An International Journal
computers & mathematics with applications

PERGAMON Computers and Mathematics with Applications 41 (2001) 135–147
www.elsevier.nl/locate/camwa

Hyperbolic Trigonometry and its Application in the Poincaré Ball Model of Hyperbolic Geometry

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(Received November 1999; accepted January 2000)

Abstract—Hyperbolic trigonometry is developed and illustrated in this article along lines parallel to Euclidean trigonometry by exposing the hyperbolic trigonometric law of cosines and of sines in the Poincaré ball model of n-dimensional hyperbolic geometry, as well as their application. The Poincaré ball model of three-dimensional hyperbolic geometry is becoming increasingly important in the construction of *hyperbolic browsers* in computer graphics. These allow in computer graphics the exploitation of hyperbolic geometry in the development of visualization techniques. It is, therefore, clear that hyperbolic trigonometry in the Poincaré ball model of hyperbolic geometry, as presented here, will prove useful in the development of efficient hyperbolic browsers in computer graphics. Hyperbolic trigonometry is governed by gyrovector spaces in the same way that Euclidean trigonometry is governed by vector spaces. The capability of gyrovector space theory to capture analogies and its powerful elegance is thus demonstrated once more.

Keywords—Hyperbolic geometry, Gyrovector spaces, Poincaré disc model, Trigonometry in hyperbolic geometry, Hyperbolic law of sines and cosines, Hyperbolic Pythagorean theorem.

1. INTRODUCTION

Structure of sections of the world wide web can be visualized by the construction of graphical representations in three-dimensional hyperbolic space. The remarkable property that hyperbolic space has *"more room"* than Euclidean space allows more information to be seen on the computer's screen amid less clutter; and motion by hyperbolic isometries, that is

(i) *left gyrotranslations* and
(ii) rotations,

provides for mathematically elegant and efficient navigation. For the construction and manipulation of the three-dimensional hyperbolic representations, hyperbolic trigonometry proves useful. The technique to visualize the hyperbolic structures, inspired by the Escher woodcuts [1–3], is called the *hyperbolic browser*.

Following the increased interest in practical applications of hyperbolic geometry, the need to develop computational techniques in hyperbolic geometry clearly arises. Accordingly, we present

0898-1221/01/$ - see front matter Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX
PII: S0898-1221(00)00262-5

in this article the hyperbolic trigonometry and its use in the Poincaré ball model of hyperbolic geometry, illustrated by a simple numerical example that demonstrates the novel remarkable analogies shared by Euclidean and hyperbolic geometry.

To achieve our goal, we present in this article the Möbius gyrovector spaces, which form the setting for the Poincaré ball model of hyperbolic geometry in the same way that vector spaces form the setting for Euclidean geometry. These, in turn, enable hyperbolic trigonometry of the Poincaré ball model of hyperbolic geometry to be unified with the familiar trigonometry of Euclidean geometry. The two laws that govern Euclidean trigonometry are

(i) the law of cosines, which includes the Pythagorean theorem as a special case, and
(ii) the law of sines.

These turn out to be a special case of the hyperbolic law of cosines and of sines that are presented in this article. Finally, the application of hyperbolic trigonometry for solving hyperbolic triangle problems in a way fully analogous to the application of Euclidean trigonometry for solving Euclidean triangle problems is demonstrated by a numerical example.

2. THE MÖBIUS GYROVECTOR SPACE

Let $\mathbb{V}_\infty$ be any real inner product space, and let $\mathbb{V}_c$ be the open ball of $\mathbb{V}_\infty$,

$$\mathbb{V}_c = \{\mathbf{v} \in \mathbb{V}_\infty : \|\mathbf{v}\| < c\} \tag{2.1}$$

with radius $c > 0$. Möbius addition is a binary operation in $\mathbb{V}_c$ given by the equation

$$\mathbf{u} \oplus \mathbf{v} = \frac{\left(1 + (2/c^2)\,\mathbf{u}\cdot\mathbf{v} + (1/c^2)\,\|\mathbf{v}\|^2\right)\mathbf{u} + \left(1 - (1/c^2)\,\|\mathbf{u}\|^2\right)\mathbf{v}}{1 + (2/c^2)\mathbf{u}\cdot\mathbf{v} + (1/c^4)\,\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{2.2}$$

satisfying

$$\lim_{c\to\infty} \mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v} \tag{2.3}$$

in

$$\lim_{c\to\infty} \mathbb{V}_c = \mathbb{V}_\infty. \tag{2.4}$$

To justify calling $\oplus$ a Möbius addition, let us consider the unit open disc

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \tag{2.5}$$

of a complex plane $\mathbb{C}$. The most general Möbius transformation of the complex open unit disc $\mathbb{D}$ [3–5],

$$z \mapsto e^{i\theta}\,\frac{z_0 + z}{1 + \bar{z}_0 z} = e^{i\theta}(z_0 \oplus z), \tag{2.6}$$

$z_0 \in \mathbb{D}$, $\theta \in \mathbb{R}$, defines the Möbius addition $\oplus$ in the disc, allowing the Möbius transformation of the disc to be viewed as a Möbius *left gyrotranslation*,

$$z \mapsto z_0 \oplus z = \frac{z_0 + z}{1 + \bar{z}_0 z} \tag{2.7}$$

followed by a rotation.

Identifying vectors $\mathbf{u}$ in $\mathbb{R}^2$ with complex numbers $u \in \mathbb{C}$ in the usual way, we have

$$\mathbf{u} = (u_1, u_2) = u_1 + iu_2 = u. \tag{2.8}$$

The inner product and the norm in $\mathbb{R}^2$ then become the real numbers

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= \mathrm{Re}(\bar{u}v) = \frac{\bar{u}v + u\bar{v}}{2}, \\ \|\mathbf{u}\| &= |u|, \end{aligned} \tag{2.9}$$

where $\bar{u}$ is the complex conjugate of u.

