Radio-Interferometric Measurement Equation

Introductory Radio Interferometry Course

Centre for Radio Astronomy Techniques and Technologies (RATT)
Rhodes University

Modhurita Mitra

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Radio-Interferometric Measurement Equation (RIME)

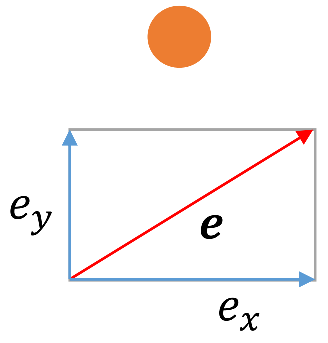
- Compact, intuitive, matrix-based way of representing propagation effects in radio interferometry.
- Useful for calibration (solving for and correcting these propagation effects).

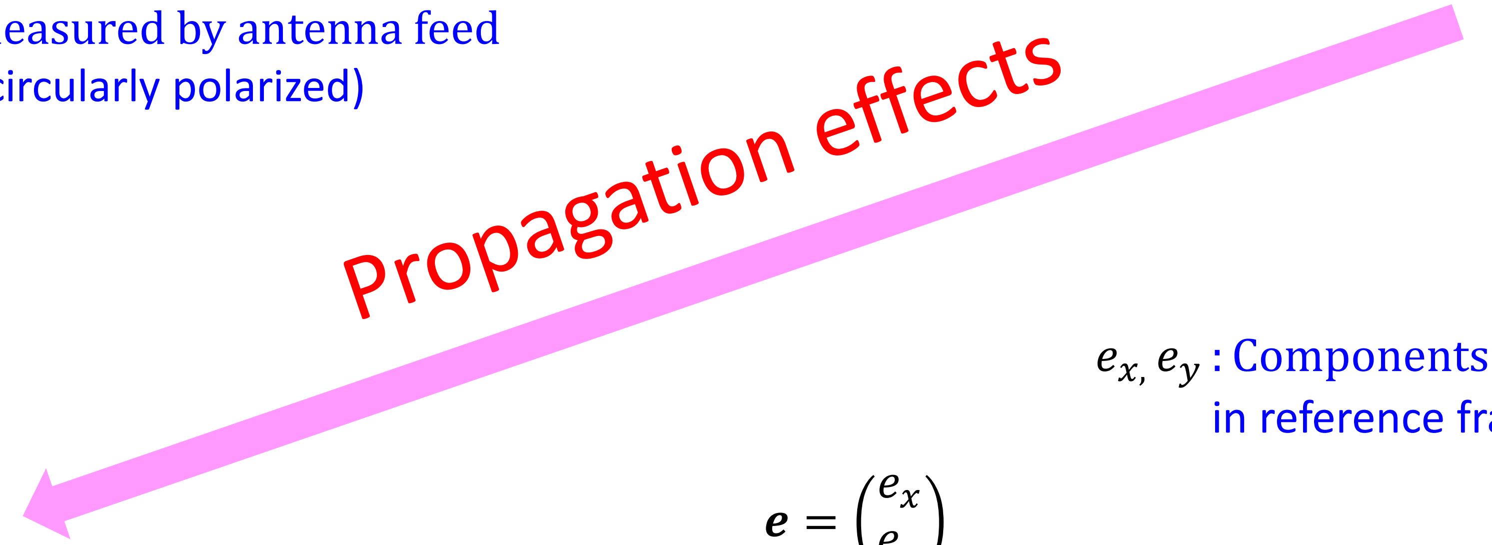
Introduction

$e'_x\,, e'_y$: Components of electric field vector
in reference frame of sky, at the observer

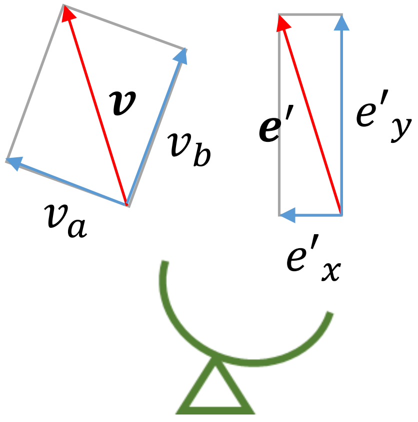
$v_a\,, v_b$: Voltages measured by antenna feed
(linearly or circularly polarized)

Source





e_x, e_y : Components of electric field vector
in reference frame of sky, at the source



Antenna

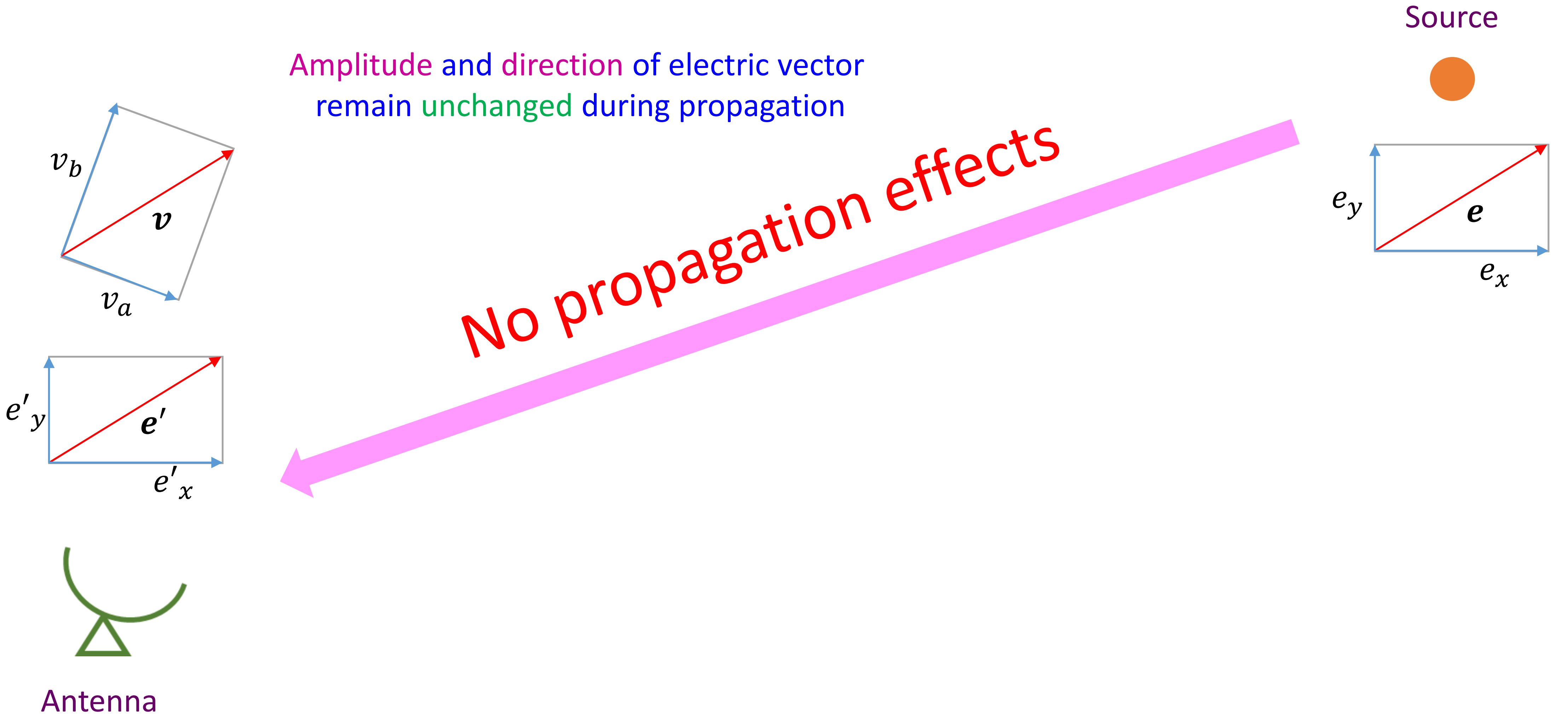
Can be represented
as vectors:

