Radio-Interferometric Measurement Equation

Introductory Radio Interferometry Course

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Radio-Interferometric Measurement Equation (RIME)

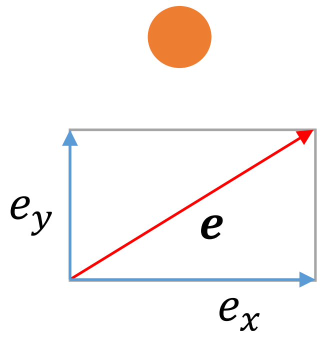
- Compact, intuitive, matrix-based way of representing propagation effects in radio interferometry.
- Useful for calibration (solving for and correcting these propagation effects).

Introduction

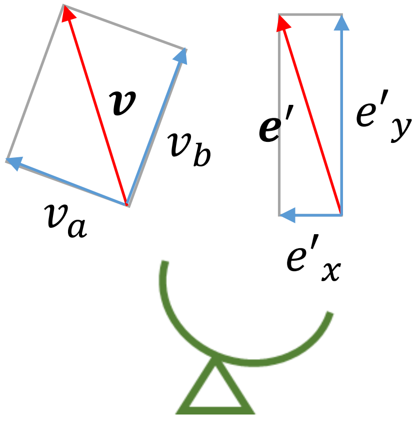
e'_x, e'_y : Components of electric field vector in reference frame of sky, at the observer

v_a, v_b : Voltages measured by antenna feed (linearly or circularly polarized)

Source



Propagation effects



Antenna

e_x, e_y : Components of electric field vector in reference frame of sky, at the source

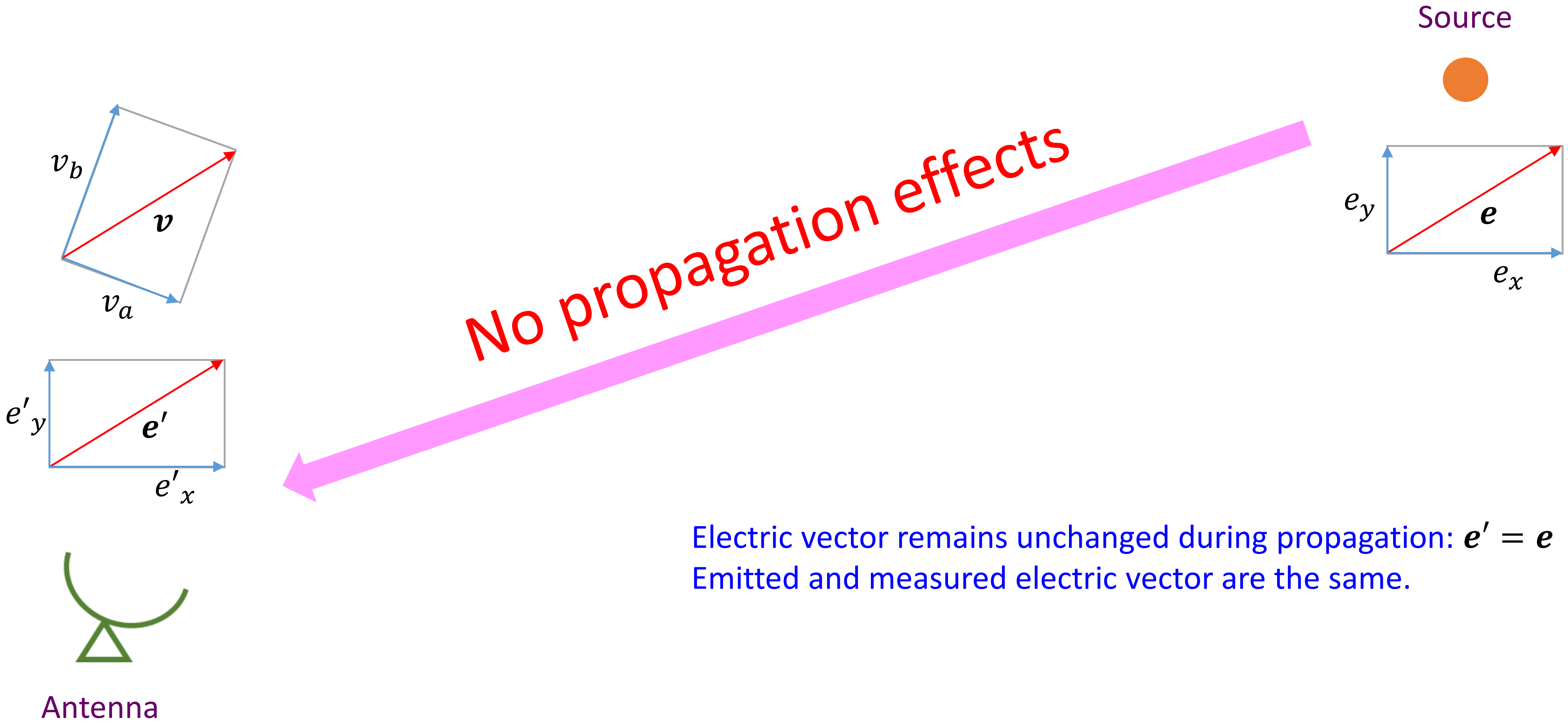
Can be represented as vectors:

$$\boldsymbol{e} = \begin{pmatrix} e_x \\ e_y \end{pmatrix}$$

$$\boldsymbol{e}' = \begin{pmatrix} e'_x \\ e'_y \end{pmatrix}$$

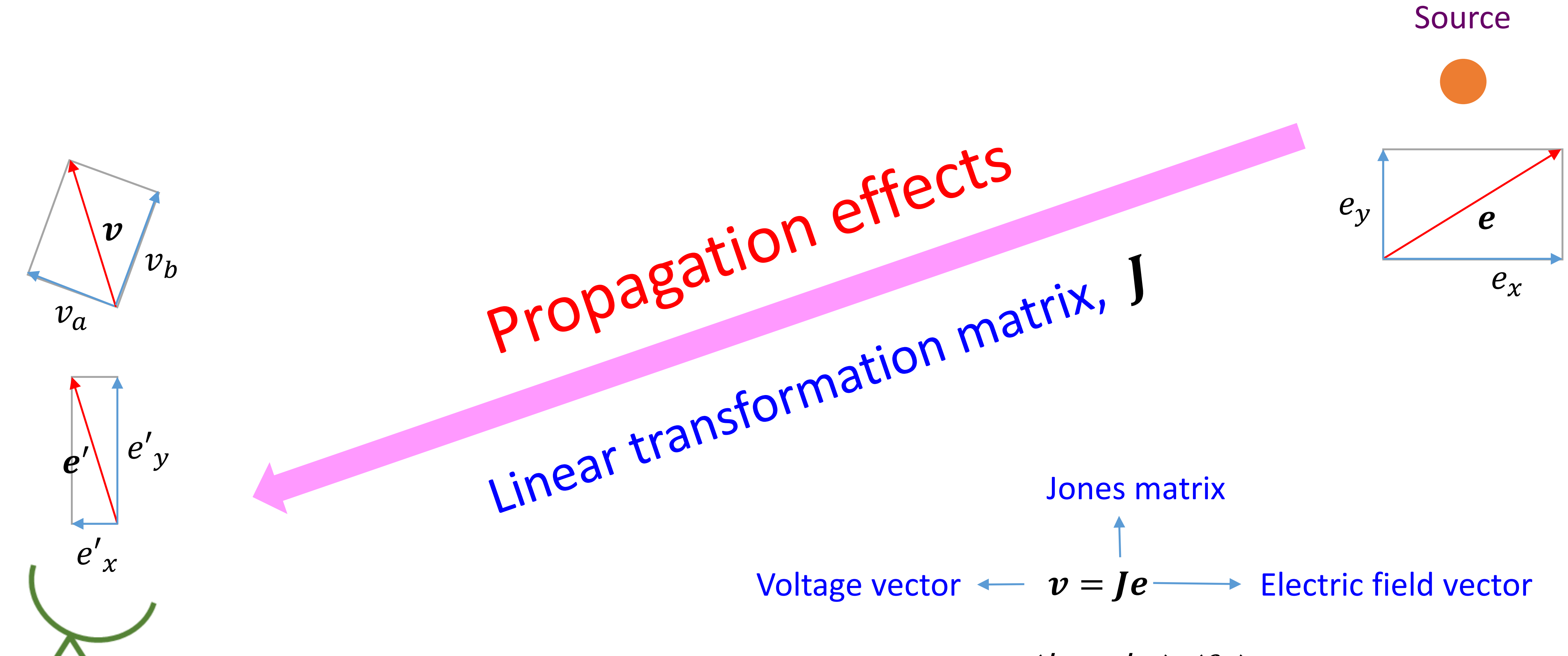
$$\boldsymbol{v} = \begin{pmatrix} v_a \\ v_b \end{pmatrix}$$

