Fundamentals of radio interferometry II

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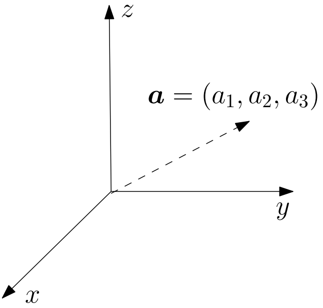
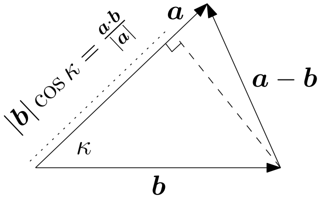
Overview

Vectors



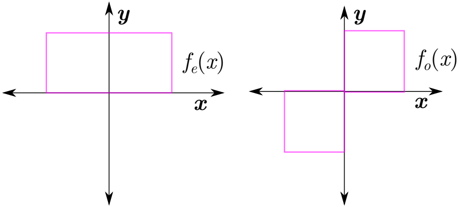


1. Vector length: $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.
2. Dot product: $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\mathbf{a}||\mathbf{b}| \cos \kappa$.
3. Scalar projection of $\mathbf{b}$ onto $\mathbf{a}$: $\mathrm{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = |\mathbf{b}| \cos \kappa$.
4. Commutativity: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

Delayed Product Identities

$$\begin{aligned}
\cos(x-y)\cos(x) &= \frac{1}{2}\cos(y)+\frac{1}{2}\cos(2x-y) \\
\sin(x-y)\sin(x) &= \frac{1}{2}\cos(y)-\frac{1}{2}\cos(2x-y) \\
\sin(x-y)\cos(x) &= -\frac{1}{2}\sin(y)+\frac{1}{2}\sin(2x-y) \\
\cos(x-y)\sin(x) &= \frac{1}{2}\sin(y)+\frac{1}{2}\sin(2x-y)
\end{aligned}$$

Even and Odd Functions



Any function $f(x)$ can be uniquely written as

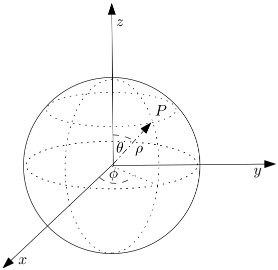
$$f(x) = f_e(x) + f_o(x)$$

where $f_e(x) = \frac{1}{2}[f(x) + f(-x)]$ and $f_o(x) = \frac{1}{2}[f(x) - f(-x)]$.

Even and Odd Functions: Properties

1. The product of an even and an odd function is an odd function
2. The product of two even functions is an even function
3. The product of two odd functions is an even function
4. $\int_{-a}^{a} f_o(x)\ dx = 0$

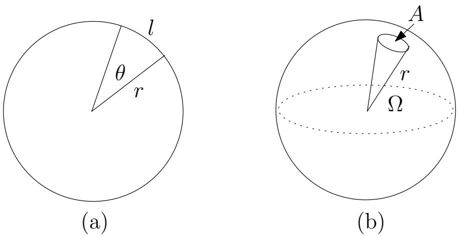
Spherical Coordinates



Relation between spherical and cartesian coordinates

$$
\begin{aligned}
x &= \rho \sin\theta \cos\phi \\
y &= \rho \sin\theta \sin\phi \\
z &= \rho \cos\phi
\end{aligned}
$$

Solid Angle



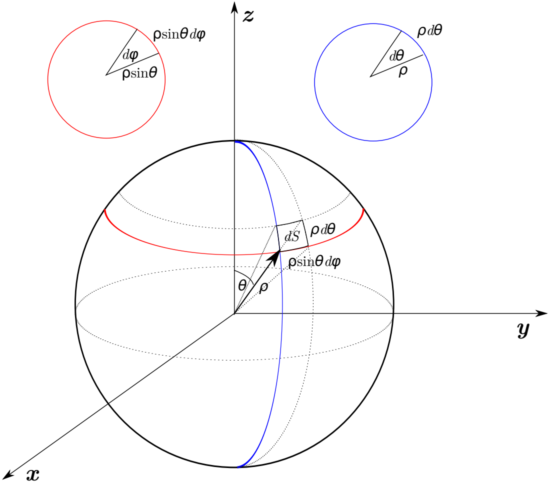
(a) $\theta = \frac{l}{r}$ radians (rad) (b) $\Omega = \frac{A}{r^2}$ steradians (sr)

$$\Omega_{\mathrm{sphere}} = \frac{4\pi r^2}{r^2} = 4\pi \ \mathrm{sr}$$

Conversion

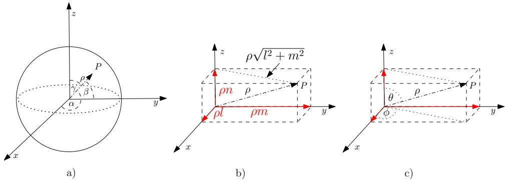
1 steradian (sr) = 1 rad^2 = 3282.8 deg^2 = 4.25 x 10^{10} arcsec^2, where 4π sr = sphere.

