, Poincaré's disc. Figure 1 illustrates the parallel axiom of hyperbolic geometry in this model.

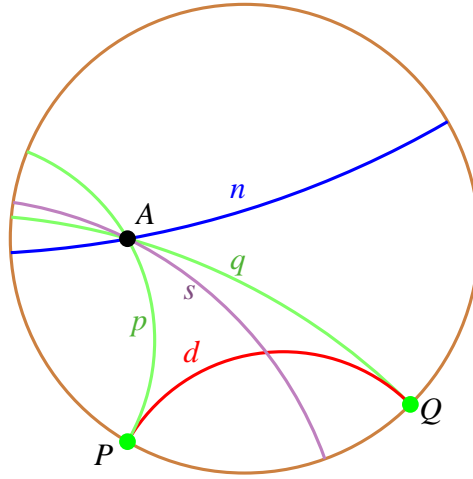


Figure 1 *Illustration of the parallel axiom of hyperbolic geometry. The lines p and q are the parallels to m which pass through A .*

In Poincaré's disc model, the points of the hyperbolic plane are exactly those which are inside a once for all fixed open disc of the Euclidean plane. The border of the disc is called the set of **points at infinity**. Note that the points at infinity do not belong to the hyperbolic plane. Lines are trace of diameters or circles orthogonal to the border of the disc, see the line m in Figure 1. In this figure, we have a point A not on m and we can see a line s which cuts m . Two lines passing through A play a particular role: the lines p and q , which touch m in the model at P and Q which are points at infinity. These lines are called **parallel** to m . More generally, in the model, two lines are parallel if and only if they have a common point at infinity.

Now, Figure 1 shows that there is another important case which has no counterpart in the Euclidean plane: the line n also passes through A without cutting m , neither inside the disc nor outside it. Such a line is called **non-secant** with m . Non-secant lines are characterized by the fact that they have a

unique common perpendicular. This is also a specific property of the hyperbolic plane where there cannot be rectangles. Another important property, equivalent to the parallel axiom of this geometry is that the sum of the interior angles of a triangle is always less than π , the measure of the straight angle.

2.2 Tessellations in the hyperbolic plane

Henri Poincaré also established an important theorem from which we know that there is an infinite family of tilings in the hyperbolic plane when in the Euclidean plane the same definition leads to three tilings only, up to similarity.

To state the theorem, we have to remember the definition of a **tessellation**. Consider the following process. We start from a single polygon P and we replicate it by reflection in its sides. Then, recursively, we replicate the images of P by reflection in their sides. If the images do not overlap and if any point of the plane is contained in at least one image, we say that we have a tiling which is generated from P by tessellation. We also call the tiling a tessellation generated by P .

In the late 19th century, Henri Poincaré proved the following:

Theorem 1 *A triangle T of the hyperbolic plane generates a tessellation if and only if its angles are of the form $\frac{2\pi}{p}$, $\frac{2\pi}{q}$ and $\frac{2\pi}{r}$, where p , q and r are positive integers satisfying $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < \frac{1}{2}$.*

Note that the condition merely says that T is a triangle of the hyperbolic plane.

An important particular case of this theorem is the case when P is a regular polygon. This case can be derived from the theorem by considering the rectangular triangle T which is constructed from P by taking as vertices the centre of P and the end-points of a half-side. If p is the number of sides of P and if $\frac{2\pi}{q}$ is its interior angle, then the angles of T are $\frac{\pi}{q}$, $\frac{\pi}{p}$ and $\frac{\pi}{2}$. Accordingly, the condition is now $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$.

Remark that, in the Euclidean plane, we can perform the same construction starting from a rectangular triangle T . This time, as the sum of angles of a triangle in the Euclidean plane is π , we get that necessarily, $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, which gives three solutions exactly: $p = q = 4$, $p = 6$ with $q = 3$ and $p = 3$ with $q = 6$. This gives us the square, the regular hexagon and the equilateral triangle respectively.

In the hyperbolic plane, we have an infinite family. Moreover, it is defined with $p = 3$ and then $q \geq 7$, or with $q = 3$ and then $p \geq 7$ or when $p \geq 4$ and then $q \geq 5$ or when $q \geq 4$ and then $p \geq 5$.

Remember that in the case $\{5, 4\}$ called the **pentagrid**, very simple tools to navigate in this tiling were devised, see [2]. This was extended to the tilings

$\{p, 4\}$ in [7], then to the tiling $\{7, 3\}$ called the heptagrid in [1], later to the tilings $\{p, q\}$ in [8]. All these tilings fall under the class of **combinatoric tilings** defined in [3, 4]. In this class, we have an algorithmic way to study the tiling which is based on the construction of a tree which spans the tiling. The tree is associated to a process of partition of the hyperbolic plane which generates several tiles at each step of the construction. The construction produces the whole tiling in infinite time. It is important to notice that there is a bijection between the tree and the tiles of the tiling. A particular way to number the nodes of the tree provides us with efficient tools of navigation in the tiling initially established in [2] for the pentagrid and then generalized as above indicated.

A precise description of these results and their proofs together with the references can be found in [6]. What we remind in Section 3 will give an insight in the method and its results.

In our next section, we turn to the general case of the tilings $\{p, q\}$ when q is even. Later, in Section 4, we introduce the new idea and its application to an appropriate splitting of the hyperbolic plane leading to another construction of the tilings $\{p, q\}$ when q is odd.

3 The case $\{p, q\}$ when q is even

Here, we remind the splitting of the hyperbolic plane thoroughly explained and analyzed in [6]. It leads to an algorithmic construction, in infinite time, of the tiling $\{p, q\}$ when q is even. The case when q is odd, which is more complex, will be dealt with in the next section.

When q is even, we define a **sector** \mathcal{S}_0 as the angular sector defined by taking a vertex V of the polygon P on which the tiling is constructed and the rays issued from V which supports the two edges of P which meet at V . We call V the **vertex** of \mathcal{S}_0 and P is its **head**. It is easy to see that the whole tiling is the union of q copies of \mathcal{S}_0 which share the same vertex.