Propagation effects absent



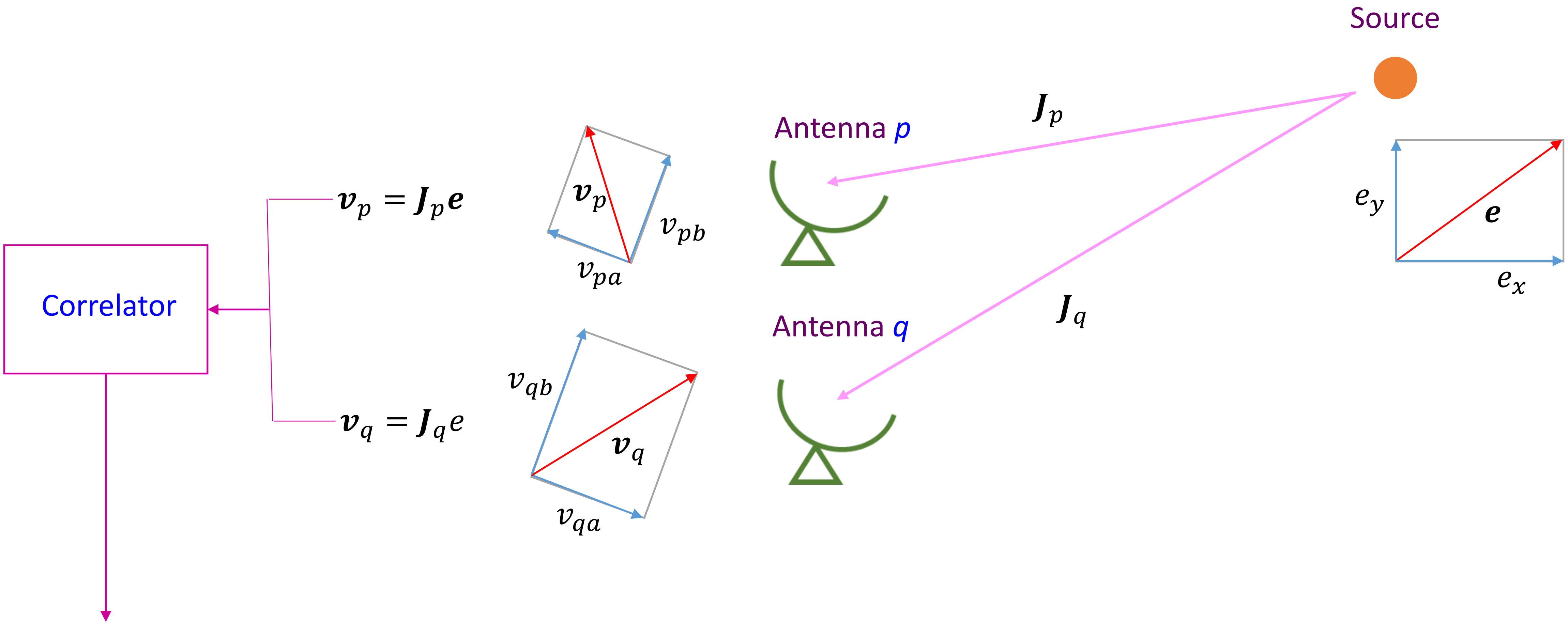
Electric vector remains unchanged during propagation: $\boldsymbol{e}' = \boldsymbol{e}$
Emitted and measured electric vector are the same.

Propagation effects present



$$\begin{pmatrix} v_a \\ v_b \end{pmatrix} = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix} \begin{pmatrix} e_x \\ e_y \end{pmatrix}$$

Correlation



$$\mathbf{V}_{pq} = 2\langle \boldsymbol{v}_p \boldsymbol{v}_q^H \rangle$$

Visibility

- The correlator computes the visibility, $\mathbf{V}_{pq}$, on the baseline pq:

Hermitian conjugate

$$\mathbf{V}_{pq} = 2\underbrace{\langle \overbrace{\boldsymbol{v}_p \boldsymbol{v}_q^H}^{\text{Outer product}} \rangle}_{\text{Average}}$$

These 4 quantities are the outputs from the correlator

$$= 2\left\langle \begin{pmatrix} v_{pa} \\ v_{pb} \end{pmatrix} \begin{pmatrix} v_{qa}^* & v_{qb}^* \end{pmatrix} \right\rangle$$

$$= 2 \overbrace{\begin{pmatrix} \langle v_{pa} v_{qa}^* \rangle & \langle v_{pa} v_{qb}^* \rangle \\ \langle v_{pb} v_{qa}^* \rangle & \langle v_{pb} v_{qb}^* \rangle \end{pmatrix}}$$

Correlation

$$\boldsymbol{v}_p = \boldsymbol{J}_p\boldsymbol{e} \quad , \ \boldsymbol{v}_q = \boldsymbol{J}_q\boldsymbol{e}$$

$$\mathbf{V}_{pq} = 2\langle\, \boldsymbol{v}_p\boldsymbol{v}_q^H \,\rangle$$

$$= 2\langle(\boldsymbol{J}_p\boldsymbol{e})(\boldsymbol{J}_q\boldsymbol{e})^H\rangle$$

$$= 2\langle\boldsymbol{J}_p(\boldsymbol{e}\boldsymbol{e}^H)\boldsymbol{J}_q^H\rangle$$

$$= \langle\boldsymbol{J}_p(2\boldsymbol{e}\boldsymbol{e}^H)\boldsymbol{J}_q^H\rangle$$

Coherency, or Brightness

$$\mathbf{V}_{pq} = \langle \boldsymbol{J}_p (2\boldsymbol{e}\boldsymbol{e}^H) \boldsymbol{J}_q^H \rangle$$

By definition, the coherency, or brightness, B, is given by:

$$\mathbf{B} = \langle 2\boldsymbol{e}\boldsymbol{e}^H \rangle = \begin{pmatrix} I+Q & U+iV \\ U-iV & I-Q \end{pmatrix}$$

