# Coordinates for a new triangular tiling of the hyperbolic plane

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# Coordinates for a new triangular tiling of the hyperbolic plane

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**Abstract**

In this paper we define an infinite family of triangular tilings of the hyperbolic plane defined by two parameters ranging in the natural numbers and we give a uniform way to define coordinates for locating the triangles of the tiling.

## 1  Introduction

In [4], the author introduced a system of coordinates for the points of the hyperbolic plane. The starting point of the system is a tessellation of the hyperbolic plane defined by a regular convex polygon $P$ with $p$ sides and with interior angle $\dfrac{2\pi}{q}$. This latter expression means that it is possible to exactly cover a neighbourhood of a point $V$ by placing $q$ copies of $P$ with $V$ as a common vertex, each copy being an image of the initial one by a suitable rotation around $V$ involving a multiple of the above angle. In Section 2 we explain how to define coordinates for the tiling defined by $P$. In Section 3, remembering the construction of [4], we shall define a family of triangular tilings together with a coordinate system for each member of the family. In Subsection 3.2, we give an algorithm to compute the coordinates of the neighbours of a tile from the coordinate of this tile.

## 2  The tilings $\{p, q\}$

In order to be as self-contained as possible, we start by reminding in what consists hyperbolic geometry, also introducing a model in which we can see something, this is the goal os Sub-section 2.1. However, it is the place here to

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warn the reader that despite the model we can see a very little only. In fact,we are in the situation of the pilot of a plane with no visibility, when the flight has to be processed with instruments only.

Then, in Sub-section 2.2, we remind the tessellations of the hyperbolic plane constructed from a regular convex polygon and, in Sub-section 2.3 we remember the definition of the coordinates for the tiles in these tessellations.

## 2.1 Hyperbolic geometry

Hyperbolic geometry appeared in the first half of the 19<sup>th</sup> century, proving the independence of the parallel axiom of Euclidean geometry. Models were devised in the second half of the 19<sup>th</sup> century and we shall use here one of the most popular ones, Poincaré's disc. This model is represented by Figure 1.
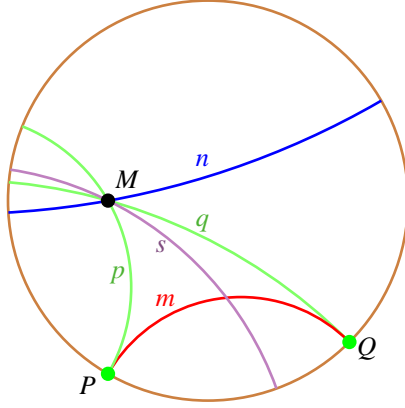


**Figure 1** *Poincaré's disc model.*

Inside the open disc represented in the figure we have the points of the hyperbolic plane. Note that by definition, the points on the border of the disc do not belong to the hyperbolic plane. However, these points play an important role in this geometry and are called **points at infinity**. Lines are trace of diameters or circles orthogonal to the border of the disc, e.g. the line $m$. In this model, two lines which meet in the open disc are called **secant** and two lines which meet at infinity, *i.e.* at a point at infinity are called **parallel**. In the figure, we can see a line $s$ through the point $A$ which cuts $m$. Now, we can see that two lines pass through $A$ which are parallel to $m$: $p$ and $q$. They touch $m$ in the model at $P$ and $Q$ respectively which are points at infinity. At last, and not the least: the line $n$ also passes through $A$ without cutting $m$, neither inside the disc nor outside it. This line is called **non-secant** with $m$.

## 2.2   The tilings $\{p, q\}$

From a famous theorem established by Poincaré in the late 19$^{\text{th}}$ century, it is known that there are infinitely many tilings in the hyperbolic plane, each one generated by the reflection of a regular convex polygon $P$ in its sides and, recursively, by the reflection of the images in their sides, provided that the number $p$ of sides of $P$ and the number $q$ of copies of $P$ which can be put around a point $A$ and exactly covering a neighbourhood of $A$ without overlapping satisfy the relation: $\dfrac{1}{p} + \dfrac{1}{q} < \dfrac{1}{2}$. The numbers $p$ and $q$ characterize the tiling which is denoted $\{p, q\}$ and the condition says that the considered polygons live in the hyperbolic plane. We call $P$ the **basic polygon** of the tessellation. Note that the three tilings of the Euclidean plane which can be defined up to similarities can be characterized by the relation obtained by replacing $<$ with $=$ in the above expression. In this way, the basic polynomial of these tessellations are the square for $\{4, 4\}$, the equilateral triangle for $\{3, 6\}$ and the regular hexagon for $\{6, 3\}$.