Infinitesimal Solid Angle $d\Omega$: Spherical Coordinates



$$d\Omega = \frac{dS}{\rho^2} = \frac{\rho^2 \sin\theta d\theta d\phi}{\rho^2} = \sin\theta d\theta d\phi$$

Infinitesimal Solid Angle $d\Omega$: Direction Cosine Coordinates



$$\phi = \tan^{-1}\frac{l}{m} = f(l,m) \mid \theta = \sin^{-1}\sqrt{l^2+m^2} = g(l,m)$$

$$\begin{aligned} d\Omega &= \sin\theta d\theta d\phi = \sin(g(l,m))|\mathbf{J}|dldm \\ &= \frac{dldm}{\sqrt{1-l^2-m^2}} = \frac{dldm}{n} \end{aligned}$$

$$|\mathbf{J}| = \begin{vmatrix} \frac{d\theta}{dl} & \frac{d\theta}{dm} \\ \frac{d\phi}{dl} & \frac{d\phi}{dm} \end{vmatrix}$$

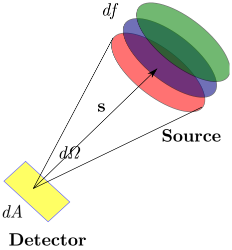
Why interferometry?

1. Resolution with one dish: $\frac{\lambda}{D}$, where D is the diameter of the dish.
2. Resolution with an interferometer: $\frac{\lambda}{|\mathbf{b}|}$, where $|\mathbf{b}|$ is the length of the longest baseline.

Number of baselines

$$N_b = \frac{N_a^2 - N_a}{2}$$

What are we trying to measure with an interferometer: Spectral radiance



1. Brightness or **spectral radiance**: is the total power received per unit area, per unit solid angle in the direction **s**, per unit frequency at frequency f.
2. The unit of spectral radiance is : $\mathrm{Wm^{-2}sr^{-1}Hz^{-1}}$.
3. We denote the brightness in the direction **s** at frequency f with $I(\mathbf{s})$.
4. We denote the effective collecting area in the direction **s** at frequency f with $A(\mathbf{s})$ [measured in m^2].

Flux density

1. The flux density F at frequency f is defined as

$$F = \int_S I(\mathbf{s})d\Omega$$

The above integral is taken over the entire surface S of the celestial sphere, which subtends 4π steradians.

2. The above integral is a **surface integral**.
3. The unit of flux is: $\mathrm{Wm}^{-2}\mathrm{Hz}^{-1}$.
4. Jansky: $1\ \mathrm{Jy} = 10^{-26}\ \mathrm{Wm}^{-2}\mathrm{Hz}^{-1}$.

Power in bandwidth Δf per $d\Omega$

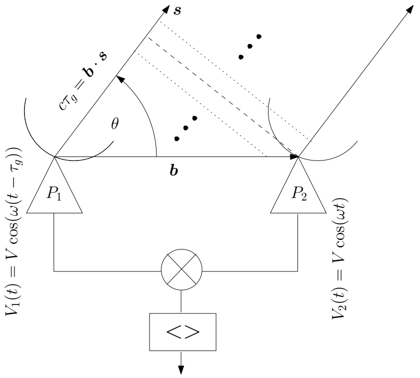
$$\Delta f A(\mathbf{s})I(\mathbf{s})d\Omega$$

NB ASSUMPTIONS

- WE ASSUME A STATIONARY SKY AND INTERFEROMETRIC REFERENCE FRAME.
  Neither the sky nor the interferometer is moving.
- Quasi-monochromatic observation. Assume we are observing in a narrow bandwidth.
- Consider the electric fields from a solid angle $d\Omega$ in the direction $\mathbf{s}$, in some small bandwidth Δf at f.
- We can express the temporal dependence of this field with

$$V(t) = V \cos(2\pi f t + \phi).$$

Two element interferometer $d\Omega$



1. $c\tau_g = \Delta s$: $v \times \Delta t = \Delta s$, where c is the speed of light and τ_g is the difference in the arrival times of the plane waves.
2. $\mathbf{s} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{s} = \Delta s$: the scalar projection of $\mathbf{b}$ onto $\mathbf{s}$, with $|\mathbf{s}| = 1$.

