TURBULENT FLOW OVER A WAVY BOUNDARY

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Abstract. The linearized, two-dimensional flow of an incompressible fully turbulent fluid over a sinusoidal boundary is solved using the method of matched asymptotic expansions in the limit of vanishing skin-friction.

A phenomenological turbulence model due to Saffman (1970, 1974) is utilized to incorporate the effects of the wavy boundary on the turbulence structure.

Arbitrary lowest-order wave speed is allowed in order to consider both the stationary wavy wall, and the water wave moving with arbitrary positive or negative velocity.

Good agreement is found with measured tangential velocity profiles and surface normal stress coefficients. The phase shift of the surface normal stress exhibits correct qualitative behavior with both positive and negative wave speeds, although predicted values are low.

1. Introduction

The problem of the generation of water waves by wind has received considerable attention over the past 100 years. The classical Kelvin–Helmholtz treatment (Lamb, 1945), which assumes uniform flow in the upper and lower fluids and yields a surface perturbation pressure in perfect anti-phase with the wave elevation (for stable waves), is incapable of accurately predicting wave growth. Jeffreys' "sheltering" model (Lamb, 1945) assumed the existence of a component of surface perturbation pressure in phase with the wave slope, presumed to arise by means of a separation of the flow on the leeward side of the wave with respect to the flow at infinity; however, the criterion established for wave growth contains a scalar (β, the 'sheltering coefficient') whose dependence on the flow characteristics is unspecified.

The work of Miles (1957, 1959) provided a mechanism to account for the component of surface perturbation pressure in phase with the wave slope for a wave progressing in the direction of the wind. Linearizing the equations of motion and assuming there was no effect of the wave on the turbulent Reynolds stresses anywhere within the flow field, Miles obtained the Orr-Sommerfeld equation as a description of the perturbation dynamics. Viscous stresses governed the flow near the wave surface and 'critical layer' (where the $O(1)$ wind velocity equals the wave velocity). The work of Benjamin (1959) extended the problem to arbitrary wind profiles and wave speeds.

Later, Miles (1967) concluded that this model underestimated the energy transfer from wind to waves, and conjectured that the wave-induced Reynolds stresses were not negligible over a significant portion of the gravity-wave spectrum.

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There have been several attempts since 1967 to account for the effect of the wavy boundary on the Reynolds stresses. Davis (1972) examined numerically two different turbulence models for the linearized problem, one patterned after the method of Bradshaw *et al.* (1967), and the other manifesting a viscoelastic constitutive relation for the turbulence. His results were inconclusive due to difficulties experienced in accounting for the boundary condition on the tangential velocity near the surface. Townsend (1972) treated the linearized problem using a turbulence model similar to that of Bradshaw *et al.* (1967) and obtained rates of wave growth, based on calculated distribution of surface pressure, considerably less than experimental values. He argued, in addition, that the linearized approach would be valid only for $ka < 0.10$, where a is the amplitude and k is the wave number. Manton (1972) proposed a viscoelastic turbulence model and found reasonable agreement with the data of Dobson (1969).

In the present case, the turbulence model equations of Saffman (1970) were utilized (for other applications of these equations, see Saffman (1974), Saffman and Wilcox (1974), and Wilcox (1973)). In addition to the practical problem of wind-wave generation, the flow configuration provides a valuable context in which to examine the ability of phenomenological turbulence models to predict the effects of mean streamline curvature – an important area of current research (Bradshaw, 1973).

2. Statement of the Problem and Assumptions

The problem considered is the two-dimensional flow of an incompressible, fully turbulent fluid over a single Fourier component of amplitude a, wave number k and velocity c where $c = \pm(g/k)^{1/2}$ for deep-water gravity waves and $g = 9.8 \text{ m s}^{-2}$. The coordinate system is taken attached to the wave moving at speed c with respect to the lower flow (see Figure 1). The surface is described by $y_s = a \cos kx$.

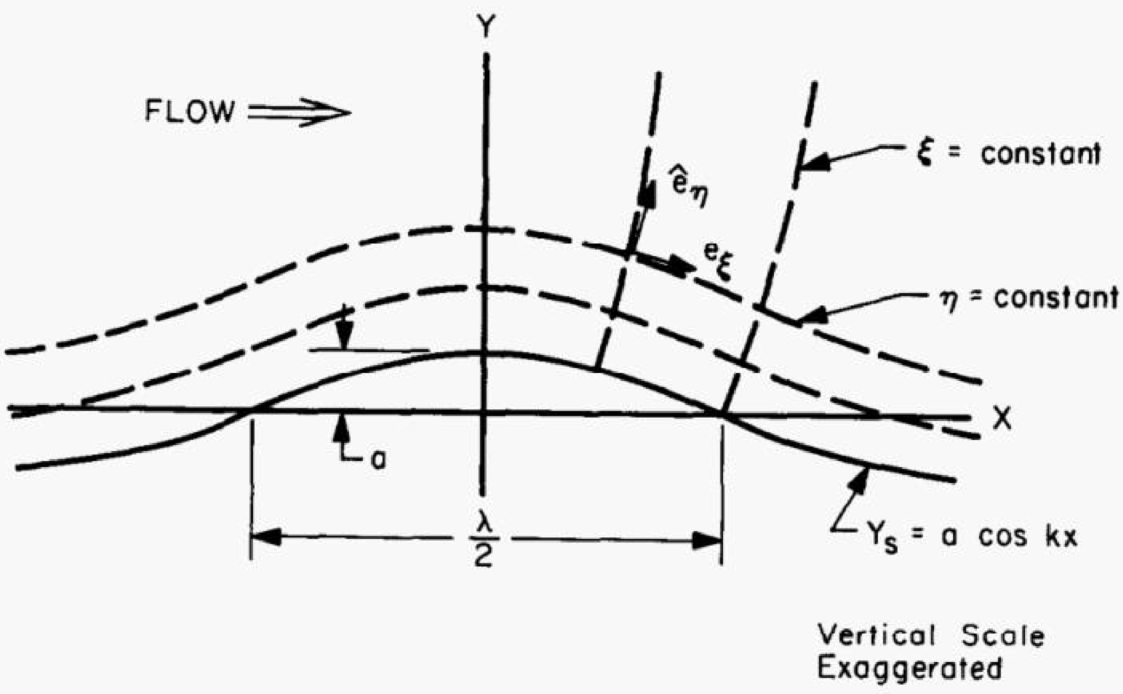

Fig. 1. Flow geometry and coordinate system.

To facilitate handling of the boundary conditions at the wave surface, an orthogonal curvilinear coordinate system (ξ, η, z) introduced by Benjamin (1959) is used where

$$\xi = x + ae^{-ky} \sin kx$$
$$\eta = y - ae^{-ky} \cos kx.$$

Note that at the surface, $k\eta = 0$ to $O(ka)$.

