# Radio-Interferometric Measurement Equation

## Introductory Radio Interferometry Course

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# Radio-Interferometric Measurement Equation (RIME)

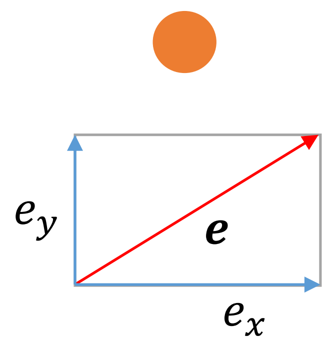
- Compact, intuitive, matrix-based way of representing propagation effects in radio interferometry.
- Useful for calibration (solving for and correcting these propagation effects).

# Introduction

$e'_x, e'_y$ : Components of electric field vector in reference frame of sky, at the observer

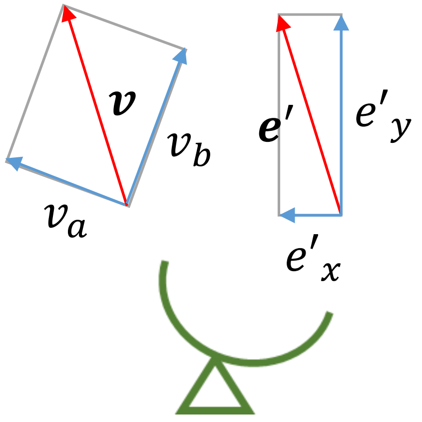
$v_a, v_b$ : Voltages measured by antenna feed (linearly or circularly polarized)

Source




$e_x, e_y$ : Components of electric field vector in reference frame of sky, at the source

Propagation effects




Antenna

Can be represented as vectors:

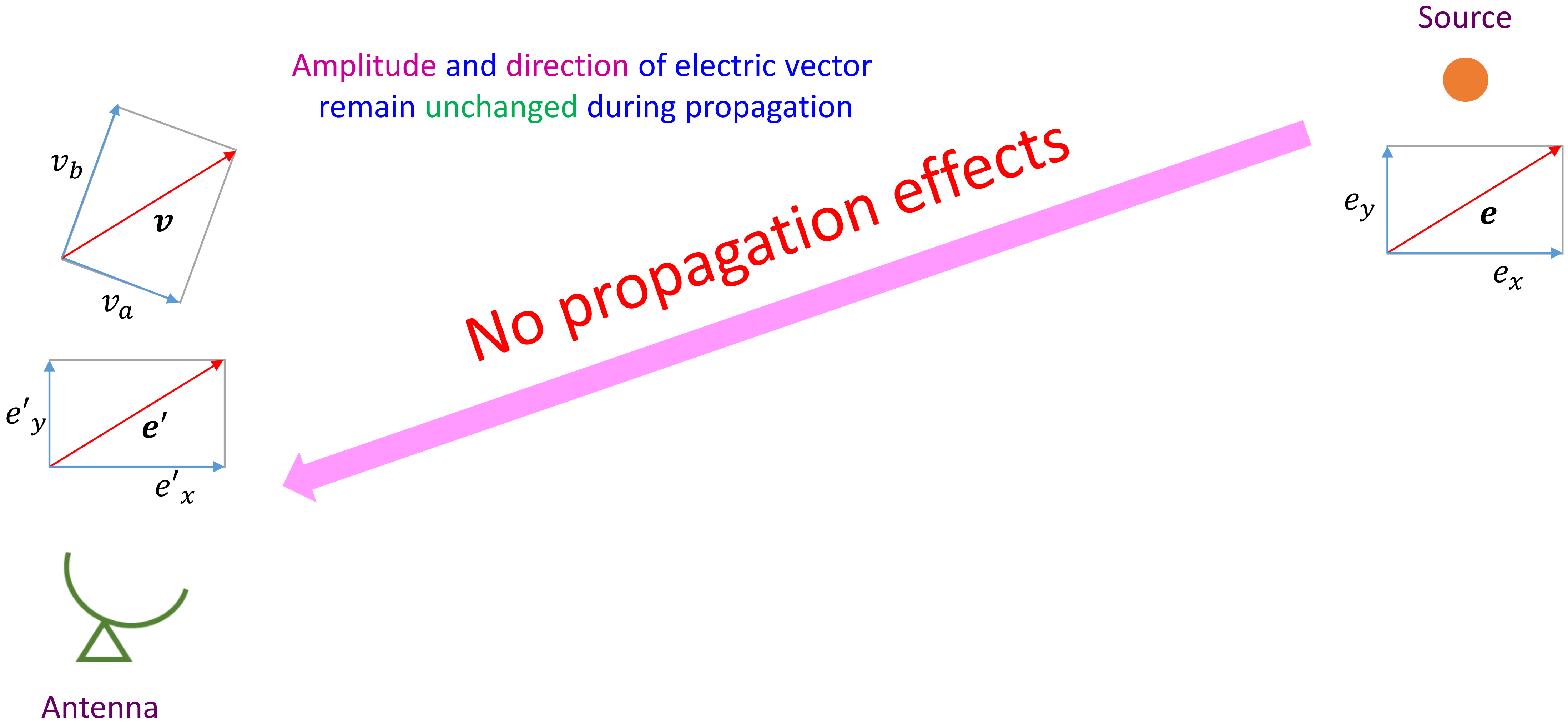
$$\boldsymbol{e} = \begin{pmatrix} e_x \\ e_y \end{pmatrix}$$