In the paper, the figures which illustrate the explanations of the text are based on two tilings which are the simplest ones which can be defined in this way in the hyperbolic plane: $\{5, 4\}$ and $\{7, 3\}$. We call these tilings the **pentagrid** and the **heptagrid** respectively. They are illustrated by Figures 2 and 3. On the right-hand side of the figure, we can see the tree which is in bijection with an angular sector, a basic structure of the heptagrid, also see Figure 7.
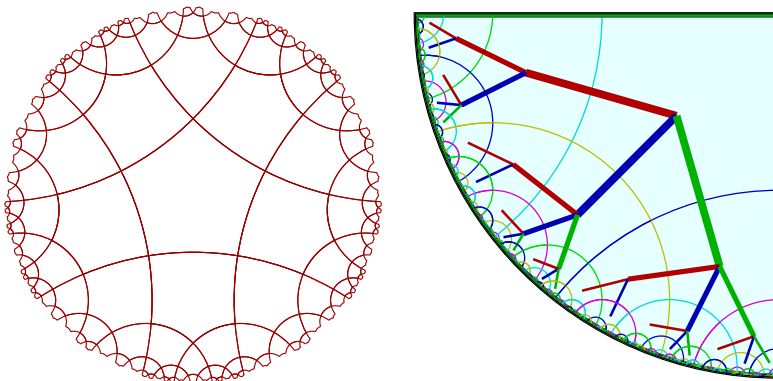


**Figure 2** *The pentagrid. On the right-hand side: the key structure to explore the tiling.*
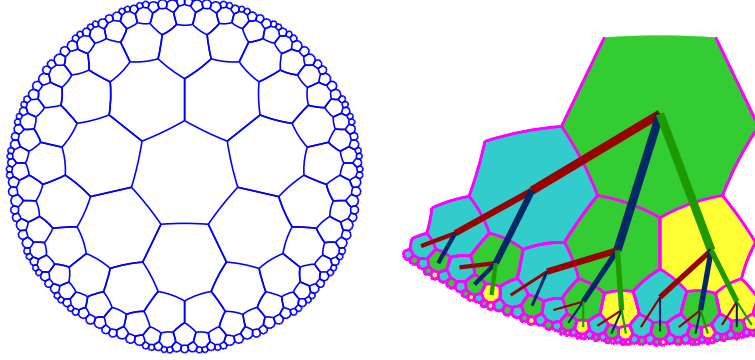
**Figure 3** *The heptagrid. On the right-hand side: the key structure to explore the tiling.*

This property is not specific to the heptagrid, it is a general property of the tilings $\{p, q\}$. We refer the reader to [3, 4, 5] for a detailed analysis and detailed explanations of these tools. However, here we give a summary of these properties which allow the reader to better understand the further constructions.

The angular sector which we put in bijection with a tree is defined in a way which depends on the parity of $q$.

When $q$ is even, we fix a copy $P$ of the basic polygon of the tessellation. We fix a vertex $V$ of $P$ and we consider $\ell$ and $r$ the two rays issued from $V$ which support the sides of $P$ ending at $V$. If we turn around $V$ in the counter-clockwise orientation and remaining in $P$, we can first see $\ell$ and then $r$. The angle defined by $V$, $\ell$ and $r$ taken in this order in the above mentioned orientation, constitutes an angular sector and we say that $P$ is its **leading tile** and that $V$ is its **summit**. As proved in [3], the restriction of the tiling to this angular sector is in bijection with a tree, and this tree looks like the one which illustrated by Figure 2 when $q = 4$.

When $q$ is odd, we again fix a copy $P$ of the basic polygon to be the central cell. However, this time we cannot take the sides which abut to a vertex in order to define a sector. The reason is that the line which supports this side cuts another tile along one of its reflection axes and this leads to further complications. A simpler way is the following one, explained in [5]. It is illustrated by Figure 4.

First, we define the type of lines which we shall use to delimit the angular sector. Consider a copy of $P$ and fix a vertex $V$ of $P$. We know that exactly $q$ sides of copies of $P$ have $V$ as an end. There is a single one $e$ which is supported by the bisector of the angle defined by the two edges of $P$ which meet at $V$. Consider the mid-point $M$ of $e$. Consider the edge of a copy of $P$ which makes the angle $\vartheta = \dfrac{2\pi.h}{q}$ with $e$, the angle being measured while turning counter-clockwise around $V$. The over end of this edge is $W$. Next, repeat the construction from $W$ with the just considered edge and considering the edge

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defined by the angle $\vartheta$ with $VN$ measured while turning around $W$ clockwise this time. This defines a new edge whose other end is $X$. Now, the mid-points $M$, $N$ and $O$ of the just considered edges are on a same ray $\delta$ issued from $M$. Next, we can endlessly repeat these two steps of construction starting from $X$. The line which support this ray is called a $h$-**mid-point line** and the ray itself is a $h$-**mid-point ray**. Note that another $h$-mid-point ray is issued from $M$: it is obtained by taking the reflection of $\delta$ in the line which supports $e$.