The following assumptions are made:

(1) The flow is exactly periodic; i.e., for any dependent variable $f(\xi, \eta)$, $f(\xi + \lambda, \eta) = f(\xi, \eta)$, $\lambda = 2\pi/k$. The dependent variables may therefore be expanded in terms of the harmonics $e^{ink\xi}$, $n = 0, 1, 2, \ldots$, and the partial differential equations governing the flow reduce to the more tractable case of an infinite hierarchy of ordinary differential equations in η. To lowest order in ka,

$$f(\xi, \eta) = f_0(\eta) + o(1) \quad \text{as} \quad ka \to 0,$$

implying that the $O(1)$ flow (i.e., the flow in the limit $ka \to 0$) is independent of streamwise position, a reasonable assumption in view of the slight downstream variation of quantities in a non-separating turbulent boundary layer.

(2) $ka \ll 1$: Only the $O(ka)$ perturbation is obtained in this analysis, hence ka should be small for application. Observations of non-breaking oceanic gravity waves give values of ka between 0.025 and 0.30, with the maximum predicted by Stokes being 0.45 (Kinsman, 1965).

(3) $\eta_{*_0} \ll a$, where η_{*_0} is the sublayer scale of the upper flow evident in the formula for the $O(1)$ mean velocity in the ξ-direction:

$$U = \frac{u_{*_0}}{\kappa} \ln(\eta/\eta_{*_0}) - c,$$

where $u_{*_0} = (\tau_{w_0}/\rho)^{1/2}$, τ_{w_0} being the $O(1)$ surface shear and ρ being the upper fluid density, c = wave speed, and κ = von Kármán's constant, taken to be 0.41 (Coles and Hirst, 1968).

For a smooth surface, η_{*_0} is proportional to the viscous sublayer thickness, and is given by

$$\eta_{*_0} = \frac{\nu}{u_{*_0}} \bar{e}^{B\kappa}$$

where ν = kinematic viscosity of upper fluid, and $B = 5.0$ (Coles and Hirst 1968). For a rough surface, η_{*_0} is the "roughness height".

Throughout this paper, the term "sublayer scale" will be used to denote η_{*_0}, since both smooth and rough surfaces are considered.

The assumption that $\eta_{*_0} \ll a$ implies that the variation in the normal velocity, total shear stress (viscous plus turbulent) and normal stress across the sublayer is negligible (for details, see Knight, 1974).

(4) $\lambda \ll \delta$, where δ = turbulent boundary-layer thickness. The $O(1)$ velocity profile used is logarithmic, which is reasonably accurate provided the scale of the decay of the wave-induced perturbations (i.e., λ) is small compared to the boundary-layer thickness.

(5) For $c \neq 0$, the upper flow imposes only a slight perturbation on the wave speed, i.e., Kelvin–Helmholtz instability is not considered. Also the lower flow is taken to be laminar, two-dimensional, incompressible and viscous and is analyzed according to classical techniques by linearization in ka (Lamb, 1945).

3. Turbulence Model, Equations of Motion and Boundary Conditions

A phenomenological turbulence model developed by Saffman (1970) is employed. The model specifies the following constitutive behavior for the Reynolds stress (Cartesian tensor notation)

$$-\overline{u_i'u_j'} = 2\frac{e}{\omega} S_{ij} - \tfrac{1}{3}\overline{q^2}\delta_{ij}, \tag{1}$$

where

u_i = total instantaneous velocity in the ith direction;

$= \bar{u}_i + u_i'$, where the overbar represents an ensemble average at a fixed point (ξ, η, z, t);

$S_{ij} = \frac{1}{2}(\partial_j \bar{u}_i + \partial_i \bar{u}_j)$ = rate-of-strain tensor where $\partial_j \equiv \partial/\partial x_j$;

e = "pseudo-turbulence energy";

ω = "pseudo-vorticity", resembling the root-mean-square fluctuating vorticity of the large eddies;

$\overline{q^2}$ = twice the turbulence kinetic energy = $\sum_{i=1}^{3} \overline{u_i'u_i'}$.

The transport equations for the turbulence quantities introduced above are the following (Saffman, 1970), where Cartesian tensor notation and the Einstein summation convention are employed:

$$\partial_t e + \bar{u}_k \partial_k e = \tilde{\alpha} e\{2S_{ij}S_{ij}\}^{1/2} - e\omega + \partial_k\left(\tilde{\sigma}\frac{e}{\omega}\partial_k e\right\} \tag{2}$$

$$\partial_t \omega + \bar{u}_k \partial_k \omega = \alpha\omega\{\partial_j \bar{u}_i \partial_j \bar{u}_i\}^{1/2} - \beta\omega^2 + \partial_k\left(\sigma\frac{e}{\omega}\partial_k \omega\right). \tag{3}$$

In the rate equations for e and ω, the terms on the right-hand side represent, respectively, generation, dissipation and diffusion. The constants α, $\tilde{\alpha}$, β, σ and $\tilde{\sigma}$

have been evaluated by simple considerations (Saffman, 1970), and were not optimized for any particular set of flows.* Their values are restricted to

$$\tilde{\alpha} = 0.3, \quad 0.075 < \alpha < 0.106$$
$$\tfrac{5}{6} < \beta < 1, \quad \tilde{\sigma} = 0.5, \quad \sigma = 1.0.$$

Furthermore, matching the law-of-the-wall boundary condition on the mean velocity requires

$$\left(\frac{\beta\tilde{\alpha} - \alpha}{\sigma}\right)^{1/2} = \kappa,$$

where κ = von Kármán's constant. The above range of values yields

$$0.380 < \left(\frac{\beta\tilde{\alpha} - \alpha}{\sigma}\right)^{1/2} < 0.475.$$

The accepted value of 0.41 lies within this interval.

The remaining equations are the conservation of mass

$$\partial_k \bar{u}_k = 0 \tag{4}$$

and conservation of momentum

$$\partial_t \bar{u}_i + \bar{u}_k \partial_k \bar{u}_i = -\frac{1}{\rho} \partial_i \Phi + \partial_k[-\overline{u_i' u_k'}], \tag{5}$$

where $\Phi = p + \frac{1}{3}\rho\overline{q^2} + \rho g y$ and $\bar{p}$ = mean static pressure. The equations are valid only within the fully turbulent region of the upper fluid, as viscous diffusion has been neglected.

The equations of motion (1) to (5) are written in the (ξ, η) coordinate system (Knight, 1974), and a streamfunction $\psi(\xi, \eta)$ introduced, where the total mean velocities in the ξ and η directions are given by, respectively,

$$(u, v) = J^{1/2}(\psi_\eta, -\psi_\xi),$$

where $J = (\partial(\xi, \eta)/\partial(x, y)) = 1 + 2kae^{-k\eta} \cos k\xi + O(ka)^2$ is the Jacobian of the transformation.

In order to gain additional insight into the effects of the wavy surface on the turbulence structure, the problem was also solved with a relaxation tensor incorporated into the equation for the Reynolds stress (1). An additional simple rate equation was postulated for the behavior of the relaxation tensor. With certain exceptions, this preliminary relaxation model yielded no improvement over the no-relaxation model detailed above, and the results have not been presented herein (for details, see Knight, 1974). A more refined relaxation model may be found in Saffman (1974, 1976).

* In the original paper (Saffman, 1970), a rate equation was written for ω^2: for the convenience of this analysis, the equation was written for ω and all constants re-evaluated according to the procedure indicated in the original paper.