$$\boldsymbol{e}' = \begin{pmatrix} e'_x \\ e'_y \end{pmatrix}$$

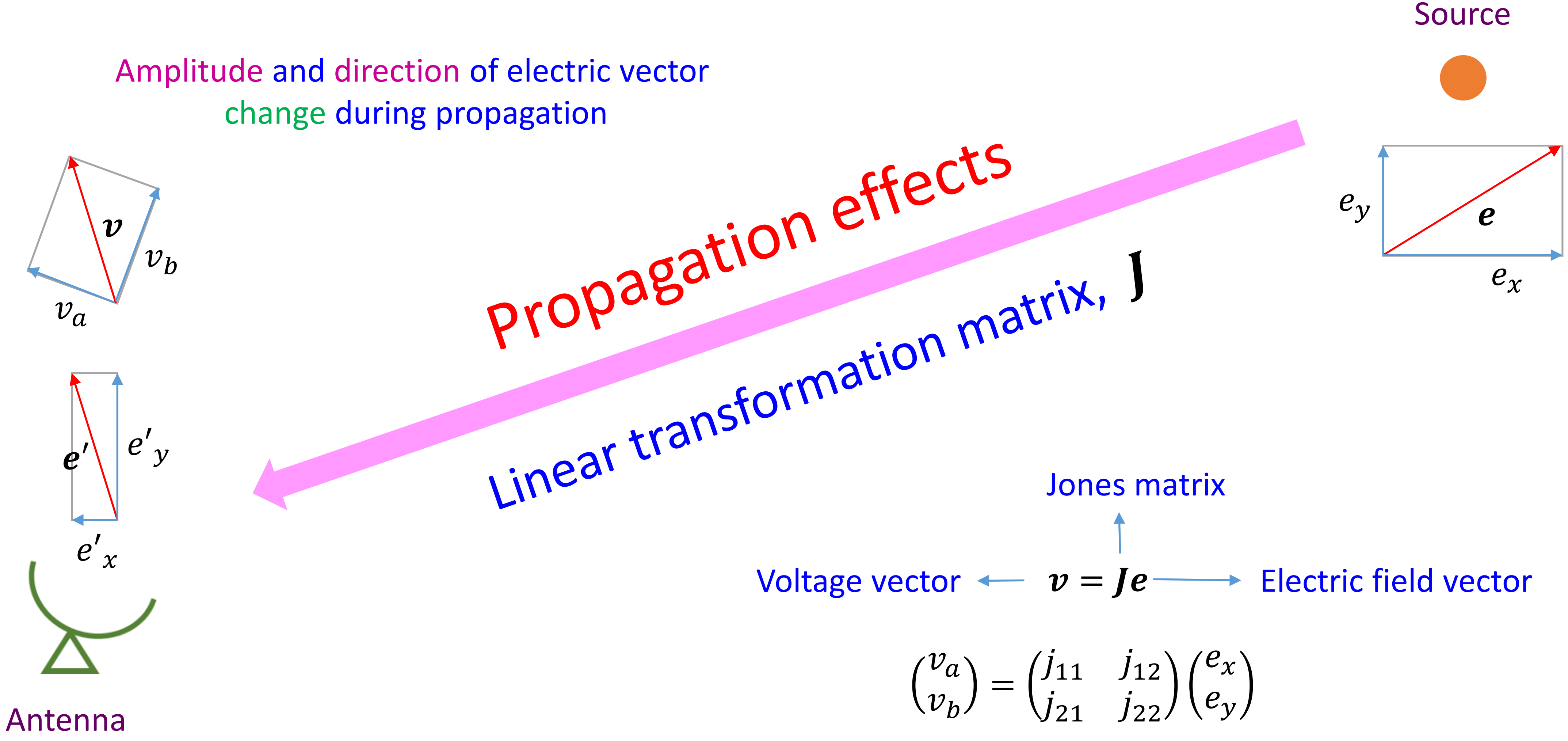
$$\boldsymbol{v} = \begin{pmatrix} v_a \\ v_b \end{pmatrix}$$

# Propagation effects absent

Amplitude and direction of electric vector
remain unchanged during propagation



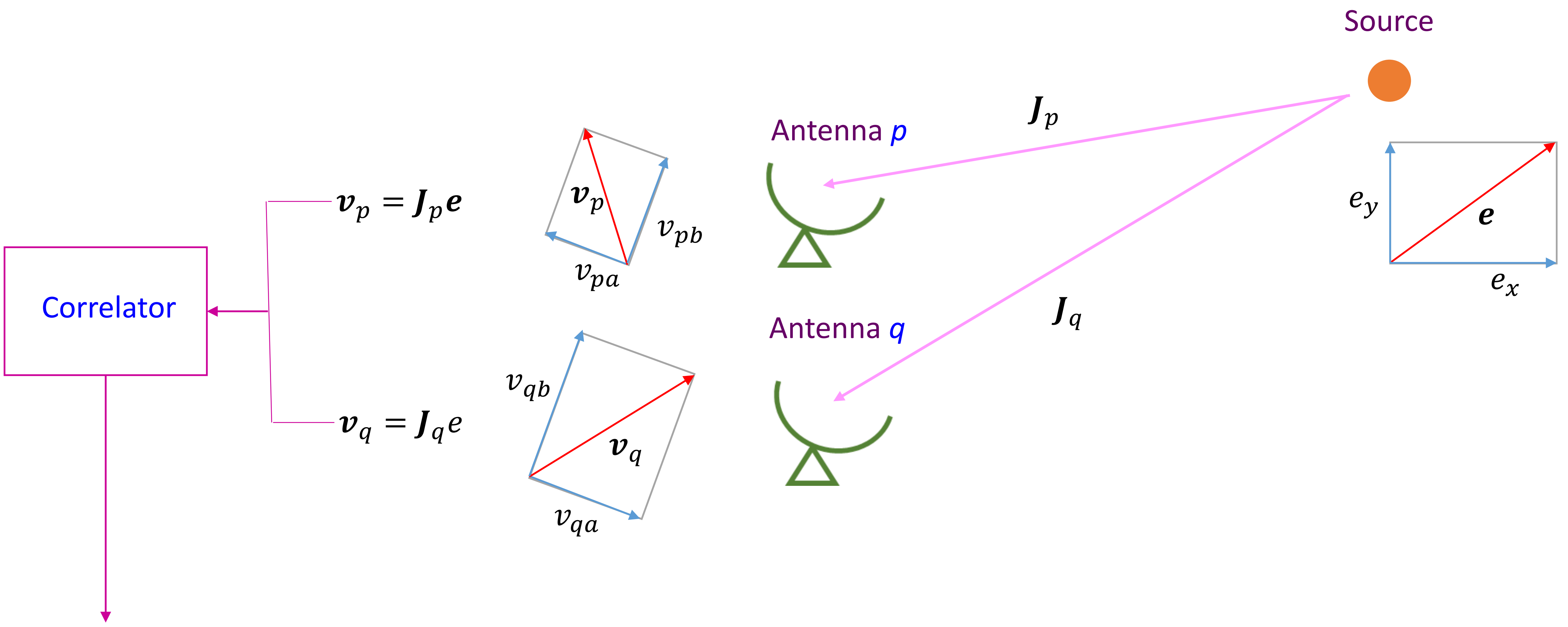
# Propagation effects present




$$\boldsymbol{v} = \boldsymbol{J}\boldsymbol{e}$$

$$\begin{pmatrix} v_a \\ v_b \end{pmatrix} = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix} \begin{pmatrix} e_x \\ e_y \end{pmatrix}$$

# Correlation




$$\mathbf{V}_{pq} = 2\langle \boldsymbol{v}_p \boldsymbol{v}_q^H \rangle$$

# Visibility

- The correlator computes the visibility, $\mathbf{V}_{pq}$, on the baseline $pq$:

Hermitian conjugate

$$\mathbf{V}_{pq} = 2\underbrace{\langle \overbrace{\boldsymbol{v}_p \boldsymbol{v}_q^H}^{\text{Outer product}} \rangle}_{\text{Average}}$$

$$= 2\left\langle \begin{pmatrix} v_{pa} \\ v_{pb} \end{pmatrix} \begin{pmatrix} v_{qa}^* & v_{qb}^* \end{pmatrix} \right\rangle$$

These 4 quantities are the outputs from the correlator

$$= 2 \overbrace{\begin{pmatrix} \langle v_{pa} v_{qa}^* \rangle & \langle v_{pa} v_{qb}^* \rangle \\ \langle v_{pb} v_{qa}^* \rangle & \langle v_{pb} v_{qb}^* \rangle \end{pmatrix}}$$

# Correlation

$$\boldsymbol{v}_p = \boldsymbol{J}_p \boldsymbol{e} \quad , \quad \boldsymbol{v}_q = \boldsymbol{J}_q \boldsymbol{e}$$

$$\mathbf{V}_{pq} = 2\langle\, \boldsymbol{v}_p \boldsymbol{v}_q^H \,\rangle$$

$$= 2\langle (\boldsymbol{J}_p \boldsymbol{e})(\boldsymbol{J}_q \boldsymbol{e})^H \rangle$$

$$= 2\langle \boldsymbol{J}_p (\boldsymbol{e}\boldsymbol{e}^H) \boldsymbol{J}_q^H \rangle$$

$$= \langle \boldsymbol{J}_p (2\boldsymbol{e}\boldsymbol{e}^H) \boldsymbol{J}_q^H \rangle$$

