Coordinates for vertices of regular honeycombs in hyperbolic space

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(*Communicated by H. S. M. Coxeter, F.R.S.—Received* 16 *December* 1965)

Schlegel (1883) enumerated all regular honeycombs of hyperbolic spaces of three or more dimensions, having finite cells and vertex figures. Coxeter (1954) extended this enumeration to include honeycombs with infinite cells and/or infinite vertex figures, the fundamental region of the symmetry group still being finite. One of these, $\{4, 4, 3\}$, was shown to have for its vertices the points whose coordinates are proportional to the integral solutions of a quadratic Diophantine equation (Coxeter & Whitrow 1950).

In the present paper, certain quadratic Diophantine equations are found whose solutions provide homogeneous coordinates for the vertices of $\{6, 3, 3\}$, $\{6, 3, 4\}$, $\{4, 3, 4, 3\}$ and $\{3, 4, 3, 3, 3\}$. A method is also given for finding coordinates for the vertices of the remaining honeycombs (with finite vertex figures), and the simplest of these are listed.

1. Introduction

In n-dimensional Euclidean space E^n, or hyperbolic space H^n, we define a *honeycomb* to be an infinite set of n-dimensional polytopes Π_n fitting together to fill the space just once, so that every cell Π_{n-1} of each Π_n belongs to just one other Π_n. The vertex figure at a vertex 0 is the n-dimensional polytope whose vertices are those vertices of the honeycomb which are joined to 0 by edges of the honeycomb. A honeycomb whose cells Π_n are equal regular polytopes is said to be *regular*. If the Π_n has the Schläfli symbol $\{p, q, \ldots, r\}$ and its dihedral angle is $2\pi/s$, so that s Π_n's surround each element Π_{n-2}, the honeycomb has the Schläfli symbol

$$\{p, q, \ldots, r, s\}$$

(Coxeter 1963, pp. 127–130); its dual is $\{s, r, \ldots, q, p\}$. A two-dimensional honeycomb is usually called a *tessellation* (Coxeter 1964, p. 147).

In H^3 there are eight regular honeycombs whose vertices are not at infinity, namely

$$\{3,5,3\},\quad \{4,3,5\},\quad \{5,3,4\},\quad \{5,3,5\},\quad \{4,4,3\},\quad \{6,3,3\},\quad \{6,3,4\},\quad \{6,3,5\}$$

(Coxeter 1954, p. 157). In H^4 there are six regular honeycombs whose vertices are not at infinity:

$$\{3,3,3,5\},\quad \{4,3,3,5\},\quad \{5,3,3,5\},\quad \{5,3,3,4\},\quad \{5,3,3,3\},\quad \{4,3,4,3\};$$

the only regular honeycomb in H^5 whose vertices are not at infinity is

$$\{3,4,3,3,3\}$$

(Coxeter 1954, p. 160) and there are no regular honeycombs in H^n for $n \geqslant 6$ (Schlegel 1883, p. 455).

In § 2 we shall consider certain quadratic Diophantine equations whose solutions provide homogeneous coordinates for the vertices of

$$\{4,4,3\},\quad \{6,3,3\},\quad \{6,3,4\},\quad \{4,3,4,3\},\quad \{3,4,3,3,3\}.$$

For the remaining honeycombs in H^4 and H^5, whose Schläfli symbols contain at least one 5, the search for integral coordinates would be just as futile as in the case of the pentagon $\{5\}$. Instead, we shall give in §3 a method for deriving coordinates for a finite number of vertices, work out the coordinates for the first few vertices of $\{5, 3, 3, 4\}$, and state analogous results for the others.

2. Certain quadratic diophantine equations

The honeycombs

$$\{4,4,3\}, \quad \{6,3,5\}, \quad \{6,3,4\}, \quad \{6,3,3\}, \quad \{4,3,4,3\}, \quad \{3,4,3,3,3\},$$

are quite different from the other regular honeycombs, as their cells are not finite polytopes but 'Euclidean' honeycombs, inscribed in horospheres. In this section we shall obtain quadratic Diophantine equations whose solutions provide homogeneous coordinates for the vertices of the last four honeycombs. The problem has already been solved for $\{4, 4, 3\}$ (Coxeter & Whitrow 1950, p. 428); its vertices are represented by the integral solutions of

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = 1.$$

This is analogous to the obvious statement that the vertices of a cube $\{4, 3\}$ in Euclidean space, or of the corresponding spherical tessellation, are represented by the integral solutions of

$$x_1^2 + x_2^2 + x_3^2 = 3,$$

or that the vertices of the 24-cell $\{3, 4, 3\}$ are represented by the integral solutions of

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 2.$$

The essential difference is that, for spherical honeycombs, the quadratic forms are definite, while for hyperbolic honeycombs they are indefinite. We shall sometimes have occasion to insert congruences as well as equations, as when one says that the vertices of a tetrahedron $\{3, 3\}$ are represented by those integral solutions of

$$x_1^2 + x_2^2 + x_3^2 = 3,$$

which satisfy

$$x_1 + x_2 + x_3 \equiv 1 \pmod 4.$$

We shall use $(x_0; x_1, x_2, \ldots, x_l)^S$ to denote all points obtained by permutations of the coordinates to the right of the semicolon, and $(x_0; x_1, x_2, \ldots, x_l)^A$ to denote all points obtained by even permutations of the coordinates to the right of the semicolon. (S for symmetric; A for alternating.)

Theorem 1. *All the integral solutions of*

$$x_0^2 - 3(x_1^2 + x_2^2 + x_3^2) = 1 \tag{1}$$

are homogeneous coordinates for all the vertices of the honeycomb $\{6, 3, 4\}$ *in* H^3 *with absolute quadric*

$$x_0^2 - 3(x_1^2 + x_2^2 + x_3^2) = 0. \tag{2}$$

Proof. Let the integral solutions of (1) be regarded as homogeneous coordinates for points (x_0, x_1, x_2, x_3) in H^3 with absolute quadric (2). The points nearest to

$(1, 0, 0, 0)$ are $(2;\ \pm 1, 0, 0)^S$. These six points are the vertices of a regular octahedron of edge-length arg cosh 4 (Coxeter 1957, p. 209).