As the equations are valid within the fully turbulent region only, boundary conditions must be applied asymptotically as η approaches the edge of the sublayer, i.e., as $\eta \to 0$. Neglecting terms of $O(ka)^2$, they are as follows (Knight, 1974):

(1) $v=0$

(2) $u=\dfrac{u_*(\xi)}{\kappa}\ln(\zeta/\eta_*(\xi))-c+c\Omega\, kae^{ik\xi}$,

where

$u_*(\xi)=(\tau_w(\xi)/\rho)^{1/2}$, $\tau_w(\xi)=$ local surface shear;
$\eta_*(\xi)=$ local sublayer scale;
Real $(c\Omega\, kae^{ik\xi})=$ orbital velocity of lower fluid at surface to $O(ka)$;
$\zeta=$ distance normal to surface $=\int_0^\eta J^{-1/2}\,d\eta+O(ka)^2$.
The above is the familiar Law of the Wall (Coles and Hirst, 1968).

(3) $e=\tilde{a}u_*^2(\xi)$, $\quad \omega=\tilde{a}u_*(\xi)/\kappa\zeta$.

(4) $-\overline{u'v'}=u_*^2(\xi)$

(5) In addition, all wave-induced perturbations are required to vanish as $\eta\to\infty$.

4. Solution of the Equations

The assumption of exact periodicity motivates a solution for the various dependent variables in terms of the harmonics $e^{\pm ink\xi}$, $n=0,1,2,\ldots$. The effect of the wavy boundary, manifested in the derivatives of the Jacobian $J(\xi,\eta)$, indicates the following expansions:

$$\psi=\psi_0(\eta)+kae^{ik\xi}\psi_1(\eta)+O(ka)^2,$$

where

$$\begin{aligned}
&\frac{d\psi_0}{d\eta}=U\\
&e=e_0(\eta)+kae^{ik\xi}e_1(\eta)+\cdots\\
&\omega=\omega_0(\eta)+kae^{ik\xi}\omega_1(\eta)+\cdots\\
&\Phi=\Phi_0(\eta)+kae^{ik\xi}\Phi_1(\eta)+\cdots\\
&-\overline{u_i'u_j'}={}_0\tau_{ij}(\eta)+kae^{ik\xi}\,{}_1\tau_{ij}(\eta)+\cdots\\
&u_*(\xi)=u_{*_0}+kae^{ik\xi}u_{*_1}+\cdots\\
&\eta_*(\xi)=\eta_{*_0}+kae^{ik\xi}\eta_{*_1}+\cdots
\end{aligned}\tag{6}$$

and so forth, where the real part of the right side is implied, the quantities ψ_1, e_1, etc., being complex in general.

Substitution of the expansions (6) into the equations of motion yields a hierarchy of ordinary differential equations in η (for details, see Knight, 1974).

The solution of the $O(1)$ equations represents turbulent flow over a flat plate, and is given by

$$U = \frac{u_{*_0}}{\kappa} \ln(\eta/\eta_{*_0}) - c \tag{7}$$

$$e_0 = \tilde{\alpha} u_{*_0}^2, \quad \omega_0 = \tilde{\alpha} u_{*_0}/\kappa\eta, \quad \Phi_0(\eta) = \text{constant}.$$

The solution of the $O(ka)$ equations yields the first-order correction to the mean flow due to the presence of the wavy boundary. It is advantageous (Benjamin, 1959) to introduce the quantity $S(\eta)$ where

$$\psi_1(\eta) = S(\eta) + Ue^{-k\eta}/k.$$

The second term on the right is a measure of the curvature effect on the perturbation streamfunction ψ_1. With the use of Equations (7), the $O(ka)$ equations reduce to a pair of linear, fourth-order, coupled homogeneous differential equations for $S(\eta)$ and $e_1(\eta)$, an explicit relationship being found between ω_1 and the derivatives of S and e_1.

An asymptotic solution of the $O(ka)$ equations was obtained in the limit of $k\eta_* \to 0$. The requirement that the mean velocity U (see Equation (7)) remain finite on the scale of λ in the limit of $k\eta_{*_0} \to 0$ implies that the $O(1)$ friction velocity u_{*_0} must vanish as $k\eta_{*_0} \to 0$. Defining a small parameter ε as

$$\varepsilon = [-\ln(k\eta_{*_0} e^{c\kappa/u_{*_0}})]^{-1},$$

the mean velocity U becomes

$$U = \frac{u_{*_0}}{\kappa}\left(\frac{1}{\varepsilon} + \ln k\eta\right)$$

and thus $k\eta_{*_0} \to 0$ implies $\varepsilon \to 0$ with

$$\lim_{\varepsilon \to 0} U = V_0 = \text{constant} = U_0 - c_0.$$

The asymptotic relation between u_{*_0} and ε is thus

$$u_{*_0} \approx \kappa V_0 \varepsilon \quad \text{as} \quad \varepsilon \to 0. \tag{8}$$

We note that the limit $\varepsilon \to 0$ can be interpreted as the limit of vanishing surface shear.

The mathematical importance of the relation (8) between u_{*_0} and ε for solving the $O(ka)$ equations in the limit $k\eta_{*_0} \to 0$ may be illuminated as follows. From the equations and boundary conditions, it is clear that a dependent variable, such as S, may be expressed as

$$S = \lambda u_{*_0} F(k\eta, \varepsilon, c/u_{*_0};\ \alpha, \tilde{\alpha}, \beta, \ldots).$$

For $c \neq 0$, there are two small parameters: ε and u_{*_0}/c. To obtain a limit process expansion of F on the scale of λ, for example, to all orders in a single parameter (say ε), the relationship between u_{*_0}/c and ε must be known. Since (8) is an asymptotic, not an exact, relation, the solution for $c \neq 0$ cannot be uniquely determined to all orders in ε.

The $O(ka)$ equations and their boundary conditions form a singular perturbation problem in the limit $\varepsilon \to 0$. The inner and outer scales for the dependent and independent variables are as follows (Knight, 1974):

Inner region:

$$\eta^* = k\eta/|\varepsilon|$$
$$\bar{S}^i = k\varepsilon S/u_{*_0}$$
$$\bar{e}_1^i = e_1/u_{*_0}^2.$$

Outer region:

$$\bar{\eta} = k\eta$$
$$\bar{S}^0 = k\varepsilon S/u_{*_0}$$
$$\bar{e}_1^0 = e_1/u_{*_0}^2,$$

where the dimensionless variables $\bar{S}^i(\eta^*, \varepsilon)$ and $\bar{e}_1^i(\eta^*, \varepsilon)$ are $O(1)$ in the inner limit, defined as

$$\lim_{\substack{\varepsilon \to 0 \\ \eta^* \text{ fixed}}}$$

and similarily for the outer variables in the outer limit.

The scalar parameters of the flow are also functions of ε, and are expanded as follows:

$$\begin{aligned} u_{*_1} &= u_{*_0}(\delta_0 + o(1)) \\ c &= c_0 + o(1) \\ c\Omega &= c_0\Omega_0 + o(1) \\ u_{*_0} &= \kappa V_0 \varepsilon(1 + o(1)) \end{aligned} \tag{9}$$

and so forth.