$$\boldsymbol{e} = \begin{pmatrix} e_x \\ e_y \end{pmatrix}$$

$$\boldsymbol{e}' = \begin{pmatrix} e'_x \\ e'_y \end{pmatrix}$$

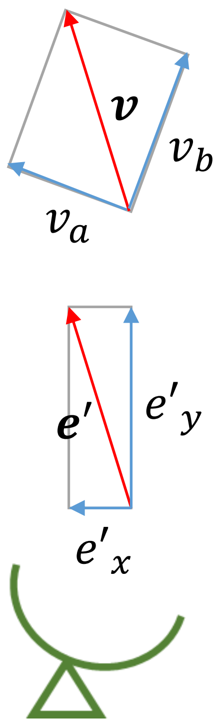
$$\boldsymbol{v} = \begin{pmatrix} v_a \\ v_b \end{pmatrix}$$

Propagation effects absent

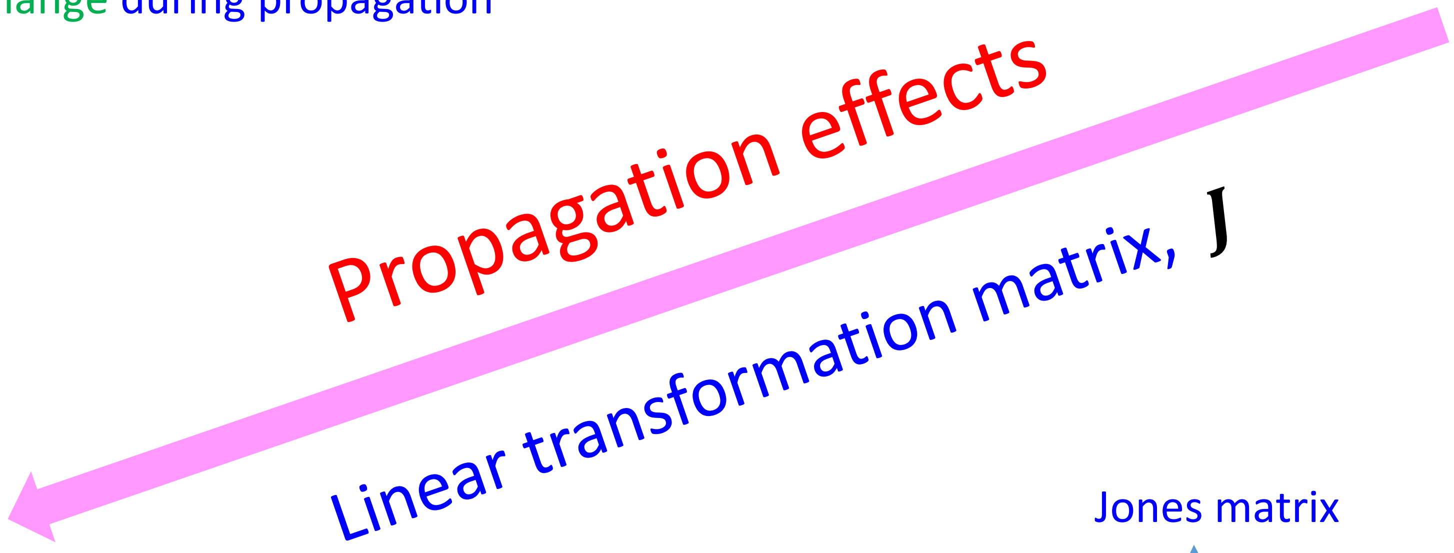


Propagation effects present

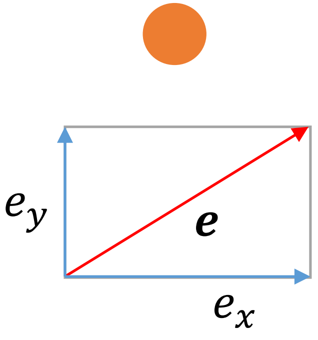
Amplitude and direction of electric vector change during propagation