Under the translation (2.9) of elements $\mathbf{u}, \mathbf{v}$ of the disc $\mathbb{R}^2_{c=1}$ of $\mathbb{R}^2$ into elements u, v of the complex unit disc $\mathbb{D}$, the Möbius addition (2.2), with $c = 1$ for simplicity, takes the form

$$\begin{aligned} \mathbf{u} \oplus_M \mathbf{v} &= \frac{\left(1 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2\right)\mathbf{u} + \left(1 - \|\mathbf{u}\|^2\right)\mathbf{v}}{1 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{u}\|^2\|\mathbf{v}\|^2} \\ &= \frac{(1 + \bar{u}v + u\bar{v} + |v|^2)u + (1 - |u|^2)v}{1 + \bar{u}v + u\bar{v} + |u|^2|v|^2} \\ &= \frac{(1 + u\bar{v})(u + v)}{(1 + \bar{u}v)(1 + u\bar{v})} \\ &= \frac{u + v}{1 + \bar{u}v}, \end{aligned} \tag{2.10}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2_{c=1}$, and all $u, v \in \mathbb{D}$, thus, recovering the special Möbius transformation of the disc which gives the Möbius addition (2.7) in the disc [7,8].

Möbius addition shares remarkable analogies with the common vector addition. These analogies for the Möbius addition in the disc follow.

If we define $\operatorname{gyr}[a, b]$ to be the complex number with modulus 1, given by the equation

$$\operatorname{gyr}[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\bar{b}}{1 + \bar{a}b}, \tag{2.11}$$

then for all $a, b, c \in \mathbb{D}$, the following group-like properties of $\oplus$ are verified by straightforward algebra:

$$\begin{aligned} a \oplus b &= \operatorname{gyr}[a, b](b \oplus a), && \text{gyrocommutative law,} \\ a \oplus (b \oplus c) &= (a \oplus b) \oplus \operatorname{gyr}[a, b]c, && \text{left gyroassociative law,} \\ (a \oplus b) \oplus c &= a \oplus (b \oplus \operatorname{gyr}[b, a]c), && \text{right gyroassociative law,} \\ \operatorname{gyr}[a, b] &= \operatorname{gyr}[a \oplus b, b], && \text{left loop property,} \\ \operatorname{gyr}[a, b] &= \operatorname{gyr}[a, b \oplus a], && \text{right loop property,} \\ \operatorname{gyr}^{-1}[a, b] &= \operatorname{gyr}[b, a], && \text{gyroautomorphism inversion.} \end{aligned}$$

The prefix "gyro" that we extensively use to emphasize analogies stems from the Thomas gyration, which is the abstract extension of the relativistic effect known as the Thomas precession [9]. The remarkable analogies that Möbius addition in the disc shares with the common vector addition remain valid in higher dimensions. In fact, the groupoid $(\mathbb{V}_c, \oplus)$ possesses a grouplike structure called a gyrogroup [9], the formal definition of which follows.

DEFINITION 2.1. GROUPOIDS, AUTOMORPHISM GROUPS. *A groupoid $(S, +)$ is a pair of a nonempty set S with a binary operation $+$. An automorphism ϕ of a groupoid $(S, +)$ is a bijective self-map of S that respects its binary operation, $\phi(s_1 + s_2) = \phi s_1 + \phi s_2$, for all $s_1, s_2 \in S$. The set of all automorphisms of a groupoid $(S, +)$ forms a group, denoted* $\operatorname{Aut}(S, +)$.

DEFINITION 2.2. GYROGROUPS. *The groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms and properties. In G, there exists a unique element, 0, called the identity, satisfying*

(g1) $$0 \oplus a = a \oplus 0 = a, \qquad \textit{identity},$$

for all $a \in G$. For each a in G, there exists a unique inverse $\ominus a$ in G, satisfying

(g2) $$\ominus a \oplus a = a \ominus a = 0, \qquad \textit{inverse},$$

where we use the notation $a \ominus b = a \oplus (\ominus b)$, $a, b \in G$. Moreover, if for any $a, b \in G$, the self-map $\mathrm{gyr}[a, b]$ *of G is given by the equation*

$$\mathrm{gyr}[a, b]z = \ominus(a \oplus b) \oplus (a \oplus (b \oplus z)), \tag{2.12}$$

for all $z \in G$, then the following hold for all $a, b, c \in G$:

(g3)	$\mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus)$,	*gyroautomorphism property,*
(g4a)	$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c$,	*left gyroassociative law,*
(g4b)	$(a \oplus b) \oplus c = a \oplus (b \oplus \mathrm{gyr}[b, a]c)$,	*right gyroassociative law,*
(g5a)	$\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]$,	*left loop property,*
(g5b)	$\mathrm{gyr}[a, b] = \mathrm{gyr}[a, b \oplus a]$,	*right loop property,*
(g6)	$\ominus (a \oplus b) = \mathrm{gyr}[a, b](\ominus b \ominus a)$,	*gyrosum inversion law,*
(g7)	$\mathrm{gyr}^{-1}[a, b] = \mathrm{gyr}[b, a]$,	*gyroautomorphism inversion.*

A gyrogroup is gyrocommutative if it satisfies

(g8)	$a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$,	*gyrocommutative law.*

The Möbius groupoid $(\mathbb{V}_c, \oplus)$, (2.1),(2.2), turns out to be a gyrocommutative gyrogroup, called the Möbius gyrogroup, where the inverse of $\mathbf{v} \in \mathbb{V}_c$ is $\ominus\mathbf{v} = -\mathbf{v}$. We will now equip it with scalar multiplication, turning it into a gyrovector space, $(\mathbb{V}_c, \oplus, \otimes)$, called the Möbius gyrovector space.