The various terms in the inner and outer expansions are determined through substitution into the governing equations (for details, see Knight, 1974). Matching of the inner and outer solutions proceeds without difficulty, yielding:

Inner Expansions:

$$\bar{S}^i = -\frac{1}{\kappa} - \frac{\varepsilon \ln|\varepsilon|}{\kappa} + \varepsilon\left\{-\frac{\ln \eta^*}{\kappa} + \frac{1}{\kappa}\eta^* \frac{\varepsilon}{|\varepsilon|}\right\}$$
$$-\varepsilon^2 \ln|\varepsilon| \left\{\frac{\varepsilon}{\kappa|\varepsilon|}\eta^*\right\} + O(\varepsilon^2).$$

Due to the complexity of the equations, the solution for $\bar{e}_1^i$ as well as terms $O(\varepsilon^2)$ in $\bar{S}^i$ were not obtained.

Outer Expansions:

$$\bar{S}^0 = -\frac{e^{-\tilde{\eta}}}{\kappa} + \varepsilon\left\{\frac{(\gamma+\ln 2)}{\kappa} e^{-\tilde{\eta}} + \frac{e^{+\tilde{\eta}}}{\kappa} E_1(2\tilde{\eta})\right\}$$

$$+\varepsilon^2\left\{\begin{array}{l} e^{-\tilde{\eta}}(C_5 - i\kappa \ln \tilde{\eta} - \kappa(2\tilde{\alpha}-\alpha)\tilde{\eta}) - \dfrac{E_1(2\tilde{\eta})e^{+\tilde{\eta}}}{\kappa}(\ln \tilde{\eta} + \gamma + \ln 2 + i\kappa^2) \\ \qquad - \dfrac{e^{+\tilde{\eta}}}{\kappa}\displaystyle\int_{2\tilde{\eta}}^{\infty} \frac{E_1(x)}{x}\,dx - \frac{e^{-\tilde{\eta}}}{\kappa}\int_{2\tilde{\eta}}^{\infty} \frac{e^{+x}E_1(x)}{x}\,dx \end{array}\right.$$

$$+ O(\varepsilon^3 \ln^3 |\varepsilon|),$$

where

$$E_1(2\tilde{\eta}) = \int_{2\tilde{\eta}}^{\infty} \frac{e^{-x}}{x}\,dx,$$

$\gamma = 0.577$ (Euler's constant) and C_5 is an undetermined constant.

$$\bar{e}_1^0 = +i2\tilde{\alpha}^2 e^{-\tilde{\eta}} + \varepsilon\left\{-\left(2\tilde{\sigma}\tilde{\alpha}^2\kappa^2 + \frac{\alpha\tilde{\alpha}^2}{\tilde{\eta}}\right)e^{-\tilde{\eta}}\right.$$
$$\left. -i2\tilde{\alpha}^2[e^{+\tilde{\eta}}E_1(2\tilde{\eta}) + (\gamma+\ln 2)e^{-\tilde{\eta}} + \ln \eta e^{-\tilde{\eta}}]\right\} + O(\varepsilon^2).$$

In addition, matching of the inner and outer solutions for S at $O(\varepsilon)$ yields

$$\delta_0 + \frac{c_0\Omega_0}{V_0} + \frac{c_0\delta_0}{V_0} = 1. \tag{10}$$

5. Expressions for Stresses

A variable of considerable interest is the $O(ka)$ normal surface stress. It can be shown (Knight, 1974) that the difference in the total mean normal stress across the sublayer is negligibly small, and thus the $O(ka)$ mean normal stress is

$$\text{Real}\,(kae^{ik\xi}\Phi_1(0;\varepsilon)) - \rho gae^{ik\xi}, \tag{11}$$

where the terms represent the aerodynamic and hydrostatic contributions, respectively. $\Phi_1(0;\varepsilon)$ may be evaluated by integration of the $\hat{e}_\eta$-momentum equation which yields:

$$\frac{\Phi_1(0;\varepsilon)}{\rho} = \int_0^\infty Uk^2 S\,d\eta - ik\int_0^\infty {}_1\tau_{12}\,d\eta + i2u_{*_0}^2. \tag{12}$$

The integrals may be determined by dividing the range of integration into regions in which the inner and outer expansions are assumed to be uniformly valid approximations. The result is,

$$\frac{\Phi_1}{\rho}(0;\varepsilon)=\left(\frac{u_{*_0}}{\varepsilon}\right)^2\left\{-\frac{1}{\kappa^2}+\varepsilon\left[\frac{2(\gamma+\ln 2)}{\kappa^2}+i2\right]+O(\varepsilon^2)\right\}, \tag{13}$$

where the imaginary term at $O(\varepsilon)$ originates in the integral of the complex amplitude of the perturbation Reynolds shear stress $({}_1\tau_{12})$. Note that the complex nature of $\Phi_1(0;\varepsilon)$ implies a phase-shift of the maximum of the $O(ka)$ normal stress away from the trough of the wave. Also, from (8),

$$\lim_{\varepsilon\to 0}\frac{\Phi_1(0;\varepsilon)}{\rho}=-V_0^2$$

which is identical to the result for a potential flow with velocity V_0 over the wavy wall. Higher-order terms in (13) were not obtained due to the complexity of the equations at higher orders.

On the outer scale, the complex amplitude of the $O(ka)$ Reynolds shear stress is,

$${}_1\tau_{12}=u_{*_0}^2\left\{\frac{-2\tilde{\eta}e^{-\tilde{\eta}}}{\varepsilon}+i(2\tilde{\alpha}-\alpha)e^{-\tilde{\eta}}+2\tilde{\eta}[e^{+\tilde{\eta}}E_1(2\tilde{\eta})+(\gamma+\ln 2)e^{-\tilde{\eta}}] +O(\varepsilon)\right\}.$$

6. Calculation of Lower Flow

In order to complete the determination of the scalar constants appearing in (9), the lower fluid must be treated. The following assumptions are made (the subscript 'w' refers to the lower fluid; similar quantities corresponding to the upper fluid are unsubscripted):

(1) The lower fluid is two-dimensional, viscous and laminar.

(2) $ka \ll 1$.

The analysis provides the wave amplitude with a temporal behavior of the form

$$a=Ae^{nt},$$

where A and n are both real. Clearly, the solution will be valid for a time interval of order n^{-1}. As with the other scalars in (9), n has an expansion

$$n+2\nu_w k^2=\varepsilon n_2+o(\varepsilon).$$

The $2\nu_w k^2$ term is the viscous decay rate for a free wave. In the limit of $\varepsilon\to 0$, the phase shift of the $O(ka)$ surface normal stress vanishes and thus, neglecting Kelvin–Helmholtz instability, there can be no wave growth. Thus, the leading term on the right is $O(\varepsilon)$.

(3) $kh \gg 1$ and $\nu_w k/c_w \ll 1$ where h = depth of lower fluid and $c_w=(g/k)^{1/2}$, i.e., only deep-water gravity waves are considered (and hence surface tension is neglected). For typical laboratory studies of wind-generated waves (Shemdin and

Hsu, 1967), $c_w/\nu_w k > 10^4$, and for deep-water gravity waves, $c_w/\nu_w k > 10^5$ for $\lambda > 1$ m. For simplification of the analysis, h is taken as infinite.