Antenna



Source

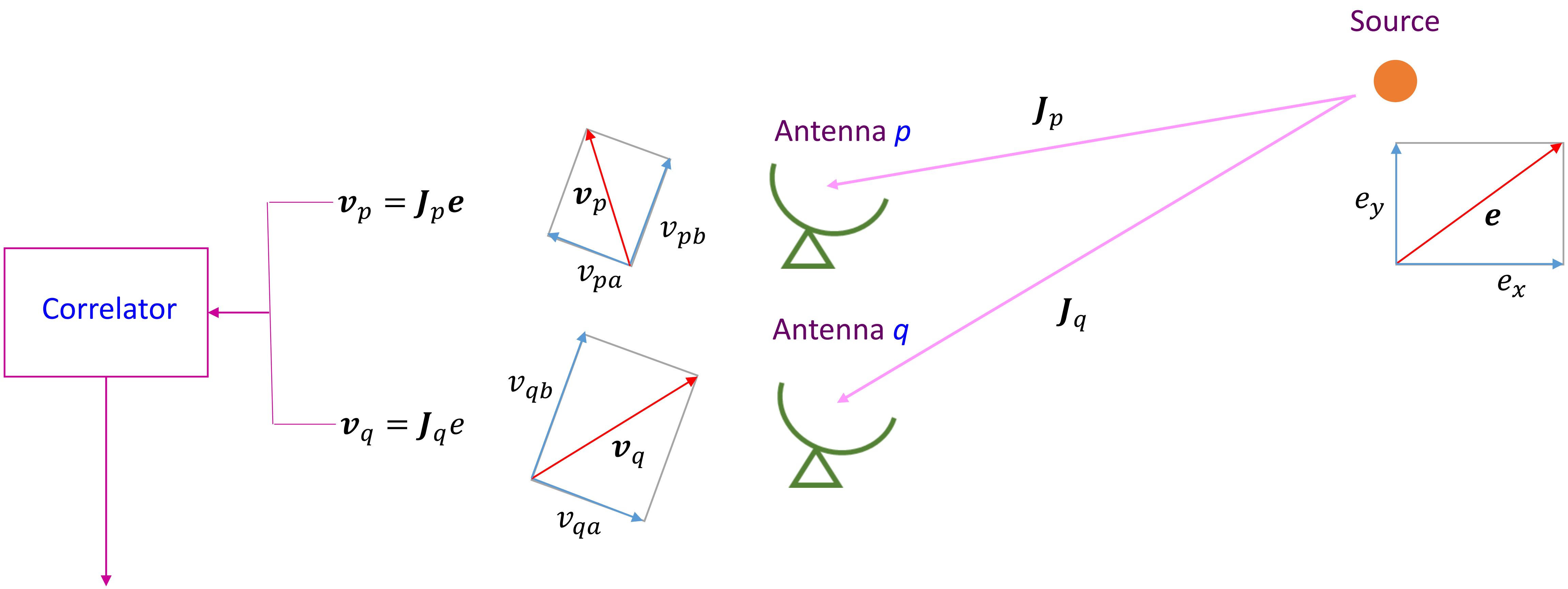


Jones matrix

Voltage vector ← $\boldsymbol{v} = \boldsymbol{J}\boldsymbol{e}$ → Electric field vector

$$\begin{pmatrix} v_a \\ v_b \end{pmatrix} = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix} \begin{pmatrix} e_x \\ e_y \end{pmatrix}$$

Correlation



$$\mathbf{V}_{pq} = 2\langle \boldsymbol{v}_p \boldsymbol{v}_q^H \rangle$$

Visibility

Visibility

- The correlator computes the visibility, $\mathbf{V}_{pq}$, on the baseline pq:

$$\mathbf{V}_{pq} = 2\underbrace{\langle \overbrace{\boldsymbol{v}_p \boldsymbol{v}_q^H}^{\text{Outer product}} \rangle}_{\text{Average}}$$

Hermitian conjugate

$$= 2\left\langle \begin{pmatrix} v_{pa} \\ v_{pb} \end{pmatrix} \begin{pmatrix} v_{qa}^* & v_{qb}^* \end{pmatrix} \right\rangle$$

These 4 quantities are the outputs from the correlator

$$= 2 \overbrace{\begin{pmatrix} \langle v_{pa} v_{qa}^* \rangle & \langle v_{pa} v_{qb}^* \rangle \\ \langle v_{pb} v_{qa}^* \rangle & \langle v_{pb} v_{qb}^* \rangle \end{pmatrix}}$$

Correlation

Voltages: $\boldsymbol{v}_p = \boldsymbol{J}_p\boldsymbol{e}$, $\boldsymbol{v}_q = \boldsymbol{J}_q\boldsymbol{e}$

Visibility: $\mathbf{V}_{pq} = 2\langle \boldsymbol{v}_p\boldsymbol{v}_q^H \rangle$

$$= 2\langle(\boldsymbol{J}_p\boldsymbol{e})(\boldsymbol{J}_q\boldsymbol{e})^H\rangle$$

$$= 2\langle\boldsymbol{J}_p(\boldsymbol{e}\boldsymbol{e}^H)\boldsymbol{J}_q^H\rangle$$

$$= \boldsymbol{J}_p\langle 2\boldsymbol{e}\boldsymbol{e}^H\rangle\boldsymbol{J}_q^H$$

Coherency, or Brightness

$$\mathbf{V}_{pq} = \boldsymbol{J}_p \langle 2\boldsymbol{e}\boldsymbol{e}^H \rangle \boldsymbol{J}_q^H$$

By definition, the coherency, or brightness, B, is given by:

$$\mathbf{B} = \langle 2\boldsymbol{e}\boldsymbol{e}^H \rangle = \begin{pmatrix} I+Q & U+iV \\ U-iV & I-Q \end{pmatrix}$$

$\langle \boldsymbol{e}\boldsymbol{e}^H \rangle$ is the coherence of the electromagnetic field with itself, and is described by the Stokes parameters I, Q, U, V

$$\mathbf{V}_{pq} = \boldsymbol{J}_p \, \mathbf{B} \, \boldsymbol{J}_q^H$$

Radio-Interferometric Measurement Equation (RIME)

Visibility

Brightness

$$\mathbf{V}_{pq} = \boldsymbol{J}_p \, \mathbf{B} \, \boldsymbol{J}_q^H$$

Jones matrices

$$\begin{pmatrix} v_{aa} & v_{ab} \\ v_{ba} & v_{bb} \end{pmatrix} = \begin{pmatrix} j_{11a} & j_{12a} \\ j_{21a} & j_{22a} \end{pmatrix} \begin{pmatrix} I + Q & U + iV \\ U - iV & I - Q \end{pmatrix} \begin{pmatrix} j_{11b}^* & j_{21b}^* \\ j_{12b}^* & j_{22b}^* \end{pmatrix}$$

Component Jones matrices

The Jones matrix for an antenna is a product of several component Jones matrices, corresponding to different corrupting effects along the signal path

Example:

$$\boldsymbol{J} = \boldsymbol{B}\ \boldsymbol{G}\ \boldsymbol{D}\ \boldsymbol{E}\ \boldsymbol{P}\ \boldsymbol{T}$$