I, Q, U, V: Stokes parameters

$$\mathbf{V}_{pq} = \boldsymbol{J}_p \, \mathbf{B} \, \boldsymbol{J}_q^H$$

Radio-Interferometric Measurement Equation (RIME)

Visibility

Brightness

$$\mathbf{V}_{pq} = \boldsymbol{J}_p \, \mathbf{B} \, \boldsymbol{J}_q^H$$

Jones matrices

$$\begin{pmatrix} v_{aa} & v_{ab} \\ v_{ba} & v_{bb} \end{pmatrix} = \begin{pmatrix} j_{11a} & j_{12a} \\ j_{21a} & j_{22a} \end{pmatrix} \begin{pmatrix} I+Q & U+iV \\ U-iV & I-Q \end{pmatrix} \begin{pmatrix} j_{11b} & j_{12b} \\ j_{21b} & j_{22b} \end{pmatrix}^H$$

Component Jones matrices

The Jones matrix for an antenna is a product of several component Jones matrices, corresponding to different corrupting effects along the signal path

$$\boldsymbol{J} = \boldsymbol{B}\ \boldsymbol{G}\ \boldsymbol{D}\ \boldsymbol{E}\ \boldsymbol{P}\ \boldsymbol{T}$$

Bandpass gain

Instrumental gain

Polarization leakage

Primary beam

Parallactic angle feed rotation

Ionospheric Faraday rotation

Component Jones matrices

Jones chain:

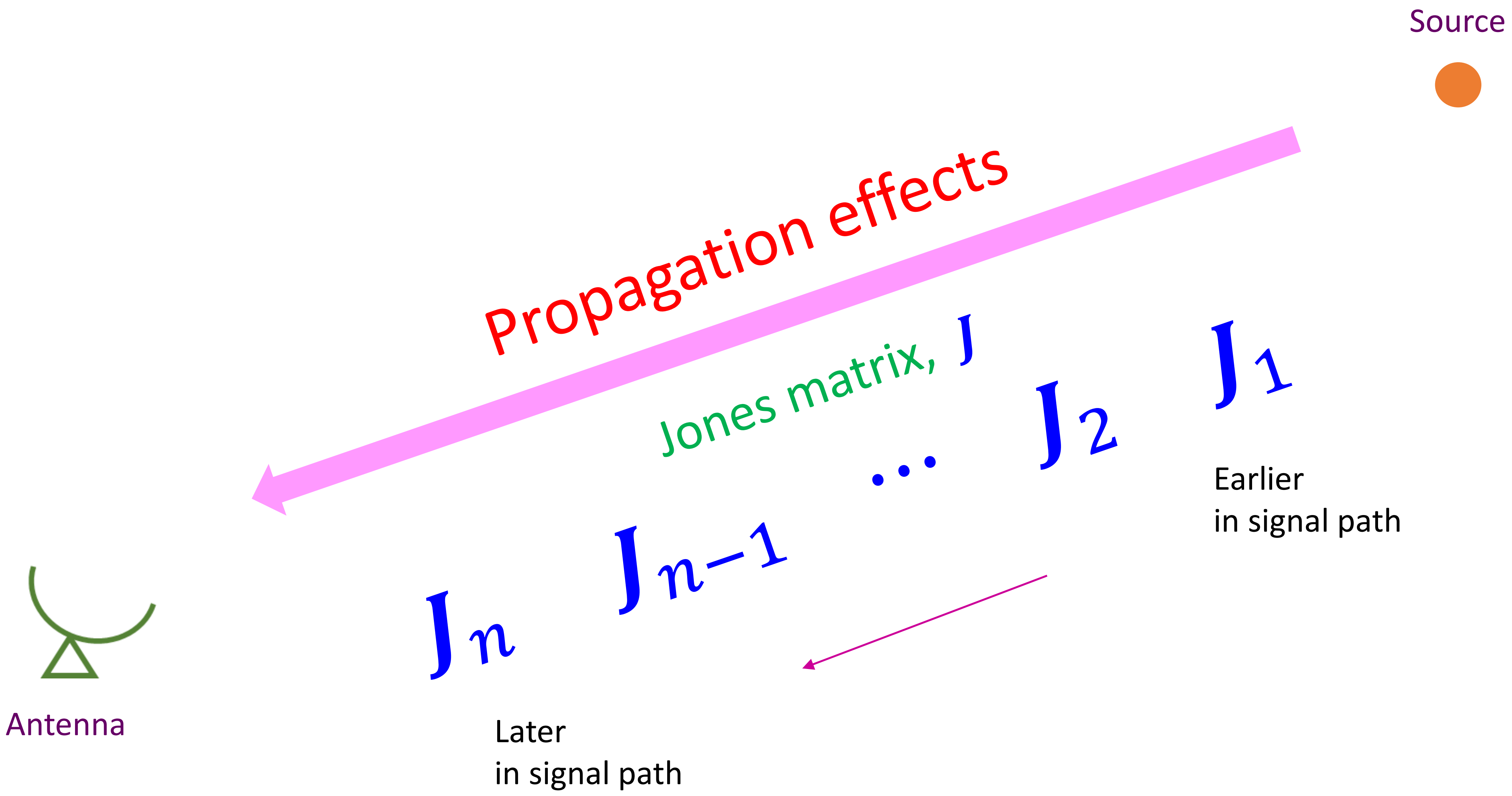
$$\boldsymbol{J} = \boldsymbol{J}_n \, \boldsymbol{J}_{n-1} \cdots \boldsymbol{J}_2 \, \boldsymbol{J}_1$$

Later
in signal path

$\longleftarrow$

Earlier
in signal path

Component Jones matrices



Component Jones matrices

Antenna p : $\quad \boldsymbol{J}_p = \boldsymbol{J}_{pn}\, \boldsymbol{J}_{p(n-1)} \cdots \boldsymbol{J}_{p2}\, \boldsymbol{J}_{p1}$

Antenna q : $\quad \boldsymbol{J}_q = \boldsymbol{J}_{qn}\, \boldsymbol{J}_{q(n-1)} \cdots \boldsymbol{J}_{q2}\, \boldsymbol{J}_{q1}$

Visibility

Brightness

$$\mathbf{V}_{pq} = \boldsymbol{J}_p\, \mathbf{B}\, \boldsymbol{J}_q^H$$

Jones matrices

$$\mathbf{V}_{pq} = \boldsymbol{J}_{pn}\, \boldsymbol{J}_{p(n-1)} \cdots \boldsymbol{J}_{p2}\, \boldsymbol{J}_{p1}\, \mathbf{B}\, \boldsymbol{J}_{q1}^H\, \boldsymbol{J}_{q2}^H \cdots \boldsymbol{J}_{q(n-1)}^H\, \boldsymbol{J}_{qn}^H$$

$$\mathbf{V}_{pq} = \boldsymbol{J}_{pn}\Big(\boldsymbol{J}_{p(n-1)}\Big(\cdots\Big(\boldsymbol{J}_{p2}\Big(\boldsymbol{J}_{p1}\, \mathbf{B}\, \boldsymbol{J}_{q1}^H\Big)\boldsymbol{J}_{q2}^H\Big)\cdots\Big)\boldsymbol{J}_{q(n-1)}^H\Big)\boldsymbol{J}_{qn}^H$$