# Coherency, or Brightness

$$\mathbf{V}_{pq} = \langle \boldsymbol{J}_p (2\boldsymbol{e}\boldsymbol{e}^H) \boldsymbol{J}_q^H \rangle$$

By definition, the coherency, or brightness, B, is given by:

$$\mathbf{B} = \langle 2\boldsymbol{e}\boldsymbol{e}^H \rangle = \begin{pmatrix} I+Q & U+iV \\ U-iV & I-Q \end{pmatrix}$$

$\langle \boldsymbol{e}\boldsymbol{e}^H \rangle$ is the coherence of the electromagnetic field with itself, and is described by the Stokes parameters $I, Q, U, V$

$$\mathbf{V}_{pq} = \boldsymbol{J}_p \, \mathbf{B} \, \boldsymbol{J}_q^H$$

# Radio-Interferometric Measurement Equation (RIME)

Visibility

Brightness

$$\mathbf{V}_{pq} = \boldsymbol{J}_p\ \mathbf{B}\ \boldsymbol{J}_q^H$$

Jones matrices

$$\begin{pmatrix} v_{aa} & v_{ab} \\ v_{ba} & v_{bb} \end{pmatrix} = \begin{pmatrix} j_{11a} & j_{12a} \\ j_{21a} & j_{22a} \end{pmatrix} \begin{pmatrix} I+Q & U+iV \\ U-iV & I-Q \end{pmatrix} \begin{pmatrix} j_{11b} & j_{12b} \\ j_{21b} & j_{22b} \end{pmatrix}^H$$

# Component Jones matrices

The Jones matrix for an antenna is a product of several component Jones matrices, corresponding to different corrupting effects along the signal path

Example:

$$\boldsymbol{J} = \underset{\substack{\text{Bandpass}\\ \text{gain}}}{\boldsymbol{B}}\ \underset{\substack{\text{Instrumental}\\ \text{gain}}}{\boldsymbol{G}}\ \underset{\substack{\text{Polarization}\\ \text{leakage}}}{\boldsymbol{D}}\ \underset{\substack{\text{Primary}\\ \text{beam}}}{\boldsymbol{E}}\ \underset{\substack{\text{Parallactic angle}\\ \text{feed rotation}}}{\boldsymbol{P}}\ \underset{\substack{\text{Ionospheric}\\ \text{Faraday rotation}}}{\boldsymbol{T}}$$

# Component Jones matrices

Jones chain:

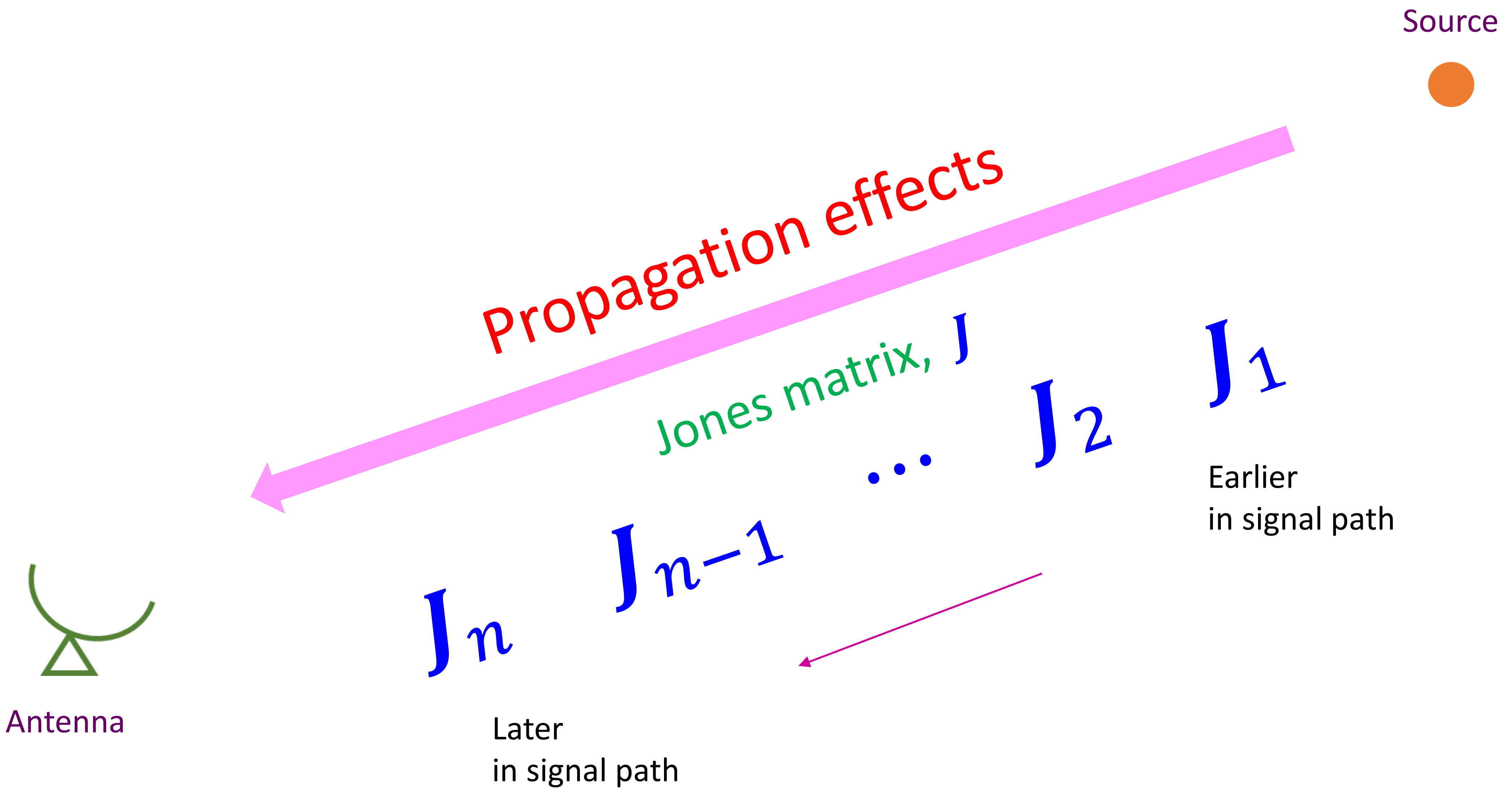
$$\boldsymbol{J} = \boldsymbol{J}_n \, \boldsymbol{J}_{n-1} \cdots \boldsymbol{J}_2 \, \boldsymbol{J}_1$$

Later
in signal path

$\longleftarrow$

Earlier
in signal path

# Component Jones matrices



# Component Jones matrices

Antenna $p$ : $$\boldsymbol{J}_p = \boldsymbol{J}_{pn}\,\boldsymbol{J}_{p(n-1)}\cdots\boldsymbol{J}_{p2}\,\boldsymbol{J}_{p1}$$

Antenna $q$ : $$\boldsymbol{J}_q = \boldsymbol{J}_{qn}\,\boldsymbol{J}_{q(n-1)}\cdots\boldsymbol{J}_{q2}\,\boldsymbol{J}_{q1}$$