(4) At $t = 0$, the $O(1)$ horizontal lower-fluid velocity is uniform and equal to $-c$.

(5) The upper fluid imposes only a slight perturbation on the water wave. As will be shown, this implies

$$\frac{\rho}{\rho_w}\left\{\left(\frac{V_0}{c_w}\right)^2 + 1\right\} \ll 1$$

and rules out Kelvin-Helmholtz instability *a priori.* (For comments on the applicability of Kelvin-Helmholtz instability to wind wave generation, see Miles (1959b).) For wind-generated water waves, $\rho/\rho_w = 1.22 \times 10^{-3}$.

The equations of motion in the upper flow neglected time variations. It can be shown (Knight, 1974) that the correction is negligible; in particular, the expression for $\Phi_1(0; \varepsilon)$ is unchanged to the order given.

The analysis proceeds in a straightforward manner (Lamb, 1945, Section 349) and the matching of the normal surface stress and the $O(ka)$ shear stress along the surface yields

$$n = -2\nu_w k^2 + \varepsilon \frac{\rho}{\rho_w}\frac{V_0^2}{c_0} k\kappa^2 + o(\varepsilon), \tag{14}$$

where the second term is proportional to the imaginary term in (13) and

$$c = c_0(1 + O(\varepsilon \ln|\varepsilon|)),$$

where

$$c_0 = c_w\left\{1 - \frac{\rho}{\rho_w}\left(\frac{V_0^2}{c_w^2} + 1\right)\right\}^{1/2}, \quad c_w = \pm(g/k)^{1/2}$$

$$c\Omega = c_0 + o(1)$$

and thus from (10)

$$\delta_0 = \left(1 - \frac{c_0}{V_0}\right)\Big/\left(1 + \frac{c_0}{V_0}\right).$$

The term

$$\varepsilon \frac{\rho}{\rho_w}\frac{V_0^2}{c_0} k\kappa^2$$

in (14) implies a generation effect for $V_0 c_w > 0$ (i.e., waves moving in the direction of the wind with $U_0 > c_0$) and a damping effect for $V_0 c_w < 0$ (i.e., waves moving faster than U_0, or waves moving in opposite direction to the wind).

7. Comparison with Experiment

The theoretical results have been compared with the experimental data of Shemdin and Hsu (1967), Kendall (1970), and Sigal (1971). The main characteris-

TABLE I
Experimental characteristics

	Experimental apparatus	ka	δ/λ	$U_\infty \lambda/\nu$
Kendall (1970)	Deformable neoprene rubber sheet	0.196	1.0	$(1.9 \rightarrow 6.4) \times 10^4$
Shemdin and Hsu (1967)	Water waves	0.100	$\ll 1$	$(1.61 \rightarrow 4.03)10^{+6}$ [a]
Sigal (1971)[b]	Stationary, rigid wall in wind tunnel ($c \equiv 0$)	0.1755	0.28	$3.6 \times 10^{+5}$

[a] Reynolds number based on 'V_{max}' in Table I of article.
[b] Model 'WWI'.

tics of these studies are listed in Table I; note especially the different apparati employed.

7.1. Total velocity in ξ-direction

In Figure 2 the total velocity in the ξ-direction at various constant values of η is compared with the data of Sigal (1971), which has been transformed from the (x, y) to the (ξ, η) coordinate system. The values $k\eta_{*_0} = 6.91 \times 10^{-5}$, $c_f = 2(u_{*_0}/U_\infty)^2 = 2.98 \times 10^{-3}$ are used, corresponding to the mean velocity profile obtained at the same Reynolds number and position in the wind tunnel with a flat lower surface.

As can be seen, the theory predicts both the phase and amplitude of the $u(\xi, \eta)$-perturbations with accuracy. These fluctuations are governed to lowest order by the convective terms in the mean vorticity equation, which is expected since to lowest order, the $O(ka)$ normal stress is that of a potential flow.

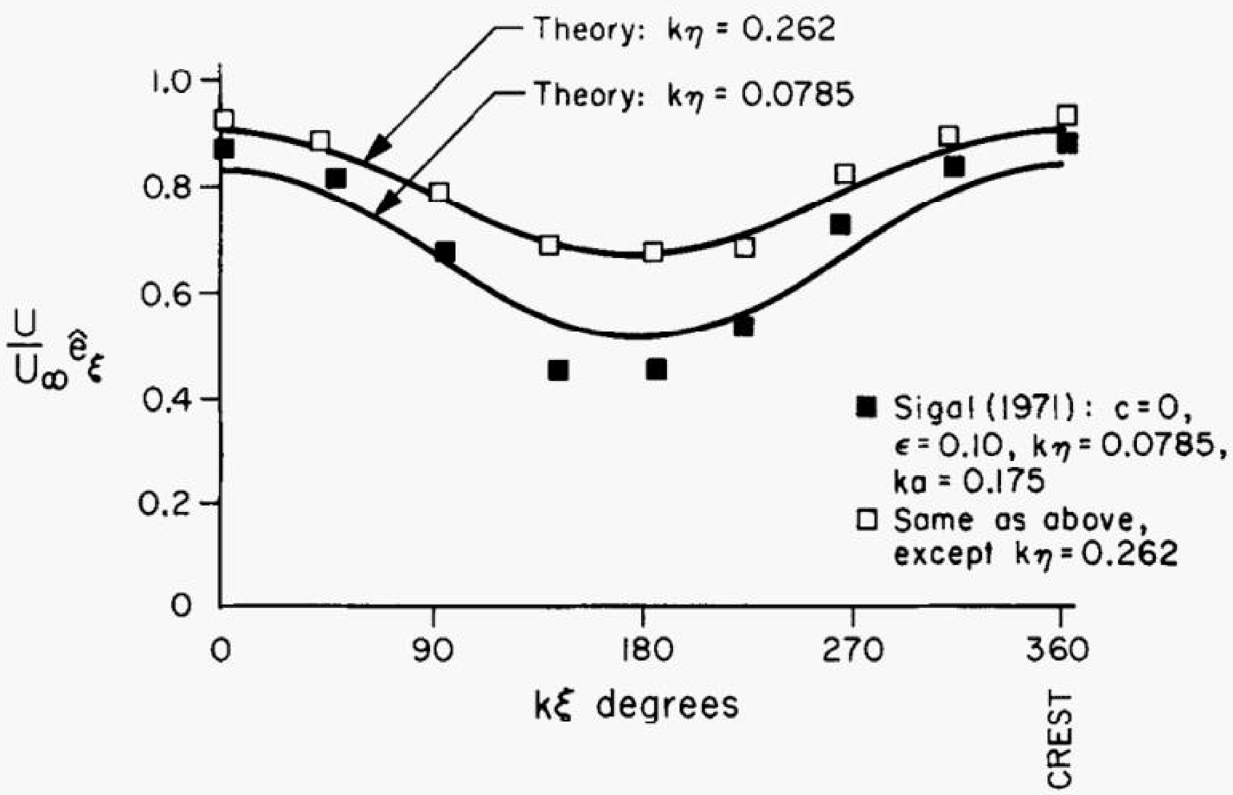

Fig. 2. Total velocity in ξ-direction at constant η.