The three vertices $(2; 1, 0, 0)^S$ of a triangular face of this octahedron belong, with $(1, 0, 0, 0)$, to some tessellation on the horosphere

$$x_0 - x_1 - x_2 - x_3 = 1. \tag{3}$$

Solving this equation along with (1), we obtain the points

$$(3a - b - c,\ a - 1,\ a - b,\ a - c),$$

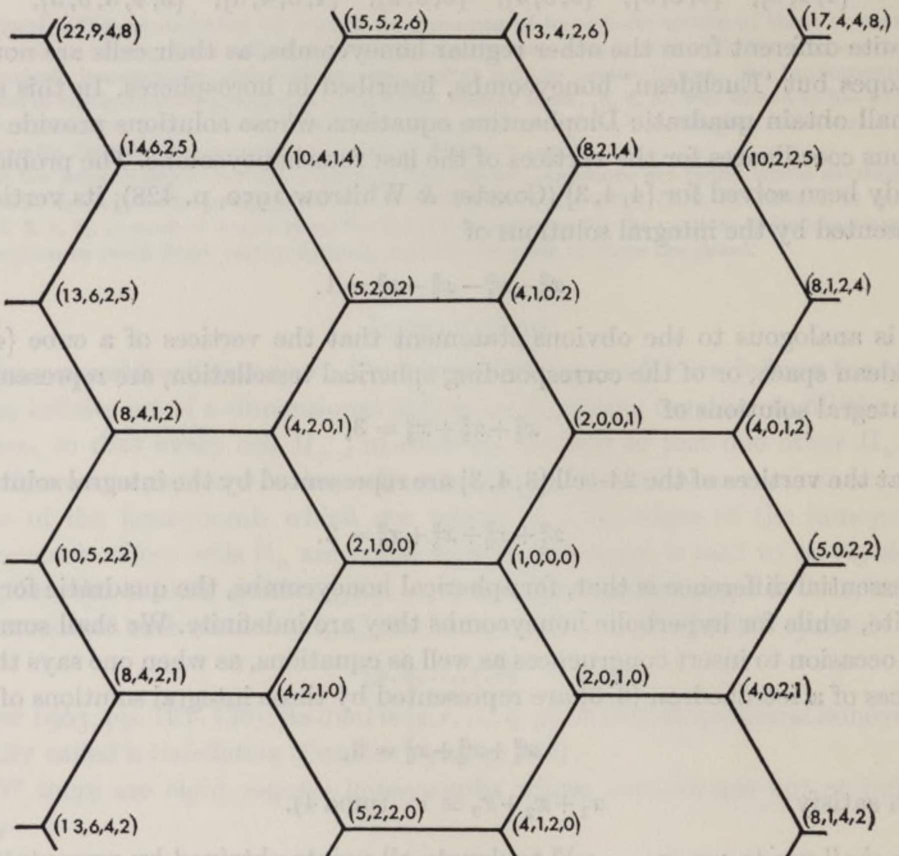

Figure 1

where $3a = b^2 - bc + c^2 + 2$ for all integers b and c satisfying $b + c \not\equiv 0 \pmod 3$. Figure 1 shows that these points are the vertices of a tessellation $\{6, 3\}$. Thus we have the beginning of a honeycomb of $\{6, 3\}$'s with vertex figure $\{3, 4\}$, namely $\{6, 3, 4\}$ (figure 1).

We must prove that every solution of (1) represents a vertex of the honeycomb, and conversely. For this purpose, we derive $\{6, 3, 4\}$ by Wythoff's construction from its characteristic tetrahedron, which is formed by the vertex $(1, 0, 0, 0)$, the mid-edge point $(3, 1, 0, 0)$, the face centre $(3, 1, 1, 0)$, and the cell centre $(3, 1, 1, 1)$ (see figure 2). The last point, satisfying (2), lies on the absolute; in fact, it is the centre of the horosphere (3). Omitting in turn each vertex of this characteristic tetrahedron, we find the equations of its face planes to be

$$x_0 - 3x_1 = 0; \quad x_1 - x_2 = 0; \quad x_2 - x_3 = 0; \quad x_3 = 0.$$

The reflexions in these planes generate the symmetry group of the honeycomb and are given by:

$$A \qquad \begin{cases} x_i' = x_i + (x_0 - 3x_1) & (i = 0, 1), \\ x_i' = x_i & (i = 2, 3), \end{cases}$$

B the transposition $(x_1 x_2)$,

C the transposition $(x_2 x_3)$,

D the change of sign of x_2.

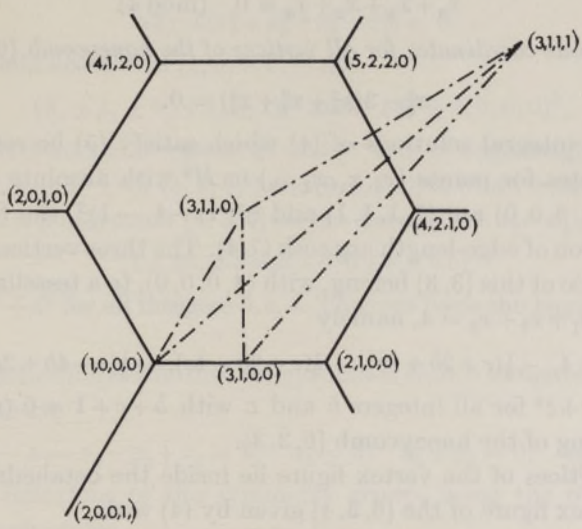

FIGURE 2

Let (x_0, x_1, x_2, x_3) be any solution of (1). Since

$$(2x_0 - 3x_1)^2 - 3[(x_0 - 2x_1)^2 + x_2^2 + x_3^2] = x_0^2 - 3(x_1^2 + x_2^2 + x_3^2),$$

we still have a solution after transforming the given solution by A. Thus, A, B, C and D are automorphs of the quadratic form $x_0^2 - 3(x_1^2 + x_2^2 + x_3^2)$. Since $(1, 0, 0, 0)$ and all its transforms by reflexion in the face planes of this characteristic tetrahedron are solutions of (1), every vertex of this honeycomb has, as homogeneous coordinates, an integral solution of (1).

Conversely, we shall show that a suitable combination of the above four transformations will transform an arbitrary solution into the solution $(1, 0, 0, 0)$.

Since the coordinates are homogeneous, we can take $x_0 > 0$. Supposing $x_0 > 1$ so that at least one of x_1, x_2, x_3 is non-zero, we can use B, C and D to reach a solution in which $x_1 \geqslant x_2 \geqslant x_3 \geqslant 0$. Now

$$x_0 = \sqrt{(1 + 3x_1^2 + 3x_2^2 + 3x_3^2)} \leqslant \sqrt{(1 + 9x_1^2)} < 1 + 3x_1,$$

and so $x_0 - 3x_1 \leqslant 0$. Since $x_0^2 - 1 = 3(x_1^2 + x_2^2 + x_3^2) \equiv 0 \pmod 3$, we have $x_0 \not\equiv 0 \pmod 3$. Thus $x_0 - 3x_1$ is strictly negative. Also

$$1 + 3x_1 = 1 + x_1 + x_1 + x_1 \leqslant \sqrt{[4(1 + x_1^2 + x_1^2 + x_1^2)]} \leqslant 2x_0.$$

Therefore $2x_0 - 3x_1 \geqslant 1$.