Calibration

Calibration: Determining and correcting for propagation effects in order to compute the brightness.

i.e., solve for Jones matrices $\boldsymbol{J}$ to compute $\mathbf{B}$:

Visibility

Brightness

$$\mathbf{V}_{pq} = \boldsymbol{J}_p \; \mathbf{B} \; \boldsymbol{J}_q^H$$

Jones matrices

$$\mathbf{B} = \boldsymbol{J}_p^{-1} \mathbf{V}_{pq} \left(\boldsymbol{J}_q^H \right)^{-1}$$

Direction-independent and direction-dependent effects

Propagation effects can be of two kinds:

- Direction-independent effects
- Direction-dependent effects

These effects can be represented by different Jones matrices:

$$\boldsymbol{J} = \boldsymbol{G}\ \boldsymbol{E}$$

Final Jones matrix

Direction-independent effects

Direction-dependent effects

Direction-independent and direction-dependent effects

$$\boldsymbol{J} = \boldsymbol{G}\ \boldsymbol{E}$$

Final Jones matrix

Direction-independent effects

Direction-dependent effects

Visibility-plane effects

Image-plane effects

$$\mathbf{V}_{pq} = \boldsymbol{J}_p\ \mathbf{B}\ \boldsymbol{J}_q^H$$

$$\mathbf{V}_{pq} = \boldsymbol{G}_p\left(\boldsymbol{E}_p\mathbf{B}\boldsymbol{E}_q^H\right)\boldsymbol{G}_q^H$$

Direction-independent and direction-dependent effects

$$\boldsymbol{J} = \boldsymbol{B}\ \ \boldsymbol{G}\ \ \boldsymbol{D}\ \ \boldsymbol{E}\ \ \boldsymbol{P}\ \ \boldsymbol{T}$$