Visibility

Brightness

$$\mathbf{V}_{pq} = \boldsymbol{J}_p\,\mathbf{B}\,\boldsymbol{J}_q^H$$

Jones matrices

$$\mathbf{V}_{pq} = \boldsymbol{J}_{pn}\,\boldsymbol{J}_{p(n-1)}\cdots\boldsymbol{J}_{p2}\,\boldsymbol{J}_{p1}\,\mathbf{B}\,\boldsymbol{J}_{q1}^H\,\boldsymbol{J}_{q2}^H\cdots\boldsymbol{J}_{q(n-1)}^H\,\boldsymbol{J}_{qn}^H$$

$$\mathbf{V}_{pq} = \boldsymbol{J}_{pn}\Big(\boldsymbol{J}_{p(n-1)}\Big(\cdots\Big(\boldsymbol{J}_{p2}\Big(\boldsymbol{J}_{p1}\,\mathbf{B}\,\boldsymbol{J}_{q1}^H\Big)\boldsymbol{J}_{q2}^H\Big)\cdots\Big)\boldsymbol{J}_{q(n-1)}^H\Big)\boldsymbol{J}_{qn}^H$$

# Calibration

Calibration: Determining and correcting for propagation effects in order to compute the brightness.

i.e., solve for Jones matrices $\boldsymbol{J}$ to compute $\mathbf{B}$:

Visibility

Brightness

$$\mathbf{V}_{pq} = \boldsymbol{J}_p \; \mathbf{B} \; \boldsymbol{J}_q^H$$

Jones matrices

$$\mathbf{B} = \boldsymbol{J}_p^{-1} \mathbf{V}_{pq} \left( \boldsymbol{J}_q^H \right)^{-1}$$

# Direction-independent and direction-dependent effects

Propagation effects can be of two kinds:

- Direction-independent effects
- Direction-dependent effects

These effects can be represented by different Jones matrices:

$$\boldsymbol{J} = \boldsymbol{G}\ \boldsymbol{E}$$

Final Jones matrix

Direction-independent effects

Direction-dependent effects

# Direction-independent and direction-dependent effects

$$\boldsymbol{J} = \boldsymbol{G}\,\boldsymbol{E}$$

Final Jones matrix

Direction-independent effects

Direction-dependent effects

Visibility-plane effects

Image-plane effects

$$\mathbf{V}_{pq} = \boldsymbol{J}_p\,\mathbf{B}\,\boldsymbol{J}_q^H$$

$$\mathbf{V}_{pq} = \boldsymbol{G}_p\left(\boldsymbol{E}_p\mathbf{B}\boldsymbol{E}_q^H\right)\boldsymbol{G}_q^H$$

# Direction-independent and direction-dependent effects

$$\boldsymbol{J} = \boldsymbol{B}\ \ \boldsymbol{G}\ \ \boldsymbol{D}\ \ \boldsymbol{E}\ \ \boldsymbol{P}\ \ \boldsymbol{T}$$

