

Combinations

Challenge Problems: What value of k will make $P(11, k)$ the biggest/smallest?

Biggest: $P(11, 11) = 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdots 2 \cdot 1$
 $P(11, 10) =$

Smallest: $P(11, 2) = 11 \cdot 10$

$$P(11, 1) = 11$$

$$P(11, 0) = 1$$

$$P(n, k) = n(n-1)(n-2)\cdots$$
$$P(11, 23) = 11 \cdot 10 \cdots 0 \cdot (-1)(-2)\cdots = 0$$

Combinations: Subset of given set without regard to arrangement

Same as permutations but order doesn't matter

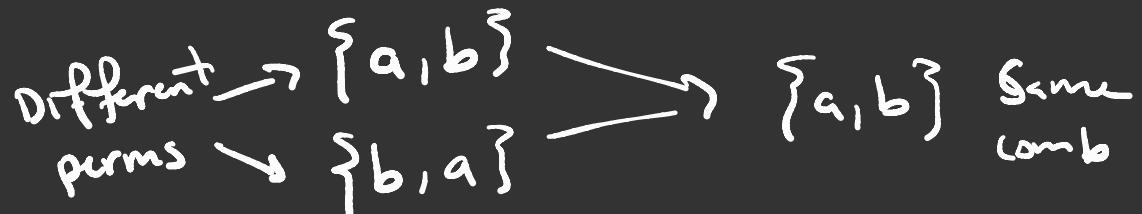
Notation: $C(n, k)$ = "combinations of n things taken k at a time"
= "n choose k "

= # ways to choose k elements
from an n -elements set

Example: $C(3, 2)$
"3"

$\{a, b, c\}$ $\xrightarrow{\text{2-element subsets}}$ $\{a, b\}$
 $\{a, c\}$
 $\{b, c\}$

Question: Which is bigger : $P(n,k)$ or $C(n,k)$?



Formula for $C(n,k)$:

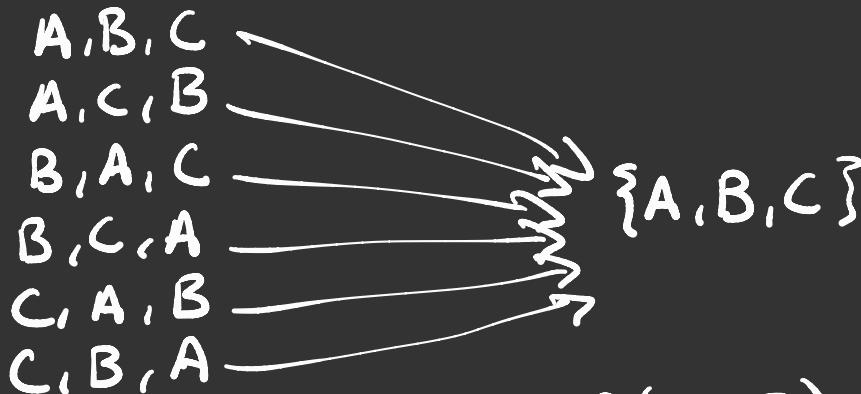
Example: $C(10,3)$

i) Count all permutation of 10 things taken 3 at a time

$$P(10,3) = \frac{10!}{7!}$$

ii) How much did we overcount by?

$$\{A, B, C, D, E, \dots, J\}$$



Overcounted by
factor of $6 = 3!$

$$\hookrightarrow \underline{C(10, 3)} = \frac{P(10, 3)}{3!} = \frac{\cancel{10 \cdot 9 \cdot 8}}{\cancel{3 \cdot 2 \cdot 1}} = 120$$

$$\hookrightarrow \text{In general, } C(n, k) = \frac{P(n, k)}{k!} = \frac{\frac{n!}{(n-k)!}}{k!} = \boxed{\frac{n!}{k!(n-k)!}}$$

Special Cases: $C(n, 1) = \frac{n!}{1!(n-1)!} = \frac{n!}{(n-1)!} = \frac{\cancel{n(n-1)(n-2)} - \cancel{(2)(1)}}{\cancel{(n-1)(n-2)} - \cancel{(2)(1)}}$

$$= \frac{n}{\cancel{1}}$$

$$C(n, 0) = \frac{n!}{0! n!} = 1$$

$$C(n, n) = 1 = \frac{n!}{n! 0!}$$

Example: Making a Tik Tok dance. All dances use same 8 dance moves

a) How many ways to choose 4 dance moves to put in dance? (Not ordering them yet)

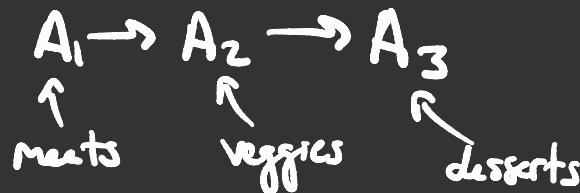
$$\frac{8!}{4!(8-4)!} = \underbrace{C(8,4)}_{\sum} = 60$$

b) How many ways to choose and order 4 moves into a dance?

$$P(8,4) = \overbrace{\frac{8!}{4!}}^{\sim} = 1680$$

Example: Lunch time

- 4 meat
 - 6 veggies
 - 5 desserts
- Choices



How many can make a meal of 2 meats, 2 veggies,
4 desserts?

$$\begin{aligned}\#\text{ such meals} &\leq (\#\text{ways to choose meats}) \times (\#\text{ways to choose veggies}) \\ &\quad \times (\#\text{ways to choose desserts}) \\ &= C(4, 2) \times C(6, 2) \times C(5, 4) \\ &= 450\end{aligned}$$

Binomial Theorem:

$$(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = \underline{x^4} + 4x^3\underline{y} + 6x^2\underline{y^2} + 4x\underline{y^3} + \underline{y^4}$$

The coefficient of x^3y in expansion is $C(4,1)$

Pattern?

$$(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$$

x^3y

How many x^3y 's show up in the expansion?

$C(4,1)$ Pick one to contribute y
4 terms to choose from

$$(x+y)^4 = \frac{1}{\uparrow} x^4 + \frac{4}{\uparrow} x^3y + \frac{6}{\uparrow} x^2y^2 + \frac{4}{\uparrow} xy^3 + \frac{1}{\uparrow} y^4$$

$C(4,0) \quad C(4,1) \quad C(4,2) \quad C(4,3) \quad C(4,4)$

In general ,

$$(x+y)^n = C(n,0)x^n + C(n,1)x^{n-1}y + C(n,2)x^{n-2}y^2 + \dots + C(n,n-1)xy^{n-1} + C(n,n)y^n$$