Second, we can now define the angular sector as follows. We fix a copy of $P$ which is the leading tile of the sector. Let $b$ and $c$ be the edges of $P$ which meet at $V$. We may assume that while counter-clockwise turning around $V$ $c$ is met before $b$. Let $a$ be the edge which has $V$ as an end and which is supported by the bisector of the angle defined by $b$ and $c$. Let $B$ and $C$ be the mid-points of $b$ and $c$ respectively. Let $r$ be the $h$-mid-point ray issued from $M$ which passes through $B$. Consider the rotation $\rho$ around $V$ which transforms $b$ into $c$: it leaves globally invariant the set of edges meeting at $V$. Accordingly, $\rho$ transforms $M$ into $M$', the mid-point of another edge meeting at $V$ which is on the bisector of the angle defined by $c = \rho(b)$ and $\rho(c)$. It is easy to see that the $h$-mid-point ray $\ell$ issued from $M'$ which passes through $C$ is exactly $\rho(r)$, see Figure 5. The angular sector which we define has its summit at $V$, its leading tile is $P$ and it is delimited by the $h$-mid-point rays $\ell$ and $r$. Now, the set of tiles we consider as attached to this angular sector is the set of tiles whose all vertices but possibly at most one are included in the sector, its two rays being included. We again say that this set of tiles is the restriction of the tiling to the angular sector.
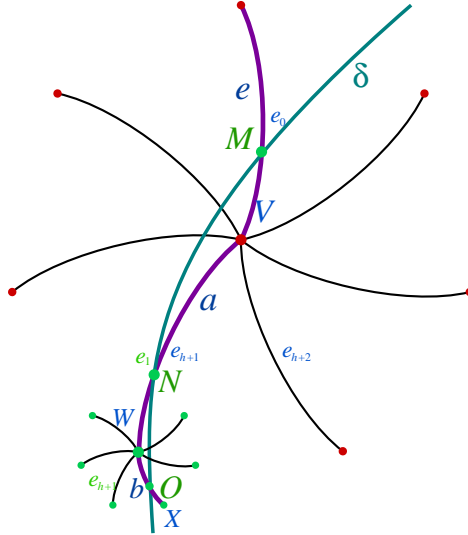


**Figure 4** *Illustration of the construction of the h-mid-point line.*

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**Figure 5** *The h-mid-point rays used for the definition of the angular sector when q is odd. The rotation around V which transforms r into ℓ also transforms M into M′ which is the mid-point of an edge abutting V of a copy of P around V.*

Now, in both cases, the whole tiling splits into the central cell and $p.(h-1)$ copies of the angular sector dispatched around the central tile.

It is important to note that the tree which is obtained in this case is a bit different from the tree which we obtain in the case when $q$ is even.

Before turning to the coordinates, we have to mention two important particular cases: the case when $q = 4$ and the case when $q = 3$. In both of them and in these cases only, the tilings $\{p, 4\}$ and $\{p+2, 3\}$ share very particular properties.

The first property is that for each $p \geq 5$, the tilings $\{p, 4\}$ and $\{p+2, 3\}$ have their angular sector spanned by the same tree.

The second property shared by all tilings $\{p, 4\}$ and $\{p+2, 3\}$ is the following. Fix a copy of $P$, the basic polygon of the tessellation. We call it the **central tile**. Say that $P$ belongs to generation 0. Now, the tiles obtained by reflection of a polygon of the generation $n$ in its sides and which do not belong to a generation $m$ with $m \leq n$ are said to belong to the generation $n+1$. Now from each tile of generation 1, we can define a copy of an angular sector, taking this tile as the leading tile of the sector, in such a way that the sectors and the central tile cover the whole plane without overlapping.

This can be obtained as follows in the case when $q = 4$. The summit of each sector is a vertex of the central tile $C$. Let $P$ be the leading tile of the angular

sector. We place the sector in such a way that the side shared by $P$ and $C$ is supported by the ray $\ell$ of the sector. We can see that the continuation of $r$ also supports a side of $C$, but this side is not contained in the sector: it is contained in another one.

In the case when $q = 3$, the $h$-mid-point lines coincide with the **mid-point lines** defined as lines joining mid-points of consecutive edges in a polygon. Then, the angular sector is defined by taking a neighbour $P$ of the central tile as leading tile of the sector. We chose $V$ a vertex belonging to $P$ and to the central tile in such a way that the ray $\ell$ passes through the mid-point of the common edge of $P$ and the central tile.

In both cases, it is not difficult to see that we have $q$ angular sectors around the central tile and that the sectors and the central cell cover the plane without overlapping, see Figure 6 for the pentagrid and the heptagrid. The levels of the tree which is in bijection with the restriction of the tiling to the sector define levels in the sector, thanks to the bijection. The leading tile is the root of the tree, whose level is 0, and it is not difficult to see that the level $n{+}1$ in the sector consists of the restriction, to the sector, of the tiles of the generation $n$.
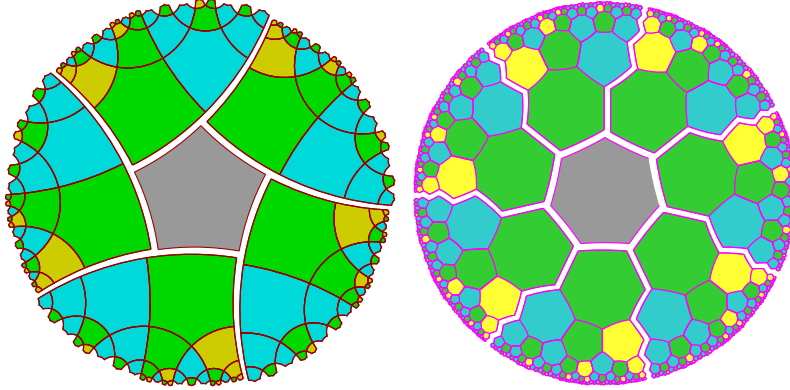


**Figure 6** *The display of $q$ sectors around the central tile. Left-hand side: in the pentagrid. Right-hand side: in the heptagrid.*

This correspondence between the generation by reflection in the sides and the levels of the tree is no more true when $q > 4$. This is due to the fact that, starting from $q = 6$ in the case when $q$ is even and from $q = 5$ in the case when $q$ is odd, one level of the tree contains tiles belonging to several generations, but not all of them, so that the correspondence between the levels and the generations is much more complex.