Number the edges of P e_1, \dots, e_p , the numbers increasing while clockwise turning around P . Similarly, denote by V_i, \dots, V_p the vertices of P , deciding that V_1 and V_2 are the end-points of e_1 . We also assume that the lines supporting e_1 and e_p also support the rays ρ_ℓ and ρ_r issued from V_1 which define \mathcal{S}_0 . As q is even, we can write $q = 2h$ and we can easily see that the angle between e_2 and ρ_r which is outside P and inside \mathcal{S}_0 is $(h-1)\frac{2\pi}{q}$, see Figure 2. And so, $h-1$ copies of \mathcal{S}_0 exactly fill up the region R_1 which is outside P , inside \mathcal{S}_0 and between ρ_ℓ and the line which supports e_2 . Say that R_1 is a **fan** of $h-1$ copies of \mathcal{S}_0 . We can define similar regions R_i which are fans of $h-1$ copies of \mathcal{S}_0 , with $i \in \{2..p-3\}$. Indeed, R_{i+1} is bordered by the continuation of e_i and the side e_{i+1} and its continuation outside P and outside R_i . Now, the complement in \mathcal{S}_0 of P and all the R_i 's we have just defined is a new region \mathcal{S}_1 . This region is defined by e_{p-2} , e_{p-1} and e_p , the angles between e_{p-1} and its neighbouring edges inside \mathcal{S}_1 being both $(h-1)\frac{2\pi}{q}$.

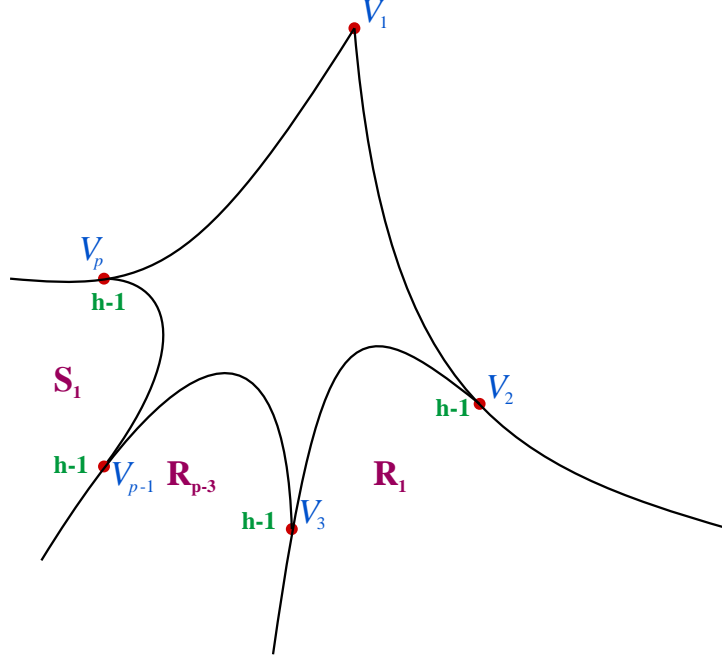


Figure 2 Case when q is even. The basic region \mathcal{S}_0 and its splitting.

Let P_1 be the reflection of P in e_{p-1} . Denote by ρ_ℓ the continuation of e_{p-2} and by ρ_r that of e_p . The splitting of \mathcal{S}_1 is a bit different from that of \mathcal{S}_0 , but it relies on the same considerations. Rename e_1 the edge of P_1 which is shared with P and denote by e_i the other sides of P_1 , $i \in \{1..p\}$, the numbering being increasing while clockwise turning around P_1 . We notice that this time, in the complement in \mathcal{S}_1 of P_1 , ρ_ℓ and e_2 define an angle which is $(h-2)\frac{2\pi}{q}$, so that we split this region, say R_1 into $(h-2)$ copies of \mathcal{S}_0 : this time, we have a fan of $(h-2)$ copies of \mathcal{S}_0 . The next regions R_i are defined by e_i and e_{i+1} in the complement in \mathcal{S}_1 of P and the regions R_j for $j < i$, e_{i-1} and e_i . But, for $i \in \{2..p-3\}$, each R_i is a fan of $h-1$ copies of \mathcal{S}_0 . Now, the other ray ρ_r and e_p define in the complement of P in \mathcal{S}_1 another region R_{p-2} which is also a fan of $(h-2)$ copies of \mathcal{S}_0 : this copies are obtained by using an odd number of reflections in lines. Now, what remains in \mathcal{S}_1 after removing R_{p-2} is a region R_{p-1} which is a copy of \mathcal{S}_1 , see Figure 3. Accordingly, we have split \mathcal{S}_1 into a copy of P , $(p-2)(h-1) - 2$ copies of \mathcal{S}_0 and one copy of \mathcal{S}_1 .

Note that we can write the splitting as follows:

$$\begin{aligned}\mathcal{S}_0 &\longrightarrow (p-3)(h-1).\mathcal{S}_0 + \mathcal{S}_1 \\ \mathcal{S}_1 &\longrightarrow ((p-2)(h-1)-2).\mathcal{S}_0 + \mathcal{S}_1,\end{aligned}$$

which is the splitting given in [6].

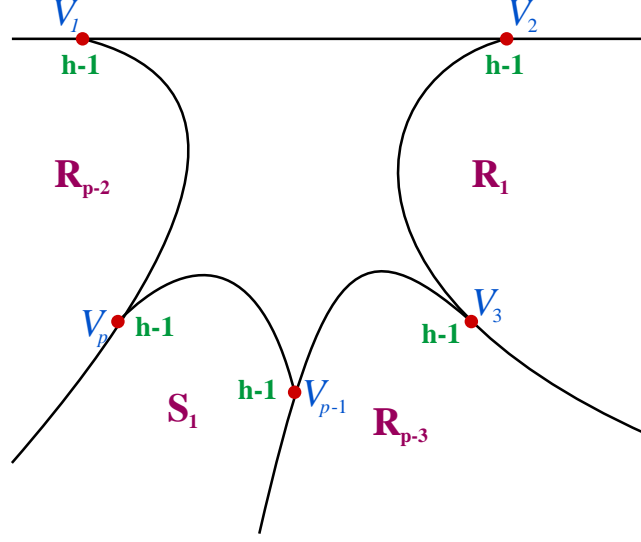


Figure 3 Case when q is even. The basic region \mathcal{S}_1 and its splitting.