Bandpass gain

Instrumental gain

Polarization leakage

Primary beam

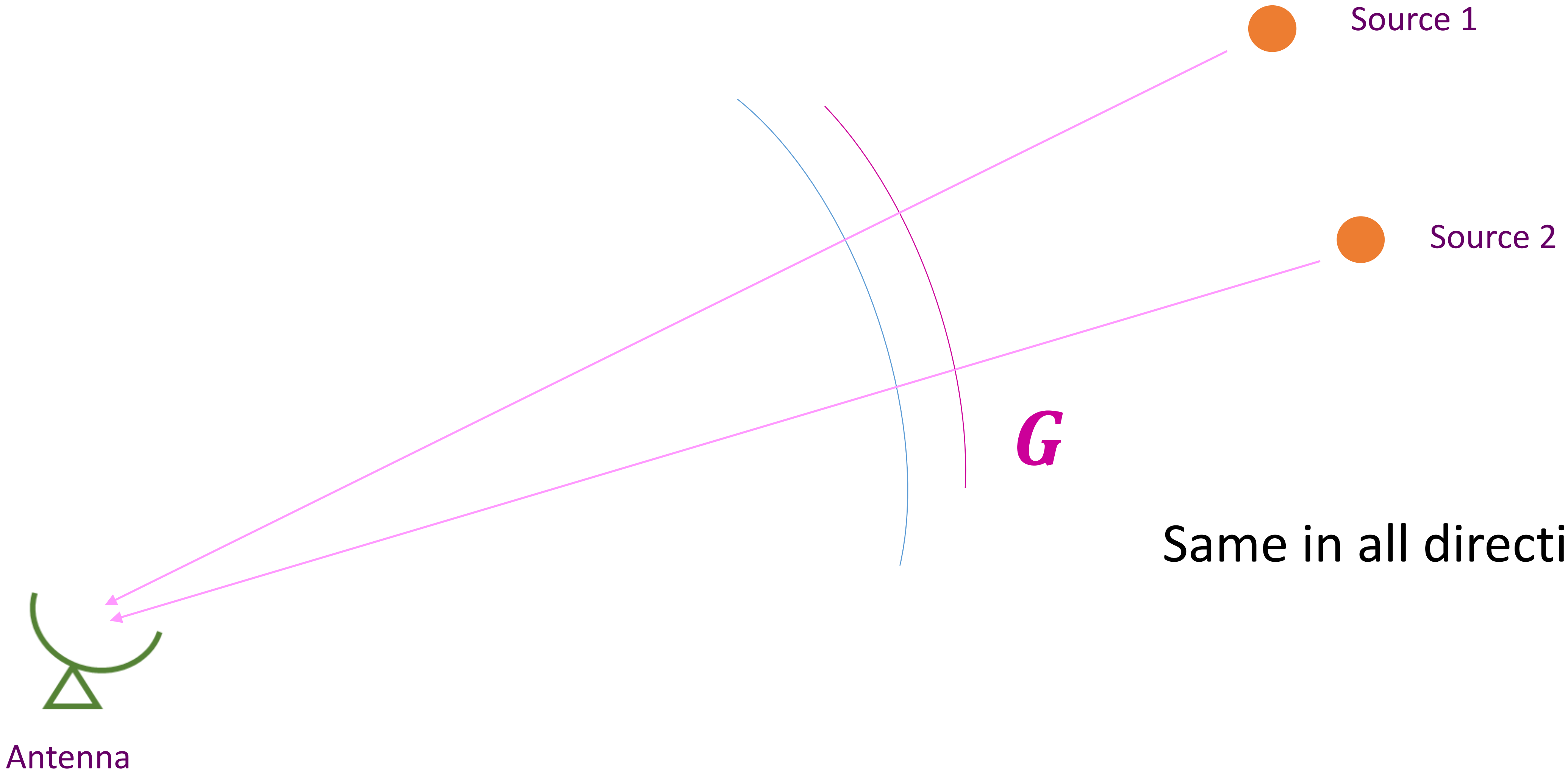
Parallactic angle feed rotation

Ionospheric Faraday rotation

Direction-dependent effect

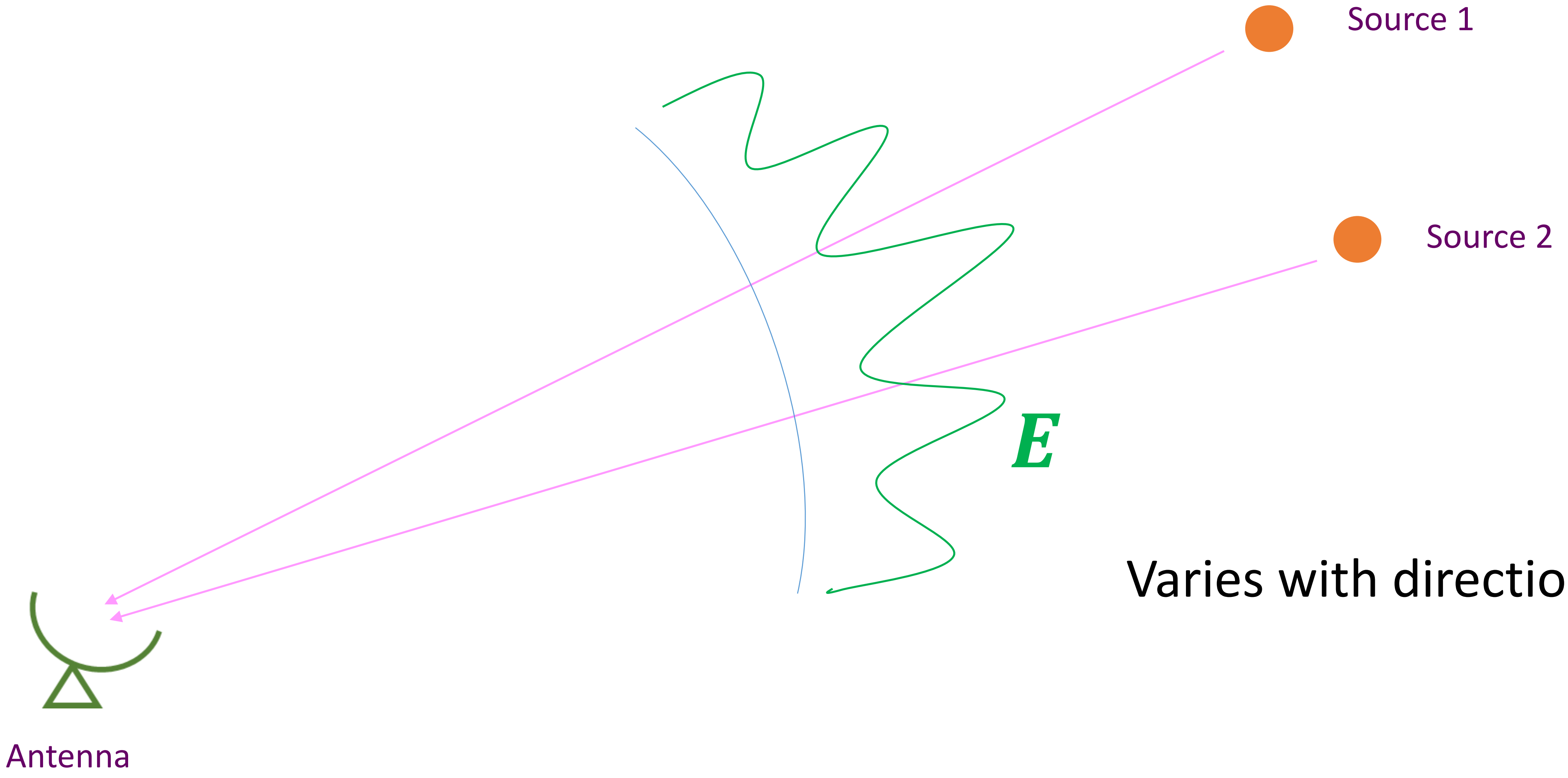
Direction-independent effects

# Direction-independent effects




Same in all directions

# Direction-dependent effects




Varies with direction

# Structure of Jones matrices

Most Jones matrices have a simple form:

- $$J_{\text{rotation}} = \begin{bmatrix} e^{j\theta} & 0 \\ 0 & e^{-j\theta} \end{bmatrix}$$
- $$J_{\text{leakage}} = \begin{bmatrix} 1 & D^R \\ D^L & 1 \end{bmatrix}$$
- $$J_{\text{gain}} = \begin{bmatrix} G^R & 0 \\ 0 & G^L \end{bmatrix}$$

(in a circularly polarized basis)

# Rotation matrices

- $\boldsymbol{J}_{\text{rotation}} = \begin{bmatrix} e^{j\theta} & 0 \\ 0 & e^{-j\theta} \end{bmatrix}$

Examples:

Parallactic angle

Parallactic angle feed rotation: $\boldsymbol{P} = \begin{bmatrix} e^{-j\alpha} & 0 \\ 0 & e^{j\alpha} \end{bmatrix}$

Ionospheric Faraday rotation: $\boldsymbol{T} = \begin{bmatrix} e^{j\chi} & 0 \\ 0 & e^{-j\chi} \end{bmatrix}$

Faraday rotation angle

# Leakage matrices

- $\boldsymbol{J}_{\text{leakage}} = \begin{bmatrix} 1 & D^R \\ D^L & 1 \end{bmatrix}$

Examples:

Polarization leakage: $\boldsymbol{D} = \begin{bmatrix} 1 & d^R \\ d^L & 1 \end{bmatrix}$

Polarization leakage terms

# Gain matrices

- $J_{\text{gain}} = \begin{bmatrix} G^R & 0 \\ 0 & G^L \end{bmatrix}$

Examples:

Instrumental gain: $\boldsymbol{G} = \begin{bmatrix} g^R & 0 \\ 0 & g^L \end{bmatrix} = \begin{bmatrix} a^R e^{j\phi^R} & 0 \\ 0 & a^L e^{j\phi^L} \end{bmatrix}$

Bandpass gain: $\boldsymbol{B} = \begin{bmatrix} B^R & 0 \\ 0 & B^L \end{bmatrix} = \begin{bmatrix} b^R(\nu) e^{j\psi^R(\nu)} & 0 \\ 0 & b^L(\nu) e^{j\psi^L(\nu)} \end{bmatrix}$

# Direction dependent effects: Primary beam

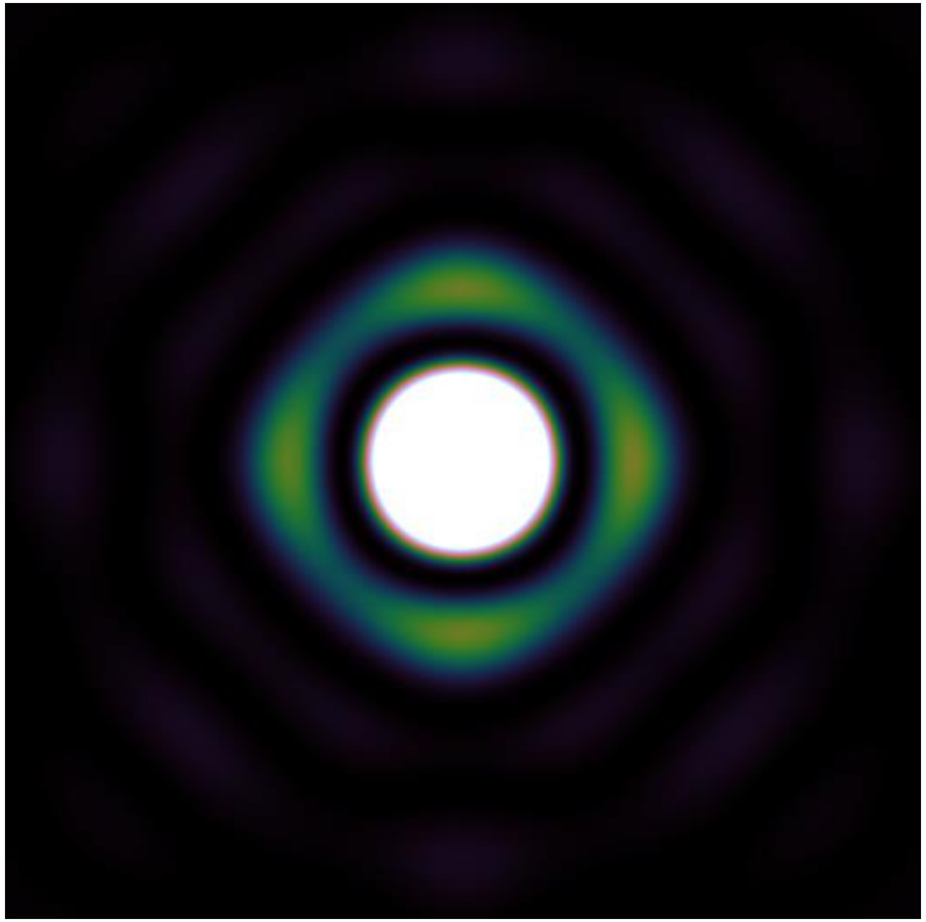
- The primary beam of the antenna is the most important direction-dependent effect.