Bandpass gain

Instrumental gain

Polarization leakage

Primary beam

Parallactic angle feed rotation

Ionospheric and tropospheric effects

Component Jones matrices

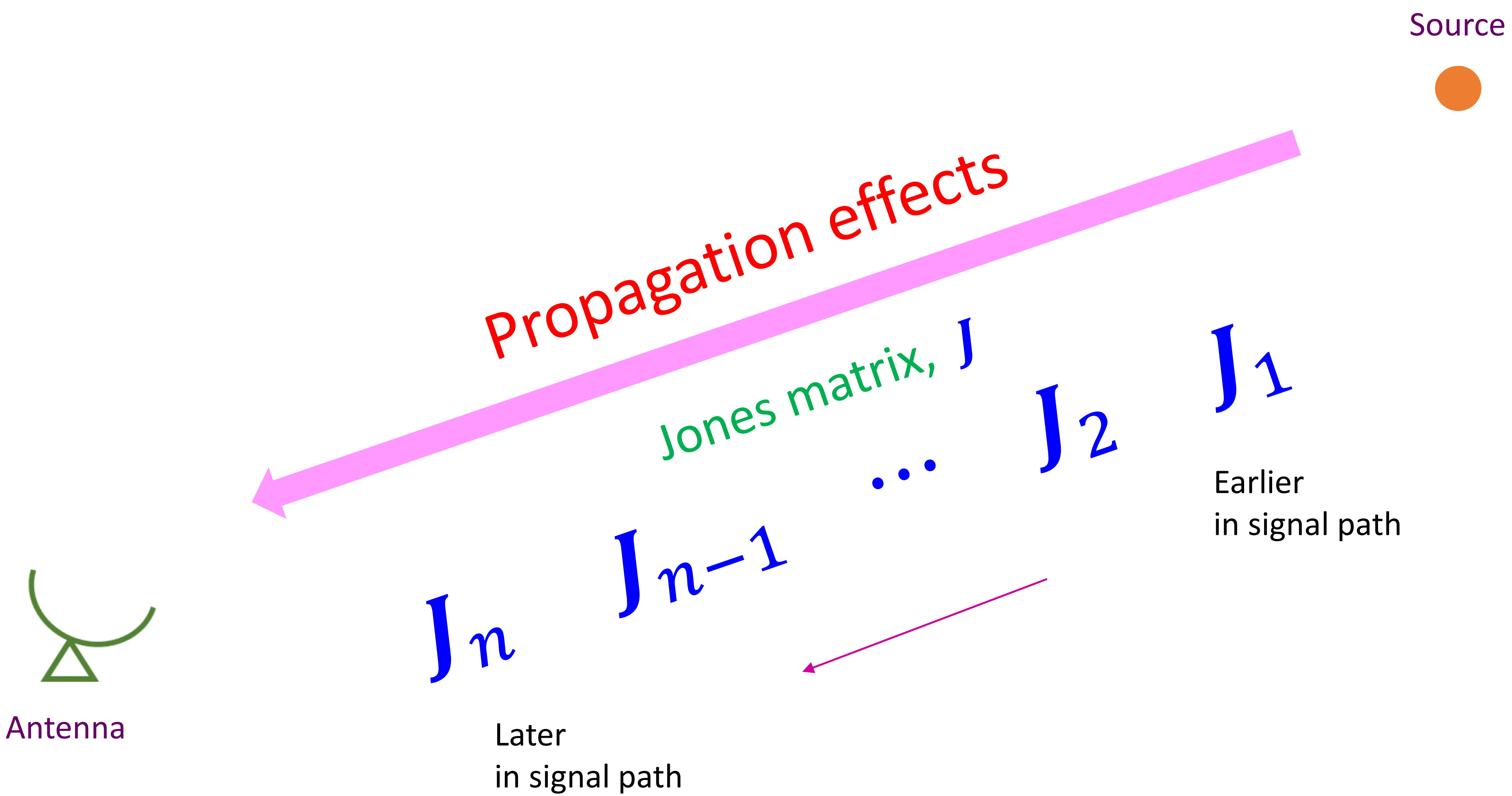
Jones chain:

$$\boldsymbol{J} = \boldsymbol{J}_n \, \boldsymbol{J}_{n-1} \cdots \boldsymbol{J}_2 \, \boldsymbol{J}_1$$

Later
in signal path

Earlier
in signal path

Component Jones matrices



Component Jones matrices

Antenna p : $\boldsymbol{J}_p = \boldsymbol{J}_{pn}\, \boldsymbol{J}_{p(n-1)} \cdots \boldsymbol{J}_{p2}\, \boldsymbol{J}_{p1}$

Antenna q : $\boldsymbol{J}_q = \boldsymbol{J}_{qn}\, \boldsymbol{J}_{q(n-1)} \cdots \boldsymbol{J}_{q2}\, \boldsymbol{J}_{q1}$

Visibility Brightness

$$\mathbf{V}_{pq} = \boldsymbol{J}_p\, \mathbf{B}\, \boldsymbol{J}_q^H$$

Jones matrices

$$\mathbf{V}_{pq} = \boldsymbol{J}_{pn}\, \boldsymbol{J}_{p(n-1)} \cdots \boldsymbol{J}_{p2}\, \boldsymbol{J}_{p1}\, \mathbf{B}\, \boldsymbol{J}_{q1}^H\, \boldsymbol{J}_{q2}^H \cdots \boldsymbol{J}_{q(n-1)}^H\, \boldsymbol{J}_{qn}^H$$

$$\mathbf{V}_{pq} = \boldsymbol{J}_{pn}\Big(\boldsymbol{J}_{p(n-1)}\Big(\cdots\Big(\boldsymbol{J}_{p2}\Big(\boldsymbol{J}_{p1}\, \mathbf{B}\, \boldsymbol{J}_{q1}^H\Big)\boldsymbol{J}_{q2}^H\Big)\cdots\Big)\boldsymbol{J}_{q(n-1)}^H\Big)\boldsymbol{J}_{qn}^H$$

Direction-independent and direction-dependent effects

Propagation effects can be of two kinds:

- Direction-independent effects
- Direction-dependent effects

These effects can be represented by different Jones matrices:

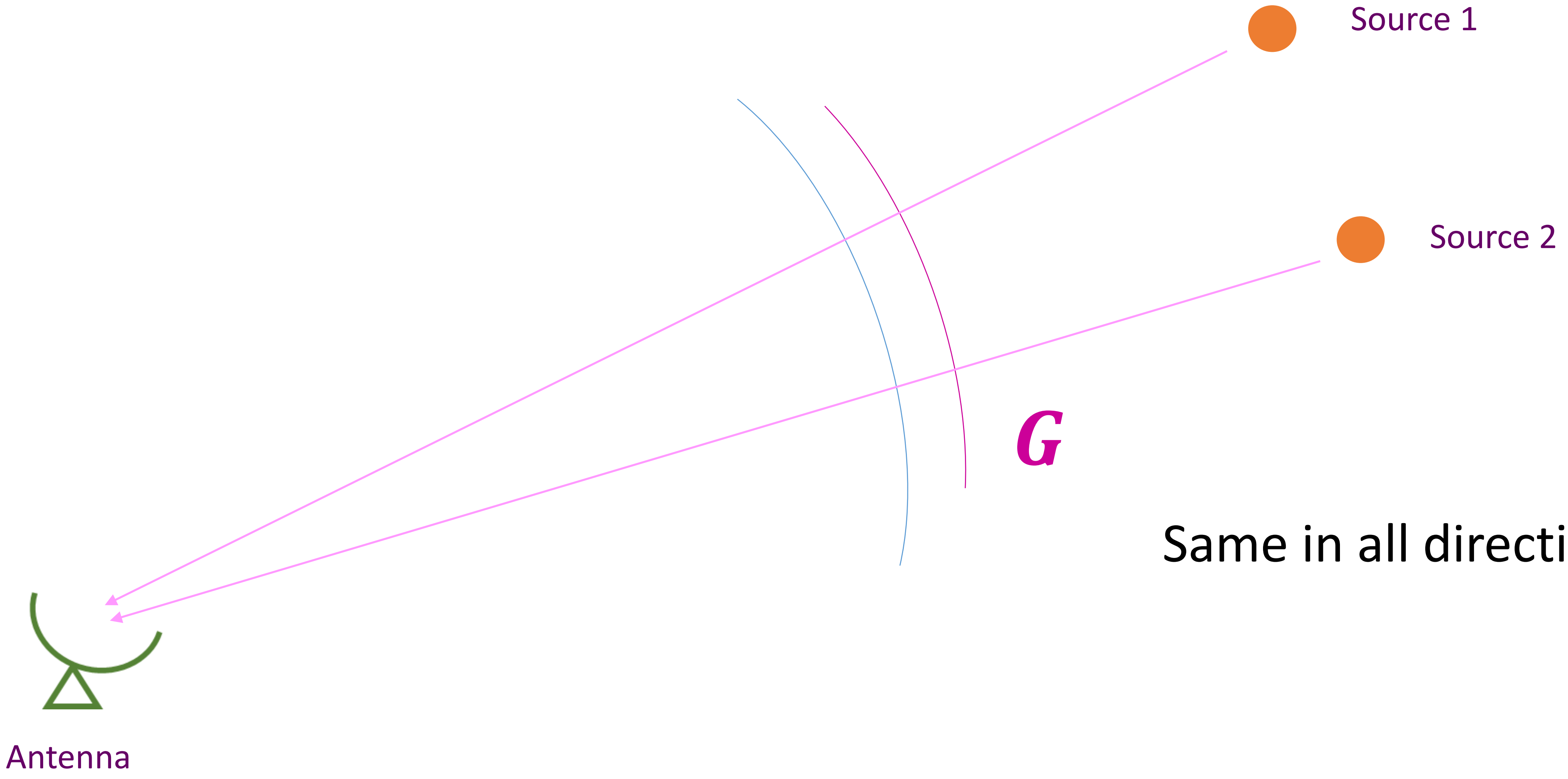
$$\boldsymbol{J} = \boldsymbol{G}\ \boldsymbol{E}$$