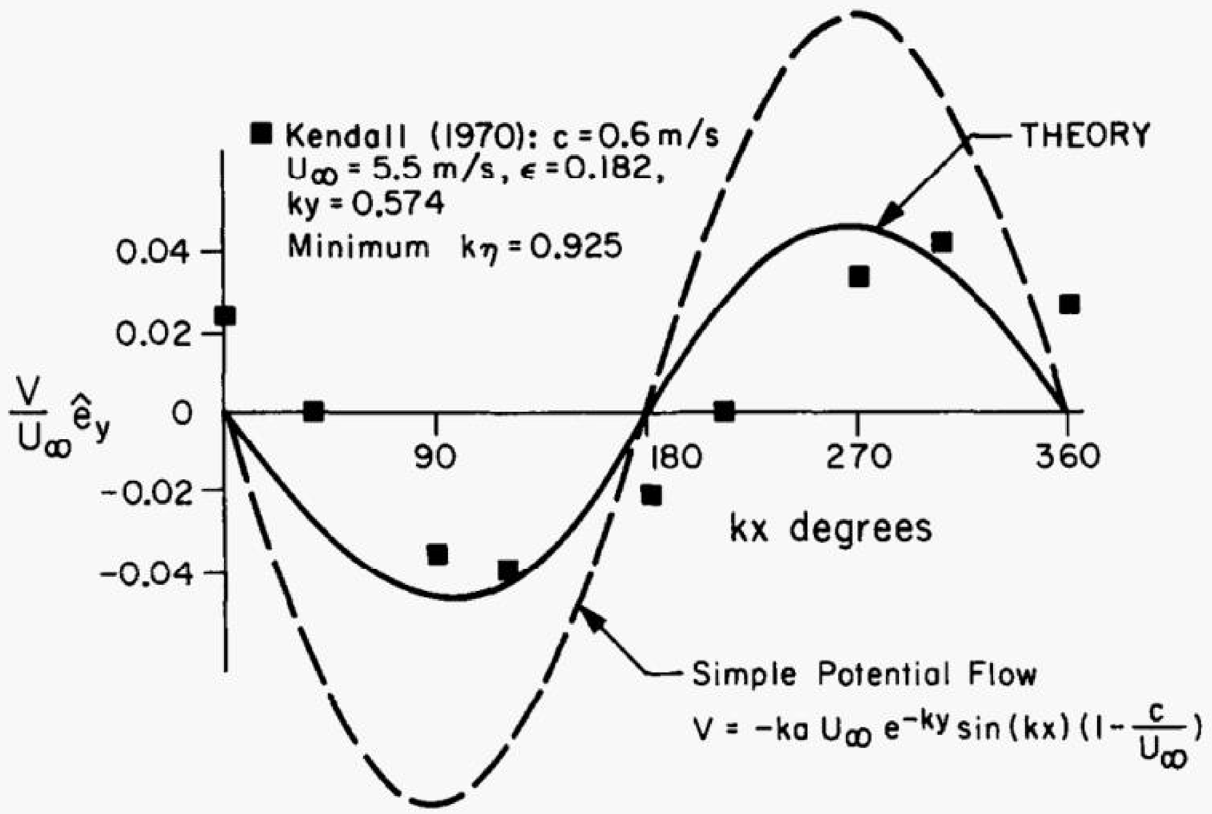

Fig. 3. Velocity in y-direction at constant η.

7.2. Total velocity in y-direction

In order to facilitate comparison with the data of Sigal (1971) and Kendall (1970), the velocity in the y-direction $\tilde{v}(x, y)$ is considered. It is straightforward to show that

$$\tilde{v}(x, y) = -ikae^{ikx}kS + O(ka)^2. \tag{15}$$

Comparison of $\tilde{v}(x, y)$ at constant values of y was made with the data of Sigal. The amplitude was predicted within 10% at all elevations, although the phase was in error by approximately 4°, the measured profile being shifted downstream with respect to the theoretical predictions.

In Figure 3, the velocity $\tilde{v}(x, y)$ at $ky = 0.574$ is compared with the data of Kendall (1970); the values of $k\eta_{*_0}$ and u_{*_0}/U_∞ are from Table 3 of Kendall. It is apparent that the amplitude is predicted within 20%, although the phase is in error by about 35°, the measured profile again being shifted downstream with respect to the theoretical predictions. Note the corresponding potential flow calculations.

Using the solution for $\bar{S}$ on the outer scale, we have from (15)

$$\tilde{v}(x, y) = -ikae^{ikx}\frac{u_{*_0}}{\varepsilon}\{\bar{S}_0^0 + \varepsilon\bar{S}_2^0 + O(\varepsilon^2)\},$$

where

$$\bar{S}_0^0 = -\frac{e^{-\bar{\eta}}}{\kappa}, \quad \bar{S}_2^0 = \frac{(\gamma + \ln 2)}{\kappa}e^{-\bar{\eta}} + \frac{e^{+\bar{\eta}}E_1(2\bar{\eta})}{\kappa}.$$

It is evident that the measured phase shift in $\tilde{v}(x, y)$ implies an imaginary term in $\bar{S}_2^0$, as no shift of the maximum of $\tilde{v}(x, y)$ in x relative to $kx = -\pi/4$ exists for $k\eta_{*_0} = 0$ (potential flow). It is apparent from the mean vorticity equation, written on the outer scale, that $\bar{S}_2^0$ is sensitive to the predictions of the Reynolds stresses. Comparison with the data of Sigal (1971) and Kendall (1970) indicates that the model predicts the magnitude of the shear stress with reasonable accuracy, although the phase prediction becomes less accurate at larger distances from the surface (Knight, 1974).

7.3. Phase shift of the surface normal stress

In the limit $\varepsilon \rightarrow 0$, the $O(ka)$ normal stress at the surface is exactly in anti-phase with the wave. We thus define the phase shift ϕ_{p_1} relative to $kx = -\pi$, a positive phase shift ϕ_{p_1} implying a shift downstream with respect to the wave trough.

From (11) and (13) we obtain,†

$$\phi_{p_1} = \tan^{-1}\left\{\frac{+2\varepsilon\kappa^2 + O(\varepsilon^2) + O(ka)^2}{1 - 2(\gamma + \ln 2)\varepsilon + O(\varepsilon^2) + O(ka)^2}\right\}.$$

Note that the net drag on the wave surface in the x-direction due to the normal surface stress (including higher harmonics) is

$$c_{D_p} = \frac{ka}{2} c_{p_1} \sin \phi_{p_1},$$

where

$$c_{p_1} = ka(\sqrt{\Phi_1(0)\Phi_1^*(0)} + O(ka)^2)/\tfrac{1}{2}\rho U^2.$$

Also, note that ϕ_{p1} depends non-linearly on ka through terms proportional to $(ka)^3 e^{ikx}$ in the expansion for Φ.