The scalar product over the real line $\mathbb{R}$ in a Möbius gyrovector space $(\mathbb{V}_c, \oplus)$ is given by the equation

$$\begin{aligned} r \otimes \mathbf{v} &= c\frac{(1 + \|\mathbf{v}\|/c)^r - (1 - \|\mathbf{v}\|/c)^r}{(1 + \|\mathbf{v}\|/c)^r + (1 - \|\mathbf{v}\|/c)^r} \frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= c \tanh\left(r \tanh^{-1} \frac{\|\mathbf{v}\|}{c^2}\right) \frac{\mathbf{v}}{\|\mathbf{v}\|}, \end{aligned} \tag{2.13}$$

where $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{V}_c$, $\mathbf{v} \neq \mathbf{0}$; and $r \otimes \mathbf{0} = \mathbf{0}$. We use the notation $r \otimes \mathbf{v} = \mathbf{v} \otimes r$.

The scalar multiplication $\otimes$ is compatible with Möbius addition, possessing the following properties. For any positive integer n and for all $r, r_1, r_2 \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{V}_c$,

$$\begin{aligned} n \otimes \mathbf{v} &= \mathbf{v} \oplus \ldots \oplus \mathbf{v}, && n \text{ terms}, \\ (r_1 + r_2) \otimes \mathbf{v} &= r_1 \otimes \mathbf{v} \oplus r_2 \otimes \mathbf{v}, && \text{scalar distributive law}, \\ (r_1 r_2) \otimes \mathbf{v} &= r_1 \otimes (r_2 \otimes \mathbf{v}), && \text{scalar associative law}, \\ r \otimes (r_1 \otimes \mathbf{v} \oplus r_2 \otimes \mathbf{v}) &= r \otimes (r_1 \otimes \mathbf{v}) \oplus r \otimes (r_2 \otimes \mathbf{v}), && \text{monodistributive law}, \\ \|r \otimes \mathbf{v}\| &= |r| \otimes \|\mathbf{v}\|, && \text{homogeneity property}, \\ \frac{|r| \otimes \mathbf{v}}{\|r \otimes \mathbf{v}\|} &= \frac{\mathbf{v}}{\|\mathbf{v}\|}, && \text{scaling property}. \end{aligned} \tag{2.14}$$

In one dimension, all vectors are parallel and, hence, the Thomas gyration vanishes, turning gyrogroups and gyrovector spaces into groups and vector spaces. Realizing the real inner product space $\mathbb{V}_\infty$ by the real line $\mathbb{R}$ one thus obtains the exotic Möbius group $(\mathbb{R}_c, \oplus)$ and Möbius vector space $(\mathbb{R}_c, \oplus, \otimes)$, where

(1) the open c-ball $\mathbb{V}_c$ of $\mathbb{V}_\infty$ becomes the open interval $\mathbb{R}_c = (-c, c)$;
(2) the Möbius gyrogroup $(\mathbb{V}_c, \oplus)$ specializes to the Möbius commutative group $(\mathbb{R}_c, \oplus)$; and
(3) the Möbius gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ specializes to the Möbius vector space $(\mathbb{R}_c, \oplus, \otimes)$.

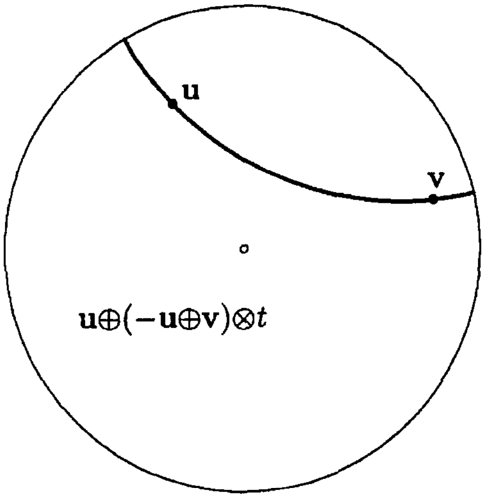

Figure 1. The unique Möbius geodesic $\mathbf{u}\oplus(-\mathbf{u}\oplus\mathbf{v})\otimes t$ $\mathbf{u},\mathbf{v}\in\mathbb{R}^2_{c=1}, t\in\mathbb{R}$, passing through two given points, $\mathbf{u}$ and $\mathbf{v}$, of the Möbius unit disc $(\mathbb{R}^2_{c=1},\oplus,\otimes)$ is shown. It is a Euclidean semicircular arc that intersects the boundary of the disc orthogonally, that one recognizes as the well-known geodesic of the Poincaré disc model of hyperbolic geometry.

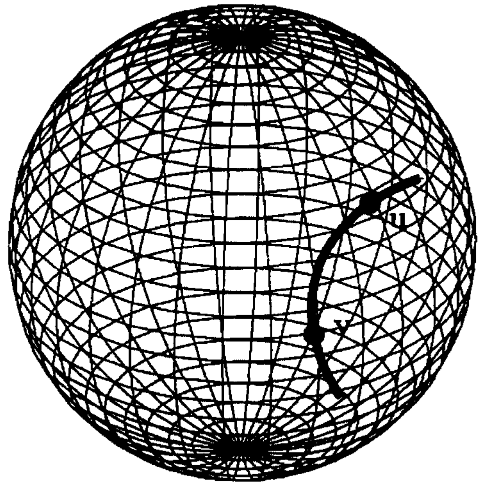

Figure 2. The unique Möbius geodesic $\mathbf{u}\oplus(-\mathbf{u}\oplus\mathbf{v})\otimes t$ $\mathbf{u},\mathbf{v}\in\mathbb{R}^3_{c=1}, t\in\mathbb{R}$, passing through two given points, $\mathbf{u}$ and $\mathbf{v}$, of the Möbius unit ball $(\mathbb{R}^3_{c=1},\oplus,\otimes)$ is shown. It is a Euclidean semicircular arc that intersects the boundary of the ball orthogonally. As in Euclidean geometry, the geodesic gyrovector equation (3.1) is dimension independent.

3. THE MÖBIUS GEODESICS AND ANGLES

In full analogy with Euclidean geometry, the unique Möbius geodesic passing thorough the two given points $\mathbf{u}$ and $\mathbf{v}$ of a Möbius gyrovector space $(\mathbb{V}_c,\oplus.\otimes)$ is represented by the parametric gyrovector equation

$$\mathbf{u}\oplus(-\mathbf{u}\oplus\mathbf{v})\otimes t \tag{3.1}$$

with parameter $t\in\mathbb{R}$. We note that in (3.1) $-\mathbf{u}=\ominus\mathbf{u}$. A Möbius two-dimensional (three-dimensional) geodesic is shown in Figure 1 (Figure 2).