$$c\tau_g = \mathbf{b} \cdot \mathbf{s}$$

Correlator

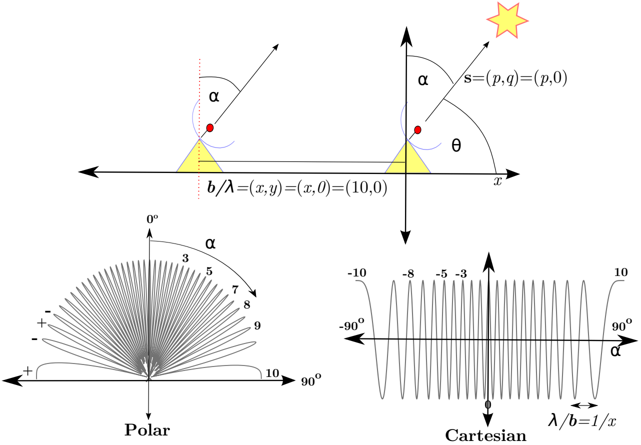
$$
\begin{aligned}
\langle V_1(t)V_2(t)\rangle &= \left\langle V^2\cos(2\pi f(t-\tau_g))\cos(2\pi ft)\right\rangle \\
&= \underbrace{\left\langle \frac{V^2}{2}\cos(2\pi f\tau_g)\right\rangle}_{\text{slowly varying}} + \underbrace{\left\langle \frac{V^2}{2}\cos(4\pi ft-2\pi f\tau_g)\right\rangle}_{\text{rapidly varying}} \\
&= \left\langle \frac{V^2}{2}\cos\left(\frac{2\pi\mathbf{b}\cdot\mathbf{s}}{\lambda}\right)\right\rangle + \left\langle \frac{V^2}{2}\cos\left(4\pi ft-\frac{2\pi\mathbf{b}\cdot\mathbf{s}}{\lambda}\right)\right\rangle \\
&\approx \frac{V^2}{2}\cos\left(\frac{2\pi\mathbf{b}\cdot\mathbf{s}}{\lambda}\right) + \cancelto{0}{\left\langle \frac{V^2}{2}\cos\left(4\pi ft-\frac{2\pi\mathbf{b}\cdot\mathbf{s}}{\lambda}\right)\right\rangle}
\end{aligned}
$$

1. The first term varies slowly as $\mathbf{b}\cdot\mathbf{s}$ depends on the rotation speed of the earth (which is slow).
2. The second term varies rapidly, since radio observations are high frequency observations – 1.4GHz.

Understanding the correlator output

1. The power in the signal $A\cos(2\pi ft)$ is equal to $\frac{A^2}{2}$.
2. We can therefore say that $V \propto \sqrt{P(\mathbf{s})}$, where $P(\mathbf{s})$ is the **power** received, per unit solid angle from **direction s**, per unit frequency at frequency f. $P(\mathbf{s})$ is measured in $\mathrm{Wsr}^{-1}\mathrm{Hz}^{-1}$.
3. $P(\mathbf{s}) = A(\mathbf{s})I(\mathbf{s})$, where A is the **effective collecting area** of the two element interferometer and I is the **sky brightness**.
4. $\cos\left(\frac{2\pi\mathbf{b}\cdot\mathbf{s}}{\lambda}\right)$ is a **modulation factor**. It changes sinusoidally with the change of source direction in the interferometer frame. These sinusoids are called **fringes**.

Understanding 1D fringes



$$\cos\left(\frac{2\pi\mathbf{b}\cdot\mathbf{s}}{\lambda}\right) = \cos(2\pi xp) = \cos(2\pi x\cos(\theta))$$
$$= \cos(2\pi x\sin(\alpha)) = \cos(2\pi(10)\sin(\alpha))$$

The response from an extend source

The response from an **extended source** is obtained by **summing the responses** at each antenna to the emmision over the **entire sky**, **multiplying** the two and **averaging**:

$$R_c(\mathbf{b}) = \left\langle \int_{S_1} V_1(t, \mathbf{s_1}, \mathbf{b}) d\Omega_1 \int_{S_2} V_2(t, \mathbf{s_2}, \mathbf{b}) d\Omega_2 \right\rangle$$