We briefly remind the consequence of the just computed relations. From them, we easily derive a matrix which we interpret as an **incident matrix** called the **matrix of the splitting**. The rows indicate how a region is split in terms of the basic regions, \mathcal{S}_0 and \mathcal{S}_1 . The columns indicate how many copies of the considered region enter the splitting of the region associated to the considered line. From the matrix, we get its characteristic polynomial which here we call the **polynomial of the splitting**. It is not difficult to see that the polynomial is $P(X) = X^2 - ((p-3).(h-1)+1).X - h + 3$. It is known that, this polynomial has a real root β which is positive and greater than 1, see [6]. It is also known that the recurrent relation obtained from the polynomial defines a sequence of increasing positive numbers in which we can decompose any natural number with sums of terms of the sequence, each term entering the sum being multiplied by an integer whose range is $0..b$, where $b = \lfloor \beta \rfloor$. It is known that, in general, the representation is not unique. However, it can be made unique by requiring that we take the longest representation in terms of number of digits. We call the set of these maximal representations the **language of the splitting**. Now, it is also known that the language is regular if and only if β is a Pisot number, which means that $\beta > 1$ and that if α is the other root, $|\alpha| < 1$. We know that most of the languages associated to a tiling $\{p, q\}$ are regular but they are very different. We refer the reader to [6] for the corresponding study.

4 The case $\{p, q\}$ when q is odd

When q is odd, we can no more split the tiling using lines which support the sides of the tiles. In [6], we solved the problem by using a zig-zag line which

follows sides of the tiles. In Subsection 4.1, we remind the main lines of this splitting and its main properties. In Subsection 4.2, we describe the new idea and a way of splitting which we can straightforward devise from it.

4.1 The previous solution

In [6] we define the basic regions by using a zig-zag line constituted of sides of the tiling. Each side is defined from the previous one by making a constant angle defined by $h\frac{2\pi}{q}$, where $h = \lfloor \frac{q}{2} \rfloor$. This angle is the biggest positive integral multiple of $\frac{2\pi}{q}$ which is less than the straight angle. More precisely, fix an edge e_1 . Let V be the vertex to which we arrive after choosing e_1 . Denote the other vertices by e_2, \dots, e_q . The angle between e_i and e_{i+1} is $\frac{2\pi}{q}$ for $i \in \{1..q-1\}$ and it is the same for the angle between e_q and e_1 . Now, it is plain that e_{h+1} and e_{h+2} are the closest to the line which supports e_1 : both these edges make an angle $h\frac{2\pi}{q}$ with e_1 . We decide to number the edges in the following way: the last chosen edge is always e_1 . The other edges are numbered from 2 up to q by counter clockwise turning around V , the vertex to which e_1 arrives and from which the edges e_i are issued, $i \in \{2..q\}$. The next vertex is e_{h+1} : it becomes the new value of e_1 and its other end becomes the new value of V . See Figure 4 which illustrates this process.

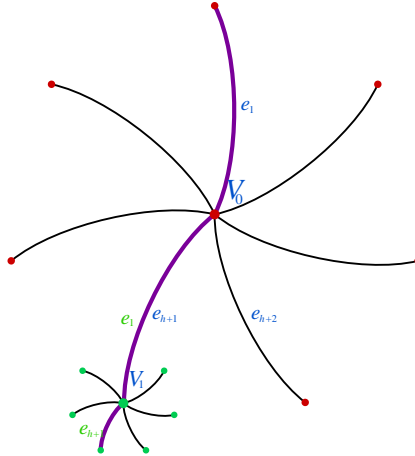


Figure 4 *Illustration of the choice of the next side in the zig-zag line devised for the case when q is odd. The resulting zig-zag line is in purple.*

With these conventions, it is possible to define two regions \mathcal{S}_0 and \mathcal{S}_1 which provide results very close to those obtained in the case when q is even. Figures 5 and 6 illustrate the situation in the case when $q = 7$. This is enough to allow

us to see how the splitting is performed in this case. In particular, we find the same splitting matrix as in the case when q is even.

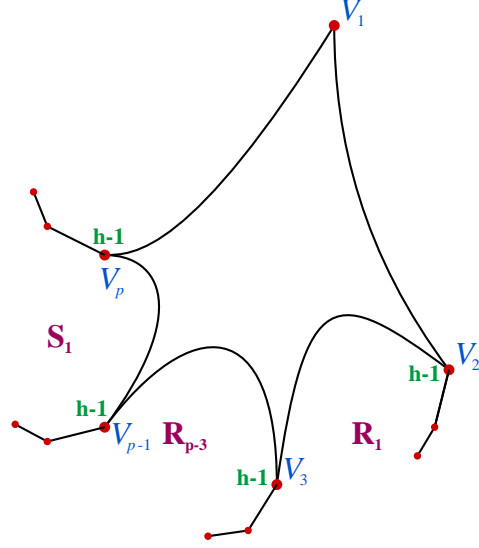


Figure 5 Case when q is odd. The basic region S_0 and its splitting in the solution given in [6].

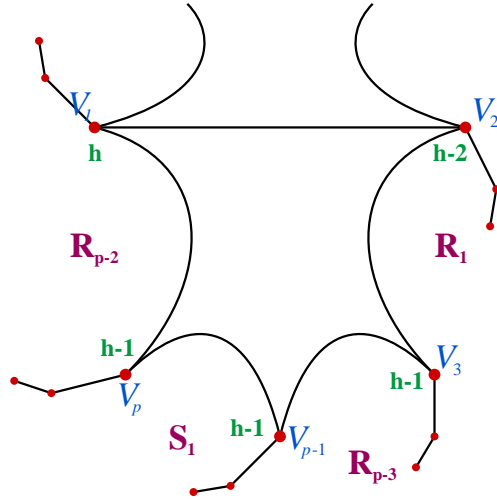


Figure 6 Case when q is odd. The basic region S_1 and its splitting in the solution given in [6].

4.2 The new splittings

Now, we turn to the new splittings announced in the abstract and in Section 1. Both the new splittings are based on the same notion of a mid-point line.

This notion was introduced in the heptagrid, in order to define the regions of the splitting giving rise to this tiling. In this case, a mid-point line is a line which joins mid-points of consecutive edges of the tiling.

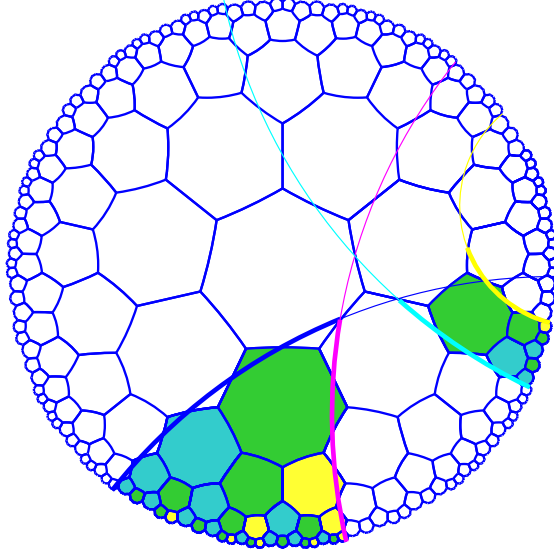


Figure 7 *The mid-point lines of the heptagrid. They allow us to define the two basic regions giving rise to this tiling.*

It is not difficult to see that when $q \geq 5$, this property of the mid-points of edges is no more true: three mid-points M_1 , M_2 and M_3 such that M_1 with M_2 and M_2 with M_3 belong to the same heptagon but M_1 and M_3 belong to different ones are never on the same line. In the next subsection, we shall see that however, there is a way to generalize the mid-points of the heptagrid. Then, in Subsubsection 4.2.2 and in Subsection 4.3 we shall see the application of this new construction to two variants of a splitting of the tiling $\{p, q\}$ in the case when q is odd. Subsubsection 4.3.1 will deal in another way for the special case when $p = 4$ and $q = 5$.