Since the Poincaré model of hyperbolic geometry is conformal to Euclidean geometry, the measure of hyperbolic angles in the model equals the Euclidean measure in the model. This helps to visualize relationships between hyperbolic angles. Since the Poincaré ball model of hyperbolic geometry is governed by the Möbius gyrovector space in the same way that Euclidean geometry is governed by the common vector space, we define the Möbius angle by analogy with the Euclidean angle as follows.

Let

$$\begin{aligned} L_{\mathbf{uv}} &= \mathbf{u}\oplus(-\mathbf{u}\oplus\mathbf{v})\otimes t, \\ L_{\mathbf{uw}} &= \mathbf{u}\oplus(-\mathbf{u}\oplus\mathbf{w})\otimes t, \end{aligned} \tag{3.2}$$

$t\in\mathbb{R}^+$, be two geodesic rays in a Möbius gyrovector space $(\mathbb{V}_c,\oplus,\otimes)$ that emanate from a common point $\mathbf{u}$, Figure 3. The cosine of the Möbius angles α and $2\pi-\alpha$ between these geodesic rays is defined by the equation

$$\cos\alpha = \frac{-\mathbf{u}\oplus\mathbf{v}}{\|-\mathbf{u}\oplus\mathbf{v}\|}\cdot\frac{-\mathbf{u}\oplus\mathbf{w}}{\|-\mathbf{u}\oplus\mathbf{w}\|} \tag{3.3}$$

and accordingly,

$$\sin\alpha = \sqrt{1-\cos^2\alpha}. \tag{3.4}$$

The definition of the angle α in (3.3) as a property of its generating intersecting geodesics is legitimate since it is independent of the choice of the points $\mathbf{v}$ and $\mathbf{w}$ on their geodesics, as

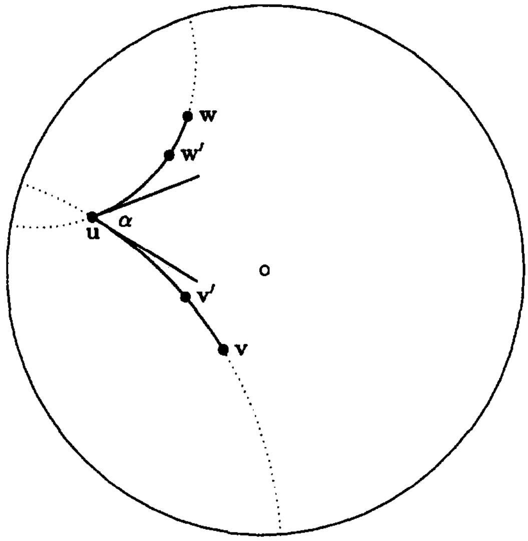

Figure 3. A Möbius angle α generated by the two intersecting Möbius geodesic rays (3.2), given by (3.3), is shown. The value of $\cos\alpha$ is independent of the choice of the points **v** and **w** on the two geodesic rays that emanate from **u** [10]. Hence, one may select the points **v** and **w** on the geodesics in an arbitrarily small neighborhood of the point **u**. Being conformal, a small neighborhood of any point of the Poincaré model of hyperbolic geometry approximates a small neighborhood in Euclidean geometry. Hence, the measure of a Möbius angle between two intersecting geodesic rays equals the measure of the Euclidean angle between corresponding intersecting tangent line. As such, the hyperbolic angle α in the Möbius gyrovector plane $(\mathbb{R}^2_{c=1}, \oplus, \otimes)$, as given by (3.3), is coincident with the well-known hyperbolic angle of the Poincaré disc model of hyperbolic geometry.

shown and explained in the caption of Figure 3. Moreover, while the definition of the hyperbolic angle in (3.3) has a novel form that exhibits analogies with the Euclidean angle, it is coincident with the standard, well-known hyperbolic angle in the Poincaré model of hyperbolic geometry, as explained in Figure 3. The hyperbolic angle is invariant under the "rigid motions" of hyperbolic geometry, that is, under

(i) left gyrotranslations and
(ii) rotations [10].

4. THE HYPERBOLIC TRIGONOMETRIC LAWS

The hyperbolic trigonometric laws,

(i) the law of sines and
(ii) the law of cosines,

in a form fully analogous to the form of their Euclidean counterparts, are presented for any hyperbolic triangle $\Delta\mathbf{abc}$, Figure 4, in a Möbius gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$. We use the notation

$$\|\mathbf{a}\|_M = \gamma_{\mathbf{a}}^2 \|\mathbf{a}\| \tag{4.1}$$

so that, conversely,

$$\frac{\|\mathbf{a}\|}{c} = \frac{2\,(\|\mathbf{a}\|_M/c)}{1+\sqrt{1+4\,(\|\mathbf{a}\|_M/c)^2}}. \tag{4.2}$$

THEOREM 4.1. THE HYPERBOLIC LAW OF SINES. *Let* $\Delta\mathbf{abc}$ *be a triangle in a Möbius gyrovector space* $(\mathbb{V}_c, \oplus, \otimes)$ *with vertices* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}_c$, *and sides*

$$\begin{aligned} A &= -\mathbf{b} \oplus \mathbf{c}, \\ B &= -\mathbf{c} \oplus \mathbf{a}, \\ C &= -\mathbf{a} \oplus \mathbf{b}, \end{aligned} \tag{4.3}$$