Final Jones matrix

Direction-independent effects

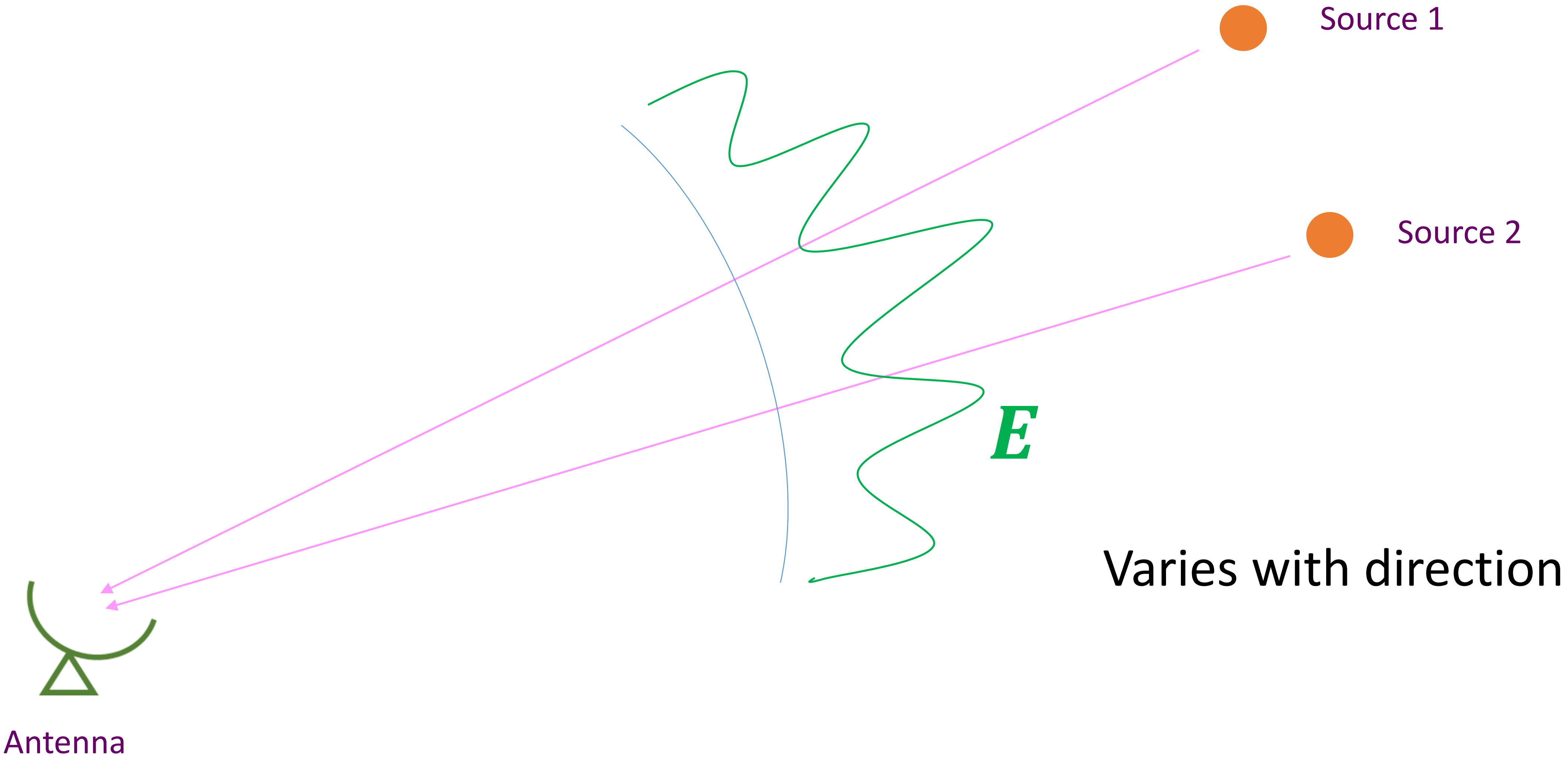
Direction-dependent effects

Direction-independent effects



Same in all directions

Direction-dependent effects



Direction-independent and direction-dependent effects

$$\boldsymbol{J} = \boldsymbol{G}\ \boldsymbol{E}$$

Final Jones matrix

Direction-independent effects

Direction-dependent effects

Visibility-plane effects

Image-plane effects

$$\mathbf{V}_{pq} = \boldsymbol{J}_p\ \mathbf{B}\ \boldsymbol{J}_q^H$$

$$\mathbf{V}_{pq} = \boldsymbol{G}_p\left(\boldsymbol{E}_p\mathbf{B}\boldsymbol{E}_q^H\right)\boldsymbol{G}_q^H$$

Direction-independent and direction-dependent effects

Example:

$$\boldsymbol{J} = \boldsymbol{B}\ \boldsymbol{G}\ \boldsymbol{D}\ \boldsymbol{E}\ \boldsymbol{P}\ \boldsymbol{T}$$

Bandpass gain

Instrumental gain

Polarization leakage

Primary beam

Parallactic angle feed rotation

Ionospheric and tropospheric effects

Direction-dependent effects

Direction-independent effects

Explicit RIME and phenomenological RIME

- Explicit RIME:

$$\boldsymbol{J} = \boldsymbol{B} \;\; \boldsymbol{G} \;\; \boldsymbol{D} \;\; \boldsymbol{E} \;\; \boldsymbol{P} \;\; \boldsymbol{T}$$

Useful for understanding the component corrupting effects along the propagation path

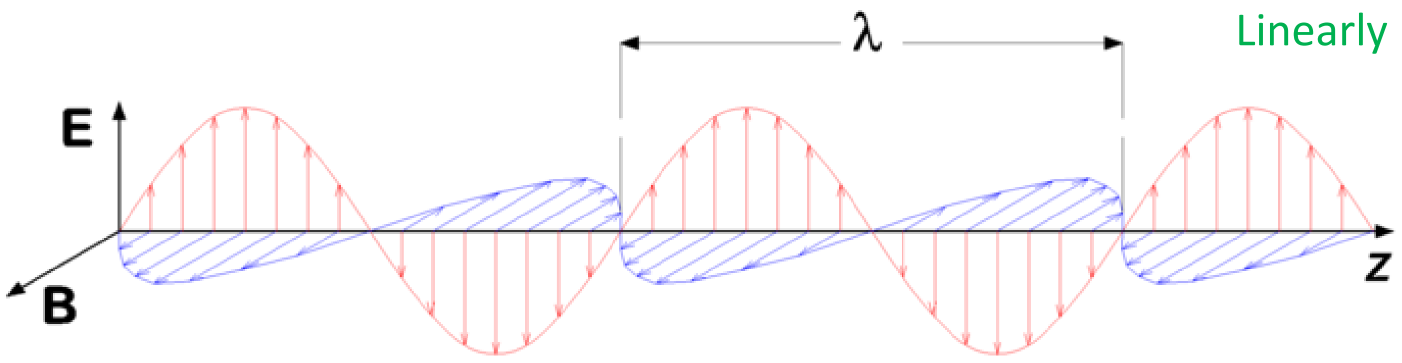
- Phenomenological RIME:

$$\boldsymbol{J} = \boldsymbol{G}\,\boldsymbol{E}$$

Useful for calibration, as these matrices are easier to solve for

Polarization

- Polarization of an electromagnetic wave describes the direction of oscillation of the electric field.



Linearly polarized wave

(From https://commons.wikimedia.org/wiki/File:Electromagnetic_wave.png, author: User:P.wormer)

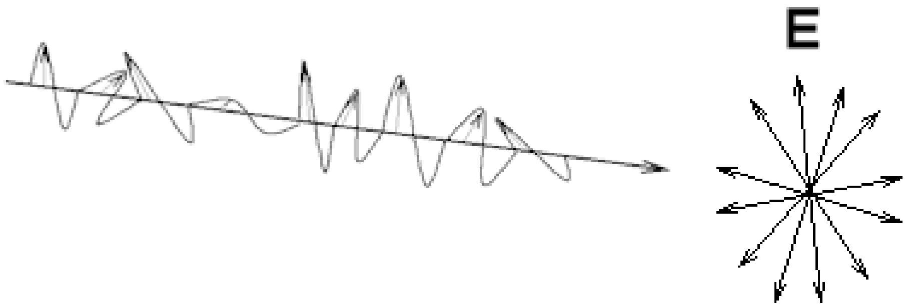
- The plane of polarization is perpendicular to the direction of propagation of the wave.