## 2.3 Coordinates for the tilings $\{p, q\}$

The existence of a tree which is in bijection with the restriction of the tiling to an appropriate angular sector is a key property. It is the basis of a very efficient navigation tool to locate tiles in these tilings. This tool is even more efficient in

the cases of the pentagrid and of the heptagrid.

In the case when $q$ is even, the coordinates of the tiles are obtained by the greatest positive root of the following polynomial:

$$P(X) = X^2 - ((p-3).(h-1)+1).X - h + 3,$$

where $h$ is defined by $q = 2h$. There are two kind of nodes: ordinary ones with $(p-3).(h-1)+1$ nodes and special ones with $(p-2).(h-1)-1$ nodes. In the case when $q$ is odd the coordinates are obtained from the greatest real root of another polynomial:

$$P(X) = X^3 - ((p-3)(h-1)+1)X^2 - ((p-2)(h-1)-2)X - h+3,$$

where $h$ is defined by $q = 2h+1$.

This time, the number of sons for the ordinary nodes is $(p-3).(h-1)+2$ nodes and among them, we have two nodes with special types, say 0 and 1, one node in each type. For the special nodes of type 0 they have $(p-3).(h-1)+1$ sons where one of them is of type 1 and all others are ordinary nodes. For the special nodes of type 1 they have $(p-2).(h-1)-1$ nodes with a single node of special type among them, namely the type 1.

In both cases, the greatest real root $\beta$ of $P(X)$ is positive. In fact it satisfies the following relation: $\beta > (p-3)(h-1)+1$ when $q$ is even and $\beta > p-2$ when $q$ is odd, assuming in both cases that $p \geq 5$ and $h \geq 4$. Now, for the remaining three cases, easy and direct computations show that $\beta > 1$. From $P(X)$, we define an induction equation:

$$u_{n+2} = ((p-3).(h-1)+1).u_{n+1} + (h-3).u_n,$$

for even $q$ with $u_0 = 1$ and $u_1 = (p-3).(h-1)+1$. When $q$ is odd, we define:

$$u_{n+3} = ((p-3)(h-1)+1).u_{n+2} + ((p-2)(h-1)-2).u_{n+1} + (h-3).u_n,$$

with $u_0 = 1$, $u_1 = (p-3)(h-1)+2$ and $u_2 = ((p-3)(h-1))^2 + (4p-11)(h-1)$. Now we can notice that if we take $u_{-1} = 0$, we obtain that:

$$u_2 = ((p-3)(h-1)+1).u_1 + ((p-2)(h-1)-2).u_0 + (h-3).u_{-1}.$$

We have that $u_n$ is the number of nodes of the level $n$ in the tree $\mathcal{T}$ in bijection with the restriction of the tiling to the angular sector.

Introducing $b = \lfloor \beta \rfloor$, it is known that any positive natural number $n$ can be represented as a sum of the form $n = \sum_{i=0}^{k} a_i u_i$, where $\{u_n\}_{n \in N}$ is defined by the equation associated to $P(X)$. In general, the representation is not unique. However, it can be made unique by requiring the representation which has the greatest number of digits, or in terms of the above formula, where $k$ is maximal. We call this representation the **greedy** representation of $n$ in the basis $\beta$. Numbering the nodes of $\mathcal{T}$ from the root and then level by level and, on each level from the left to the right, we call **coordinate** of a tile $T$ of an angular sector the greedy representation of the number attached to $T$ under the bijection between $\mathcal{T}$ and the sector. Now, the central tile has 0 as coordinate and any other tile is in a single sector $\sigma$ with $\sigma \in [1..max]$, where $max = p(h-1)$. Accordingly, we call **coordinate** of a tile $T$, either 0, for the central tile, or $(\sigma, \nu)$ where $\sigma$ is the number of the sector containing $T$ and $\nu$ is its coordinate in $\mathcal{T}$.

Now, the language of the coordinates is regular in all cases but one. The

reason is the following. It is not difficult to see that the other roots of $P(X)$ have a modulus less than 1 so that we always have a Pisot polynomial, except in the case when $q = 5$ and $p = 4$. Now, it is known that the language of the greedy representation of the positive natural numbers in the basis $\beta$ is regular if and only if $\beta$ is the greatest real root of a Pisot polynomial. However, in the exceptional case, we can introduce coordinates using the dual graph of $\{4, 5\}$ which is the pentagrid: as the pentagrid has a coordinate system with much more convenient properties we can use this system for $\{4, 5\}$, see [5] for more details.