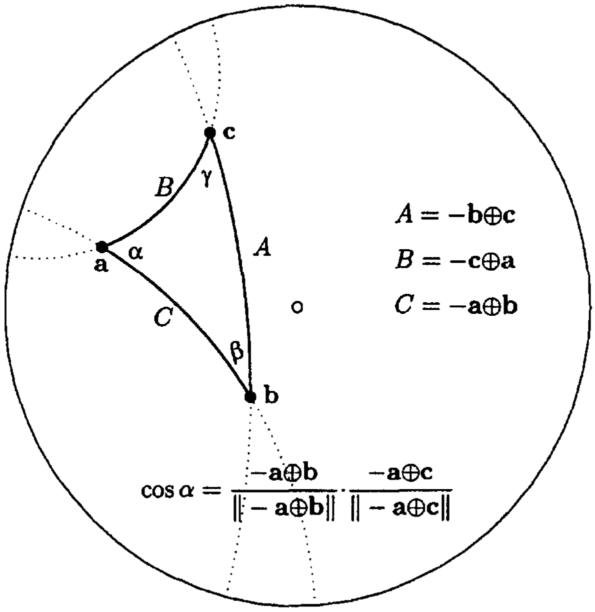

Figure 4. A Möbius triangle $\Delta\mathbf{abc}$ in the Möbius gyrovector plane $(\mathbb{R}^2_{c=1}, \oplus, \otimes)$ is shown. Its sides are formed by geometric gyrovectors that link its vertices, in full analogy with Euclidean triangles.

and with hyperbolic angles α, β, and γ at the vertices $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$, *Figure 4. Then*

$$\frac{\|A\|_M}{\sin\alpha} = \frac{\|B\|_M}{\sin\beta} = \frac{\|C\|_M}{\sin\gamma}. \tag{4.4}$$

In the special case when $\gamma = \pi/2$, corresponding to a hyperbolic right-angled triangle, Figure 5, the hyperbolic law of sines is of particular interest, giving rise to the relations

$$\begin{aligned} \sin\alpha &= \frac{\|A\|_M}{\|C\|_M}, \\ \sin\beta &= \frac{\|B\|_M}{\|C\|_M}. \end{aligned} \tag{4.5}$$

THEOREM 4.2. THE HYPERBOLIC LAW OF COSINES. *Let* $\Delta\mathbf{abc}$ *be a triangle in a Möbius gyrovector space* $(\mathbb{V}_c, \oplus, \otimes)$ *with vertices* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}_c$, *and sides*

$$\begin{aligned} A &= -\mathbf{b} \oplus \mathbf{c}, \\ B &= -\mathbf{c} \oplus \mathbf{a}, \\ C &= -\mathbf{a} \oplus \mathbf{b}, \end{aligned} \tag{4.6}$$

and with hyperbolic angles α, β, and γ at the vertices $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$, *Figure 4. Then*

$$\frac{1}{c}\|C\|^2 = \frac{1}{c}\|A\|^2 \oplus \frac{1}{c}\|B\|^2 \ominus \frac{1}{c}\frac{2\|A\|\,\|B\|\,\cos\gamma}{\left(1+\frac{\|A\|^2}{c^2}\right)\left(1+\frac{\|B\|^2}{c^2}\right) - \frac{2}{c^2}\|A\|\,\|B\|\,\cos\gamma}. \tag{4.7}$$

One may note that the Möbius addition $\oplus$ in (4.6) is a gyrogroup operation in the Möbius gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$, while the Möbius addition $\oplus$ in (4.7) is a group operation in the Möbius group $(\mathbb{R}_c, \oplus)$.

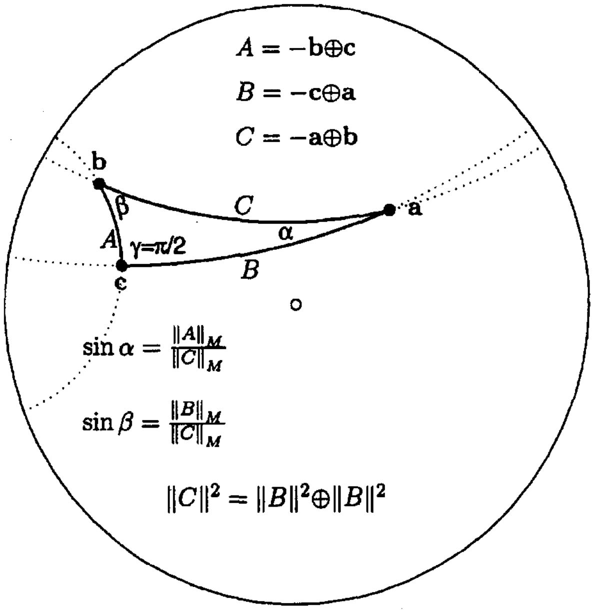

Figure 5. A Möbius right-angled triangle $\Delta\mathbf{abc}$ in the Möbius gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ is shown for the special case when $\mathbb{V}_c = \mathbb{R}^2_{c=1}$. Its sides, formed by the geometric gyrovectors A, B, and C that link its vertices, satisfy the Möbius hyperbolic Pythagorean identity (4.7).

The hyperbolic law of cosines (4.7) is an identity in the Möbius vector space $(\mathbb{R}_c, \oplus, \otimes)$. To solve it for $\cos\gamma$, we use the notation

$$\begin{aligned} P_{ABC} &= \frac{1}{c}\|A\|^2 \oplus \frac{1}{c}\|B\|^2 \ominus \frac{1}{c}\|C\|^2, \\ Q_{AB} &= \frac{2}{c^2}\|A\|\|B\|, \\ R_{AB} &= \left(1+\frac{\|A\|^2}{c^2}\right)\left(1+\frac{\|B\|^2}{c^2}\right), \end{aligned} \tag{4.8}$$

so that (4.7) can be written as

$$c\frac{Q_{AB}\cos\gamma}{R_{AB} - Q_{AB}\cos\gamma} = P_{ABC} \tag{4.9}$$

implying

$$\cos\gamma = \frac{P_{ABC}R_{AB}}{(c+P_{ABC})Q_{AB}} \tag{4.10}$$

and similarly by cyclic permutations,

$$\cos\alpha = \frac{P_{BCA}R_{BC}}{(c+P_{BCA})Q_{BC}}, \tag{4.11}$$

$$\cos\beta = \frac{P_{CAB}R_{CA}}{(c+P_{CAB})Q_{CA}}. \tag{4.12}$$

In the special case when $\gamma = \pi/2$, corresponding to a hyperbolic right-angled triangle, Figure 5, the hyperbolic law of cosines is of particular interest, giving rise to the hyperbolic Pythagorean theorem in the Poincaré ball model of hyperbolic geometry.

