A notation note on seasonal auto-regressive integrated moving-average (sarima) models

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From time to time I find myself forgetting what things look like and have to make a personal note somewhere so that I can look it up. Here is my note for the general notation used for seasonal auto-regressive integrated moving-average models, and I thought they may be able to benefit somebody at some time.

The general form of a sarima model is given by

$$\frac{(1-L^S)^D(1-L)^d\phi(L)\Phi(L)}{\theta(L)\Theta(L)}(y_t-\mu_t)=\epsilon_t, \tag{1}$$

where L is the backward shift operator, which works as follows,

$$L^p y_t = y_{t-p},$$

S is the seasonal period of the time series y_t, D is the order of seasonal differences, d is the order of (first) differencing, $\phi(\cdot)$ is the auto-regressive polynomial, $\Phi(\cdot)$ is the seasonal auto-regressive polynomial, $\theta(\cdot)$ is the moving average polynomial, $\Theta(\cdot)$ is the seaosnal moving average polynomial and ϵ_t is a the white noise series. Above we have not assumed the time series y_t is first-order stationary, that is the mean is time dependent, hence the subscript t in μ_t. For a time series $\{\epsilon_t\}$ to be white noise, the following conditions need to hold,

- $\mathbb{E}(\epsilon_t)=0$,
- $\mathbb{V}(\epsilon_t)=\sigma_\epsilon^s<\infty$,
- $\mathrm{Cov}(\epsilon_t,\epsilon_s)=\begin{cases}0 & \text{if } s\neq t\\ 1 & \text{if } s=t.\end{cases}$

The standard notation to identify the order of a seasonal arima model is

$$\mathrm{ARIMA}(p,d,q)(P,D,Q)_S$$

where p,d,q are the orders of the auto-regressive polynomial, first difference and moving-average polynomial respectively, and, P,D,Q are respectively, the seasonal auto-regressive polynomial, seasonal difference and seasonal moving-average polynomial orders. The (seasonal) auto-regressive and (seasonal) moving-average polynomials give a concise way of writing these parts of the model. They are, as their names suggests, just polynomials, and we treat them like so. Assume we have an auto-regressive term in the model of order p and seasonal moving-average component of order Q then the polynomials are written in the following way,

$$\phi(L) = 1 - \phi_1 L - \ldots - \phi_p L^p, \quad \text{and} \quad \Theta(L) = 1 - \Theta_1 L^s - \ldots \Theta_Q L^{SQ}.$$

We use the convention here that all polynomial terms are displayed with a minus, in front of the coefficient. Some authors and software do not use this convention. **Notice** the powers of the backward shift operator appear as multiples of the season S. Some authors stress this by writing the seasonal polynomials as $\Theta(L^S)$ and $\Phi(L^S)$.

So what does a sarima model look like?

Let's look at an ARIMA$(1,0,0)(1,1,1)_7$ model. The beauty of the backward shift operator is that we can just substitute values for p, d, q, P, D, Q and S into (1) and use standard algebraic techniques on the polynomial to determine the model as follows,

$$\begin{aligned}
\epsilon_t &= \frac{(1-L^7)(1-\phi_1 L)(1-\Phi_1 L^7)}{(1-\Theta_1 L^7)}(y_t - \mu_t), \\
&= \frac{(1-\phi_1 L - L^7 + \phi_1 L^8)(1-\Phi_1 L^7)}{(1-\Theta_1 L^7)}(y_t - \mu_t), \\
&= \frac{(1-\phi_1 L - L^7 + \phi_1 L^8 - \Phi_1 L^7 + \Phi_1 L^{14} + \Phi_1\phi_1 L^8 - \phi_1\Phi_1 L^{15})}{(1-\Theta_1 L^7)}(y_t - \mu_t).
\end{aligned} \tag{2}$$

We can multiply both sides of (2) by $(1-\Theta_1 L^7)$ so that,

$$\begin{aligned}
(1-\Theta_1 L^7)\epsilon_t = (1-\phi_1 L - L^7 + \phi_1 L^8 - \Phi_1 L^7 + \Phi_1 L^{14} + \Phi_1\phi_1 L^8 - \phi_1\Phi_1 L^{15})(y_t - \mu_t). \\
=(y_t - \phi_1 y_{t-1} - y_{t-7} + \phi_1 y_{t-8} - \Phi_1 y_{t-7} + \Phi_1 y_{t-14} + \Phi_1\phi_1 y_{t-8} - \phi_1\Phi_1 y_{t-15}) - \\
(\mu_t - \phi_1 \mu_{t-1} - \mu_{t-7} + \phi_1 \mu_{t-8} - \Phi_1 \mu_{t-7} + \Phi_1 \mu_{t-14} + \Phi_1\phi_1 \mu_{t-8} - \phi_1\Phi_1 \mu_{t-15})
\end{aligned}$$

and then,

$$\begin{aligned}
&(y_t - \phi_1 y_{t-1} - y_{t-7} + \phi_1 y_{t-8} - \Phi_1 y_{t-7} + \Phi_1 y_{t-14} + \Phi_1\phi_1 y_{t-8} - \phi_1\Phi_1 y_{t-15}) - \\
&(\mu_t - \phi_1 \mu_{t-1} - \mu_{t-7} + \phi_1 \mu_{t-8} - \Phi_1 \mu_{t-7} + \Phi_1 \mu_{t-14} + \Phi_1\phi_1 \mu_{t-8} - \phi_1\Phi_1 \mu_{t-15}) = \epsilon_t - \Theta_1 \epsilon_{t-7}.
\end{aligned}$$