Hence A decreases x_0 but leaves it positive. By repeated application we must eventually reach a solution with $x_0 = 1$, namely $(1, 0, 0, 0)$. Reversing the procedure, we have a sequence of symmetry operations leading from $(1, 0, 0, 0)$ to the given solution (x_0, x_1, x_2, x_3). Thus this arbitrary solution does represent a vertex of the honeycomb, and the theorem is proved.

THEOREM 2. *All the integral solutions of*

$$x_0^2 - 3(x_1^2 + x_2^2 + x_3^2) = 16, \tag{4}$$

which satisfy

$$x_0 + x_1 + x_2 + x_3 \equiv 0 \pmod 4 \tag{5}$$

provide homogeneous coordinates for all vertices of the honeycomb $\{6, 3, 3\}$ *in* H^3 *with absolute quadric*

$$x_0^2 - 3(x_1^2 + x_2^2 + x_3^2) = 0. \tag{2}$$

Proof. Let the integral solutions of (4) which satisfy (5) be regarded as homogeneous coordinates for points (x_0, x_1, x_2, x_3) in H^3 with absolute quadric (2). The points nearest $(4, 0, 0, 0)$ are $(5, 1, 1, 1)$ and $(5; 1, -1, -1)^S$, the four vertices of a regular tetrahedron of edge-length arg cosh $(7/4)$. The three vertices $(5; 1, -1, -1)^S$ of a triangular face of this $\{3, 3\}$ belong, with $(4, 0, 0, 0)$, to a tessellation $\{6, 3\}$ on the horosphere $x_0 + x_1 + x_2 + x_3 = 4$, namely

$$(r+4,\ -\tfrac{1}{3}(r+2b+2c),\ -\tfrac{1}{3}(r+2b-4c),\ -\tfrac{1}{3}(r-4b+2c)), \tag{6}$$

where $r = b^2 - bc + c^2$ for all integers b and c with $b + c + 1 \not\equiv 0 \pmod 3$. Thus we have the beginning of the honeycomb $\{6, 3, 3\}$.

These four vertices of the vertex figure lie inside the octahedron $(8; \pm 4, 0, 0)^S$ which is the vertex figure of the $\{6, 3, 4\}$ given by (4) with

$$x_0 \equiv x_1 \equiv x_2 \equiv x_3 \equiv 0 \pmod 4$$

(cf. theorem 1). The reflexion which interchanges $(4, 0, 0, 0)$ and $(5, 1, 1, 1)$ is

$$E \qquad x_i' = \tfrac{1}{4}[4x_i + (x_0 - 3x_1 - 3x_2 - 3x_3)] \quad (i = 0, 1, 2, 3).$$

This is an automorph of (4) and satisfies the congruence (5), since

$$\begin{aligned} x_0' + x_1' + x_2' + x_3' &= \tfrac{1}{4}[4(x_0 + x_1 + x_2 + x_3) + 4x_0 - 12x_1 - 12x_2 - 12x_3] \\ &= x_0 + x_1 + x_2 + x_3 + x_0 - 3x_1 - 3x_2 - 3x_3 \equiv 0 \pmod 4. \end{aligned}$$

Thus the generating reflexions of the symmetry group of this honeycomb $\{6, 3, 3\}$ are the A, B, C, D of theorem 1 and E. Any vertex of the honeycomb can be transformed into $(4, 0, 0, 0)$ by means of these five transformations, and so has as coordinates a solution of (4) satisfying (5).

Conversely, let (x_0, x_1, x_2, x_3) be any solution of (4) satisfying (5). Using B, C, D as needed, we may assume that x_1, x_2, x_3 are all positive. Since the coordinates are homogeneous, we may take $x_0 > 4$. Then

$$x_0 = \sqrt{(16 + 3x_1^2 + 3x_2^2 + 3x_3^2)} < 4 + 3x_1 + 3x_2 + 3x_3 \leqslant \sqrt{\{25(16 + 3x_1^2 + 3x_2^2 + 3x_3^2)\}} = 5x_0$$

and so $\quad x_0 - 3x_1 - 3x_2 - 3x_3 < 4 \quad$ and $\quad 5x_0 - 3x_1 - 3x_2 - 3x_3 \geqslant 0.$

By (5), $x_0-3x_1-3x_2-3x_3 \equiv 0 \pmod 4$, and since $x_0 \not\equiv 0 \pmod 3$, $x_0-3x_1-3x_2-3x_3$ is strictly negative. Hence E can be used to transform this arbitrary solution into another solution with smaller, yet still positive, x_0. After a finite number of steps we reach a solution with $x_0 = 4$, namely $(4,0,0,0)$. Reversing the procedure, we have a sequence of symmetry operations transforming $(4,0,0,0)$ into (x_0,x_1,x_2,x_3). Thus any solution of (4) satisfying (5) provides homogeneous coordinates for a vertex of $\{6,3,3\}$.

The proofs of the following two theorems follow the same line of reasoning as that of Theorem 1 and will be presented here as briefly as possible.

THEOREM 3. *All the integral solutions of*

$$x_0^2-2(x_1^2+x_2^2+x_3^2+x_4^2) = 1 \tag{7}$$

provide homogeneous coordinates for the vertices of $\{4,3,4,3\}$ *in* H^4 *with absolute quadric*

$$x_0^2-2(x_1^2+x_2^2+x_3^2+x_4^2) = 0. \tag{8}$$

Proof. The points nearest to $(1,0,0,0,0)$ are

$$(3, \pm 1, \pm 1, \pm 1, \pm 1) \quad \text{and} \quad (3; \pm 2,0,0,0)^S,$$

the twenty-four vertices of a regular 24-cell $\{3,4,3\}$ of edge-length arg cosh 5. The six vertices $(3,1,1,\pm 1,\pm 1)$, $(3,2,0,0,0)$, $(3,0,2,0,0)$ of an octahedral cell of this $\{3,4,3\}$ belong to a honeycomb $\{4,3,4\}$ on the horosphere $x_0-x_1-x_2 = 1$, namely

$$(1+2a, a+d, a-d, b+c, b-c), \tag{9}$$

where $a = b^2+c^2+d^2$ for all integers b, c, d. Thus we have the beginning of a honeycomb $\{4,3,4,3\}$.

The characteristic simplex of this honeycomb is shown in figure 3. The equations of its hyperplanes are

$$x_0-2x_1 = 0; \quad x_1-x_2-x_3+x_4 = 0; \quad x_4 = 0; \quad x_3+x_4 = 0; \quad x_2-x_3 = 0.$$

The generating reflexions of the symmetry group, being the reflexions in these hyperplanes, are given by

$$A \qquad \begin{cases} x_i' = x_i+2(x_0-2x_1) & (i=0,1), \\ x_i' = x_i & (i=2,3,4), \end{cases}$$

$$B \qquad \begin{cases} x_0' = x_0 \\ x_i' = x_i-\tfrac{1}{2}(x_1-x_2-x_3+x_4) & (i=1,4), \\ x_i' = x_i+\tfrac{1}{2}(x_1-x_2-x_3+x_4) & (i=2,3), \end{cases}$$

C the reversal of sign of x_4,

D the transposition $(x_3 x_4)$ with change of signs,

E the transposition $(x_2 x_3)$.