We refer the reader to [3] for a detailed study of the properties of the co-ordinates of the tiles which allows us there to obtain the computation of the neighbours of a tile. Remember that by neighbour of a tile $T$, we mean a tile which shares a side with $T$.

We also attach numbers to the sides of a tile $T$. Assuming that side 1 is fixed, the other sides are obtained in increasing number while counter-clockwise turning around $T$. If $T$ is the central tile, side 1 is fixed once and for all. If $T$ is another tile, side 1 is the side shared with its father, assuming that the father of the leading tile of a sector is the central tile. This numbering of the sides allows to number the neighbours of a tile $T$: by definition, the neighbour $i$ of $T$ is the tile which shares with $T$ the side numbered $i$ in $T$.

For the following sections, we assume that we have the function $v(\sigma, \nu, \tau)$ at our disposal, which computes the neighbour $\tau$ of the tile $T$ whose coordinate is $(\sigma, \nu)$. Most often, the sector of $v(\sigma, \nu, \tau)$ is again the sector $\sigma$. Now, if $\nu$ is the coordinate of a node on an extremal branch of the tree, the sector of $v(\sigma, \nu, \tau)$ is different: if may be $\sigma \oplus 1$ or $\sigma \ominus 1$, where $\sigma \oplus 1 = \sigma + 1$ when $\sigma \in [1..max{-}1]$ and $max \oplus 1 = 1$; similarly, $\sigma \ominus 1 = \sigma - 1$ when $\sigma \in [2..max]$ and $1 \ominus 1 = max$. Then, when the sector of $v(\sigma, \nu, \tau)$ is not $\sigma$ it is either $\sigma \oplus 1$ or $\sigma \ominus 1$.

However, we give in Tables 1 and 2 the possibility to compute $v(\sigma, \nu, \tau)$ in the case of the heptagrid in terms of two auxiliary functions: $f(\nu)$, which gives the number of the father of a node belonging to the tree, and $\sigma(\nu)$ which is the number of neighbour 4 in all the cases. As in most cases, the sector of $v(\sigma, \nu, \tau)$ is also $\sigma$, we shall consider only the number of the tree for $v(\nu, \tau)$, the neighbour $\tau$ of the node $\nu$. In Table 2, we mention the computation of the new sector when the neighbour is outside the sector of the tile.

The computation of these expressions can be found in [3, 4] but we repeat them for the self-containedness of the paper. The computation of $v(\tau, \nu)$ involves the already mentioned auxiliary functions, $f(\nu)$ and $\sigma(\nu)$. As mentioned from [1, 2], the tree which we consider has two kinds of nodes: black and white ones with two and three sons respectively. The sons can be deduced from the node by the following rules $B \rightarrow B_* W$ and $W \rightarrow BW_* W$ in easy notations, where the star indicates the place of neighbour 4. We assume here that the functions $f(\nu)$ and $\sigma(\nu)$ are known. Their computation is efficient but it requires notions which we have not mentioned here. We refer the reader to [2, 3, 4] for additional information.

Clearly, the coordinate of a neighbour $N$ of a tile $T$ with coordinate $\nu$ de-

pends on the side $\tau$ shared by $T$ and $N$. Now, notice that this side numbered by $\tau$ in $T$ does not receive the same number in $N$ and we shall say that $N$ is the neighbour $\tau$ of $T$. The correspondence between these numbers gives the value of the function $t(\nu, \tau)$ and, for completeness, we give it in Table 1. Note that the sides of the central cell are all numbered by 1 in its neighbours. For the other cells, the correspondence depends on the status of $T$ and it may also depend on that of $N$. Side 7 is always the side shared by a neighbour which is on the same level of the tree, even when there is a change of tree by the change of sector. If $T$ is black, its side 7 is numbered 2 on the other side. If $T$ is white, the number of its side 7 in the other neighbour $N$ depends on the status of $N$ as indicated in the table.

**Table 1** *Correspondence between the numbers of a side shared by two heptagons, $T$ and $N$. Note that if $T$ is white, the other number of side 1 may be 4 or 5 when $N$ is white and that it is always 5 when $N$ is black.*

| | black $T$ | | white $T$ | |
| :---: | :---: | :---: | :---: | :---: |
| in $T$ | | in $N$ | in $T$ | in $N$ |
| 1 | | $3^{wN},4^{bN}$ | 1 | $4^{wN}, 5$ |
| 2 | | 6 | 2 | 7 |
| 3 | | 7 | 3 | 1 |
| 4 | | 1 | 4 | 1 |
| 5 | | 1 | 5 | 1 |
| 6 | | 2 | 6 | 2 |
| 7 | | 2 | 7 | $2^{wN}, 3^{bN}$ |