Bandpass gain

Instrumental gain

Polarization leakage

Primary beam

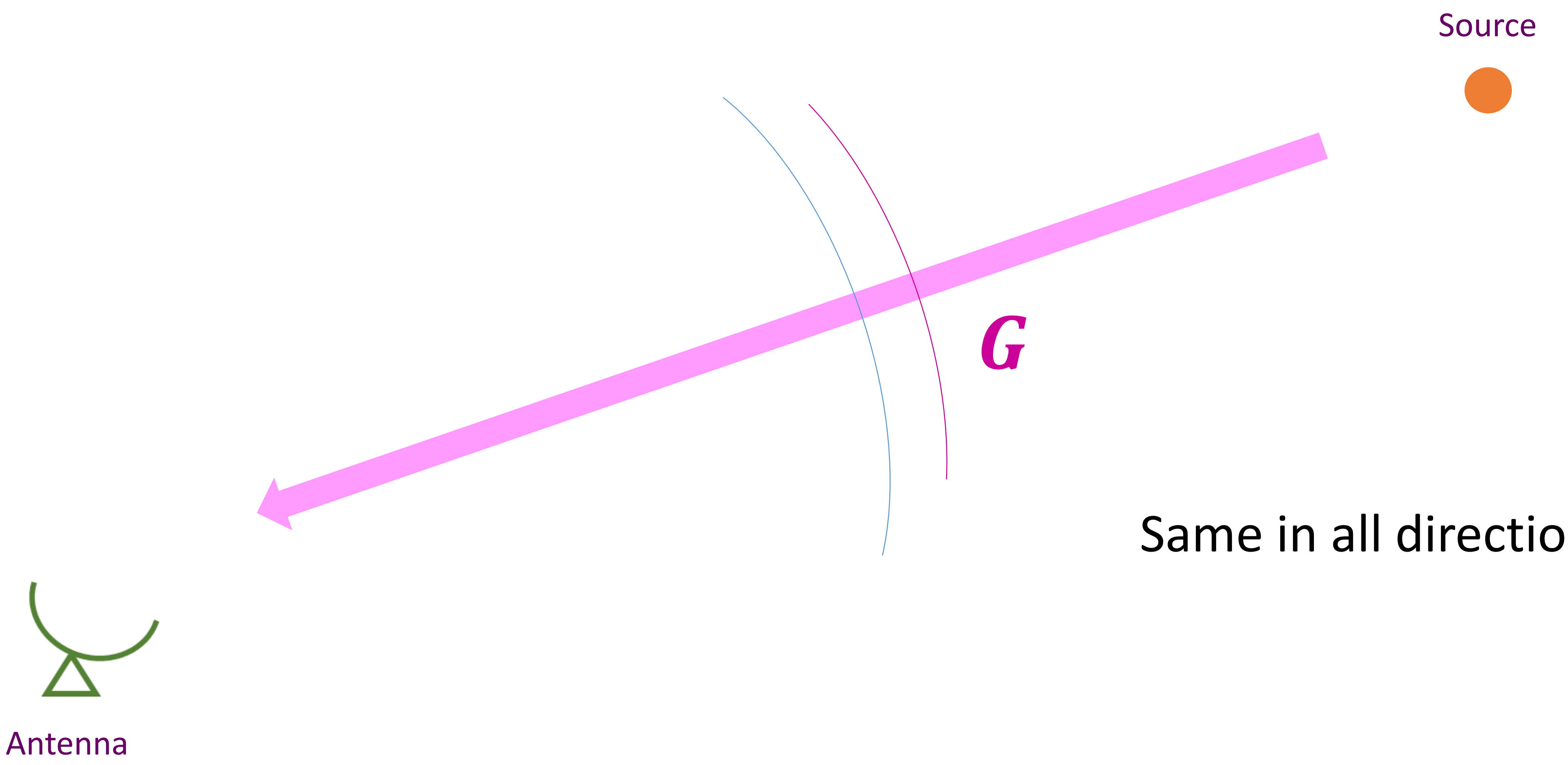
Parallactic angle feed rotation

Ionospheric Faraday rotation

Direction-dependent effect

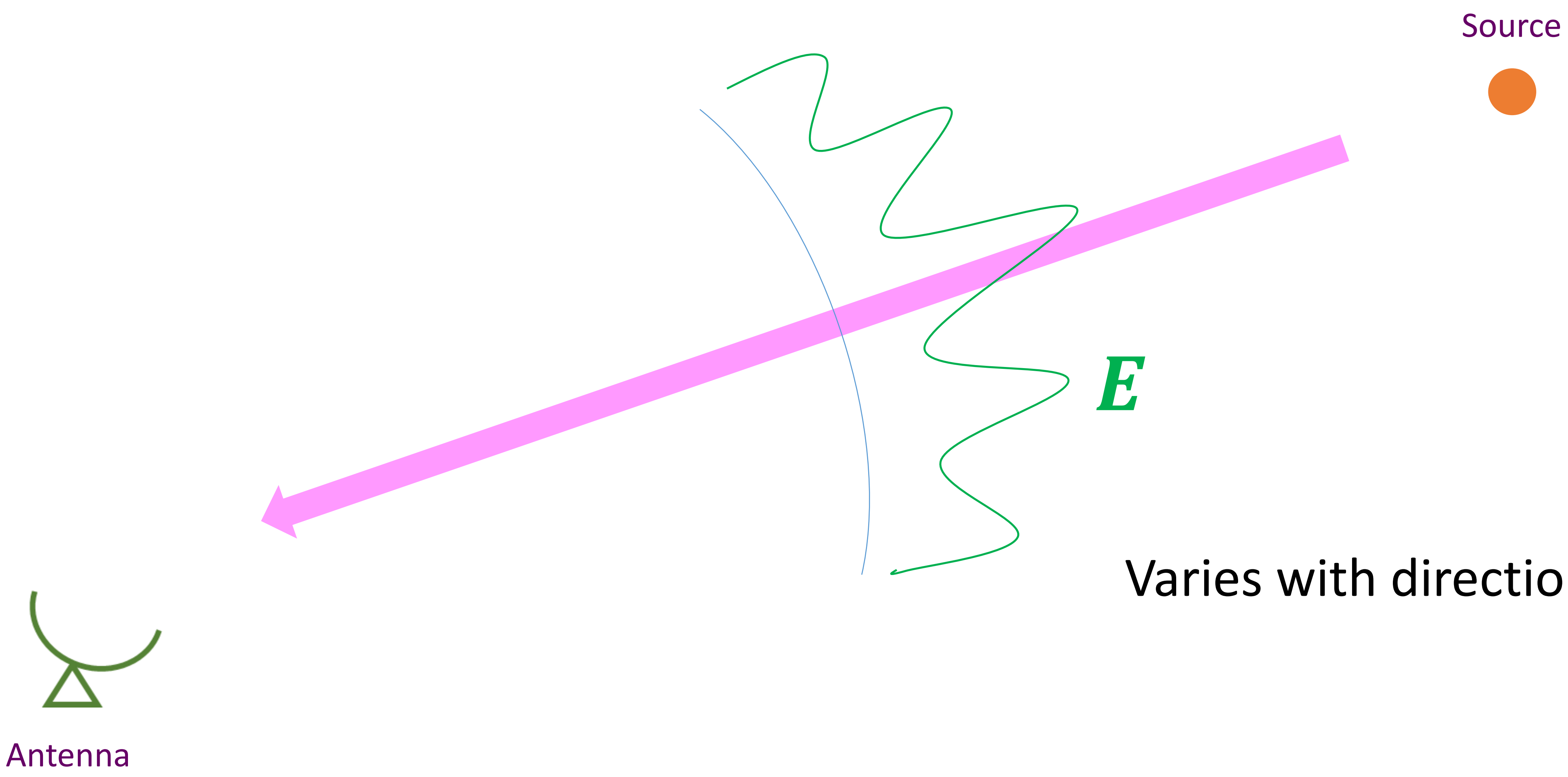
Direction-independent effects

Direction-independent effects



Same in all directions

Direction-dependent effects



Varies with direction

Structure of Jones matrices

Most Jones matrices have a simple form:

- $$J_{\text{rotation}} = \begin{bmatrix} e^{j\theta} & 0 \\ 0 & e^{-j\theta} \end{bmatrix}$$
- $$J_{\text{leakage}} = \begin{bmatrix} 1 & D^R \\ D^L & 1 \end{bmatrix}$$
- $$J_{\text{gain}} = \begin{bmatrix} G^R & 0 \\ 0 & G^L \end{bmatrix}$$

(in a circularly polarized basis)

Rotation matrices

- $$J_{\text{rotation}} = \begin{bmatrix} e^{j\theta} & 0 \\ 0 & e^{-j\theta} \end{bmatrix}$$

Examples:

Parallactic angle

Parallactic angle feed rotation: $$\boldsymbol{P} = \begin{bmatrix} e^{-j\alpha} & 0 \\ 0 & e^{j\alpha} \end{bmatrix}$$

Ionospheric Faraday rotation: $$\boldsymbol{T} = \begin{bmatrix} e^{j\chi} & 0 \\ 0 & e^{-j\chi} \end{bmatrix}$$

Faraday rotation angle

Leakage matrices

- $\boldsymbol{J}_{\text{leakage}} = \begin{bmatrix} 1 & D^R \\ D^L & 1 \end{bmatrix}$

Examples:

Polarization leakage: $\boldsymbol{D} = \begin{bmatrix} 1 & d^R \\ d^L & 1 \end{bmatrix}$

Polarization leakage terms

Gain matrices

- $\boldsymbol{J}_{\text{gain}} = \begin{bmatrix} G^R & 0 \\ 0 & G^L \end{bmatrix}$

Examples:

Instrumental gain: $\boldsymbol{G} = \begin{bmatrix} g^R & 0 \\ 0 & g^L \end{bmatrix} = \begin{bmatrix} a^R e^{j\phi^R} & 0 \\ 0 & a^L e^{j\phi^L} \end{bmatrix}$

Bandpass gain: $\boldsymbol{B} = \begin{bmatrix} B^R & 0 \\ 0 & B^L \end{bmatrix} = \begin{bmatrix} b^R(\nu) e^{j\psi^R(\nu)} & 0 \\ 0 & b^L(\nu) e^{j\phi^L(\nu)} \end{bmatrix}$

Direction dependent effect: Primary beam

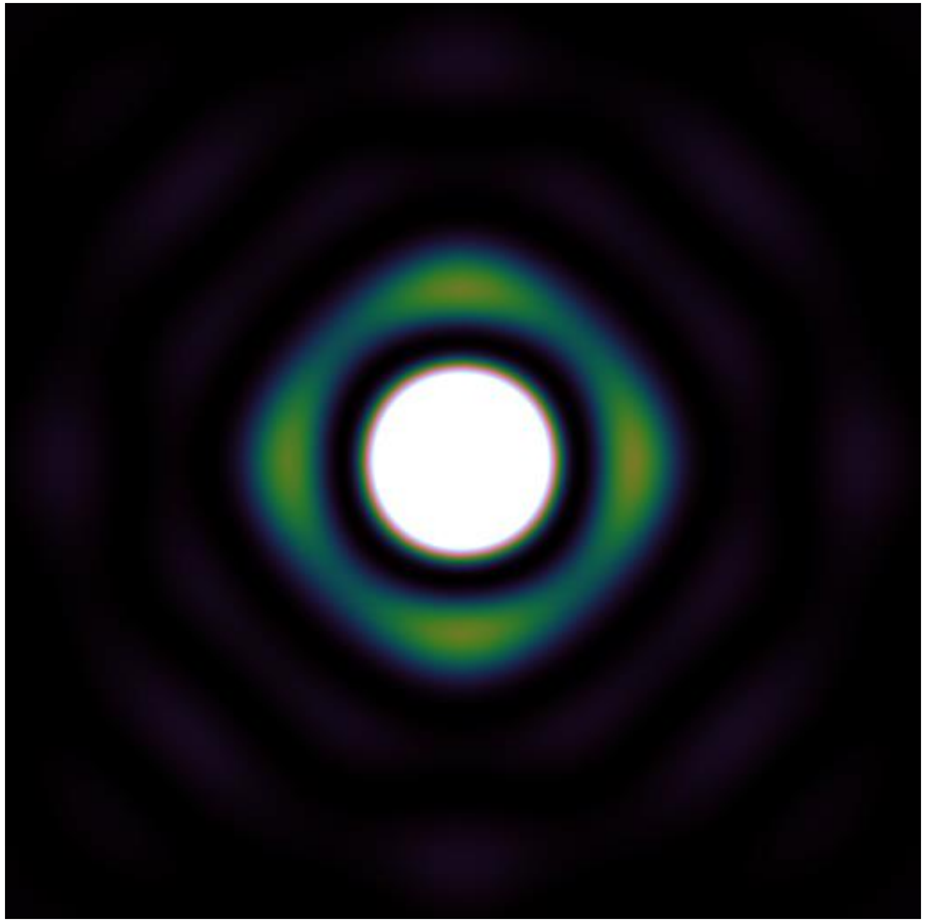
- The primary beam of the antenna is the most important direction-dependent effect