The transformation BA is a rotation about the plane common to the first two hyperplanes and is given by

$$BA \qquad \begin{cases} x_0' = x_0+2(x_0-x_1-x_2-x_3+x_4), \\ x_1' = \tfrac{1}{2}x_0+\tfrac{3}{2}(x_0-x_1-x_2-x_3+x_4), \\ x_i' = x_i+\tfrac{1}{2}(x_1-x_2-x_3+x_4) \quad (i=2,3), \\ x_4' = x_4-\tfrac{1}{2}(x_1-x_2-x_3+x_4). \end{cases}$$

Since $(1, 0, 0, 0, 0)$ is a solution of (7), and since the transformations A, B, C, D, E can be shown to be automorphs of the quadratic form $x_0^2 - 2(x_1^2 + x_2^2 + x_3^2 + x_4^2)$, all the vertices of this honeycomb $\{4, 3, 4, 3\}$ have, as coordinates, solutions of (7).

Assuming an arbitrary solution $(x_0, x_1, x_2, x_3, x_4)$ with $x_0 > 1$, we shall show that this can be transformed into another solution with $x_0 = 1$. Using B, C, D and E as needed, we can reach a solution with $-x_4 \geqslant x_3 \geqslant x_2 \geqslant 0$.

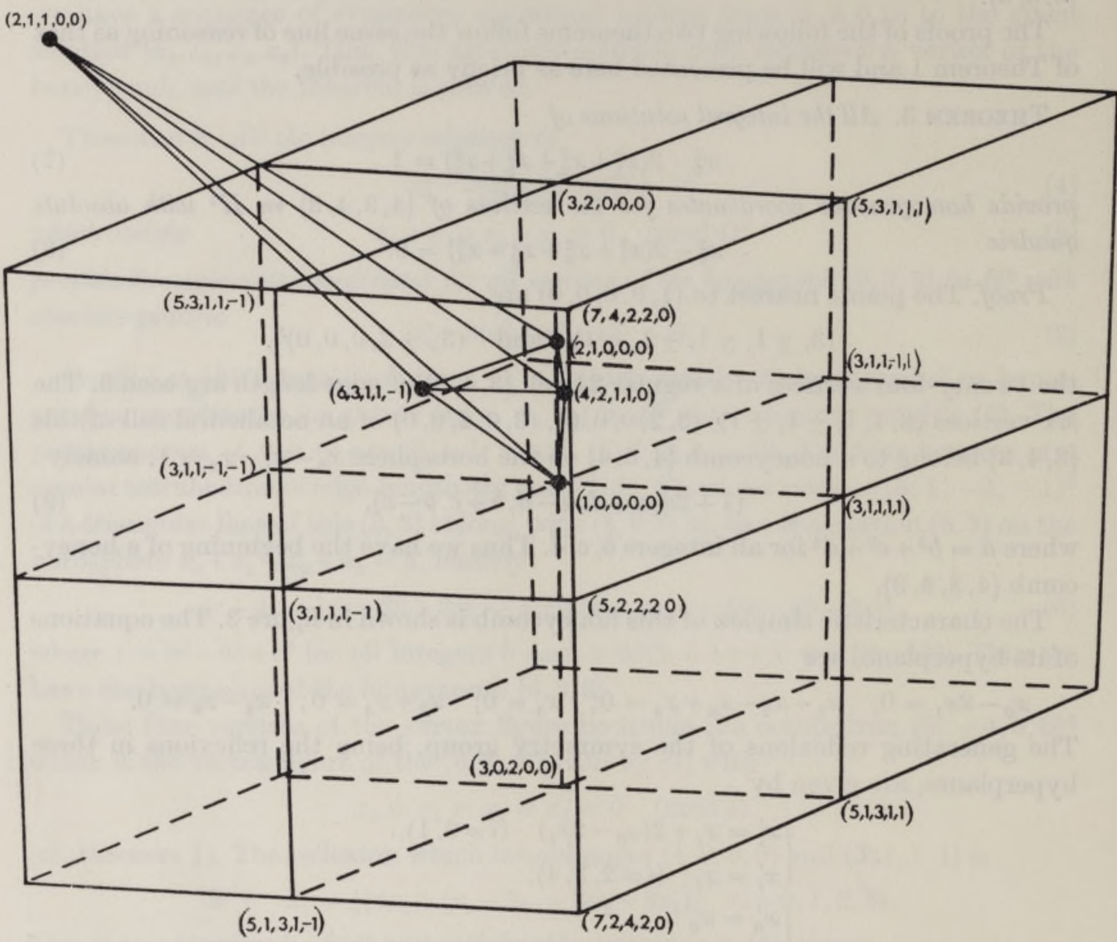

Figure 3

Further, we may suppose $x_1 \geqslant 0$. For suppose $x_1 < 0$. Transforming by B, we obtain

$$(x_0', x_1', x_2', x_3', x_4') = (x_0, \tfrac{1}{2}(x_1 + x_2 + x_3 - x_4), \tfrac{1}{2}(x_1 + x_2 - x_3 + x_4),$$
$$\tfrac{1}{2}(x_1 - x_2 + x_3 + x_4), \tfrac{1}{2}(-x_1 + x_2 + x_3 + x_4)).$$

If $x_1' \geqslant 0$, the assertion is proved. But if $x_1' < 0$, $x_2 + x_3 - x_4 < -x_1$. Then x_2' and x_3' are negative, and x_4' positive. Applying the transformation $CDECDC$ we obtain†

$$(x_0'', x_1'', x_2'', x_3'', x_4'') = (x_0, \tfrac{1}{2}(x_1 + x_2 + x_3 - x_4), \tfrac{1}{2}(-x_1 + x_2 + x_3 + x_4),$$
$$\tfrac{1}{2}(-x_1 + x_2 - x_3 - x_4), \tfrac{1}{2}(x_1 + x_2 - x_3 + x_4)).$$

† $XY \ldots Z$ means: apply X first, then $Y \ldots$, then Z.

Under B this becomes

$$(x_0''', x_1''', x_2''', x_3''', x_4''') = (x_0, \tfrac{1}{2}(-x_1+x_2+x_3-x_4), \tfrac{1}{2}(x_1+x_2+x_3+x_4), \tfrac{1}{2}(x_1+x_2-x_3-x_4), \tfrac{1}{2}(-x_1+x_2-x_3+x_4))$$

and $x_1''' > 0$. In the following, we shall assume the arbitrary solution $(x_0, x_1, x_2, x_3, x_4)$ is such that $x_1 \geqslant 0$ and $-x_4 \geqslant x_3 \geqslant x_2 \geqslant 0$.