The data of Sigal indicate a value of ϕ_{p_1} between +3° and +4.5°, based on his calculated value of c_{Dp} and the maximum and minimum measured values of c_{p_1}. The model predicts +2.6°.

The data of Shemdin and Hsu for water waves are shown in Figure 4. The theoretical predictions of Miles (1959a) are also shown. Although the model predicts the correct qualitative behavior with c, it clearly underestimates ϕ_{p_1} by 10° to 20° for $1.5 < c\kappa/u_{*_0} < 4.0$. Part of this discrepancy can perhaps be attributed to higher order terms in ε; indeed, ε increases from 0.216 to 0.347 as $c\kappa/u_*$ grows from 1.5 to 4.0.

The data of Kendall are plotted in Figure 5, along with the theoretical predictions of Miles (1959a), and the calculations of the model, using the values of $k\eta_{*_0}$ and u_{*_0}/U_∞ from Table 3 of Kendall. The theory manifests the correct

† The contribution of the perturbation hydrostatic pressure to ϕ_{p_1} and c_{p_1} has been omitted. In the experiments of Kendall (1970) and Sigal (1971), the contribution is entirely negligible. In the experiments of Shemdin and Hsu (1967), the contribution was removed by their measuring apparatus, and in that case ϕ_{p_1} represents the phase shift of the *aerodynamic component* of surface normal stress.

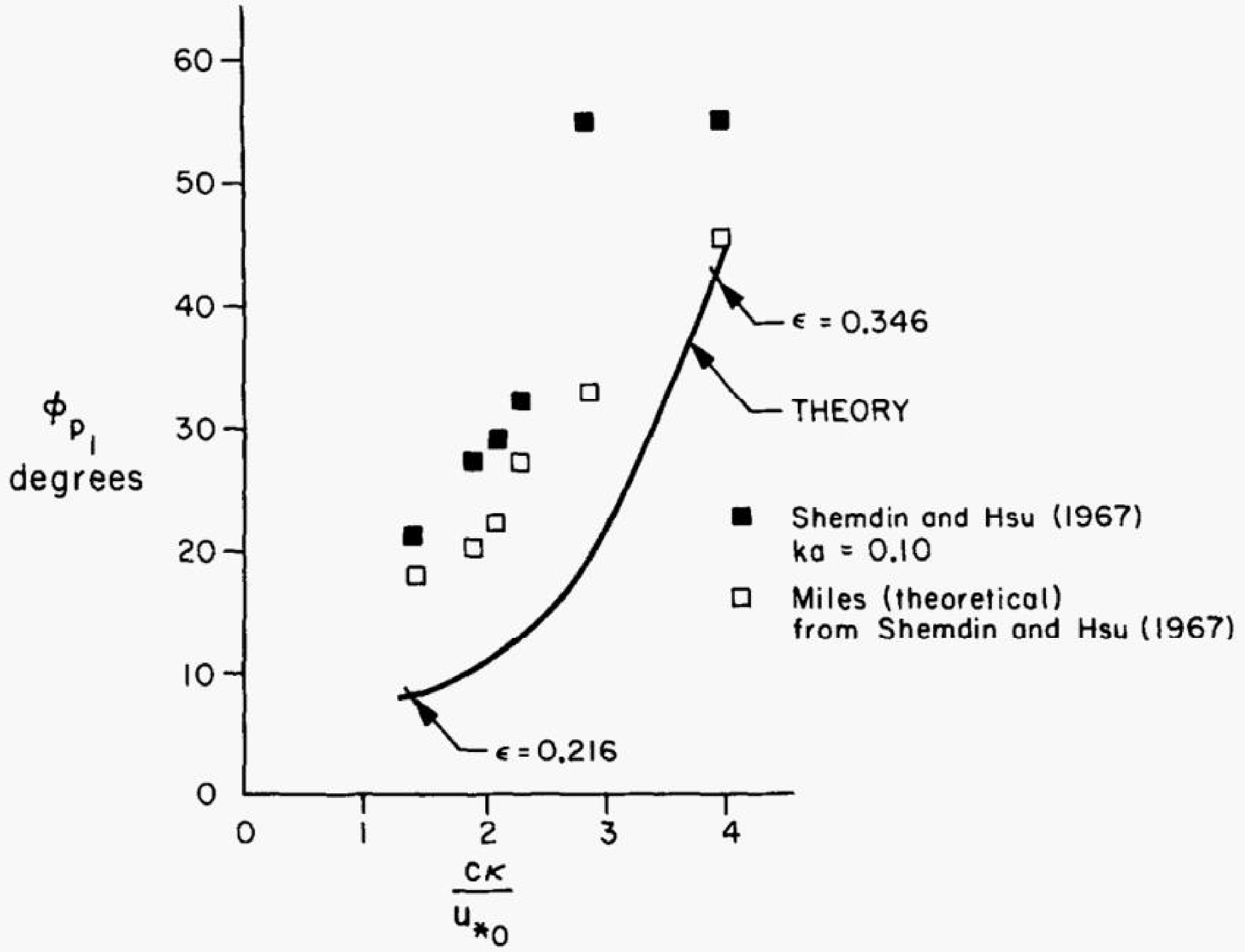

Fig. 4. Phase shift of surface perturbation pressure (comparison with data of Shemdin and Hsu (1967)).