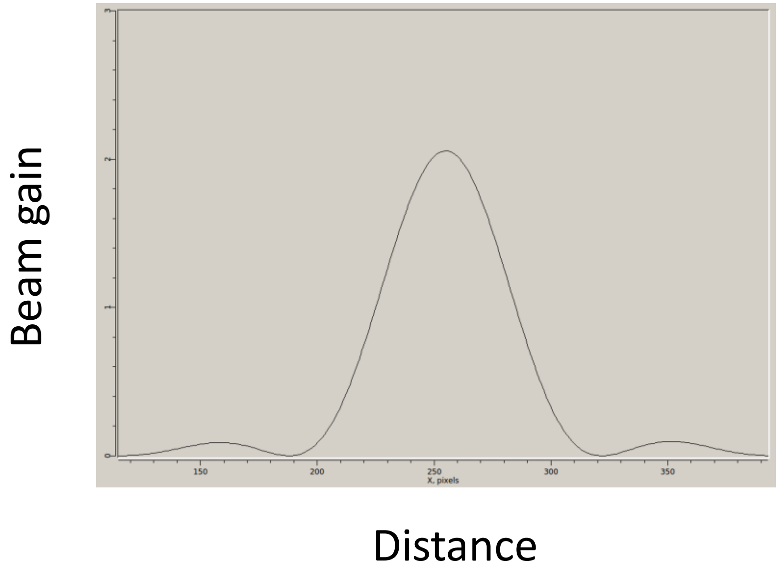
- Becomes important in wide-field, wide-band observations

- The primary beam pattern has a multiplicative effect in the image plane, convolutional effect in the visibility plane

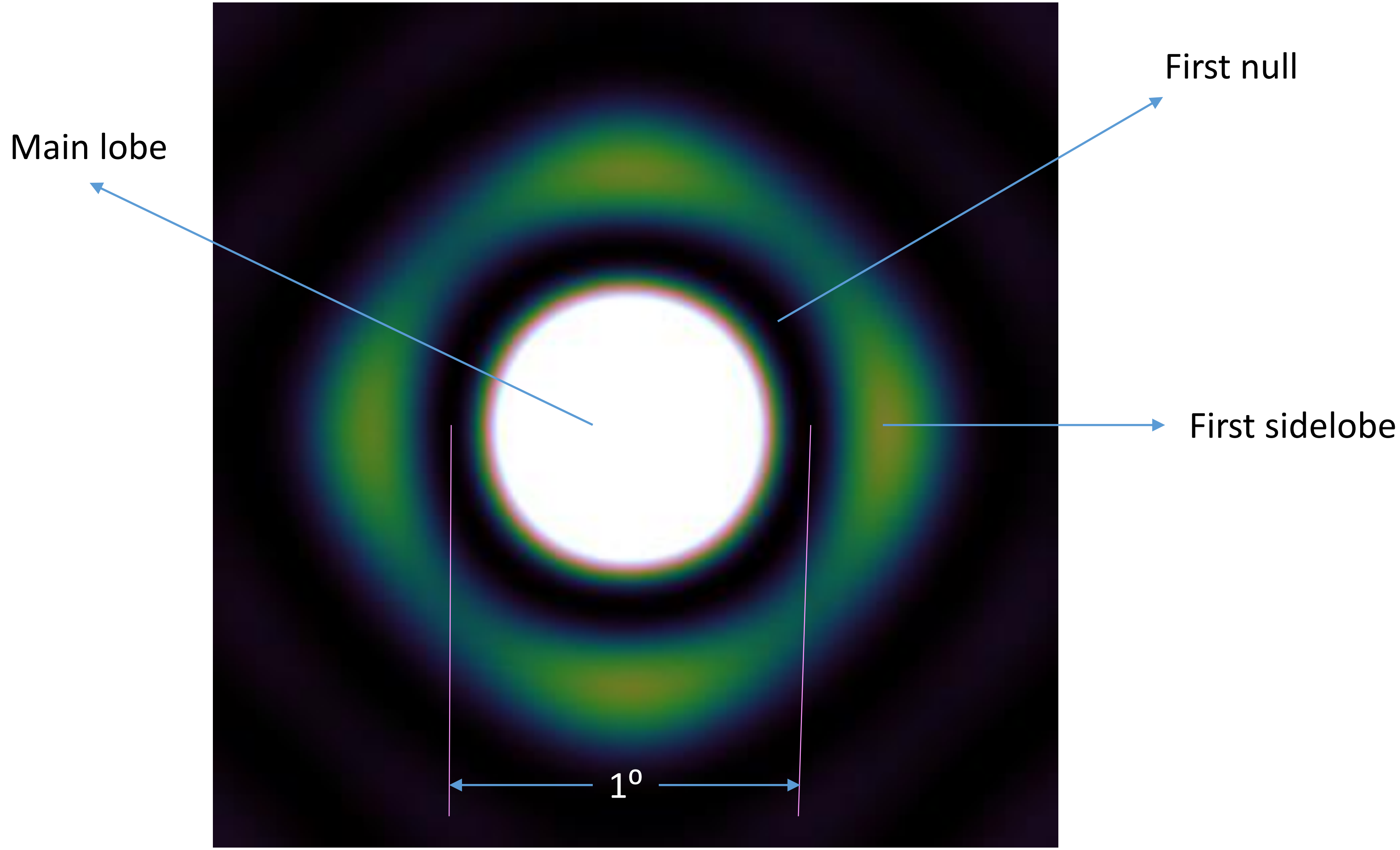
- We will consider the example of an EVLA (Expanded Very Large Array) antenna here