4.2.1 The mid-point lines

Let $h = \lfloor \frac{q}{2} \rfloor$, as this number will play an important role. In our new setting, we shall again use a mid-point line, but this time, it will be defined by the angle $h \frac{2\pi}{q}$. Note that when $q = 3$, $h = 1$, so that this definition is a natural generalization of what we did in the heptagrid. Indeed, consider V a vertex of P , a

regular polygon with p sides and q copies of P exactly covering a neighbourhood of a vertex. Let Q_i be the copies around V covering one of its neighbourhoods. Assume that $Q_1 = P$ and let the others, Q_2, \dots, Q_p be increasingly numbered while counter clockwise turning around V . Let e_0 be the side of P abutting V which is not shared by Q_2 . The continuation of e cuts Q_{h+1} into two parts which are the reflection of each other in the continuation of e . Let a be the side of Q_h which makes the angle $h \frac{2\pi}{q}$ with e at V . Let M be the mid-point of e and let N be that of a . Let δ be the line joining M and N . Let W be the other end-point of a . Consider the angle at W defined by a which is $h \frac{2\pi}{q}$, but defined clockwise. This defines the side b of one copy of P among those which, dispatched around W exactly cover a neighbourhood of this point. Now, from the above definition of angles, it is not difficult to see that with respect to the line d supporting a we have that e lies in one of the half-plane delimited by d and b lies in the other. Let O be the mid-point of b . The triangles MVN and NWO are equal and this shows that δ also joins N to O , see Figure 8 as V and W are not on the same side of d . In this way, we can see that δ is a line of mid-points of sides of the tiles, but instead of joining mid-points of consecutive edges, it joins mid-points of edges of the tiles which make the angle $h \frac{2\pi}{q}$: we shall say that the line joins **h -consecutive mid-points**. From now on, we shall call **h -mid-point lines** the lines which joins h -consecutive mid-points.

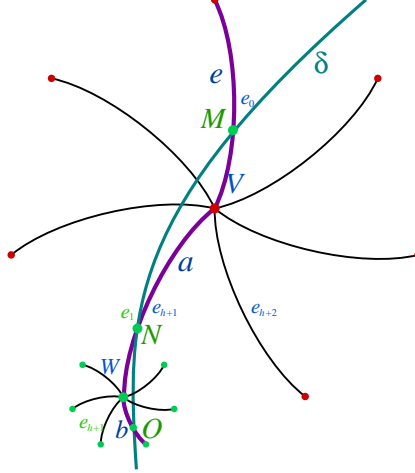


Figure 8 *Illustration of the construction of a h -mid-point line. Here, $q = 7$ and so, $h = 3$.*

4.2.2 The splitting

Now, we are ready to define the splitting.

First, define a sector \mathcal{S}_0 to be delimited by a vertex V , called the vertex

of \mathcal{S}_0 . The head of the sector is a copy of P for which V is a vertex. Let b and c denote the sides of P which meet at V . The bisector of the angle between b and c at V which is outside P is the edge a of a copy of P among the q copies of P which can be put around V . Let C , B and A be the mid-points of c , b and a respectively. We may assume that we go from c to b by counter clockwise turning around V . Let ρ_B be the ray issued from B whose supporting line goes through A . We take the ray which does not contain A . Note that the ray is supported by a h -mid-point line. Now, we define ρ_C by taking the image of ρ_B under the rotation around V which transforms B into C . Clearly, ρ_C is also a h -mid-point line: the next mid-point on this ray, starting from C belongs to an edge of a tile which is outside P , see Figure 9. We define \mathcal{S}_0 as the region delimited by V , B , C and both rays ρ_B and ρ_C . From the definition of ρ_B , we easily conclude that the tiling $\{p, q\}$ can exactly be split into q copies of \mathcal{S}_0 .

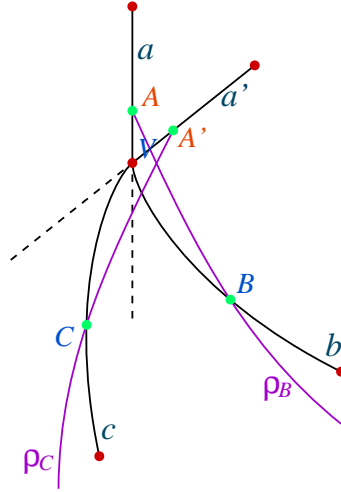


Figure 9 The h -mid-point rays used for the definition of the region \mathcal{S}_0 .

Consider the region \mathcal{S}_0 , as illustrated in Figure 10. The figure indicates how we split this region. There is a small difference with the case when q is even: we have to introduce three regions in order to get a combinatoric splitting.

First, consider the delimitation of the sector. On the figure, the mid-points are called M_1, M_2, \dots, M_p as most of them are used in the splitting. Here, M_2 plays the role of C and M_1 that of B . On the figure, for each vertex V_i , we have represented the side s_i of the polygon having V_i among its vertices which makes the angle $h \frac{2\pi}{q}$ with the edge e_{i-1} of P , with $i \in \{2..p\}$ and we consider e_p when $i = 1$. We consider the ray issued from M_i and which goes through the mid-point m_i of s_i . On Figure 10, we can see that V_2 is outside \mathcal{S}_0 . However, the $h-1$ copies of P which share V_2 and which are inside the angle between e_1 and s_2 are considered as belonging to \mathcal{S}_0 : for these polygons, V_2 is the single vertex among those of the polygon which is not in \mathcal{S}_0 . This is why we define the

set of tiles spanned by \mathcal{S}_0 as those which have at most one vertex outside \mathcal{S}_0 .