**Table 2** *The values of $v(\tau, \nu)$.*

| $\tau$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| black | $f(\nu)$ | $f(\nu)-1$ | $\nu-1$ | $\sigma(\nu)$ | $\sigma(\nu)+1$ | $\sigma(\nu)+2$ | $\nu+1$ |
| left | $f(\nu)$ | $\nu-1$ | $\sigma(\nu)-1$ | $\sigma(\nu)$ | $\sigma(\nu)+1$ | $\sigma(\nu)+2$ | $\nu+1$ |
| white | $f(\nu)$ | $\nu-1$ | $\sigma(\nu)-1$ | $\sigma(\nu)$ | $\sigma(\nu)+1$ | $\sigma(\nu)+2$ | $\nu+1$ |
| right | $f(\nu)$ | $\nu-1$ | $\sigma(\nu)-1$ | $\sigma(\nu)$ | $\sigma(\nu)+1$ | $\nu+1$ | $f(\nu)+1$ |
| root | 0 | 1 | $\sigma(\nu)-1$ | $\sigma(\nu)$ | $\sigma(\nu)+1$ | $\nu+1$ | 1 |

NOTE: left denotes the leftmost branch of the tree, right denotes its rightmost one.

In the case of left, the sector for neighbours 2 and 3 is $\sigma \ominus 1$. In the case of right, the sector for neighbours 6 and 7 is $\sigma \oplus 1$.

From this table, we can indicate the values of $v(\tau, \nu)$ which are given in Table 2. The basic point is that $v(1, \nu)$ for the heptagon $H$ defined by $\nu$ is always $f(\nu)$, as its neighbour 1 is the father of $H$. Similarly, we have that $v(4, \nu)$ is always $\sigma(\nu)$ by definition, regardless of the status of $H$. Note that in

the case of a black heptagon $H$ on the leftmost branch of the tree, two of its neighbours belong to the other tree on this side of the sector of $H$: neighbours 2 and 3. Neighbour 2 is still $\nu-1$ and, consequently, neighbour 3 is the rightmost son of neighbour 2, hence it is $\sigma(\nu-1)+1 = \sigma(\nu-1)$. A symmetrical remark holds for a white $H$ standing on the rightmost branch of the tree: neighbours 6 and 7 belong to another tree, the one which spans the other sector than that of $H$. Now, this time, neighbour 6 is numbered $\nu+1$ and so, as neighbour 7 is the father of neighbour 6, neighbour 7 is numbered $f(\nu+1) = f(\nu)+1$. At last, the root, which is a white node, belongs to both the left- and the rightmost branches of the tree. This is why it has a specific profile, different from both a standard white node and from a node on the rightmost branch of the tree.

It remains to indicate that in the case of a heptagon $H$ which is on the left- or the rightmost branch, it is easy to define the number of the sector to which belongs the neighbours which do not belong to the tree of $H$. The new number is $\sigma \ominus 1$ for neighbours 2 and 3 of a tile which is on the leftmost branch of the tree, it is $\sigma \oplus 1$ for neighbours 6 and 7 of a tile which is on the rightmost branch, see Table 2.

## 3 The triangular grids

Now, we can define the new family of tilings constructed upon the tilings $\{p, q\}$. This is the goal of Subsection 3.1. Then, in Subsection 3.2, we construct coordinates for the tiles of the tilings which we define in Subsection 3.1.

### 3.1 Defining the trigrids

In [4], the author gives a system of coordinates for the points of the hyperbolic plane based on the heptagrid. The idea, illustrated by Figure 7, is to iterate a simple construction. In the first step, we split each heptagon $H$ in seven equal triangles, each one based on an edge of $H$ and having its summit at the centre of $H$. This first step of the process defines **1-triangles**. Now, considering that the $n$-**triangles** have been defined, for each $n$-triangle $T$, we obtain four $n+1$-triangles as follows: three of these triangles are defined by a vertex $V$ of $T$ and by the mid-points of the two edges of $T$ which meet at $V$; the fourth $n+1$-triangle is defined by the mid-points of the edges of $T$. In Figure 7, we can see the 1-triangles and the 2-triangles obtained from a heptagon.

We can describe this process more precisely and generalize it as follows.

We fix a tiling $\{p, q\}$ of the hyperbolic plane. Accordingly, each tile has a coordinate $(\sigma, \nu)$, with $\sigma$ the number of the sector where the tile lies and $\nu$ its coordinate in the sector, the coordinate being 0 for the central tile.

Each polygon $P$ defines $p$ 1-triangles, each one being numbered from 1 up to $p$. The 1-triangle $i$ has the side $i$ of $P$ as its basis and the centre of $P$ as the vertex, opposite to the basis. In each triangle, we number the vertices from 0 to 2 with 2 being given to the centre of $P$ and 0 with 1 associated to the ends of the side $i$. We fix the positions of 0 and 1 in such a way that 0,1

