信息论作业 1

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题目 1

1. 两枚骰子总点数之和为 7 的情况有 $\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$ 6 种情况，总共可能产生 $6\times 6=36$ 种情况。所以投掷总点数之和为 7 的概率为 $\frac{3}{18}$，其自信息为 $\log_2\frac{18}{3}\approx 2.58bit$。

2. 两枚骰子总点数之和为 12 的情况有 $\{(1,1)\}$ 1 种情况。所以投掷总点数之和为 12 的概率为 $\frac{1}{36}$，其自信息为 $\log_2 36\approx 5.17bit$。

题目 2 本题目如果直接计算多个离散概率的互信息，脑补了一下计算复杂度，可能会比较高，查阅了部分资料后 [1]，得到如下结论（论文上说 Elements of Information Theory 上面有，但是我没找到）：

Corollary 1 *Let $Y=X_1+X_2$, where X_1 and X_2 are independent, then $I(X_1;Y)=H(Y)-H(X_2)$*

$$
\begin{aligned}
I(X_1;X_1+X_2) &= H(X_1+X_2)-H(X_1+X_2|X_1) && (1)\\
&= H(X_1+X_2)-H(X_2|X_1) && (2)\\
&= H(X_1+X_2)-H(X_2) && (3)\\
&= H(Y)-H(X_2) && (4)
\end{aligned}
$$

其中 (1) 到 (2) 的步骤不是很明白，想请老师答疑解惑一下。本题目的解答中很大程度上依靠了 (1) 到 (2) 化简方法。这里令 X_1 表示第一颗骰子的结果，X_2 表示第二颗骰子的结果，X_3 表示第三颗骰子的结果，所以 $X=X_1,Y=X_1+X_2,Z=X_1+X_2+X_3$。

$$
\begin{aligned}
H(X) &= H(X_1)=H(X_2)=H(X_3) && (5)\\
&= \sum_{1}^{6}\frac{1}{6}\log_2 6 && (6)\\
&= \log_2 6bit && (7)\\
&= 2.5850bit && (8)
\end{aligned}
$$

$$
\begin{aligned}
H(Y) &= H(X_1 + H_2) = H(X_2 + X_3) = H(X_1 + X_3) && (9)\\
&= \sum_p^{\Omega} p \log_2 \frac{1}{p} && (10)\\
&= 3.2744 bit && (11)\\
\Omega &= \{\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}\} && (12)
\end{aligned}
$$

1.

$$
\begin{aligned}
H(Z|Y) &= H(X_1 + X_2 + X_3 | X_1 + X_2) && (13)\\
&= H(X_3 | X_1 + X_2) && (14)\\
&= H(X_3) && (15)\\
&= 2.5850 bit && (16)
\end{aligned}
$$

2.

$$
\begin{aligned}
H(X|Y) &= H(X_1 | X_1 + X_2) && (17)\\
&= H(X_1) - I(X_1; X_1 + X_2) && (18)\\
&= H(X_1) - H(X_1 + X_2) + H(X_2) && (19)\\
&= 1.8955 bit && (20)
\end{aligned}
$$

3.

$$
\begin{aligned}
H(Z|X, Y) &= H(X_1 + X_2 + X_3 | X_1, X_1 + X_2) && (21)\\
&= H(X_3 | X_1, X_1 + X_2) && (22)\\
&= H(X_3) && (23)\\
&= 2.5850 bit && (24)
\end{aligned}
$$

4.

$$
\begin{aligned}
H(X, Z|Y) &= H(X|Y) + H(Z|X, Y) && (25)\\
&= 4.4805 bit && (26)
\end{aligned}
$$

5.

$$
\begin{aligned}
H(Z|X) &= H(X_1 + X_2 + X_3 | X_1) && (27)\\
&= H(X_2 + X_3 | X_1) && (28)\\
&= H(X_2 + X_3) && (29)\\
&= 3.2744 bit && (30)
\end{aligned}
$$

题目 3

1.

$$\begin{aligned}
P(0) &= \sum_{i=1}^{8} P(u_i)P(0|u_i) && (31)\\
&= \frac{4(1-p)}{8} + \frac{4p}{8} && (32)\\
&= \frac{1}{2} && (33)\\
P(u_1|0) &= \frac{P(u_1)P(0|u_1)}{P(0)} && (34)\\
&= \frac{\frac{1}{8}(1-p)}{\frac{1}{2}} && (35)\\
&= \frac{1-p}{4} && (36)\\
I(u_1;0) &= \log_2 \frac{P(u_1|0)}{P(u_1)} && (37)\\
&= \log_2 \frac{\frac{1-p}{4}}{\frac{1}{8}} && (38)\\
&= 1 + \log_2(1-p) bit && (39)
\end{aligned}$$

2.

$$\begin{aligned}
P(00) &= \sum_{i=1}^{8} P(u_i)P(00|u_i) && (40)\\
&= \frac{1}{4} && (41)\\
P(u_1|00) &= \frac{P(u_1)P(00|u_1)}{P(00)} && (42)\\
&= \frac{1-p^2}{2} && (43)\\
I(u_1;00) &= \log_2 \frac{P(u_1|00)}{P(u_1)} && (44)\\
&= 2 + 2\log_2(1-p) bit && (45)
\end{aligned}$$

3.

$$P(000) = \sum_{i=1}^{8} P(u_i)P(000|u_i) \tag{46}$$
$$= \frac{1}{8} \tag{47}$$
$$P(u_1|000) = \frac{P(u_1)P(000|u_1)}{P(000)} \tag{48}$$
$$= (1-p)^3 \tag{49}$$
$$I(u_1;000) = \log_2 \frac{P(u_1|000)}{P(u_1)} \tag{50}$$
$$= 3 + 3\log_2(1-p) bit \tag{51}$$

4.

$$P(0000) = \sum_{i=1}^{8} P(u_i)P(0000|u_i) \tag{52}$$
$$= \frac{(1-p)^4 + p^4 + 6(1-p)^2p^2}{8} \tag{53}$$
$$P(u_1|0000) = \frac{P(u_1)P(0000|u_1)}{P(0000)} \tag{54}$$
$$= \frac{(1-p)^4}{(1-p)^4 + p^4 + 6(1-p)^2p^2} \tag{55}$$
$$I(u_1;0000) = \log_2 \frac{P(u_1|0000)}{P(u_1)} \tag{56}$$
$$= \log_2 \frac{8(1-p)^4}{(1-p)^4 + p^4 + 6(1-p)^2p^2} bit \tag{57}$$

参考文献

[1] Mokshay Madiman. On the entropy of sums. In *2008 IEEE Information Theory Workshop*, pages 303–307. IEEE, 2008.