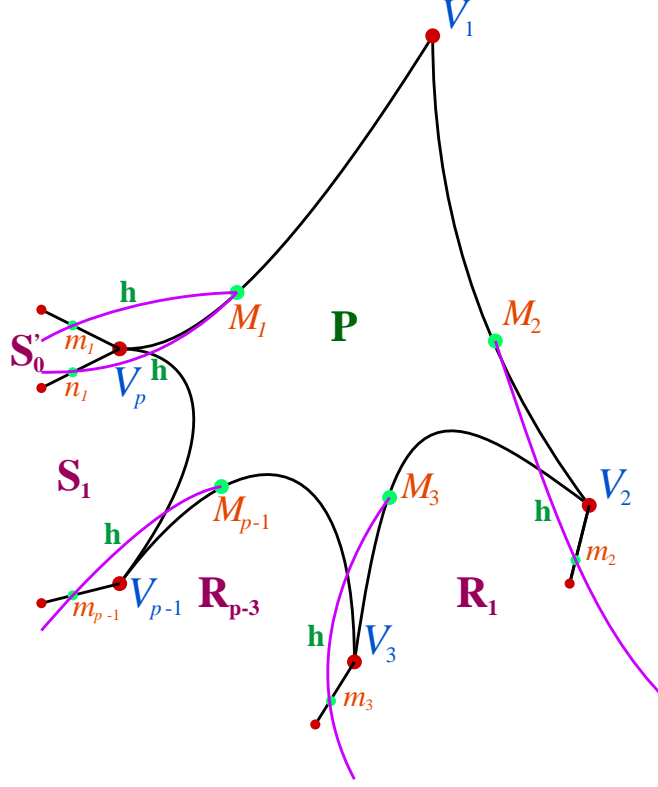


Figure 10 *Splitting the region \mathcal{S}_0 .*

Now, from this remark, we can say that there is a fan of $h-1$ copies of \mathcal{S}_0 delimited by V_2 , the ray issued from m_2 and supported by that joining M_2 to m_2 and the ray issued from M_3 and passing through m_3 . The just described region R_1 consists of $h-1$ copies of \mathcal{S}_0 , V_2 being their common vertex. Similar regions R_2, \dots, R_{p-3} can successively be defined in the complement in \mathcal{S}_0 of P and the regions already defined.

When we arrive to M_{p-1} , the ray issued from M_{p-1} passing through m_{p-1} defines the left-hand side of R_{p-3} . What remains from \mathcal{S}_0 is a region which we split as indicated in Figure 10. From M_1 we draw the two rays ρ_m and ρ_n which are supported by the h -mid-point lines passing through M_1 . One ray goes through m_1 and the other from n_1 . Now, the region delimited by M_1 and the rays ρ_m and ρ_n is a new type of region which we call \mathcal{S}'_0 as it looks like \mathcal{S}_0 . Now, what remains from \mathcal{S}_0 once we removed \mathcal{S}'_0 is by definition \mathcal{S}_1 .

And so, we can summarize the splitting of \mathcal{S}_0 by writing:

$$\mathcal{S}_0 \longrightarrow (p-3)(h-1)\mathcal{S}_0 + \mathcal{S}'_0 + \mathcal{S}_1.$$

Now, as \mathcal{S}'_0 is obtained from \mathcal{S}_0 by just removing a copy of \mathcal{S}'_0 , we immediately get that the splitting of \mathcal{S}'_0 , illustrated by Figure 11 can be summarized by the following formula:

$$\mathcal{S}'_0 \longrightarrow (p-3)(h-1)\mathcal{S}_0 + \mathcal{S}_1.$$

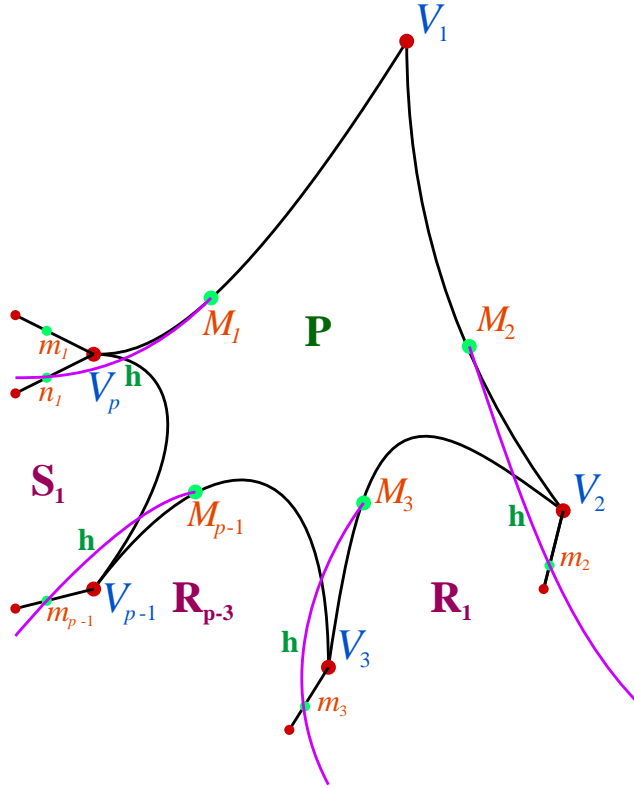


Figure 11 *Splitting the region S'_0 .*

Presently, we arrive at the splitting of \mathcal{S}_1 . Everything goes as for \mathcal{S}_0 , taking into account the following: for defining \mathcal{S}_1 , we need $p-1$ sides of P , but we have to take into account the mid-points of the edges of the reflection of P in V_1V_2 which have V_1 and V_2 as end-points, see Figure 12. We start the splitting both from V_2 and from V_1 . From V_2 , we define a region R_1 as previously. It is a fan of $h-2$ copies of \mathcal{S}_0 instead of $h-1$ because \mathcal{S}_1 is defined from M_2 which, in this case, is not on an edge of P but on an edge of the reflection of P in V_1V_2 . The $h-2$ polygons having V_2 as a vertex are counted as included in \mathcal{S}_1 for the same reason as we did in the case of \mathcal{S}_0 . Now, this remark also holds for V_1 : from there we define a fan of $h-2$ copies of \mathcal{S}_0 , defining a region R_{p-2} , whose right-hand side limit is defined by the ray issued from M_p and passing through m_p . Note that the copies of \mathcal{S}_0 which constitute R_{p-2} are obtained from those which

constitute R_1 by a reflection in the bisector of V_1V_2 .