- Becomes important in wide-field, wide-band observations.

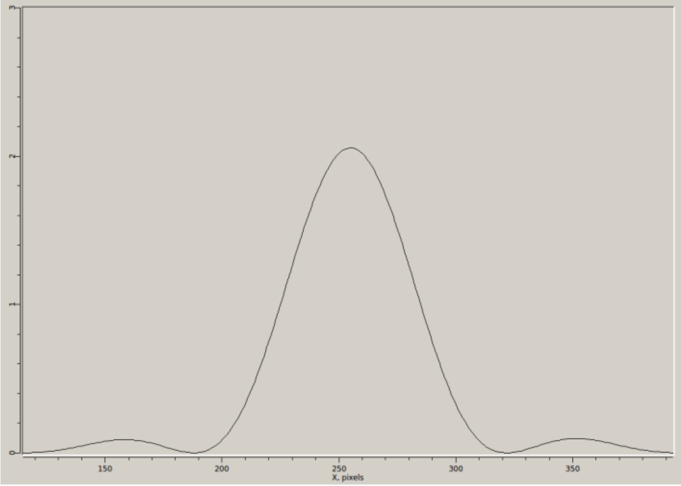
- The primary beam pattern has a multiplicative effect in the image plane, convolutional effect in the visibility plane.

- We will consider the example of a JVLA (Jansky Very Large Array) antenna here.