If we evaluate the above under the assumption that the emission is **spatially incoherent** we obtain:

Interferometer response R_c

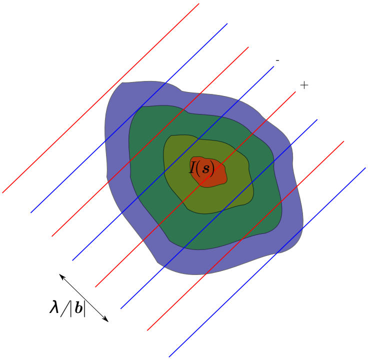
$$R_c^N(\mathbf{b}) = \int_S A_N(\mathbf{s}) I(\mathbf{s}) \cos\left(\frac{2\pi \mathbf{b}\cdot\mathbf{s}}{\lambda}\right) d\Omega$$

where $A_N(\mathbf{s}) = \frac{A(\mathbf{s})}{A_0}$ is the normalized primary beam. We now assume for the sake of simplicity that $A_N(\mathbf{s}) = 1$.

The response from an extended source: minor details

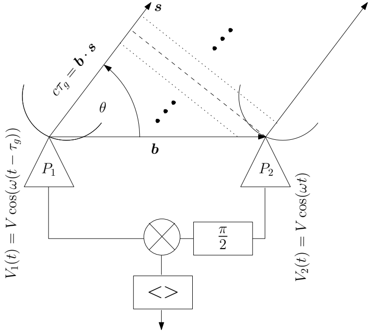
$$
\begin{aligned}
R_c(\mathbf{b}) &= \left\langle \int_{S_1} V_1(t,\mathbf{s_1},\mathbf{b})d\Omega_1 \int_{S_2} V_2(t,\mathbf{s_2},\mathbf{b})d\Omega_2 \right\rangle \\
&= \int_{S_1}\int_{S_2} \big\langle V_1(t,\mathbf{s_1},\mathbf{b}) \cdot V_2(t,\mathbf{s_2},\mathbf{b})\big\rangle d\Omega_1 d\Omega_2 \\
&= \int\int_{\mathbf{s}_1=\mathbf{s}_2} \big\langle V_1(t,\mathbf{s_1},\mathbf{b}) \cdot V_2(t,\mathbf{s_2},\mathbf{b})\big\rangle d\Omega_1 d\Omega_2 + \cdots \\
\cdots &\quad \cancelto{0}{\int\int_{\mathbf{s}_1\neq\mathbf{s}_2} \big\langle V_1(t,\mathbf{s_1}) \cdot V_2(t,\mathbf{s_2},\mathbf{b})\big\rangle} \quad d\Omega_1 d\Omega_2 \\
&= \int_S \big\langle V_1(t,\mathbf{s},\mathbf{b}) \cdot V_2(t,\mathbf{s},\mathbf{b})\big\rangle d\Omega \\
&= \int_S P(\mathbf{s})\cos(\omega\tau_g)d\Omega \\
&= \int_S A(\mathbf{s})\cdot I(\mathbf{s})\cos\left(\frac{2\pi f\mathbf{b}\cdot\mathbf{s}}{c}\right)d\Omega.
\end{aligned}
$$

2D fringes



- The correlator **casts** a **sinusoidal coherence pattern** with an angular scale $\sim \frac{\lambda}{|\mathbf{b}|}$ onto the sky and then **multiplies** it with the **sky brightness**.
- The correlator then **multiplies** this coherence pattern with the sky brightness and then **integrates** over the entire sky.

SIN Correlator



$$\text{Output: } \frac{V^2}{2}\sin\left(\frac{2\pi\mathbf{b}\cdot\mathbf{s}}{\lambda}\right)$$

$$R_s^N(\mathbf{b}) = \int_S I(\mathbf{s})\sin\left(\frac{2\pi\mathbf{b}\cdot\mathbf{s}}{\lambda}\right)d\Omega$$

Why the SIN correlator?