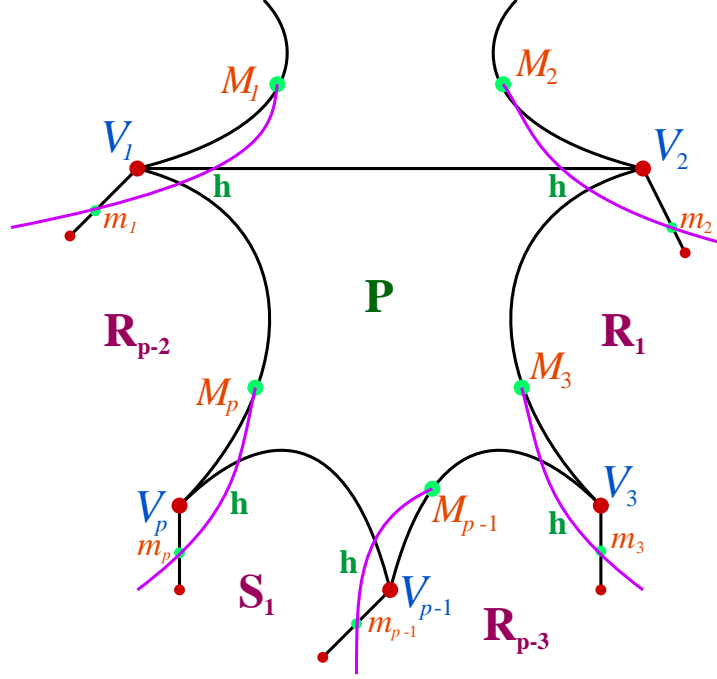


Figure 12 *Splitting the region S_1 .*

As we did in the case of \mathcal{S}_0 , we can define the regions R_2, \dots, R_{p-3} which are fans of $h-1$ copies of \mathcal{S}_0 . Now, when we arrive at R_{p-3} , what we have between this latter region, P and R_{p-2} is a copy of \mathcal{S}_1 . Accordingly, we can summarize the splitting by:

$$\mathcal{S}_1 \longrightarrow ((p-2)(h-1) - 2)\mathcal{S}_0 + \mathcal{S}_1.$$

Indeed, in this formula, the term $(p-2)(h-1) - 2$ comes from $p-4$ fans with $h-1$ copies of \mathcal{S}_0 , and the two fans with $h-2$ copies of \mathcal{S}_0 defined by R_{p-2} and R_1 .

Accordingly, the polynomial of the splitting is:

$$P(X) = X^3 - ((p-3)(h-1)+1)X^2 - ((p-2)(h-1)-2)X - h+3.$$

From this, we know that the number u_n of nodes which are on the same level of the spanning tree of the tiling is defined by:

$$u_{n+3} = ((p-3)(h-1)+1)u_{n+2} + ((p-2)(h-1)-2)u_{n+1} - (h-3)u_n.$$

Note that easy computations give us that $P(-1) = -2$, $P(0) = -h+3$ and $P(-\frac{1}{2}) = \frac{(p-5)}{4}(h-1) + \frac{5}{8}$. Accordingly, when $h > 3$ and $p \geq 5$, we have that $P(-\frac{1}{2}) > 0$ and $P(0) < 0$. This shows that P has three real roots, that two of

them are in $] -1, 1[$. Let $a = (p-3)(h-1) + 1$. Then, an easy computation shows that $P(a) < 0$. This tells us that $\beta > a$ where β is the greatest real root of P . Now, under the assumption that $h > 1$, $a > 1$, so that $\beta > 1$ too. Accordingly, when $h \geq 4$ and $p \geq 5$, P is a Pisot polynomial which means that the language of the splitting is regular.

We remain with the study of the cases when $h = 2$ and $h = 3$ on one hand and the case when $p = 4$ on the other hand.

First, assume that $p \geq 5$.

When $h = 3$, $P(X)$ can be divided by X and so we can replace P by a polynomial of degree 2 which we again call P . We have:

$$P(X) = X^2 - (2p-5)X - (2p-6).$$

Note that the splitting assumes that we have $p \geq 4$. In this case, we have two real roots, α and β , assuming $\alpha < \beta$. Now, $P((2p-5)) = -2p+6 < 0$, so that $\beta > 2p-5 > 1$. On another hand, $P(0) = -2p+6 < 0$ too. Now, $P(-1) = 1 + 2p-5 - (2p-6) = 2$, which means that $-1 < \alpha < 0$. Accordingly, P is a Pisot polynomial and the language of the splitting is again regular.

When $h = 2$, the polynomial is now:

$$P(X) = X^3 - (p-2)X^2 - (p-4)X + 1.$$

This time we have that $P(0) = 1 > 0$. Remember that $P(-1) = -2 < 0$ as the computation does not depend neither on h nor on p . An easy computation gives us $P(1) = 8 - 2p < 0$ when $p \geq 5$. Now, $P(p-2) = -(p-4)(p-2) + 1 < 0$ when $p \geq 5$, so that $\beta > p-2 > 1$, where β is the greatest real root of P : again, P is a Pisot polynomial.

This allows us to conclude that in all cases when $p \geq 5$, we have that P has three real roots, that two of them have a modulus which is less than 1 and that the biggest real root is positive and greater than 1. We have that P is a Pisot polynomial and, consequently, the language of the splitting is always regular in this case.

Let us look at the case when $p = 4$.

This time we have that $P(X) = X^3 - hX^2 - 2(h-2)X - h + 3$.

Denote by β , x_1 and x_2 the roots of P , β being the greatest real root, which is positive. Indeed: $P(h) = -2(h-2)h - h + 3 = -2h^2 + 3h + 3 < 0$ when $h \geq 3$, which implies that $h < \beta$. Now, $\beta x_1 x_2 = -h+3$ and, on another hand, $x_2 = \overline{x_1}$ as the coefficients of P are real numbers. This gives the following computation: $|x_1|^2 = \frac{h-3}{\beta} < \frac{h-3}{h} = 1 - \frac{3}{h}$. Accordingly, $|x_1| < 1$ so that P is a Pisot polynomial when $h \geq 3$.

We remain with the single case $h = 2$ under the assumption that $p = 4$.

In this case $P(X) = X^3 - 2X^2 + 1$. Clearly, 1 is a root of this polynomial. The other roots are those of $X^2 - X - 1$ as $P(X) = (X-1)(X^2 - X - 1)$. The other roots of P are α and β with $\beta > 1$ and $\alpha < 1$, as $\beta = \frac{1 + \sqrt{5}}{2}$ and, consequently, $\alpha = \frac{\sqrt{5} - 1}{2}$. But, as 1 is a root of P , P is no more a Pisot polynomial. As the language of the splitting is regular if and only if its polynomial is a Pisot polynomial or the product of a Pisot polynomial with polynomials of the form

$\sum_{k=0}^m X^k$, we have that the language of the splitting is not regular when $p = 4$ and $h = 2$.