# Primary beam amplitude variation with distance from center



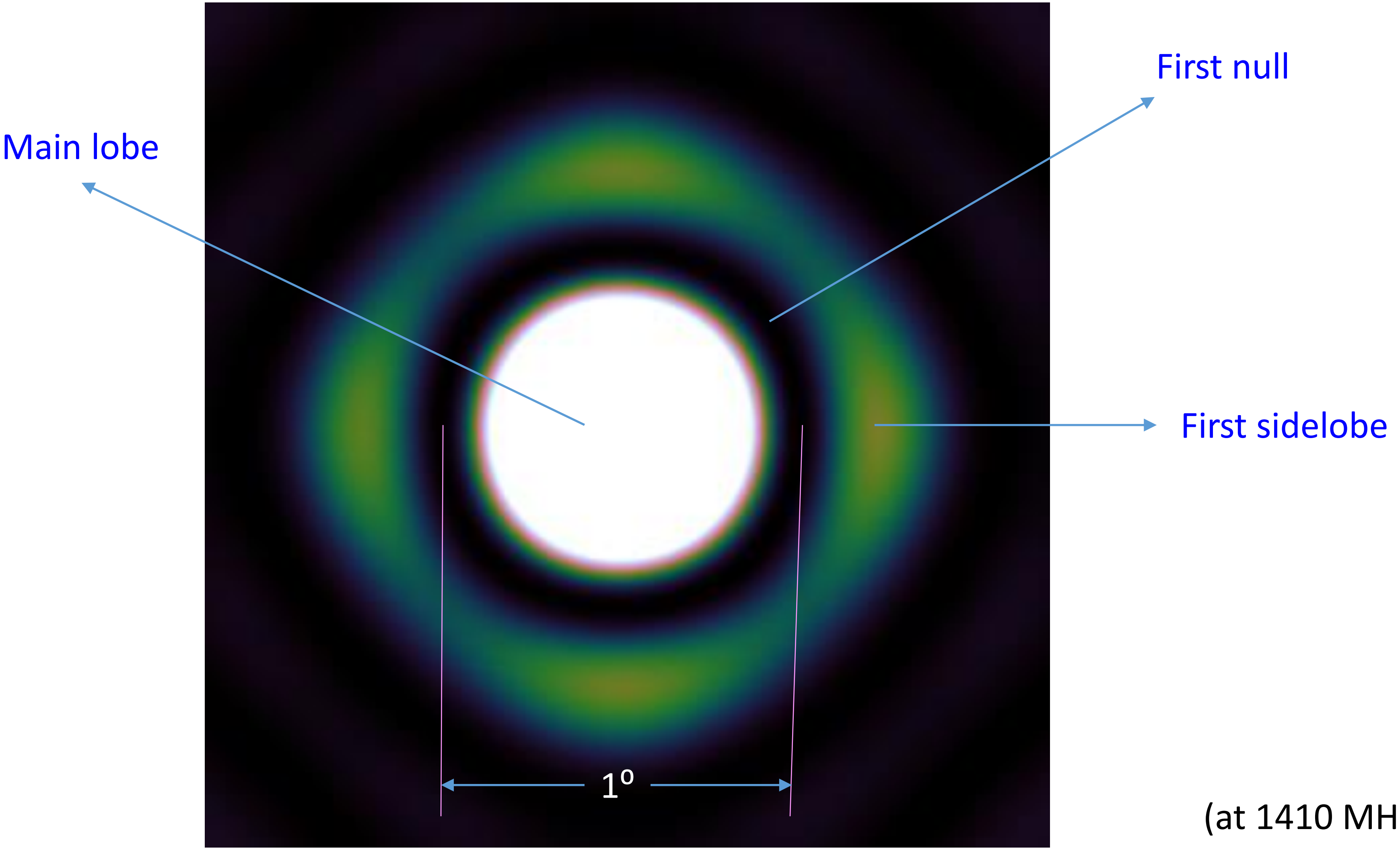
Beam gain




Distance

Horizontal cross-section of the beam through the center

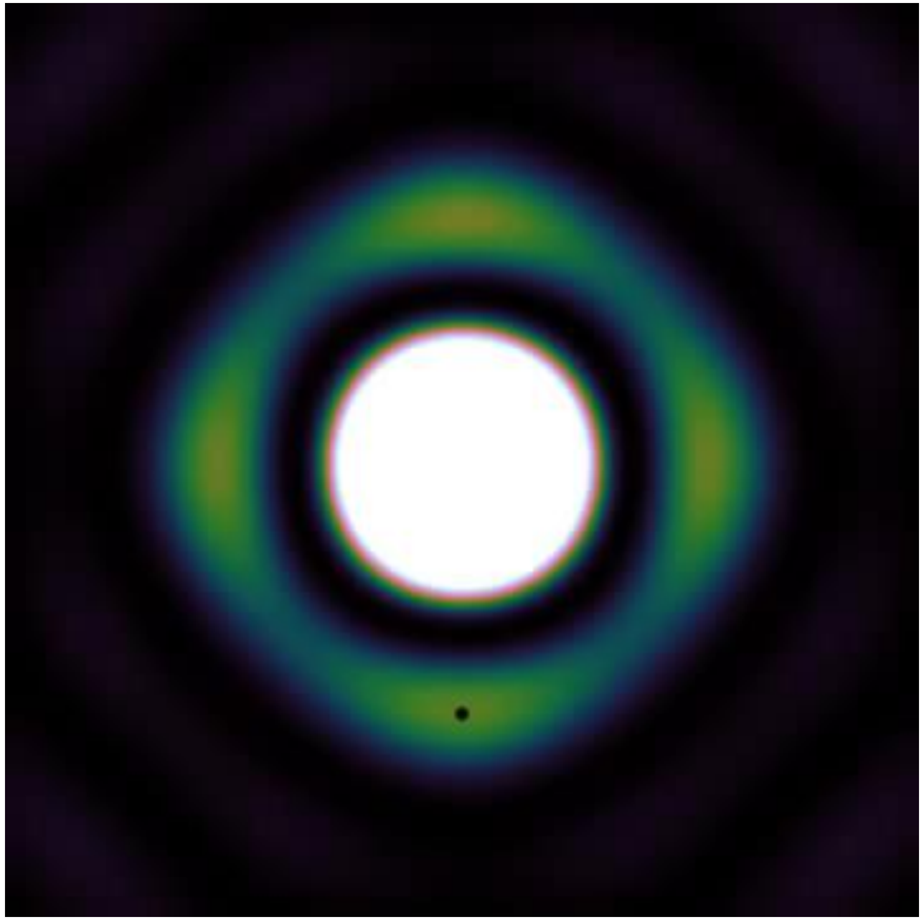
# JVLA primary beam



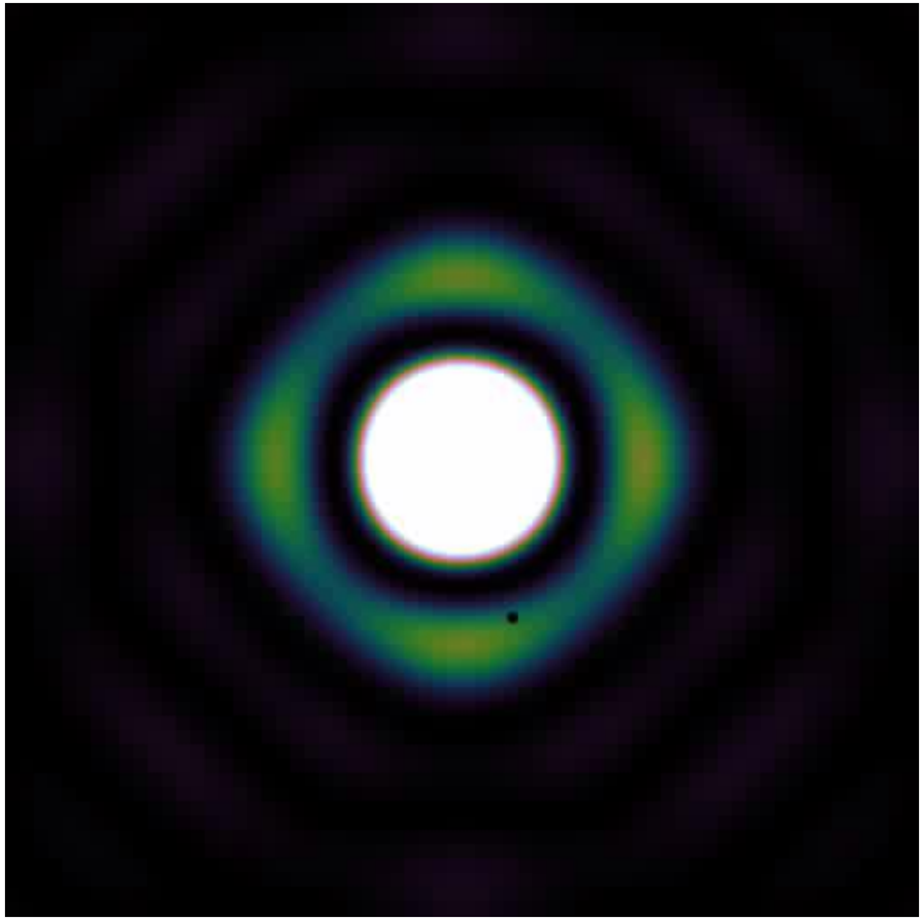

(at 1410 MHz)

# Primary beam rotation

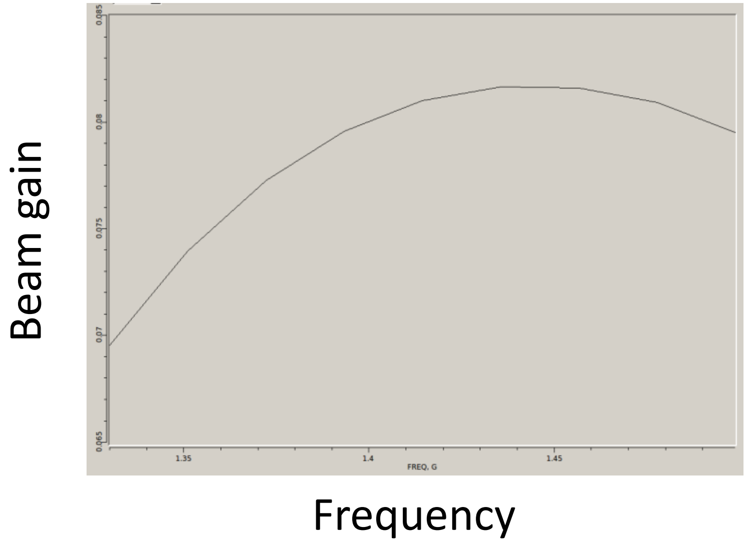
An EVLA antenna has an alt-azimuth mount;
the primary beam rotates during the course of an observation



# Variation of primary beam with frequency



The beam pattern becomes more compact with increasing frequency




Beam-induced spectral variation for the source represented by a dot

# Incorporating primary beam in calibration

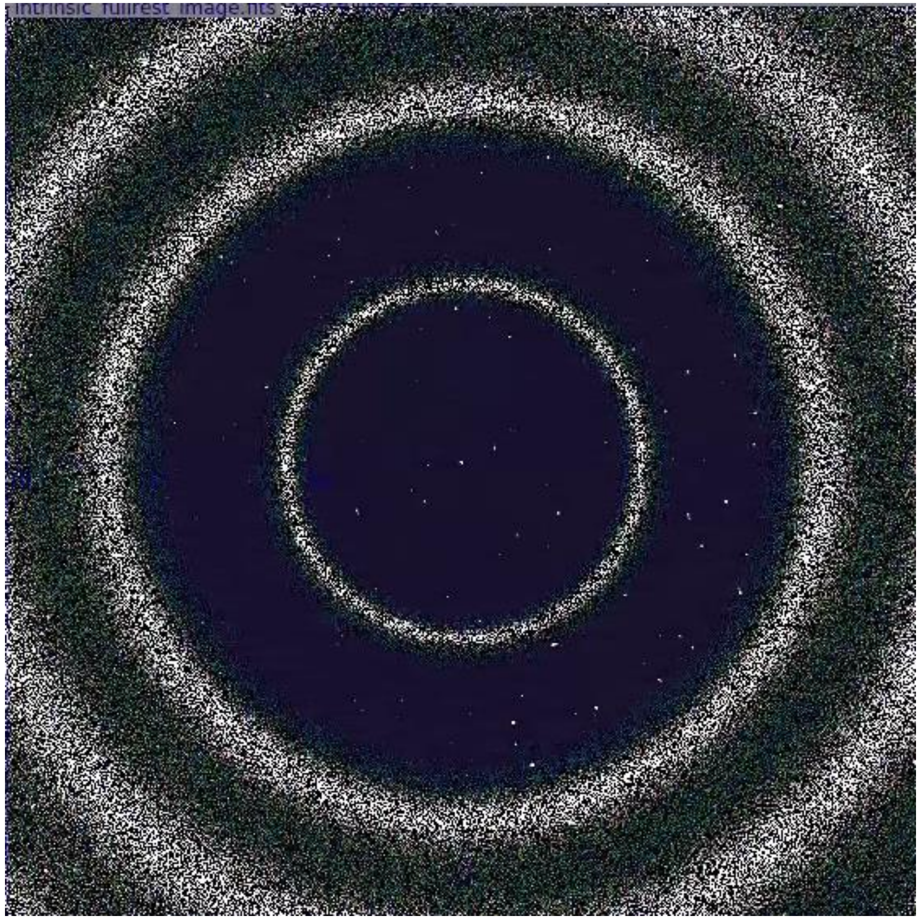
EVLA image of the field around the radio source 3C147