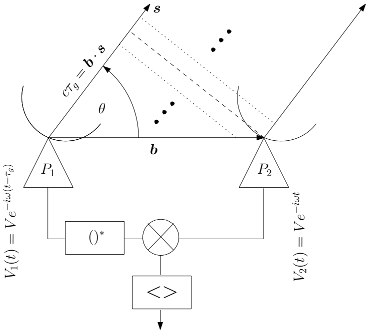
- The cosine correlator can only detect the even part of $I(\mathbf{s})$

$$
\begin{aligned}
R_c^N(\mathbf{b}) &= \int_S I(\mathbf{s}) \cos\left(\frac{2\pi \mathbf{b}\cdot\mathbf{s}}{\lambda}\right) d\Omega \\
&= \int_S I_e(s) \cos\left(\frac{2\pi \mathbf{b}\cdot\mathbf{s}}{\lambda}\right) d\Omega
\end{aligned}
$$

- Similarly

$$
\begin{aligned}
R_s^N(\mathbf{b}) &= \int_S I(s) \sin\left(\frac{2\pi \mathbf{b}\cdot\mathbf{s}}{\lambda}\right) d\Omega \\
&= \int_S I_o(s) \sin\left(\frac{2\pi \mathbf{b}\cdot\mathbf{s}}{\lambda}\right) d\Omega
\end{aligned}
$$

Complex correlator

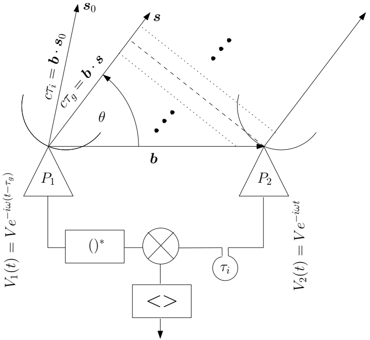


Complex response

$$\text{Output: } V^2 e^{-2\pi i \frac{\mathbf{b}\cdot\mathbf{s}}{\lambda}}$$

$$\tilde{V}(\mathbf{b}) = R_c^N - iR_s^N = \int_S I(\mathbf{s}) e^{-2\pi i \frac{\mathbf{b}\cdot\mathbf{s}}{\lambda}} d\Omega$$

Fringe stopping/tracking

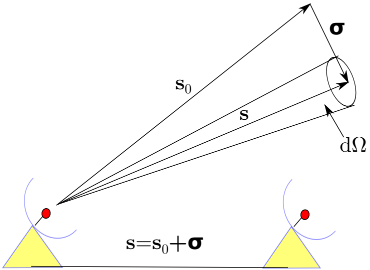


We can stop the fringes at one particular position in the sky $\mathbf{s}_0$ by adding a time delay τ_i.

Complex response

$$\text{Output: } V^2 e^{-2\pi i \frac{\mathbf{b}\cdot(\mathbf{s}-\mathbf{s}_0)}{\lambda}}$$

Complex Visibilities

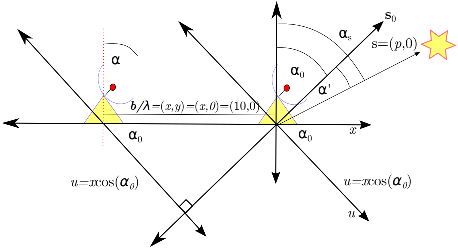


Complex Visibilites

$$V(\mathbf{b}) = \int_S I(\boldsymbol{\sigma}) e^{-2\pi i \frac{\mathbf{b}\cdot\boldsymbol{\sigma}}{\lambda}} d\Omega$$

Units: Jy

Understanding fringe stopping/tracking



$$
\begin{aligned}
\cos\left(2\pi\frac{\mathbf{b}\cdot\boldsymbol{\sigma}}{\lambda}\right) &= \cos\left(2\pi\frac{\mathbf{b}\cdot(\mathbf{s}-\mathbf{s}_0)}{\lambda}\right)\\
&= \cos(2\pi x(\sin\alpha_s - \sin\alpha_0)) = \cos(2\pi x(\sin(\alpha_0+\alpha') - \sin\alpha_0)))\\
&= \cos(2\pi x(\sin\alpha_0\cos\alpha' + \cos\alpha_0\sin\alpha' - \sin\alpha_0)\\
&\approx \cos(2\pi x\cos\alpha_0\sin\alpha') = \cos(2\pi u\sin\alpha')
\end{aligned}
$$

We have therefore effectively shifted the fringe pattern to the field center.