Accordingly, in all cases when the splitting holds, its language is regular, except in the case when $p = 4$ and $h = 2$.

4.3 A variant of the new splitting

A variant of the previous splitting consists in replacing the region \mathcal{S}_0 by the region \mathcal{S}'_0 and by keeping the region \mathcal{S}_1 unchanged. We have two basic regions instead of three, but the coefficients of the splitting matrix are different.

In order to see this point, let us again have a look at the situation around a vertex. Figure 13 represents the rays which are used to delimit the region \mathcal{S}'_0 in a way which can be seen as a local zoom on the part of Figure 11.

Now, Figure 14 shows two sectors \mathcal{S}'_0 headed by adjacent polygons around a common vertex. In the figure, the right-hand side sector is delimited by the h -mid-point lines ρ_{AB} and ρ_{AC} while the left-hand side one is delimited by the h -mid-point lines ρ_{GC} and ρ_{GF} . It can be seen that the continuation of the rays ρ_{AC} and ρ_{GC} issued from C define again a copy of \mathcal{S}'_0 which contains exactly the tiles contained in this angular sector and which are not contained neither in the sector defined by ρ_{AB} and ρ_{AC} nor that which is defined by ρ_{GC} and ρ_{GF} .

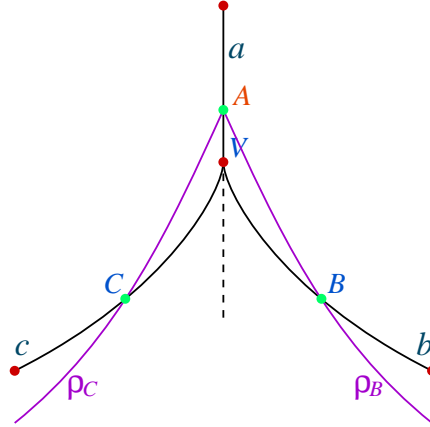


Figure 13 The h -mid-point rays used for the definition of the region \mathcal{S}'_0 .

This means that, around a vertex, the q polygons which meet at this vertex define q copies of \mathcal{S}'_0 which do not cover the hyperbolic plane. In between two consecutive copies of \mathcal{S}'_0 , there is room for exactly one copy. Accordingly, we can split the hyperbolic plane into $2q$ copies of \mathcal{S}'_0 , reproducing q times the scheme represented by Figure 14.

Now, consider Figure 14. As the side g is a rotated image of the side a , we get that ρ_{GC} is a rotated image of ρ_{AB} under the same rotation. Now, we have

just seen that the continuation of the rays ρ_{AC} and ρ_{GC} issued from C define again a copy of \mathcal{S}'_0 . This means that \mathcal{S}_0 can be split into two copies of \mathcal{S}'_0 exactly. Accordingly, the splittings of \mathcal{S}'_0 and \mathcal{S}_1 illustrated by Figures 11 and 12 can be adapted to the regions \mathcal{S}'_0 and \mathcal{S}_1 only.

This gives us the following formulas:

$$\begin{aligned}\mathcal{S}'_0 &\longrightarrow 2(p-3)(h-1)\mathcal{S}'_0 + \mathcal{S}_1. \\ \mathcal{S}_1 &\longrightarrow 2((p-2)(h-1) - 2)\mathcal{S}'_0 + \mathcal{S}_1.\end{aligned}$$

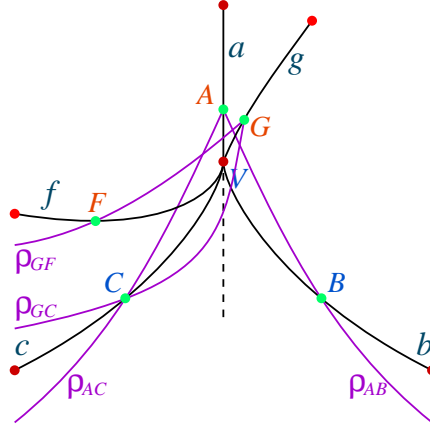


Figure 14 Two consecutive copies of the region \mathcal{S}'_0 .

This time, the polynomial of the splitting is:

$$P(X) = X^2 - (2(p-3)(h-1)+1)X - 2h+6.$$

As $q = 2h+1$, this can be rewritten as

$$P(X) = X^2 - ((p-3)(q-3)+1)X - q+7.$$

When $q \geq 7$, the last coefficient of this polynomial of degree 2 is negative and so it has a positive real root β . We remark that

$$P((p-3)(q-3)) = -(p-3)(q-3) - q + 7 < 0,$$

as $q > 7$ and $(p-3)(q-3) > 0$ when $p \geq 4$ and $q > 7$. so that $\beta > (p-3)(q-3)$. Consequently, if α is the other root of the polynomial, it is also a real number and we have that

$$|\alpha|\beta = q-7 < q-3, \text{ and so } |\alpha| < \frac{q-3}{(p-3)(q-3)} = \frac{1}{p-3} \leq 1,$$

as $p \geq 4$. And so, $|\alpha| < 1$ which means that P is a Pisot polynomial.

We remain with the cases when $h = 2$ and $h = 3$.

When $h = 2$, the polynomial is now:

$$P(X) = X^2 - (2(p-3) + 1)X + 2.$$

$P(2(p-3)) = -2(p-3) + 2 < 0$ when $p \geq 5$. Accordingly, P has a real root

β , with $\beta > 2(p-3) > 1$ when $p \geq 5$. There is another root α for which we have $\alpha\beta = 2$, so that $\alpha = \frac{2}{\beta} < \frac{2}{2(p-3)} = \frac{1}{p-3} < 1$ as $p \geq 5$.

When $p = 4$ and $h = 2$, we have that $P(X) = X^2 - 3X + 2 = (X-1)(X-2)$. As 1 is a root of P , it cannot be a Pisot polynomial and so, the language of the splitting is not regular in this case, as it is also the case with the splitting of Subsection 4.2.

When $h = 3$, the polynomial becomes $P(X) = X^2 - (2(p-3)+1)X$, so that the roots are now $X = 0$ and $X = 2(p-3)+1 > 1$ when $p \geq 4$. And so, in this case, P is a Pisot polynomial.