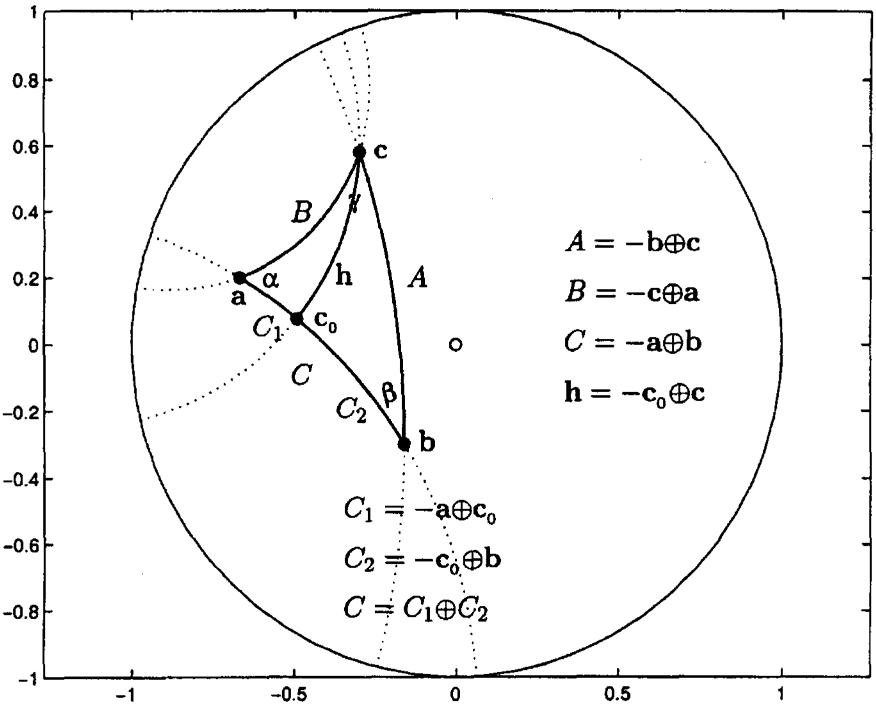

Figure 6. A Möbius triangle $\Delta\mathbf{abc}$ in the Möbius gyrovector plane $(\mathbb{R}^2_{c=1}, \oplus, \otimes)$ is shown. Its sides A, B, and C are formed by geometric gyrovectors that link its vertices $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. The height $\|\mathbf{h}\|$ of the triangle relative to its side C is the hyperbolic length of the geometric gyrovector $\mathbf{h}$ of the hyperbolic triangle, drawn from its vertex $\mathbf{c}$ orthogonal to the side C opposite to $\mathbf{c}$. The determination of $\mathbf{h}$ in terms of the triangle vertices $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ by means of hyperbolic trigonometry is presented in Section 5, illustrating numerically the analogies shared by Euclidean and hyperbolic trigonometry.

THEOREM 4.3. THE MÖBIUS HYPERBOLIC PYTHAGOREAN THEOREM. *Let $\Delta\mathbf{abc}$ be a triangle in a Möbius gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ with vertices $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}_c$, and sides*

$$\begin{aligned} A &= -\mathbf{b} \oplus \mathbf{c}, \\ B &= -\mathbf{c} \oplus \mathbf{a}, \\ C &= -\mathbf{a} \oplus \mathbf{b}, \end{aligned} \tag{4.13}$$

and with hyperbolic angles α, β, and γ at the vertices $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. If $\gamma = \pi/2$, Figure 5, then

$$\frac{1}{c}\,\|C\|^2 = \frac{1}{c}\,\|A\|^2 \oplus \frac{1}{c}\,\|B\|^2. \tag{4.14}$$

5. NUMERICAL DEMONSTRATION

In order to demonstrate the use of the hyperbolic trigonometry laws of sines and cosines, we present a numerical example of solving a hyperbolic triangle problem. Without loss of generality, we select, for simplicity, $c = 1$. Figure 6 presents a hyperbolic triangle $\Delta\mathbf{abc}$ in the Möbius gyrovector plane $(\mathbb{R}^2_{c=1}, \oplus, \otimes)$ with given vertices

$$\begin{aligned} \mathbf{a} &= (-0.67000000000000, 0.20000000000000), \\ \mathbf{b} &= (-0.16000000000000, -0.30000000000000), \\ \mathbf{c} &= (-0.30000000000000, 0.57950149830724), \end{aligned} \tag{5.1}$$

so that

$$\begin{aligned} A &= -\mathbf{b} \oplus \mathbf{c} = (0.00237035916884, 0.78080386796326), \\ B &= -\mathbf{c} \oplus \mathbf{a} = (-0.22314253210515, -0.66278993445542), \\ C &= -\mathbf{a} \oplus \mathbf{b} = (0.62614848317175, -0.37164924792294). \end{aligned} \tag{5.2}$$

Accordingly,

$$\begin{aligned} \|A\| &= 0.78080746591524, & \|A\|^2 &= 0.60966029882898, \\ \|B\| &= 0.69934475536012, & \|B\|^2 &= 0.48908308684971, \\ \|C\| &= 0.72813809573458, & \|C\|^2 &= 0.53018508645998, \end{aligned} \tag{5.3}$$

and by (4.1),

$$\begin{aligned} \|A\|_M &= 2.00032808236727, \\ \|B\|_M &= 1.36880329728761, \\ \|C\|_M &= 1.54984031955926, \end{aligned} \tag{5.4}$$

and by (5.3),

$$\begin{aligned} 2\|A\|\|B\| &= 1.09210721246770, \\ 1 + \|A\|^2 &= 1.60966029882898, \\ 1 + \|B\|^2 &= 1.48908308684971. \end{aligned} \tag{5.5}$$