First, suppose $x_0 - x_1 - x_2 - x_3 + x_4 < 0$. Then BA decreases x_0, and since

$$3x_0 = \sqrt{\{18(x_1^2+x_2^2+x_3^2+x_4^2)+9\}} > \sqrt{\{4(4x_1^2+4x_2^2+4x_3^2+4x_4^2)\}} \geqslant 2x_1+2x_2+2x_3-2x_4,$$

we have $3x_0 - 2x_1 - 2x_2 - 2x_3 + 2x_4 > 0$, and so the x_0 coordinate remains positive.

Secondly, suppose $x_0 - 2x_1 < 0$. Then A decreases x_0, but leaves it positive. For suppose $x_0' = 3x_0 - 4x_1 \leqslant 0$, that is, $3x_0 \leqslant 4x_1$. Then

$$16x_1^2 \geqslant 9x_0^2 = 18(x_1^2+x_2^2+x_3^2+x_4^2)+9,$$

which is impossible.

Finally, suppose neither $x_0 - x_1 - x_2 - x_3 + x_4$ nor $x_0 - 2x_1$ is negative. Then

$$x_0 \geqslant 2x_1 \tag{10}$$

and
$$x_0 \geqslant x_1 + x_2 + x_3 + x_4. \tag{11}$$

Applying B to the arbitrary solution we get the point

$$(x_0', x_1', x_2', x_3', x_4') = (x_0, \tfrac{1}{2}(x_1+x_2+x_3-x_4), \tfrac{1}{2}(x_1+x_2-x_3+x_4), \tfrac{1}{2}(x_1-x_2+x_3+x_4), \tfrac{1}{2}(-x_1+x_2+x_3+x_4)).$$

Since $x_0' - 2x_1' = x_0 - x_1 - x_2 - x_3 + x_4$, we cannot use A to decrease x_0. Instead, apply the transformation $DECDC$, obtaining

$$(x_0'', x_1'', x_2'', x_3'', x_4'') = (x_0, \tfrac{1}{2}(x_1+x_2+x_3-x_4), \tfrac{1}{2}(x_1-x_2-x_3-x_4), \tfrac{1}{2}(-x_1+x_2-x_3-x_4), \tfrac{1}{2}(x_1+x_2-x_3+x_4))$$

and $x_0'' - x_1'' - x_2'' - x_3'' + x_4'' = x_0 + 2x_4$. To show that this is negative, we shall demonstrate the inconsistency of (10), (11) and

$$x_0 \geqslant 2x_4. \tag{12}$$

For definiteness, assume $-x_4 \geqslant x_1$. From (7) and (12) we have

$$4x_4^2 \leqslant 2(x_1^2+x_2^2+x_3^2+x_4^2)+1.$$

Thus
$$x_4^2 \leqslant x_1^2+x_2^2+x_3^2. \tag{13}$$

But from (7) and (11), $(x_1+x_2+x_3-x_4)^2 \leqslant 2(x_1^2+x_2^2+x_3^2+x_4^2)+1$; from which, using (13), we get

$$x_1^2+x_2^2+x_3^2 \geqslant x_1x_2 + x_1x_3 - x_1x_4 + x_2x_3 - x_2x_4 - x_3x_4 - \tfrac{1}{2},$$

or
$$\tfrac{1}{2} > x_1x_2 + x_1x_3 + x_2^2.$$

This is only possible if $x_2 = 0$ and one of x_1, x_3 is zero. Thus we have a contradiction unless the point under consideration is of the form

$$(x_0, 0, 0, x, x_4) \quad \text{or} \quad (x_0, x, 0, 0, x_4) \quad (x > 0).$$

Then by (13), $x_4^2 \leqslant x^2$, and by assumption, $-x_4 \geqslant x$. Thus the point is of the form

$$(x_0, 0, 0, x, -x) \quad \text{or} \quad (x_0, x, 0, 0, -x) \quad (x>0).$$

By (7), $x_0^2 = 4x^2+1$, which is impossible with $x \neq 0$. Thus

$$x_0'' - x_1'' - x_2'' - x_3'' + x_4'' = x_0 + 2x_4$$

is negative, and BA can be applied to decrease x_0''. We have seen that if $x_2'', x_3'' \geqslant 0$ and $x_4'' < 0$, then x_0'' will remain positive. It is easily seen that this is still the case if any number of these conditions is violated. For example, if $x_2'', x_3'' < 0$ and $x_4'' \geqslant 0$, then, as above,

$$3x_0'' > \sqrt{\{4(4x_1''^2 + 4x_2''^2 + 4x_3''^2 + 4x_4''^2)\}} \geqslant 2x_1'' - 2x_2'' - 2x_3'' + 2x_4'',$$

and so $3x_0'' - 2x_1'' + 2x_2'' + 2x_3'' - 2x_4'' > 0$. Therefore

$$3x_0'' - 2x_1'' - 2x_2'' - 2x_3'' + 2x_4'' + 4(x_2'' + x_3'' - x_4'') > 0.$$

Since $x_2'' + x_3'' - x_4''$ is negative, $3x_0'' - 2x_1'' - 2x_2'' - 2x_3'' + 2x_4'' > 0$.

Thus any solution can be transformed into another solution with smaller, yet still positive, x_0 coordinate. By repeated application we must eventually reach a solution with $x_0 = 1$, namely $(1, 0, 0, 0, 0)$. Reversing the procedure, we have a sequence of symmetry operations leading from $(1, 0, 0, 0, 0)$ to the given solution, and the theorem is proved.

Finally we prove

THEOREM 4. *The integral solutions of*

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2 = 4 \tag{14}$$

which satisfy

$$x_0 \equiv x_1 \equiv x_2 \equiv x_3 \equiv x_4 \equiv x_5 \pmod 2 \tag{15}$$

provide homogeneous coordinates for the vertices of $\{3, 4, 3, 3, 3\}$ *in* H^5 *with absolute quadric*

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2 = 0. \tag{16}$$

Proof. Let the integral solutions of (14) satisfying (15) be regarded as homogeneous coordinates for points $(x_0, x_1, x_2, x_3, x_4, x_5)$ in H^5 with absolute quadric (16). The points nearest $(2, 0, 0, 0, 0, 0)$ are $(3, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$, thirty-two in all. Since the distance between $(x_0, x_1, x_2, x_3, x_4, x_5)$ and $(y_0, y_1, y_2, y_3, y_4, y_5)$ is given by

$$\operatorname{arg\,cosh}\left[\frac{x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3 - x_4 y_4 - x_5 y_5}{\sqrt{(x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2)}\sqrt{(y_0^2 - y_1^2 - y_2^2 - y_3^2 - y_4^2 - y_5^2)}}\right]$$

(Coxeter 1957, p. 204), these points are the vertices of a $\{4, 3, 3, 3\}$.