Accordingly, the language of the splitting defined by this new variant is regular in all cases, except in the same case as that induced by the splitting of Subsection 4.2, *i.e.* when $p = 4$ and $h = 2$.

4.3.1 The special case of the tiling $\{4, 5\}$

For this special case, corresponding to the case when $p = 4$ and $h = 2$ in our study of Subsections 4.2 and 4.3, it can be noted that the tiling $\{4, 5\}$ is the dual tiling of the tiling $\{5, 4\}$, the pentagrid for which the language of the splitting is regular. From this remark, we deduce that the coordinates which are used for the pentagrid can also be used for the tiling $\{4, 5\}$. In order to show how to perform this association, we remind the construction of the pentagrid using the Fibonacci tree which is in bijection with the tiles contained in a quarter, see Figure 15.

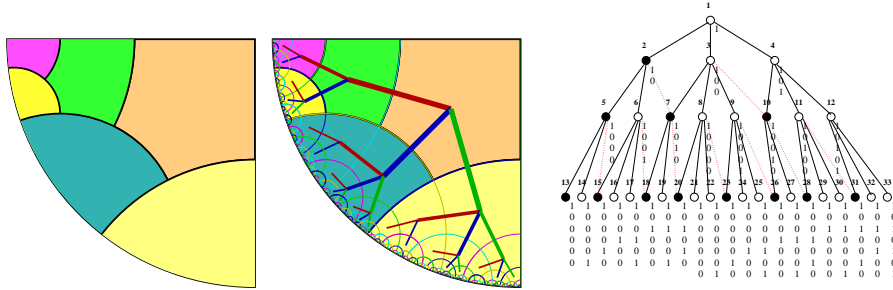


Figure 15 *Left-hand side: splitting a sector of the pentagrid. Middle: the construction of the tree generated by the splitting. Right-hand side: the tree, called **Fibonacci tree** with its black nodes, 2 sons, and its white ones, 3 sons.*

The splitting is summarized by the left-hand side picture of the figure, the middle one showing the recursive construction of the tree yielded by the splitting: the Fibonacci tree. The right-hand side represents the Fibonacci tree which can be defined for itself: it has black nodes and white ones. By definition, black nodes have two sons, white ones have three sons. For each node, exactly one of its sons is black, always the leftmost one. The left-hand side picture of Figure 16 shows how the pentagrid can be covered by five sectors around a central cell.

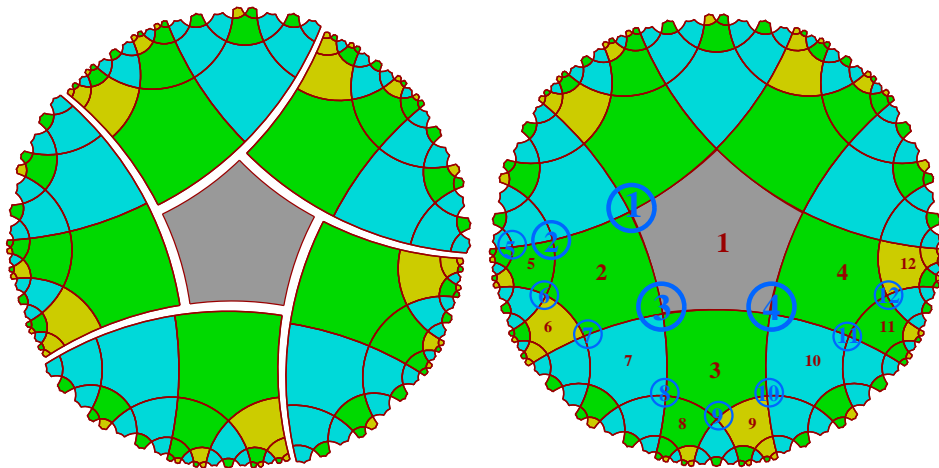


Figure 16 *The correspondence between the numbers of the pentagons and of their vertices.*

Now, we can perform the announced association as follows. We start from the splitting given by the left-hand side picture of Figure 16. On the right-hand side picture of the figure, we have the correspondence for the first three levels of the pentagons. The idea is to cover all vertices of the pentagons of a sector by numbers in a bijective way, the vertices on the rightmost branch being excluded. Number the sides of a pentagon from 1 to 5, counter clockwise turning around the tile, number 1 being given to the side shared by the father of the node. We decide that the father of the head of a sector is the central cell. With this convention, the number of a tile is given to its vertex shared by sides 1 and 2 for a white node and to its vertex shared by sides 2 and 3 for a black node. It is easy to see, by induction on the level of the Fibonacci tree, that with this process, we completely cover the vertices of all pentagons of a sector, excepted the vertices which are on the right-hand side ray which delimit the sector, including the vertex of the sector.

It is not difficult to see that this algorithm can be generalized to all tilings of the form $\{4, p\}$ for which the numbering used for $\{p, 4\}$, very close to that of the pentagrid, could be used in place of the one devised in Section 4.1.

Conclusion

It is interesting that both the splittings we defined in Subsections 4.2 and 4.3 give rise to languages which are always regular, except for the tiling $\{4, 5\}$ in both cases. As shown in [6], these languages can be very different. In particular, there are sharp differences in the forbidden patterns for the language when the constant coefficient of the polynomial of the splitting is positive and when it is negative.

It seems to me that each splitting has its own merit.

For the splitting of Subsection 4.2, we have a polynomial of degree 3, but it is still Pisot in all cases but one: when $p = 4$ and $q = 5$. The advantage of this splitting is that we have less tiles generated at each node by the splitting. So that the process is here closer to the process defined by the recursive application of the reflection in the sides of the previous generation.

The splitting of Subsection 4.3 has two regions only, giving rise to a polynomial of degree 2 which is usually easier to solve than a cubic one. Also, in this case, the language of the splitting is regular in all cases but one: again the case when $p = 4$ and $q = 5$ as with the other splitting. However, the number of tiles which are generated is bigger at each generation. Moreover, in one generation of the splitting process we get tiles which are not in contact, neither by an edge nor by a vertex, with the tiles of the current generation. This is a difference with the traditional definition of a generation. Nevertheless, it can give a formal support to the notion of influence not only on the immediate neighbourhood which may have its own, as the action at a distance in quantum physics or, in mechanics, with gravity. And so, this splitting might have more advantages than the other.

At last, the algorithm of Subsection 4.3.1 and its generalization to the tilings $\{4, p\}$ raises the question whether further generalizations are possible or not. This is a question for future research.

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