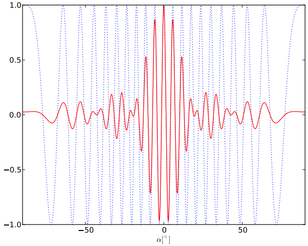
The effect of bandwidth (no fringe stopping)

To find the finite bandwidth response we integrate our fundamental response over a frequency width Δf, centered at f_0:

$$\begin{aligned} V &= \int_S \frac{1}{\Delta f} \int_{f_0 - \frac{\Delta f}{2}}^{f_0 + \frac{\Delta f}{2}} I(\mathbf{s}) e^{-2\pi f \tau_g} df d\Omega \\ &= \int_S I(\mathbf{s}) \mathrm{sinc}(\tau_g \Delta f) e^{-2\pi i f_0 \tau_g} d\Omega \end{aligned}$$

1. $\mathrm{sinc}(\tau_g \Delta f)$ is known as the **fringe wahshing function** or the **fringe attenuation function**.
2. The fringe washing function attenuates the sources far from the meridional plane.
3. $\mathrm{sinc}(x) = 0$ when $x = 1$.

1D Example



$$\mathrm{sinc}\left(\frac{\Delta f}{c} x \sin(\alpha)\right) \cos(2\pi x \sin(\alpha))$$

$$x = 10 \text{ and } \Delta f = 87.7 \text{ MHz}$$

First null

$$\frac{\Delta f}{c} x \sin(\alpha) = 1 \rightarrow \alpha_o \approx 20^\circ$$

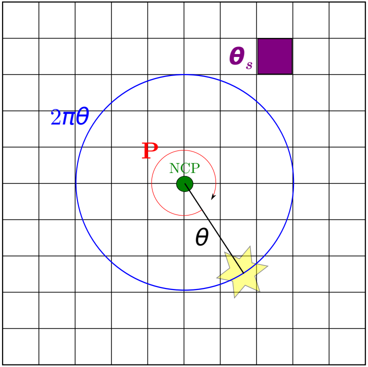
Usable field of view

1. By fringe tracking we **shift the fringe pattern** to the field center, but what is even more important is that we **simultaneously** also **shift the fringe washing function** to the field center, which means we **do not** severely **attenuate** the **sources** close to the **field center**. NB — Fringe stopping extremely important.
2. The usable field of view $2\Delta\theta$:

$$\Delta\theta\Delta f << \theta_s f_0,$$

where $\theta_s = \frac{\lambda}{|\mathbf{b}|}$ is the angular resolution of interferometer.

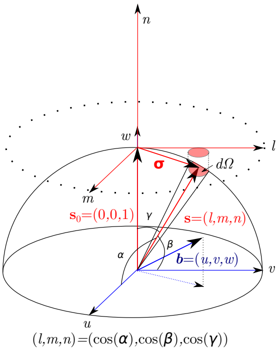
Time-smearing loss



$$\frac{2\pi\Delta\theta}{P}\Delta t << \theta_s = \frac{\lambda}{|\mathbf{b}|}$$

$$P \approx 86164s \approx 23\mathrm{h}56\mathrm{m}04\mathrm{s}$$

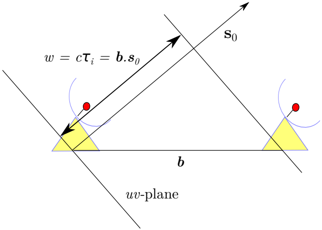
Choosing an appropriate coordinate system



Visibilities

$$V(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I(l,m)e^{-2\pi i[ul+vm+w(\sqrt{1-l^2-m^2}-1)]}\frac{dldm}{\sqrt{1-l^2-m^2}}$$

A closer look at the w-term



$$w \approx \frac{|\mathbf{b}|}{\lambda} \approx \frac{1}{\theta_s} \mid \gamma = \frac{\Delta\theta}{2}$$

$$2\pi(\sqrt{1 - l^2 - m^2} - 1)w = 2\pi(\cos(\gamma) - 1)w \approx -\pi\gamma^2 w$$

$$\pi\gamma^2 w < 0.1 \rightarrow \frac{\pi\Delta\theta_F}{4\theta_s} < 0.1$$

When is w small enough to be negligible

$$\Delta\theta_F < \frac{1}{3}\sqrt{\theta_s}$$

3D to 2D

Fourier transform relationship

$$
\begin{aligned}
V(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I(l,m)e^{-2\pi i[ul+vm+w(\sqrt{1-l^2-m^2}-1)]}\frac{dldm}{\sqrt{1-l^2-m^2}}\\
&\approx \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I(l,m)e^{-2\pi i(ul+vm)}dldm\\
&= \mathcal{F}\{I(l,m)\}\\
I(l,m) &\approx \mathcal{F}^{-1}\{V(u,v)\}
\end{aligned}
$$