The sixteen vertices $(3, 1, \pm 1, \pm 1, \pm 1, \pm 1)$ belong, with $(2, 0, 0, 0, 0, 0)$, to a honeycomb $\{3, 4, 3, 3\}$ on the horosphere $x_0 - x_1 = 2$, namely

$$(a+2, a, b, c, d, e),$$

where $4a = b^2 + c^2 + d^2 + e^2$, for all even integers b, c, d, e such that

$$b^2 + c^2 + d^2 + e^2 \equiv 0 \pmod 8$$

and for all odd integers b, c, d, e such that $b^2 + c^2 + d^2 + e^2 \equiv 4 \pmod 8$. Thus we have the beginning of a honeycomb $\{3, 4, 3, 3, 3\}$.

The characteristic simplex is formed by the vertex $(2,0,0,0,0,0)$; the mid-edge point $(5,1,1,1,1,1)$; the centre of a $\{3\}$, $(4,1,1,1,1,0)$; the centre of a $\{3,4\}$, $(3,1,1,1,0,0)$; the centre of a $\{3,4,3\}$, $(2,1,1,0,0,0)$; and the cell centre $(1,1,0,0,0,0)$, which is on the absolute. Omitting these vertices one by one, we find that the equations of the face hyperplanes are

$$x_0-x_1-x_2-x_3-x_4-x_5=0;\quad x_5=0;\quad x_4-x_5=0;$$
$$x_3-x_4=0;\quad x_2-x_3=0;\quad x_1-x_2=0.$$

The reflexions in these hyperplanes generate the symmetry group of the honeycomb and are given by

A $\quad x_i'=x_i+\frac{1}{2}(x_0-x_1-x_2-x_3-x_4-x_5)\quad (i=0,1,2,3,4,5)$,

B $\quad$ the change of sign of x_5,

C $\quad$ the transposition (x_4x_5),

D $\quad$ the transposition (x_3x_4),

E $\quad$ the transposition (x_2x_3),

F $\quad$ the transposition (x_1,x_2).

We must prove that every integral solution of (14) and (15) represents a vertex of this honeycomb, and conversely.

Let $(x_0,x_1,x_2,x_3,x_4,x_5)$ be any solution of (14) and (15). Then transforming by B,C,D,E or F, we obviously have another solution. Moreover, since

$$\begin{aligned}&\tfrac{1}{4}[(3x_0-x_1-x_2-x_3-x_4-x_5)^2-(x_0+x_1-x_2-x_3-x_4-x_5)^2\\&\quad-(x_0-x_1+x_2-x_3-x_4-x_5)^2-(x_0-x_1-x_2+x_3-x_4-x_5)^2\\&\quad-(x_0-x_1-x_2-x_3+x_4-x_5)^2-(x_0-x_1-x_2-x_3-x_4+x_5)^2]\\&=x_0^2-x_1^2-x_2^2-x_3^2-x_4^2-x_5^2,\end{aligned}$$

the point obtained after transforming by A is still a solution of (14); it also satisfies (15). Since $(2,0,0,0,0,0)$ and all its transforms by reflexion in the face hyperplanes of this characteristic simplex are solutions of (14) and (15), every vertex of this honeycomb has, as homogeneous coordinates, an integral solution of (14) and (15).

Conversely, we shall show that a suitable combination of the above four transformations will transform an arbitrary solution into one with $x_0=2$, namely $(2,0,0,0,0,0)$. Since the coordinates are homogeneous, we may take $x_0>0$, and we assume $x_0>2$. Using B,C,D,E and F as needed, we can reach a solution with $x_i\geqslant 0$ $(i=1,2,3,4,5)$. Then

$$\begin{aligned}x_0&=\sqrt{(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+4)}<x_1+x_2+x_3+x_4+x_5+2\\&\leqslant\sqrt{\{6(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+4)\}}<3x_0,\end{aligned}$$

from which we infer

$$x_0-x_1-x_2-x_3-x_4-x_5<2\quad\text{and}\quad\tfrac{1}{2}(3x_0-x_1-x_2-x_3-x_4-x_5)>1.$$

From (15), $x_0-x_1-x_2-x_3-x_4-x_5$ is even, and so it is less than or equal to zero. Supposing equality, and substituting in (14), we obtain

$$(x_1+x_2+x_3+x_4+x_5)^2=x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+4,$$

or

$$\sum_{\substack{i,j=1\\i\neq j}}^{5}x_ix_j=2.$$

But this is only possible if three of x_1, x_2, x_3, x_4, x_5 are zero and the other two are 1 and 2 respectively. Since such a solution does not satisfy (15), we have a contradiction.

Thus $x_0 - x_1 - x_2 - x_3 - x_4 - x_5$ is negative and A decreases each coordinate while leaving x_0 positive. By repeated application we eventually reach a solution with $x_0 = 2$, that is $(2, 0, 0, 0, 0, 0)$. Reversing the procedure, we have a sequence of symmetry operations leading from $(2, 0, 0, 0, 0, 0)$ to the given solution

$$(x_0, x_1, x_2, x_3, x_4, x_5),$$

and so the theorem is proved.

3. Coordinates for a finite number of vertices of a honeycomb

Although it is not always possible to obtain a quadratic Diophantine equation whose solutions provide coordinates for all the vertices of the honeycomb, another procedure can be used to find coordinates for a finite number of vertices. We shall refer to the table (Coxeter 1954, pp. 167–8) of edge-lengths of the regular honeycombs in hyperbolic space.

An n-dimensional hyperbolic honeycomb $\{p, q, \ldots, s\}$ with edge-length 2ϕ, can be regarded as a polytope $\{p, q, \ldots, s\}$, of (spacelike) edge $2l$ and (timelike) circumradius $R = l/\sinh\phi$ in a Minkowskian $(n+1)$-space with metric form $x_0^2 - x_1^2 - \ldots - x_n^2$. In the latter aspect, the vertices all lie on the 'sphere' $x_0^2 - x_1^2 - \ldots - x_n^2 = R^2$ (Coxeter 1943, pp. 221–2) and the vertex figure is the Euclidean polytope $\{q, \ldots, s\}$ of edge-length $2cl$ where $c = 2\cos(\pi/p)$. Any convenient coordinates $(x_1, x_2, \ldots, x_n)$ for the vertices of $\{q, \ldots, s\}$ can be used, in the form $(x_0, x_1, x_2, \ldots, x_n)$, for those vertices of the Minkowskian polytope which are nearest to the particular vertex $(R, 0, 0, \ldots, 0)$. By a bit of Minkowskian trigonometry, of which the Euclidean analogue is familiar (Robb 1921, p. 73), we can use the isosceles triangle formed by the centre $(0, 0, \ldots, 0)$ and an edge whose vertices are $(R, 0, \ldots, 0)$ and $(x_0, x_1, \ldots, x_n)$ to obtain

$$x_0 = R\cosh 2\phi.$$

For instance, the vertex figure of $\{5, 3, 3, 4\}$ is the $\{3, 3, 4\}$ whose eight vertices are $(x_0;\ \pm 1, 0, 0, 0)^S$. In this case,

$$c = 2\cos\tfrac{1}{5}\pi = \tau = \tfrac{1}{2}(\sqrt{5}+1), \quad 2cl = \sqrt{2}, \quad \text{and} \quad \cosh^2\phi = \tfrac{1}{2}\tau^2;$$

therefore

$$\sinh^2\phi = \tfrac{1}{2}\tau^{-1}, \quad \cosh 2\phi = \tau, \quad l = \tau^{-1}/\sqrt{2}, \quad R = \tau^{-\frac{1}{2}}, \quad x_0 = \tau^{\frac{1}{2}}.$$