Primary beam amplitude variation with distance from center



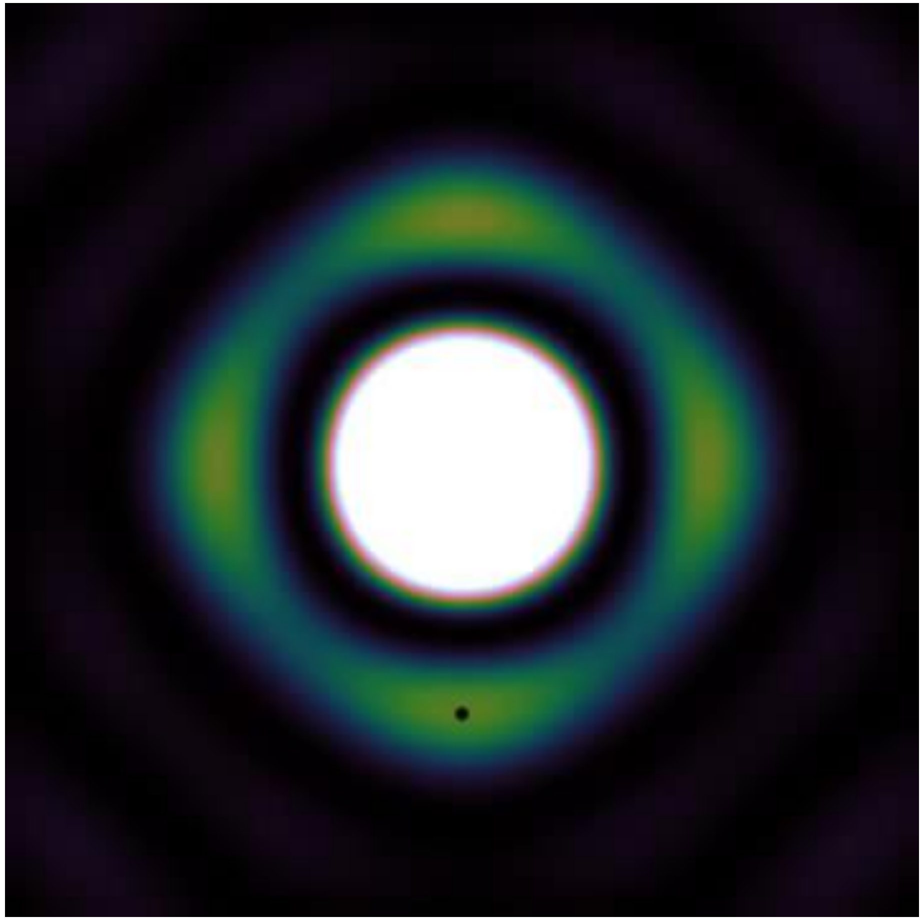


EVLA primary beam

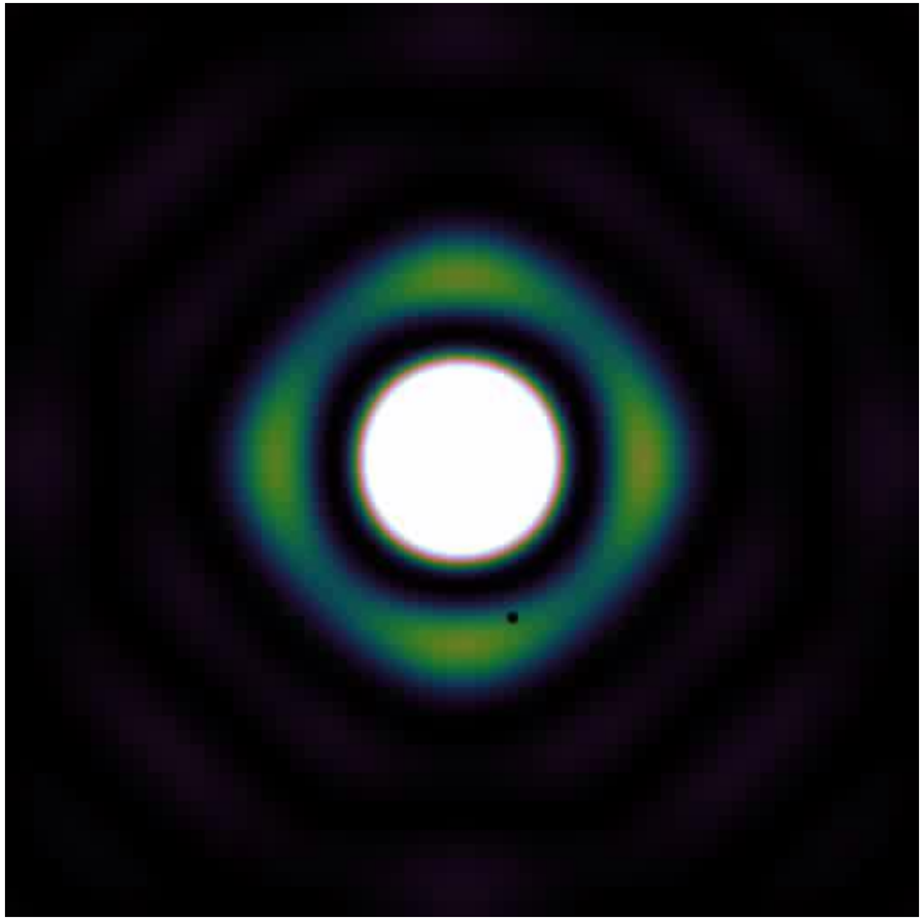


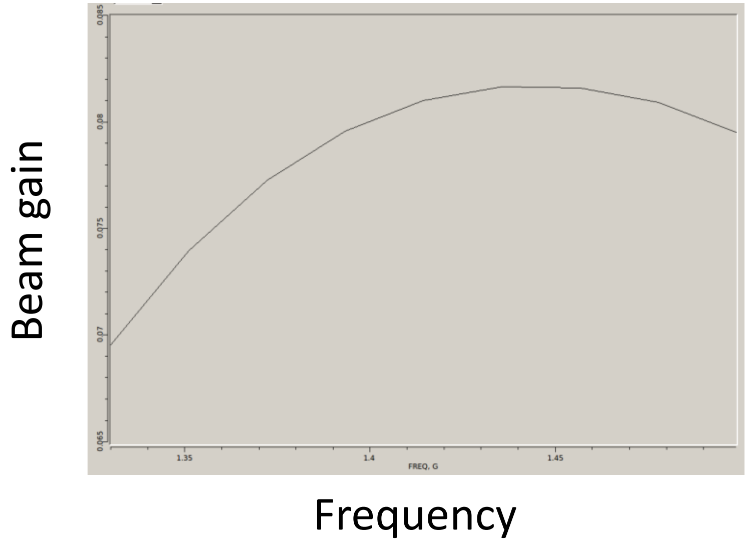
Primary beam rotation

An EVLA antenna has an alt-azimuth mount;
the primary beam rotates during the course of an observation



Variation of primary beam with frequency





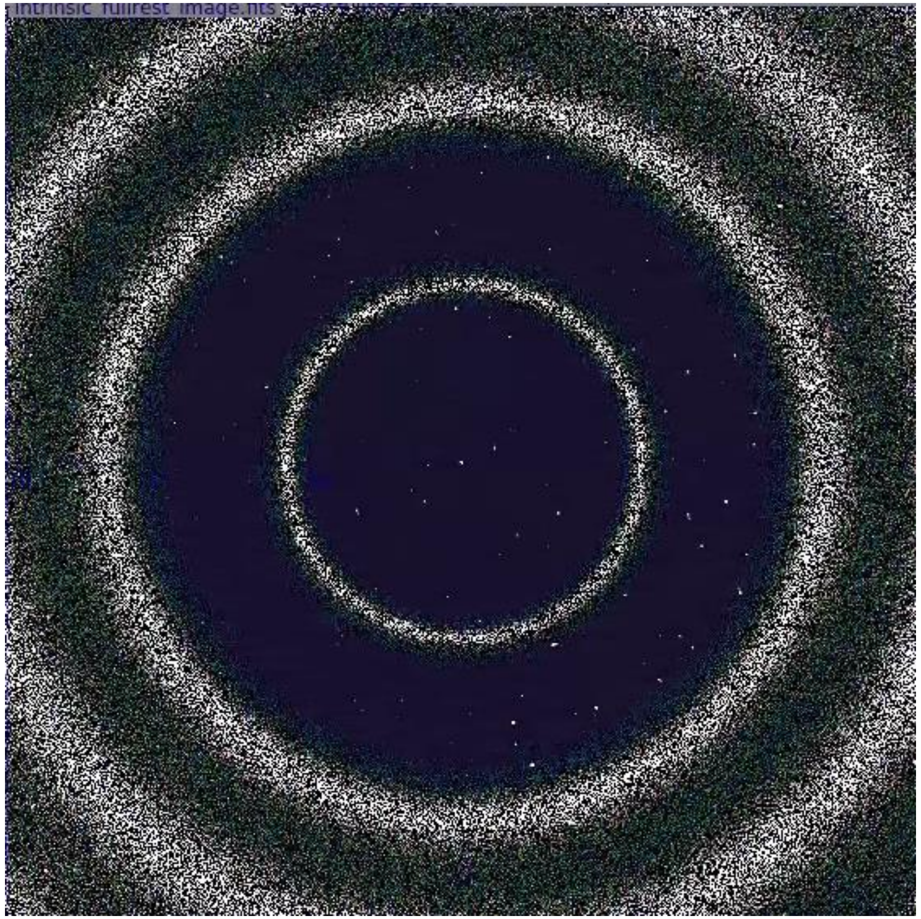
Incorporating primary beam in calibration