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and 2 are obtained by counter-clockwise turning around the 1-triangle. Once the numbering of the vertices are fixed, we use the same numbers to number the edges and the mid-points of the edges. By convention, the edge $i$ is opposite to the vertex $i$ and the mid-point of the edge $i$ is also called $i$. Once the $n$-triangles are defined with the numbering of their vertices, their edges and the mid-points of their edges are fixed, we can define the $n+1$-triangles as follows. Consider an $n$-triangle $T$. If $\alpha$ is a vertex of $T$, then the edges meeting at $\alpha$ are numbered $\beta$ and $\gamma$ with $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$. It is the same for the mid-points of these edges and so, the vertex $\alpha$ and the mid-points $\beta$ and $\gamma$ define an $n+1$-triangle. Again by definition, the mid-points of the edges of $T$ also define an $n+1$-triangle. Now, as the vertices of each $n+1$-triangle has received
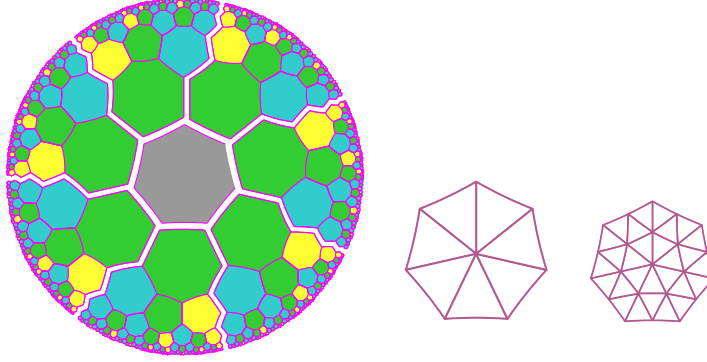


**Figure 7** *From the heptagrid to the second triangular heptagrid, the heptatrigrid.*

a number in $[0..2]$ in such a way that we have 0,1, 2 or 0,2,1 while counter-clockwise turning around the $n+1$-triangle, the edges and the mid-points of each $n+1$-triangle can also be numbered by applying the same rules as those applied for numbering the respective elements of the $n$-triangles. At last, let $T$ be an $n+1$-triangle $T$ and let $R$ be the single $n$-triangle which contains $T$. If $T$ has as a vertex the vertex $\alpha$ of $R$, $T$ is numbered $\alpha$. If all the vertices of $T$ are the mid-points of the edges of $R$, then $T$ is numbered with 3.

Accordingly, this process can be repeated endlessly as it was completed for the 1-triangles. We can notice that if the numbering of an $n$-triangle $T$ is 0, 1, 2 while counter-clockwise turning around $T$, the numbering of the $n+1$-triangles inside $T$ numbered 0, 1 and 2 has the opposite orientation, while the numbering of the $n+1$-triangle 3 has the same orientation.

For each $n$, the set of $n$-triangles constitutes a triangular tiling of the hyperbolic plane which we call the $n$-**trigrid** based on the tiling $\{p, q\}$. As already mentioned in the captions, the trigrids based on the heptagrid are called **heptatrigrids**.
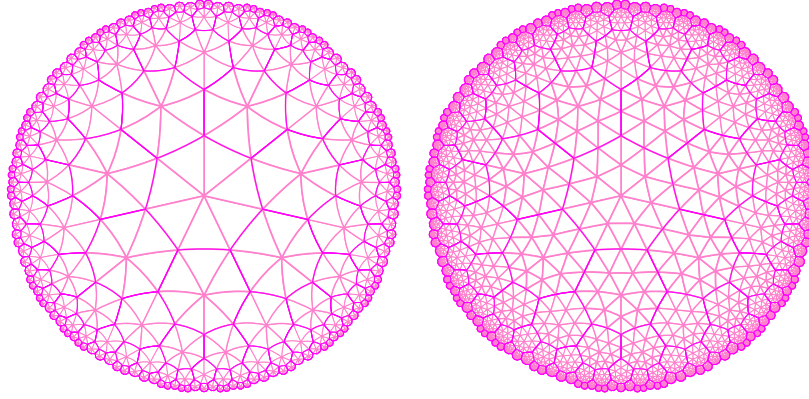
**Figure 8** *Two heptatrigrids: generation 1 on the left-hand side, generation 2 on the right-hand side.*

## 3.2 Coordinates for the trigrids

Now, we have all the elements which allow us to define coordinates for the tiles of the trigrids.

Consider an $n+1$-triangle $T$, with $n > 0$. There is a unique sequence of $n+1$ $m$-triangles $T_i$ with $m \leq n$, $T_{i+1} \subset T_i$ for $i \in [1..n]$, with $T_{n+1} = T$ and each $T_i$ being an $i$-triangle. Let $[i]$ be the number of $T_i$. Then the coordinate of $T$ is the sequence $(\sigma, \nu, [\alpha_1], .., [\alpha_{n+1}])$. We remark that $[\alpha_1] \in [1..p]$ and that $[\alpha_i] \in [0..3]$ for $i \in [2..n+1]$.

In [4] the author defined coordinates of the points of the hyperbolic plane starting from the heptagrid by considering, for each point, a sequence of $n$-triangles converging to the point. The numbers of the $n$-triangles which are defined in [4] exactly in the same way as they are defined in Subsection 3.1 are used as digits for the representation of the point. There are two big differences with the situation we consider in this paper: in [4], the coordinate of a point is an infinite sequence while here, it is a finite one; the second difference is that a point has infinitely many possible coordinates while here, an $n$-triangle has a unique coordinate.

In this sub-section, we shall see how to compute the coordinates of the neighbours of a tile.