Thus the initial vertex is conveniently taken to be $(\tau^{-\frac{1}{2}}, 0, 0, 0, 0)$ and the neighbouring vertices $(\tau^{\frac{1}{2}};\ \pm 1, 0, 0, 0)^S$.

Consider a pentagonal face $ABCDE$ with $A = (\tau^{-\frac{1}{2}}, 0, 0, 0, 0)$, $B = (\tau^{\frac{1}{2}}, 1, 0, 0, 0)$, $E = (\tau^{\frac{1}{2}}, 0, 1, 0, 0)$, lying in the plane

$$\tau^{\frac{1}{2}} x_0 - x_1 - x_2 = \tau, \quad x_3 = x_4 = 0.$$

The vertex C, being distant $\sqrt{2}\,\tau^{-1}$ from B and $\sqrt{2}$ from A and E, is $(\tau^{\frac{3}{2}}, \tau, 1, 0, 0)$. Considering all such pentagonal faces meeting at A, we conclude that the forty-eight vertices which are the extremities of first diagonals of $\{5, 3, 3, 4\}$ from the vertex $(\tau^{-\frac{1}{2}}, 0, 0, 0, 0)$ are $(\tau^{\frac{3}{2}};\ \pm\tau, \pm 1, 0, 0)^S$.

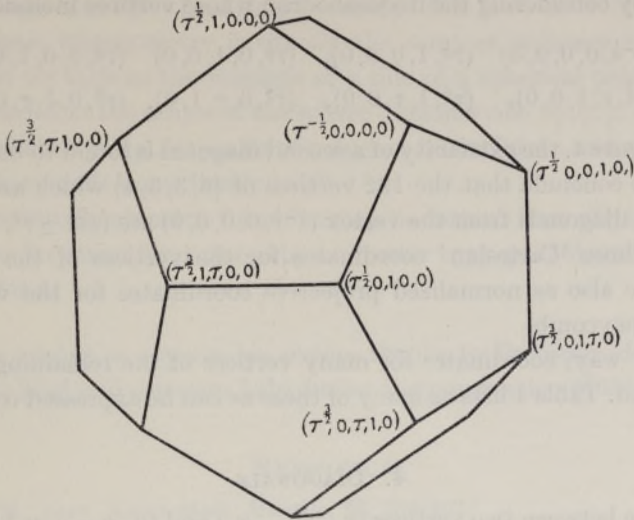

FIGURE 4

TABLE 1

polytope or honeycomb	$\sinh^2\phi$	$\cosh 2\phi$	l	R	vertices	section of polytope
{4, 3, 5}	$\frac{1}{2}\tau$	τ^2	$\sqrt{\frac{1}{2}}$	$\tau^{-\frac{1}{2}}$	$(\tau^{-\frac{1}{2}}, 0, 0, 0)$	—
					$(\tau^{\frac{3}{2}};\ \pm\tau,\ \pm 1, 0)^A$	{3, 5}
					$(\tau^{\frac{5}{2}};\ \pm 2\tau, 0, 0)^S, (\tau^{\frac{5}{2}};\ \pm\tau^2,\ \pm 1,\ \pm\tau)^A$	$\begin{Bmatrix}3\\5\end{Bmatrix}$
					$(\sqrt{5}\tau^{\frac{3}{2}};\ \pm\tau^2,\ \pm\tau^3, 0)^A$	{3, 5}
{5, 3, 4}	$\frac{1}{2}\tau^{-1}$	τ	$\sqrt{\frac{1}{2}}\tau^{-1}$	$\tau^{-\frac{1}{2}}$	$(\tau^{-\frac{1}{2}}, 0, 0, 0)$	—
					$(\tau^{\frac{1}{2}};\ \pm 1, 0, 0)^S$	{3, 4}
					$(\tau^{\frac{3}{2}};\ \pm\tau,\ \pm 1, 0)^A$	{3, 5}
					$(\tau^{\frac{5}{2}};\ \pm 2\tau, 0, 0)^S, (\tau^{\frac{5}{2}};\ \pm\tau^2,\ \pm 1,\ \pm\tau)^A$	$\begin{Bmatrix}3\\5\end{Bmatrix}$
{3, 3, 3, 5}	τ	τ^3	τ^{-1}	$\tau^{-\frac{3}{2}}$	$(\tau^{-\frac{3}{2}}, 0, 0, 0, 0)$	—
					$(\tau^{\frac{3}{2}};\ \pm 2, 0, 0, 0)^S, (\tau^{\frac{3}{2}},\ \pm 1,\ \pm 1,\ \pm 1,\ \pm 1)$	{3, 3, 5}
					$(\tau^{\frac{3}{2}};\ \pm\tau,\ \pm 1,\ \pm\tau^{-1}, 0)^A$	
					$(\sqrt{5}\tau^{\frac{3}{2}};\ \pm 2\tau,\ \pm 2\tau, 0, 0)^S$	{5, 3, 3}
					$(\sqrt{5}\tau^{\frac{3}{2}};\ \pm\sqrt{5}\tau,\ \pm\tau,\ \pm\tau,\ \pm\tau)^S$	
					$(\sqrt{5}\tau^{\frac{3}{2}};\ \pm\tau^2,\ \pm\tau^2,\ \pm\tau^2,\ \pm\tau^{-1})^S$	
					$(\sqrt{5}\tau^{\frac{3}{2}};\ \pm\tau^3,\ \pm 1,\ \pm 1,\ \pm 1)^S$	
					$(\sqrt{5}\tau^{\frac{3}{2}};\ \pm\sqrt{5}\tau,\ \pm 1,\ \pm\tau^2, 0)^A$	
					$(\sqrt{5}\tau^{\frac{3}{2}};\ \pm 2\tau,\ \pm\tau,\ \pm\tau^2,\ \pm 1)^A$	
					$(\sqrt{5}\tau^{\frac{3}{2}};\ \pm\tau^3,\ \pm\tau^{-1},\ \pm\tau, 0)^A$	
{4, 3, 3, 5}	τ^3	$\sqrt{5}\tau^3$	$\sqrt{\frac{1}{2}}\tau^{-1}$	$\sqrt{\frac{1}{2}}\tau^{-\frac{5}{2}}$	$(\sqrt{\frac{1}{2}}\tau^{-\frac{5}{2}}, 0, 0, 0, 0)$	—
					$(\sqrt{\frac{5}{2}}\tau^{\frac{1}{2}};\ \pm 2, 0, 0, 0)^S$	{3, 3, 5}
					$(\sqrt{\frac{5}{2}}\tau^{\frac{1}{2}},\ \pm 1,\ \pm 1,\ \pm 1,\ \pm 1)$	
					$(\sqrt{\frac{5}{2}}\tau^{\frac{1}{2}};\ \pm\tau,\ \pm 1,\ \pm\tau^{-1}, 0)^A$	
{5, 3, 3, 5}	$\sqrt{5}\tau^2$	$3\tau^3$	τ^{-2}	$5^{-\frac{1}{4}}\tau^{-3}$	$(5^{-\frac{1}{4}}\tau^{-3}, 0, 0, 0, 0)$	—
					$(3{\cdot}5^{-\frac{1}{4}};\ \pm 2, 0, 0, 0)^S$	{3, 3, 5}
					$(3{\cdot}5^{-\frac{1}{4}};\ \pm\tau,\ \pm 1,\ \pm\tau^{-1}, 0)^A$	
					$(3{\cdot}5^{-\frac{1}{4}},\ \pm 1,\ \pm 1,\ \pm 1,\ \pm 1)$	
{5, 3, 3, 4}	$\frac{1}{2}\tau^{-1}$	τ	$\sqrt{\frac{1}{2}}\tau^{-1}$	$\tau^{-\frac{1}{2}}$	$(\tau^{-\frac{1}{2}}, 0, 0, 0, 0)$	—
					$(\tau^{\frac{1}{2}};\ \pm 1, 0, 0, 0)^S$	{3, 3, 4}
					$(\tau^{\frac{3}{2}};\ \pm\tau,\ \pm 1, 0, 0)^S$	
					$(\tau^{\frac{5}{2}};\ \pm\tau^2,\ \pm\tau,\ \pm 1, 0)^S$	