Polarization

- Light can be:
  - Unpolarized
  - Polarized
  - Partially polarized

Unpolarized light

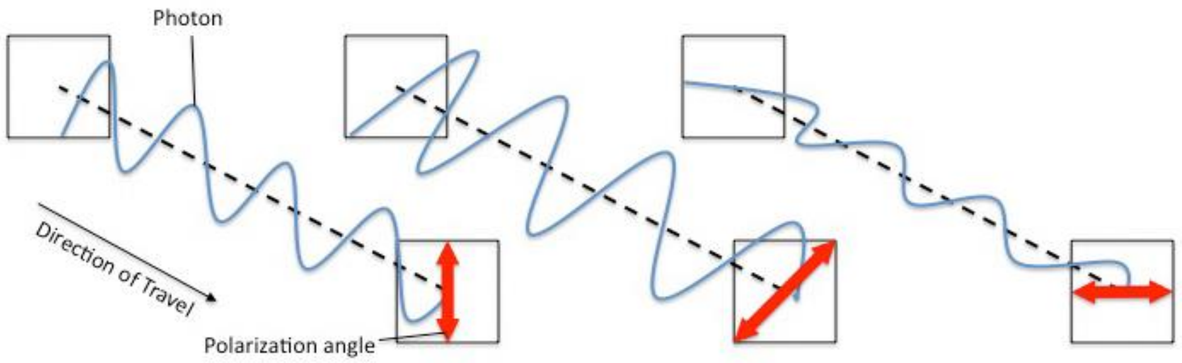
- Electric vector oscillates in random directions in the plane of polarization.
- Most naturally-occurring radiation is unpolarized.
- Also called incoherent radiation.
- Example: Thermal radiation.



(From http://labman.phys.utk.edu/phys222core/modules/m6/polarization.htm)

Polarized light

- Electric vector oscillates in a well-defined manner.
- Also called coherent radiation.
- Example: Light from a laser.



(From https://arthropoda.files.wordpress.com/2009/11/pol1.jpg)

Partially polarized light

- Electric vector oscillates randomly, but there is more power in a preferred polarization mode.
- Partially polarized light can be expressed as a superposition of its polarized and unpolarized parts:

$$I = P + U$$

Total power

Power in polarized component

Power in unpolarized component

- Fractional polarization is defined as the ratio of the power in the polarized component to the total power:

$$p = \frac{P}{I}$$

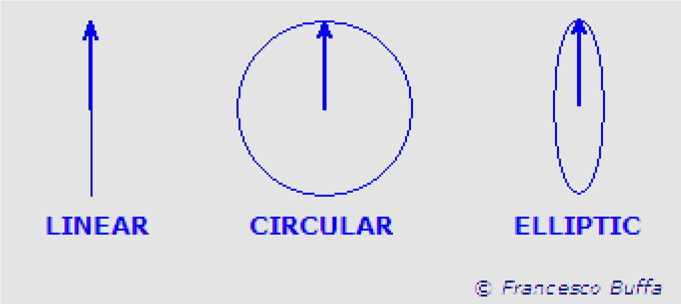
Polarization

- Polarized electromagnetic radiation can be:

  - Linearly polarized

  - Circularly polarized

  - Elliptically polarized

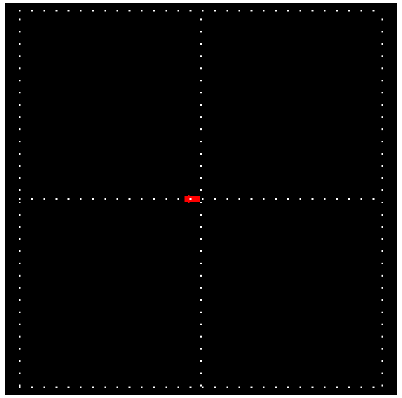


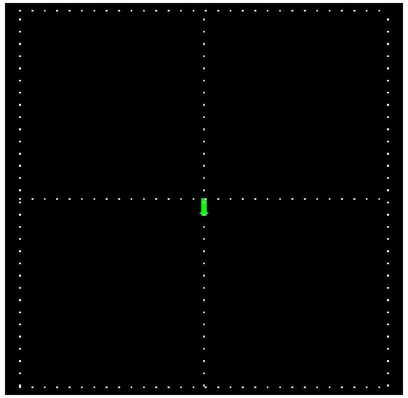
Antenna feeds

An antenna contains feeds, which detect and measure specific polarized components of an electromagnetic wave:

- Antenna with orthogonal linearly polarized feeds:

x-polarized feed

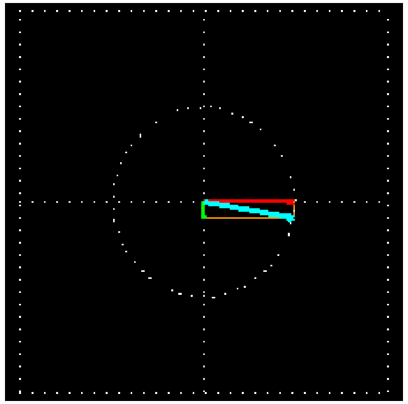


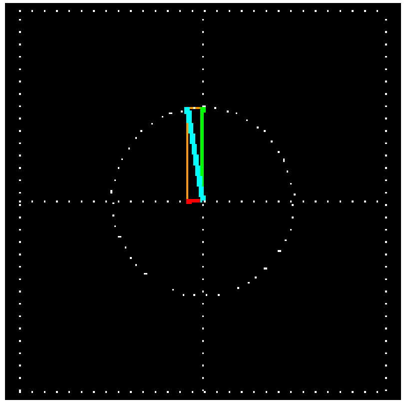


y-polarized feed

- Antenna with orthogonal circularly polarized feeds:

Right-circularly polarized (RCP) feed



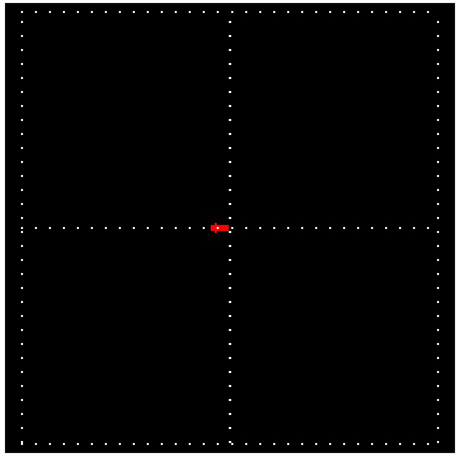


Left-circularly polarized (LCP) feed

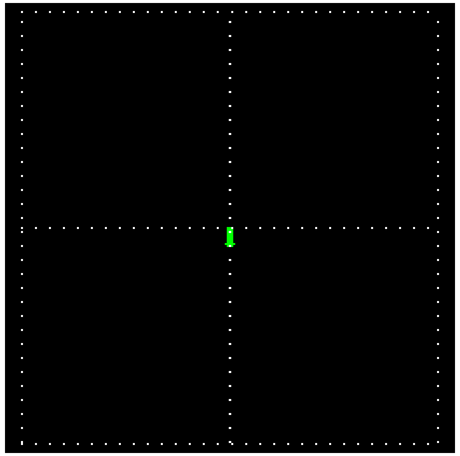
(From http://cddemo.szialab.org/)