The following quantities, defined in (4.8), can now be calculated for the triangle $\Delta\mathbf{abc}$ in Figure 6:

$$\begin{aligned} P_{ABC} &= 0.57357361200450, \\ P_{BCA} &= 0.39429577686007, \\ P_{CAB} &= 0.64338520930826, \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} Q_{AB} &= 1.09210721246770, \\ Q_{BC} &= 1.01843911685977, \\ Q_{CA} &= 1.13707132273373, \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} R_{AB} &= 2.39691792655969, \\ R_{BC} &= 2.27857273199722, \\ R_{CA} &= 2.46307818353481, \end{aligned} \tag{5.8}$$

resulting, according to (4.10)–(4.12), in

$$\begin{aligned} \cos\alpha &= 0.63269592569088, \\ \cos\beta &= 0.84805142781500, \\ \cos\gamma &= 0.80000000000000, \end{aligned} \tag{5.9}$$

and by (3.4),

$$\begin{aligned} \sin\alpha &= 0.77440032645535, \\ \sin\beta &= 0.52991393242765, \\ \sin\gamma &= 0.60000000000000. \end{aligned} \tag{5.10}$$

The three angles of the hyperbolic triangle $\Delta\mathbf{abc}$ in Figure 6 are, therefore,

$$\begin{aligned} \alpha &= 0.88576673758019 = 0.28194830942454\pi, \\ \beta &= 0.55849907350057 = 0.17777577651972\pi, \\ \gamma &= 0.64350110879328 = 0.20483276469913\pi, \end{aligned} \tag{5.11}$$

whose sum is

$$\alpha + \beta + \gamma = 2.08776691987404 = 0.66455685064339\pi < \pi. \tag{5.12}$$

Corroborating the hyperbolic trigonometric law of sines, Theorem 4.1, we find from (5.4) and (5.10) that

$$\frac{\|A\|_M}{\sin\alpha} = \frac{\|B\|_M}{\sin\beta} = \frac{\|C\|_M}{\sin\gamma} = 2.58306719926544. \tag{5.13}$$

We now wish to calculate the orthogonal projection $\mathbf{c}_0$ of the vertex $\mathbf{c}$ on its opposite side C, as well as the resulting height, $\mathbf{h}$, and the partition (C_1, C_2) of the side C of the hyperbolic triangle $\Delta\mathbf{abc}$ in Figure 6. By an application of (4.5) to the hyperbolic right-angled triangles $\Delta\mathbf{acc}_0$ and $\Delta\mathbf{bcc}_0$ that partition the triangle $\Delta\mathbf{abc}$ in Figure 6 we have, in full analogy with Euclidean trigonometry,

$$\begin{aligned} \|\mathbf{h}\|_M &= \|B\|_M \sin\alpha = 1.06000172027269, \\ \|\mathbf{h}\|_M &= \|A\|_M \sin\beta = 1.06000172027269. \end{aligned} \tag{5.14}$$

The two results in (5.14) agree with each other, as expected. It follows from (5.14) and (4.2) that

$$\|\mathbf{h}\| = \frac{2\|\mathbf{h}\|_M}{1 + \sqrt{1 + 4\|\mathbf{h}\|_M^2}} = 0.63396914263027. \tag{5.15}$$

Having the value of $\|\mathbf{h}\|$ in hand, we can now calculate $\|C_1\|$ and $\|C_2\|$ from the hyperbolic Pythagorean Theorem 4.3,

$$\begin{aligned} \|C_1\| &= \sqrt{\|B\|^2 \ominus \|\mathbf{h}\|^2} = 0.29523924712401, \\ \|C_2\| &= \sqrt{\|A\|^2 \ominus \|\mathbf{h}\|^2} = 0.45578879431335. \end{aligned} \tag{5.16}$$

Having the values of $\|\mathbf{h}\|$, $\|C_1\|$, and $\|C_2\|$ in hand, we can now calculate the point $\mathbf{c}_0$ in two equivalent ways, as indicated by Figure 6, and in full analogy with Euclidean geometry,

$$\begin{aligned} \mathbf{c}_0 &= \mathbf{a} \oplus (-\mathbf{a} \oplus \mathbf{b}) \times \frac{\|C_1\|}{\|-\mathbf{a} \oplus \mathbf{b}\|} = (-0.49248883838421,\ 0.07655787984884), \\ \mathbf{c}_0 &= \mathbf{b} \oplus (-\mathbf{b} \oplus \mathbf{a}) \times \frac{\|C_2\|}{\|-\mathbf{b} \oplus \mathbf{a}\|} = (-0.49248883838421,\ 0.07655787984884), \end{aligned} \tag{5.17}$$

where $\times$ is the common scalar multiplication in the vector space $\mathbb{R}^2$ that contains the Möbius disc $\mathbb{R}^2_{c=1}$ where the hyperbolic triangle $\Delta\mathbf{abc}$ resides, Figure 6.

Finally, let us use the calculated value of $\mathbf{c}_0$ to calculate the hyperbolic angles $\gamma_1 = \angle\mathbf{acc}_0$ and $\gamma_2 = \angle\mathbf{bcc}_0$ the sum of which must be $\gamma_1 + \gamma_2 = \gamma$, as shown in Figure 6. We have

$$\begin{aligned} \cos\gamma_1 &= \cos\angle\mathbf{acc}_0 = \frac{-\mathbf{c} \oplus \mathbf{a}}{\|-\mathbf{c} \oplus \mathbf{a}\|} \cdot \frac{-\mathbf{c} \oplus \mathbf{c}_0}{\|-\mathbf{c} \oplus \mathbf{c}_0\|} = 0.96288288079610, \\ \cos\gamma_2 &= \cos\angle\mathbf{bcc}_0 = \frac{-\mathbf{c} \oplus \mathbf{b}}{\|-\mathbf{c} \oplus \mathbf{b}\|} \cdot \frac{-\mathbf{c} \oplus \mathbf{c}_0}{\|-\mathbf{c} \oplus \mathbf{c}_0\|} = 0.93225802841014. \end{aligned} \tag{5.18}$$