Similarly, by considering the dodecahedron whose vertices include

$$(\tau^{-\frac{1}{2}},0,0,0,0)\quad(\tau^{\frac{1}{2}},1,0,0,0),\quad(\tau^{\frac{1}{2}},0,1,0,0)\quad(\tau^{\frac{1}{2}},0,0,1,0),$$
$$(\tau^{\frac{3}{2}},\tau,1,0,0),\quad(\tau^{\frac{3}{2}},1,\tau,0,0),\quad(\tau^{\frac{3}{2}},0,\tau,1,0),\quad(\tau^{\frac{3}{2}},0,1,\tau,0)$$

as shown in figure 4, the extremity of a second diagonal is found to be $(\tau^{\frac{5}{2}},\tau,\tau^2,1,0)$. We are able to conclude that the 192 vertices of $\{5,3,3,4\}$ which are the extremities of second diagonals from the vertex $(\tau^{-\frac{1}{2}},0,0,0,0)$ are $(\tau^{\frac{5}{2}};\ \pm\tau^2,\ \pm\tau,\ \pm1,0)^S$.

Of course these 'Cartesian' coordinates for the vertices of the Minkowskian polytope serve also as normalized projective coordinates for the vertices of the hyperbolic honeycomb.

In a similar way, coordinates for many vertices of the remaining honeycombs can be obtained. Table 1 lists as many of these as can be expressed concisely.

4. Diagonals

The distance between two vertices $(x_0,x_1,\ldots,x_n)$ and $(y_0,y_1,\ldots,y_n)$ of any honeycomb or polytope $\{p,q,\ldots,r,s\}$ in the above table can be read off in two different ways. The hyperbolic distance $2\phi'$ is given by

$$R^2\cosh 2\phi' = x_0y_0-x_1y_1-\ldots-x_ny_n$$

(Coxeter 1957, p. 209) and the (spacelike) distance a (which is the length of an edge or diagonal of the Euclidean cell $\{p,q,\ldots,r\}$ of the Minkowskian polytope $\{p,q,\ldots,r,s\}$) is given by

$$\begin{aligned}a^2 &= -(x_0-y_0)^2+(x_1-y_1)^2+\ldots+(x_n-y_n)^2\\ &= -(x_0^2-x_1^2-\ldots-x_n^2)-(y_0^2-y_1^2-\ldots-y_n^2)+2(x_0y_0-x_1y_1-\ldots-x_ny_n)\\ &= -2R^2+2R^2\cosh 2\phi' = 2R^2(\cosh 2\phi'-1) = 4R^2\sinh^2\phi'.\end{aligned}$$

Thus $$a = 2R\sinh\phi'.$$

But we have seen above that the (spacelike) edge length $2l$ of the Minkowskian polytope is related to the circum-radius R by $R = l/\sinh\phi$, where 2ϕ is the edge length of the hyperbolic honeycomb $\{p,q,\ldots,r,s\}$. Thus

$$\frac{a}{2l} = \frac{2R\sinh\phi'}{2R\sinh\phi},$$

and so $$\sinh\phi' = \frac{a}{2l}\sinh\phi.$$

Considering the original polytope $\{p,q,\ldots,r\}$ to have unit edge length, this gives an indirect proof of the following

Theorem 5. *Let Π_n be a regular polytope in E^n of unit edge-length, and let a be the length of some diagonal. Then if the edge-length of Π_n in H^n is 2ϕ, the length of the same diagonal is $2\phi'$, where $\sinh\phi' = a\sinh\phi$.*

Another proof of this theorem follows immediately from two well known results:

(1) Let Π_2 be a regular polygon in E^2 of unit side and let a be the length of some diagonal. Then if the side of the corresponding spherical polygon is 2ϕ, the length

of the same diagonal is $2\phi'$ where $\sin\phi' = a\sin\phi$. This is obvious by trigonometry. But since elliptic trigonometry is exactly the same as ordinary spherical trigonometry when we take as the measure of a side of a spherical polygon the angle which is subtended at the centre of the sphere (Sommerville 1919, p. 120), this same relation holds between the diagonals of polygons in Euclidean and spherical space. Moreover, this extends immediately to E^n.

(2) Hyperbolic trigonometric formulae may be derived from elliptic formulae by multiplying each symbol for distance by i (Coxeter 1957, p. 238); and so the desired result is obtained.

The author wishes to express his sincere thanks to Professor H. S. M. Coxeter, F.R.S., for his kind and generous help during the preparation of this paper.

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