Antenna feeds

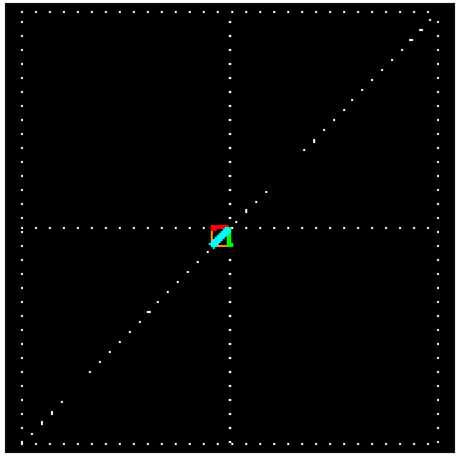
- Linearly polarized wave, measured by linearly polarized feeds:



x component



y component

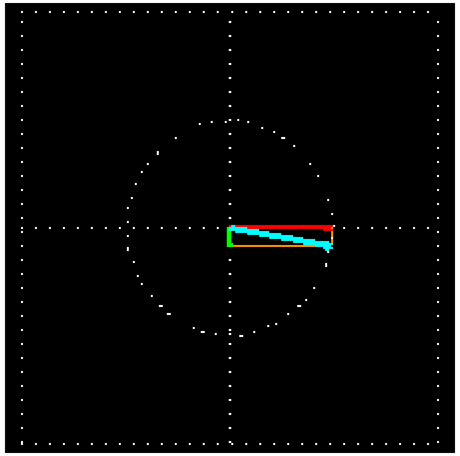


Combination of
x and y components

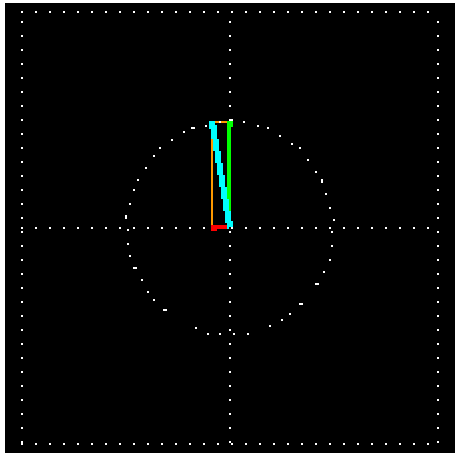
(From http://cddemo.szialab.org/)

Antenna feeds

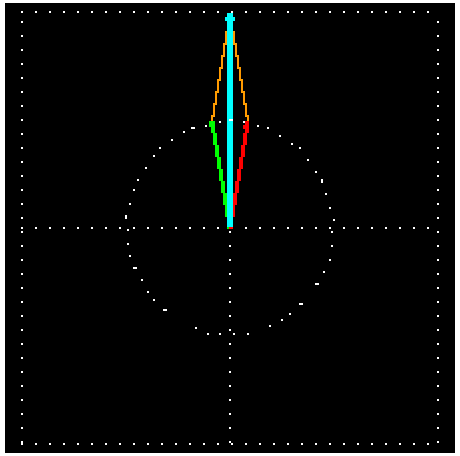
- Linearly polarized wave, measured by circularly polarized feeds:



RCP component



LCP component



Combination of
RCP and LCP components

(From http://cddemo.szialab.org/)

Structure of Jones matrices

Most Jones matrices have a simple form:

- $J_{\text{rotation}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

- $J_{\text{leakage}} = \begin{bmatrix} 1 & D_X \\ D_Y & 1 \end{bmatrix}$

- $J_{\text{gain}} = \begin{bmatrix} G_X & 0 \\ 0 & G_Y \end{bmatrix}$

(in a linearly polarized basis)

Structure of Jones matrices

- The structure of individual Jones matrices depends on the antenna feed polarization basis, but the RIME is valid independent of the polarization basis.
- For example,
  - $\boldsymbol{J}_{\text{rotation}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ in a linearly polarized basis
  - $\boldsymbol{J}_{\text{rotation}} = \begin{bmatrix} e^{j\theta} & 0 \\ 0 & e^{-j\theta} \end{bmatrix}$ in a circularly polarized basis

Rotation matrices

- $J_{\text{rotation}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Examples:

Parallactic angle

Parallactic angle feed rotation: $\boldsymbol{P} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

Ionospheric Faraday rotation: $\boldsymbol{T} = \begin{bmatrix} \cos\chi & -\sin\chi \\ \sin\chi & \cos\chi \end{bmatrix}$

Faraday rotation angle

Leakage matrices

- $J_{\text{leakage}} = \begin{bmatrix} 1 & D_X \\ D_Y & 1 \end{bmatrix}$

Examples:

Polarization leakage: $\boldsymbol{D} = \begin{bmatrix} 1 & d_x \\ d_y & 1 \end{bmatrix}$

Polarization leakage terms

Gain matrices

- $\boldsymbol{J}_{\text{gain}} = \begin{bmatrix} G_X & 0 \\ 0 & G_Y \end{bmatrix}$

Examples:

Instrumental gain: $\boldsymbol{G} = \begin{bmatrix} g_x & 0 \\ 0 & g_y \end{bmatrix} = \begin{bmatrix} a_x e^{j\phi_x} & 0 \\ 0 & a_y e^{j\phi_y} \end{bmatrix}$

Bandpass gain: $\boldsymbol{B} = \begin{bmatrix} B_x & 0 \\ 0 & B_y \end{bmatrix} = \begin{bmatrix} b_x(\nu) e^{j\psi_x(\nu)} & 0 \\ 0 & b_y(\nu) e^{j\psi_y(\nu)} \end{bmatrix}$

Order of Jones matrices in Jones chain

$$\boldsymbol{J} = \boldsymbol{J}_n \, \boldsymbol{J}_{n-1} \cdots \boldsymbol{J}_2 \, \boldsymbol{J}_1$$

- Order of matrices in the Jones chain is important, because matrix multiplication is not commutative, in general.
- However, specific kinds of matrices do commute – scalar matrices commute with all kinds of matrices, rotation matrices with each other, diagonal matrices with each other.

References

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