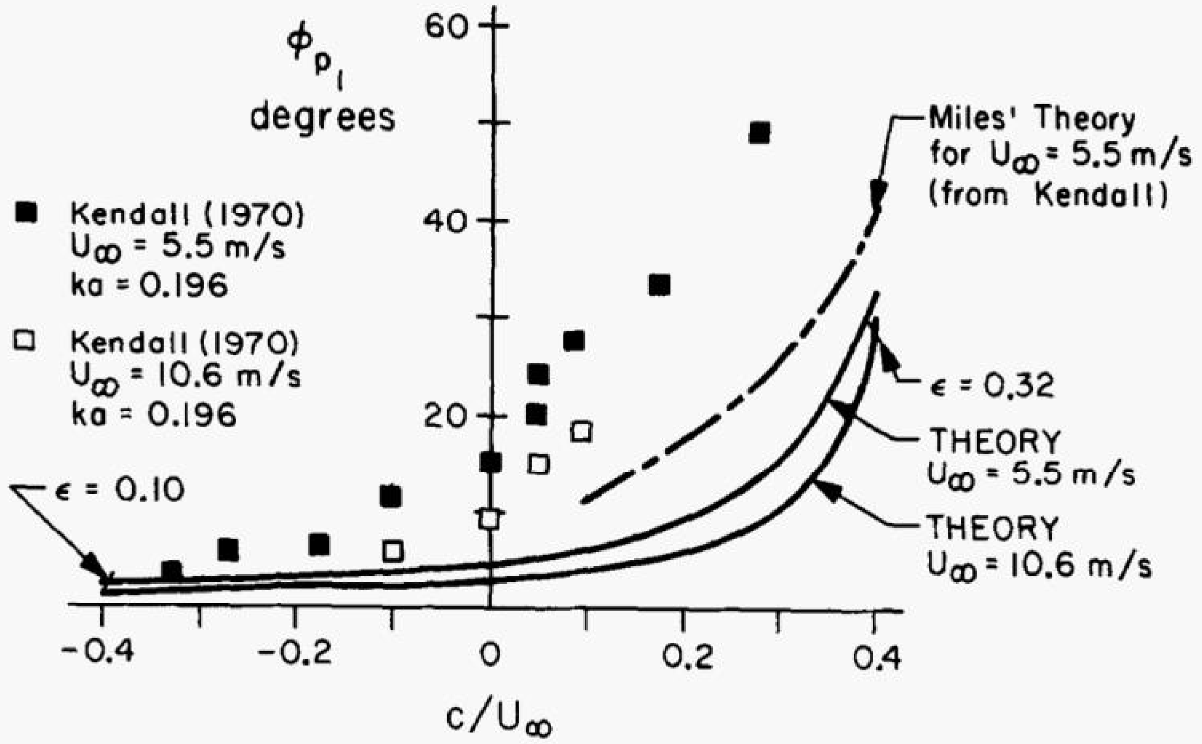

Fig. 5. Phase shift of surface perturbation pressure (comparison with data of Kendall (1970)).

qualitative behavior with $c\kappa/u_{*_0}$ over a wide range of both positive and negative wave speeds. As Miles' theory relies on the existence of a critical layer, it is inapplicable for $c \leqslant 0$. The theory also indicates qualitatively the observed downshift of the curve with increasing Reynolds number. However, it clearly underestimates the phase shift. As in the data of Shemdin and Hsu, ε increases with increasing wave speed, and, at large positive wave velocities, higher order terms in ε will be significant. Furthermore, non-linear terms may be important in Kendall's data ($ka = 0.196$); Townsend (1972) has argued that the linear approximation fails for $ka \geqslant 0.10$. Nevertheless, the data near $c = 0$ appear to indicate that the model is lacking a positive term of $O(\varepsilon)$ in tan ϕ_{p_1}. As indicated earlier, the discrepancy in the predicted and measured values of the phase of $\tilde{v}(x, y)$ could be accounted for by a term in $S_2^0(\eta)$ of form

$$+i \text{ (positive function of } \eta).$$

Such a term would also increase tan ϕ_{p1} (see (12)). Noting the remarks made in Section 7.2, it is concluded that the underestimation of ϕ_{p_1} is due, in part, to misprediction of the $O(ka)$ Reynolds stresses.

7.4. Magnitude of surface perturbation pressure

The variation of c_{p_1} with c for the data of Kendall is shown in Figure 6. The pressure coefficient is predicted within 15% for $-0.4 \leqslant c/U_\infty \leqslant 0.4$. For comparison, c_{p_1} for a potential flow of velocity $U_\infty - c$ is also shown.

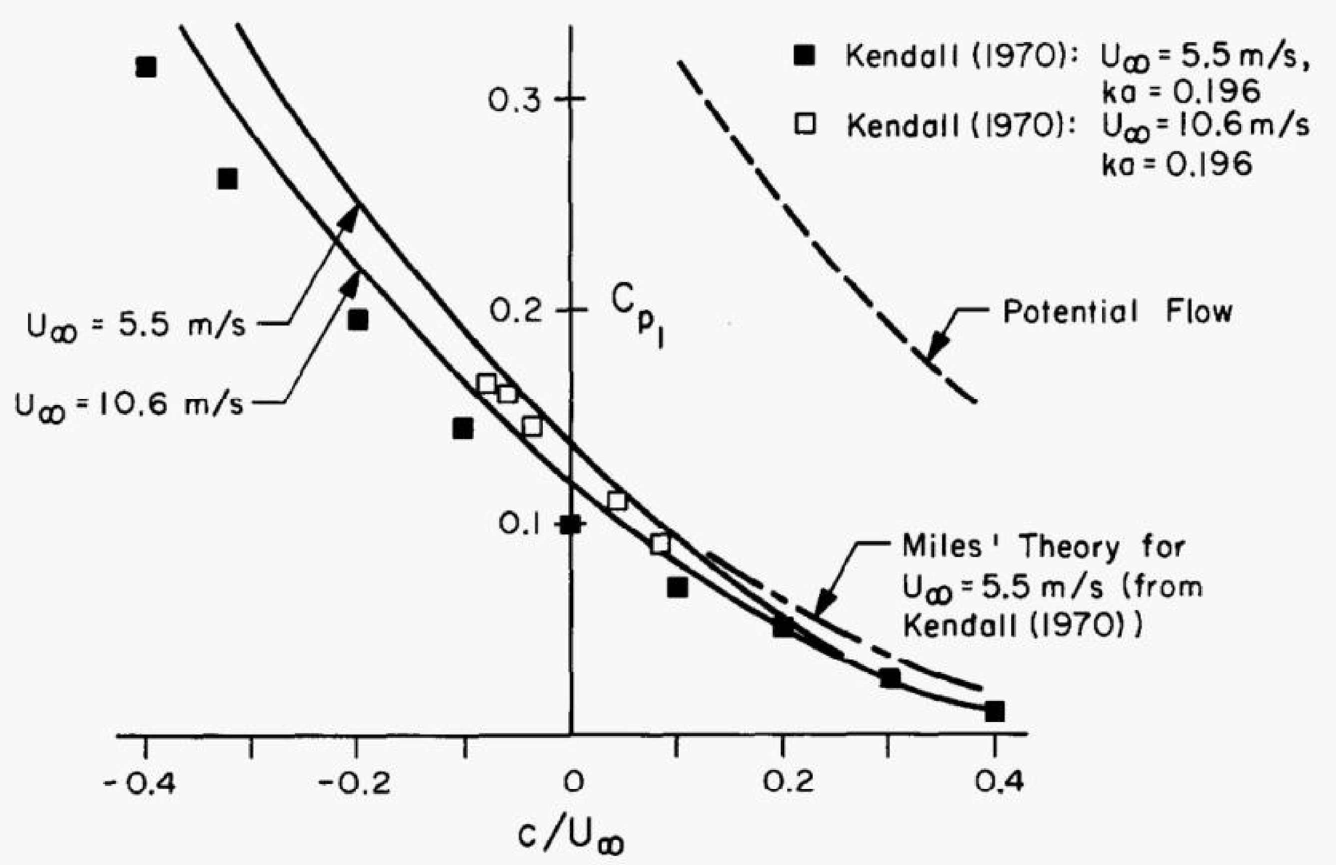

Fig. 6. Magnitude of surface perturbation pressure.

8. Conclusions

The following conclusions can be drawn:

(1) The model accurately predicts the $O(ka)$ horizontal velocity.

(2) The $O(ka)$ Reynolds stresses are not adequately represented. A more suitable relaxation hypothesis (see Saffman, 1974 and 1976) is needed.

(3) The model predicts the correct qualitative behavior for the phase shift of the surface normal stress for a wide range of positive and negative wave speeds; quantitative predictions are deficient due to inadequate representation of the turbulent Reynolds stresses and higher-order effects in ε and ka.

(4) The model accurately predicts the coefficient of the fundamental harmonic of the surface normal stress over a broad range of positive and negative wave speeds.

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