Hence,

$$\begin{aligned} \gamma_1 &= 0.27330946853165, \\ \gamma_2 &= 0.37019164026164, \end{aligned} \tag{5.19}$$

and as expected, by (5.11) and (5.19), we have

$$\gamma_1 + \gamma_2 = 0.64350110879328 = \gamma. \tag{5.20}$$

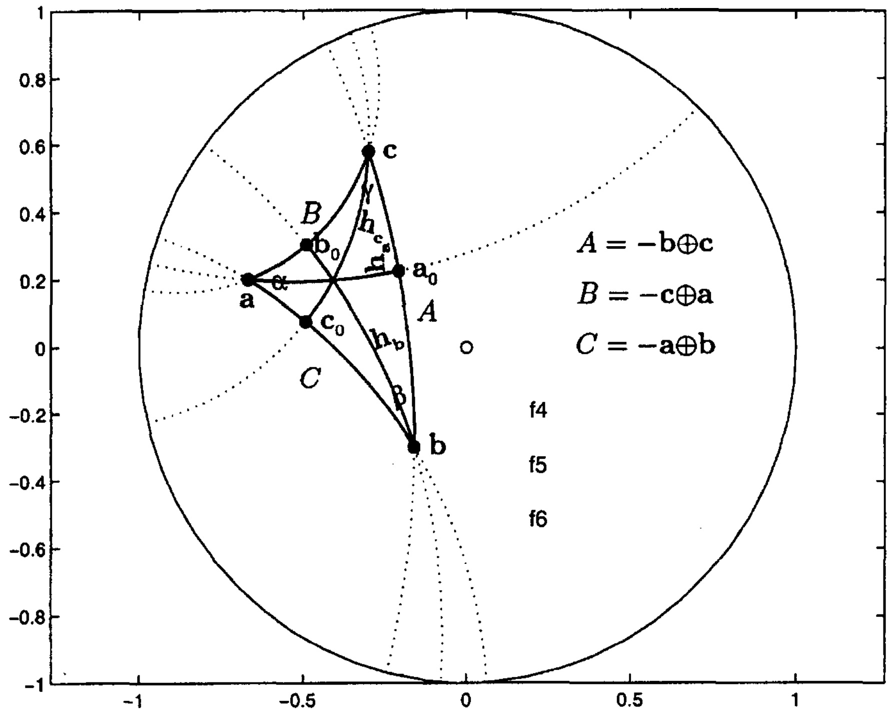

Figure 7. A Möbius triangle $\Delta \mathbf{abc}$ with its heights in the Möbius gyrovector plane $(\mathbb{R}^2_{c=1}, \oplus, \otimes)$ is shown. Its sides are formed by geometric gyrovectors that link its vertices, and its heights satisfy the equalities in (5.23).

Moreover, as expected, the angles $\angle \mathbf{a}\mathbf{c}_0\mathbf{c}$ and $\angle \mathbf{b}\mathbf{c}_0\mathbf{c}$ are right since

$$\begin{aligned}(-\mathbf{c}_0 \oplus \mathbf{a}) \cdot (-\mathbf{c}_0 \oplus \mathbf{c}) &= (-0.25619973269889, 0.20701304758399) \\ &\quad \cdot (0.39844242102025, 0.49311308128956) = 0, \\ (-\mathbf{c}_0 \oplus \mathbf{b}) \cdot (-\mathbf{c}_0 \oplus \mathbf{c}) &= (0.40801675213057, -0.32968337021312) \\ &\quad \cdot (0.39844242102025, 0.49311308128956) = 0.\end{aligned} \tag{5.21}$$

By cyclic permutations of the vertices of the triangle $\Delta \mathbf{abc}$ in Figure 6, interested readers may calculate in a similar way the orthogonal projections $\mathbf{a}_0$ and $\mathbf{b}_0$ of the vertices $\mathbf{a}$ and $\mathbf{b}$ on their respective opposite sides A and B, obtaining

$$\begin{aligned}\mathbf{a}_0 &= (-0.20839699612629, 0.22494064383527), \\ \mathbf{b}_0 &= (-0.48987957721610, 0.30327461221180), \\ \mathbf{c}_0 &= (-0.49248883838421, 0.07655787984884).\end{aligned} \tag{5.22}$$

The resulting three heights of the triangle $\Delta \mathbf{abc}$ of Figure 6 are shown in Figure 7, demonstrating the well-known fact that in hyperbolic geometry, these are concurrent, as they are in Euclidean geometry. Moreover, the product of the M-magnitude of each height of the triangle with that of its corresponding side gives a constant $S_{\mathbf{abc}}$ of the triangle $\Delta_{\mathbf{abc}}$, that reminds the double area of its Euclidean counterpart,

$$S_{\mathbf{abc}} = \|A\|_M \|\mathbf{h_a}\|_M = \|B\|_M \|\mathbf{h_b}\|_M = \|C\|_M \|\mathbf{h_c}\|_M. \tag{5.23}$$

Identity (5.23), in any Möbius gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$, is deduced from the Möbius hyperbolic law of sines in Theorem 4.1 and resulting identities, like (4.5). The numerical value of the triangle constant $S_{\mathbf{abc}}$ for the triangle $\Delta_{\mathbf{abc}}$ in Figure 7 is

$$S_{\mathbf{abc}} = 1.64283340488080. \tag{5.24}$$

For the sake of simplicity, the hyperbolic trigonometric calculations are presented in this article in the two-dimensional Möbius gyrovector space, which is the gyrovector space that governs the Poincaré disc model of hyperbolic geometry. However, hyperbolic trigonometric calculations can be performed in a similar way in the Poincaré ball model of n-dimensional hyperbolic geometry in any dimension n. The case of three dimensions is of particular interest in the development of efficient computer software for three-dimensional hyperbolic browsers.

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