Calibration without primary beam included

Calibration with primary beam included

Field around radio source 3C147, imaged with EVLA

(After taking position-dependent effect of primary beam on noise into account)



Calibration procedure

1. Start with visibility data, $\mathbf{V}_{pq}$, and initial brightness model, $\mathbf{B}$.
2. Solve $\min_{G} \left| \mathbf{V}_{pq} - \boldsymbol{J}_p \ \mathbf{B} \ \boldsymbol{J}_q^H \right|$ for $\boldsymbol{J}s$.
3. Calculate residual visibility data $\mathbf{V}_{pq}^{\text{residual}} = \mathbf{V}_{pq} - \boldsymbol{J}_p \ \mathbf{B} \ \boldsymbol{J}_q^H$.
4. Image $\mathbf{V}_{pq}^{\text{residual}}$ to create a residual image, I.
5. Perform a source-finding procedure to find sources in the residual image, and add these to the initial model $\mathbf{B}$ to form a new, updated model $\mathbf{B}^{\text{new}}$.
6. Set $\mathbf{B}$ = $\mathbf{B}^{\text{new}}$, and repeat steps 2-5 until the residual image I is noise-like.

Differential gains

- Differential gain solutions encompass the unknown and unmodelled direction-dependent effects in the signal path.
- The Jones matrix in the direction of source s is then given by:

$$\boldsymbol{J}^{(s)} = \boldsymbol{G}\,\boldsymbol{E}\,\Delta\boldsymbol{E}^{(s)}$$

Final Jones matrix in the direction of source *s*

Direction-independent effects

Known and modelled direction-dependent effects

Differential gain: Unknown/unmodelled direction-dependent effects in the direction of source *s*

Differential gains

- Differential gain solutions are computed (in the direction of a few bright sources) and applied after regular calibration in order to correct for leftover, uncalibrated effects.

Incorporating differential gains in calibration

(Without primary beam incorporated in calibration)

Without differential gain solutions

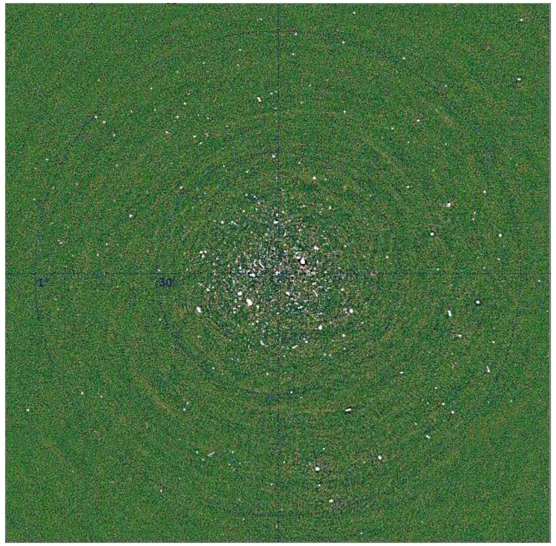
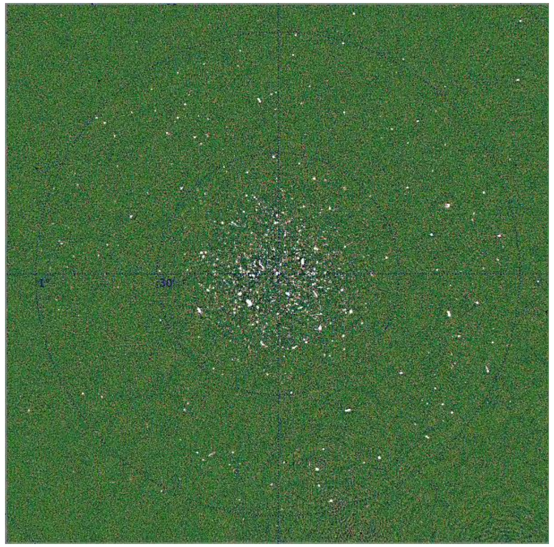
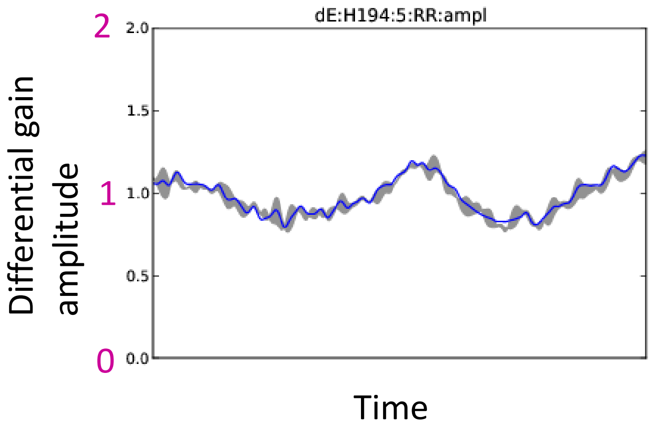
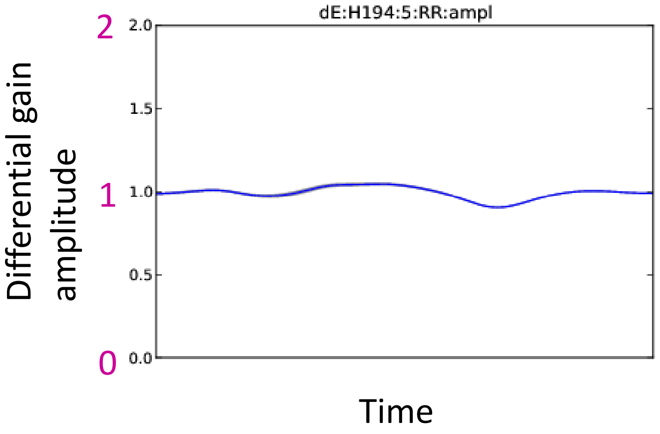
With differential gain solutions applied

Differential gains

- As more corrupting effects are modelled and accounted for, the calibration becomes more comprehensive, and differential gain solutions approach unity.

Incorporating primary beam in calibration

Differential gain plots

Without primary beam	With primary beam
	
	

- Flattened differential gain curves, nearly 1 over the whole range
- Residual variation due to remaining uncorrected direction-dependent effects (like antenna pointing errors)

References

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- *14th Synthesis Imaging Workshop* lecture slides (2014), National Radio Astronomy Observatory, Socorro, New Mexico, USA
- Oleg Smirnov’s RIME lecture from *3GC3 Workshop and Interferometry School* (2013), Port Alfred, South Africa

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