Calibration without primary beam included

Calibration with primary beam included

# Effect of primary beam on noise over the field of view

EVLA image of the field around the radio source 3C147



# Calibration procedure

1. Start with visibility data, $\mathbf{V}_{pq}$, and initial brightness model, $\mathbf{B}$.
2. Solve $\min_{\boldsymbol{J}} \left| \mathbf{V}_{pq} - \boldsymbol{J}_p \, \mathbf{B} \, \boldsymbol{J}_q^H \right|$ for $\boldsymbol{J}s$.
3. Calculate residual visibility data $\mathbf{V}_{pq}^{\mathrm{residual}} = \mathbf{V}_{pq} - \boldsymbol{J}_p \, \mathbf{B} \, \boldsymbol{J}_q^H$.
4. Image $\mathbf{V}_{pq}^{\mathrm{residual}}$ to create a residual image, $I$.
5. Perform a source-finding procedure to find sources in the residual image, and add these to the initial model $\mathbf{B}$ to form a new, updated model $\mathbf{B}^{\mathrm{new}}$.
6. Set $\mathbf{B}$ = $\mathbf{B}^{\mathrm{new}}$, and repeat steps 2-5 until the residual image $I$ is noise-like.

# Differential gains

- Differential gain solutions encompass the unknown and unmodeled direction-dependent effects in the signal path.
- The Jones matrix in the direction of source $s$ is then given by:

$$\boldsymbol{J}^{(s)} = \boldsymbol{G}\ \boldsymbol{E}\ \Delta\boldsymbol{E}^{(s)}$$

Final Jones matrix in the direction of source *s*

Direction-independent effects

Known and modeled direction-dependent effects

Differential gain: Unknown/unmodeled direction-dependent effects in the direction of source *s*

# Differential gains

- Differential gain solutions are computed (in the direction of a few bright sources) and applied after regular calibration in order to correct for leftover, uncalibrated effects.

# Incorporating differential gains in calibration

(Without primary beam incorporated in calibration)

Without differential gain solutions

With differential gain solutions applied

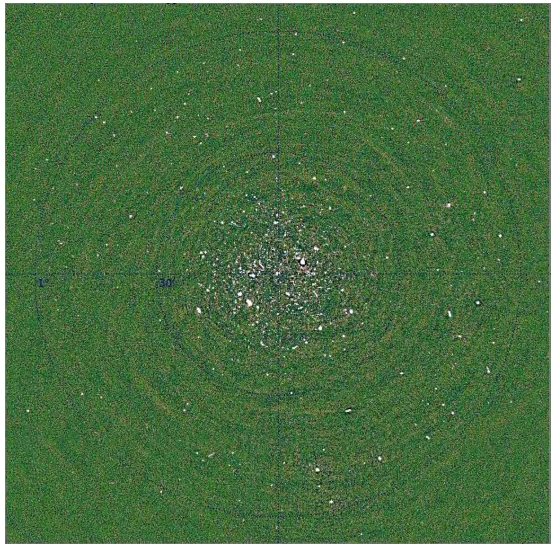
# Differential gains

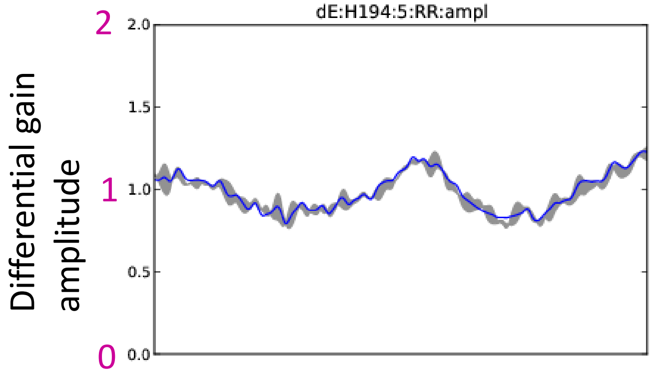
- As more corrupting effects are modeled and accounted for, the calibration becomes more comprehensive, and differential gain solutions approach unity.

# Incorporating primary beam in calibration

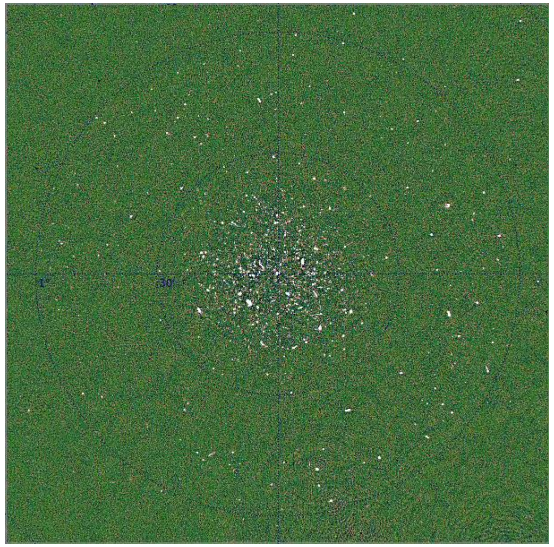
## Differential gain plots

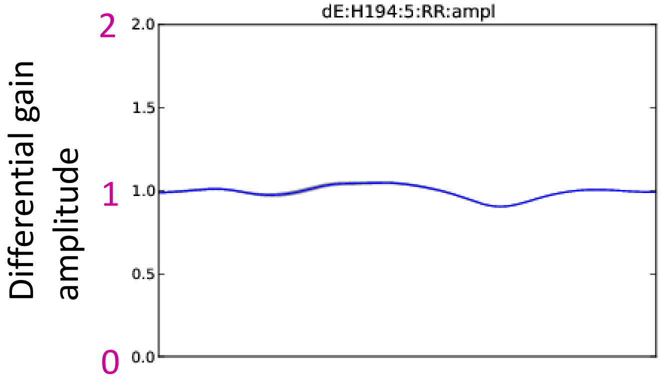
Without primary beam






With primary beam






- Flattened differential gain curves, ~1 over the whole range
- Residual variation due to remaining uncorrected direction-dependent effects (like antenna pointing errors)

# References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