Consider an $n+1$-triangle $T$. Let $S$ be the $n$-triangle which contains $T$. We shall see that if we know the coordinates of the neighbours of $S$, we can easily compute the coordinates of the neighbours of $T$.

The reason will be clear from the following study.

If [3] is the number of $T$, all its neighbours are in $S$. in the other cases, one neighbour is in $S$ while the others are not, but, as can easily be seen, they are necessarily in a neighbour of $S$. We denote the neighbours of an $n$-triangle in a way which similar to what we did for a polygon. Here, the neighbours 1, 2 and 3

of an $n$-triangle $Z$ are the $n$-triangles which share with $Z$ the side 0, 1 and 2 respectively. The exact correspondence for the neighbours of $T$ is given by the following table in which $U$, $V$ and $W$ are the neighbours 1, 2 and 3 respectively of $S$. Also, we denote by $<Z>$ the coordinate of the $n$-triangle $Z$ and the $n{+}1$-triangles contained in $Z$ have $<Z>.[0]$, $<Z>.[1]$, $<Z>.[2]$ and $<Z>.[3]$ as coordinates, where '.' denotes the concatenation which transforms a sequence with $k$ terms into a sequence with $k{+}1$ terms.

**Table 3** *Table of the coordinates of the neighbours of an $n{+}1$-triangle $T$, knowing the coordinates of the neighbours of the $n$-triangle $Z$ which contains $T$ and also knowing the coordinates of the neighbours of $Z$.*

| $Z$ | $U$ | $V$ | $W$ |
|-----|-----|-----|-----|
| $T.0$ | $<T>.[3]$ | $<W>.[1]$ | $<V>.[1]$ |
| $T.1$ | $<W>.[0]$ | $<T>.[3]$ | $<U>.[0]$ |
| $T.2$ | $<V>.[2]$ | $<U>.[2]$ | $<T>.[3]$ |
| $T.3$ | $<T>.[0]$ | $<T>.[1]$ | $<T>.[2]$ |

From this table, we can easily devise an algorithm which computes the coordinates of the neighbours of a tile $T$. The idea is simple: we start with computing the neighbours $U_1$, $V_1$ and $W_1$ of the 1-triangle $T_1$ which contains $T$. We know that $U_1$ and $V_1$ belong to the same polygon $P$ as $T_1$, they are simply based on another side of $P_1$. More precisely, if $\tau$ is the number of the side on which $T_1$ is based, then $U_1$ is based on the side $\tau \oplus 1$ and $V_1$ is based on the side $\tau \ominus 1$. Now, $W_1$ belongs to a polygon $Q$ which is the neighbour $\tau$ of $P$ and the number in $Q$ of the side which it shares with $P$ is given by a table similar to Table 1 which can be used in the case of a heptatrigrid. Next, we go from the neighbours $U_m$, $V_m$, $W_m$ of $T_m$, where $T_m$ is the $m$-triangle which contains $T$, to those, $U_{m+1}$, $V_{m+1}$, $W_{m+1}$ of $T_{m+1}$ thanks to Table 3. The exact writing of this algorithm in a pseudo-code or in a programming language is left to the reader.

Now, we can notice that each step of the considered algorithm is constant in term of resources and in term of what is appended to the already known resources. Accordingly, this algorithm is linear. Now, it was proved in [4] that the computation of the coordinates of the neighbours of a tile in a tiling $\{p, q\}$ is linear in the size of the coordinate of the tile itself. Accordingly, this proves the following result:

**Theorem 1** *For each $p$ and $q$ such that the tiling $\{p, q\}$ is a tiling of the hyperbolic plane, there is an algorithm which computes the coordinates of the neighbours of an $n$-triangle $T$ in linear time with respect to the size of the coordinate of $T$.*

It is worth noticing that this algorithm allows us to compute the size of the coordinates of a tile for each $n$-trigrid, independently of $n$. Note that, in any case, the size of $n$ is contained in unary in the size of the coordinate as the

number of numbers of $m$-triangles involved in the coordinates is $n-1$. But, of course, there is an algorithm for each $p$ and $q$.

# 4 Conclusion

With Theorem 1, we have an algorithm which allows us to efficiently compute the coordinates of the neighbours of a tile in any $n$-trigrid of a given tiling $\{p, q\}$. This allows us to implement cellular automata in such tilings. In particular, this holds in the heptatrigrid.

It might be objected that the resulting cellular automaton has not a uniform structure. This is not completely true. First of all, each triangle has three neighbours so that, as long as the neighbours are defined by the condition of sharing a side with the tile, the uniformity on the number of neighbours is preserved. What is here different is that the number of tiles around a vertex is not always the same. It is 6 in the majority of cases but it is $p$ in the others: in fact, when an $m$-triangle has the centre $C$ of a polygon as a vertex, there are $p$ $m$-triangles around $C$, regardless of $m$. But, this occurrence of vertices around which there is a number of sides which is different of 6 is also regular as the tiling $\{p, q\}$ itself is regular, being defined by the tessellation of a regular polygon.

Such an implementation of cellular automata was done in the heptatrigrid, see [6].

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