Radio-Interferometric Measurement Equation

Introductory Radio Interferometry Course

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# Radio-Interferometric Measurement Equation (RIME)

- Compact, intuitive, matrix-based way of representing propagation effects in radio interferometry.

- Useful for calibration (solving for and correcting these propagation effects).

Introduction

e'_x , e'_y : Components of electric field vector
    in reference frame of sky, at the observer

v_a , v_b : Voltages measured by antenna feed
    (linearly or circularly polarized)

Source

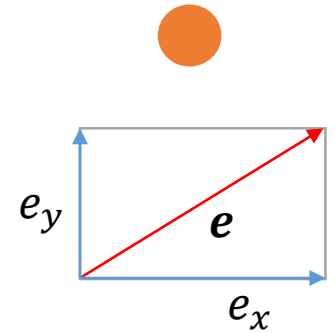e_y $\boldsymbol{e}$

e_x

Propagation effects

e_x, e_y : Components of electric field vector
    in reference frame of sky, at the source

$\boldsymbol{v}$ v_b $\boldsymbol{e'}$ e'_y

v_a e'_x
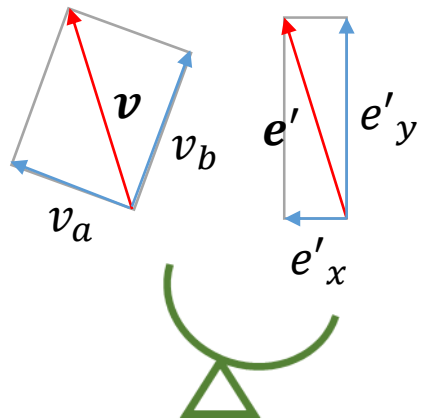
Antenna

Can be represented
as vectors:

$$\boldsymbol{e} = \begin{pmatrix} e_x \\ e_y \end{pmatrix}$$

$$\boldsymbol{e'} = \begin{pmatrix} e'_x \\ e'_y \end{pmatrix}$$

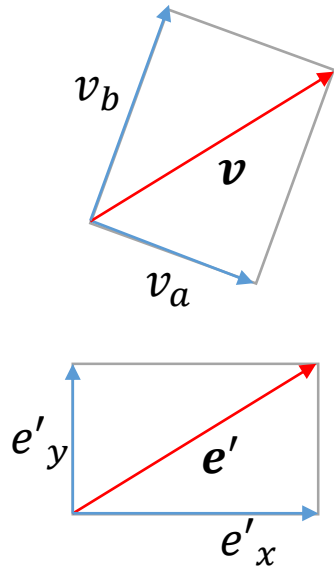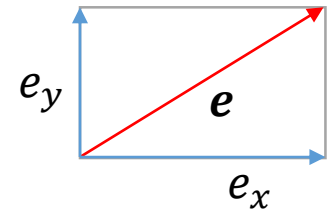$$\boldsymbol{v} = \begin{pmatrix} v_a \\ v_b \end{pmatrix}$$

3

Propagation effects absent

Amplitude and direction of electric vector
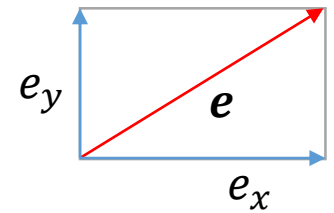remain unchanged during propagation

Source

No propagation effects

v_b

v_a

v

e_y

e_x

e

e'_y

e'_x

e'

Antenna

Propagation effects present

Source

Amplitude and direction of electric vector
change during propagation

e_y $\boldsymbol{e}$ e_x

$\boldsymbol{v}$ v_b v_a

Propagation effects

Linear transformation matrix, $\mathbf{J}$

$\boldsymbol{e'}$ e'_y e'_x
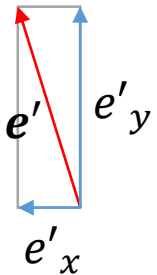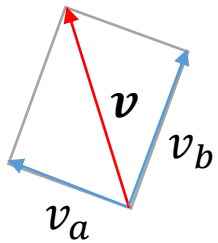
Antenna

Jones matrix

Voltage vector $\longleftarrow$ $\boldsymbol{v} = \boldsymbol{J}\boldsymbol{e}$ $\longrightarrow$ Electric field vector

$$\begin{pmatrix} v_a \\ v_b \end{pmatrix} = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix} \begin{pmatrix} e_x \\ e_y \end{pmatrix}$$

Correlation
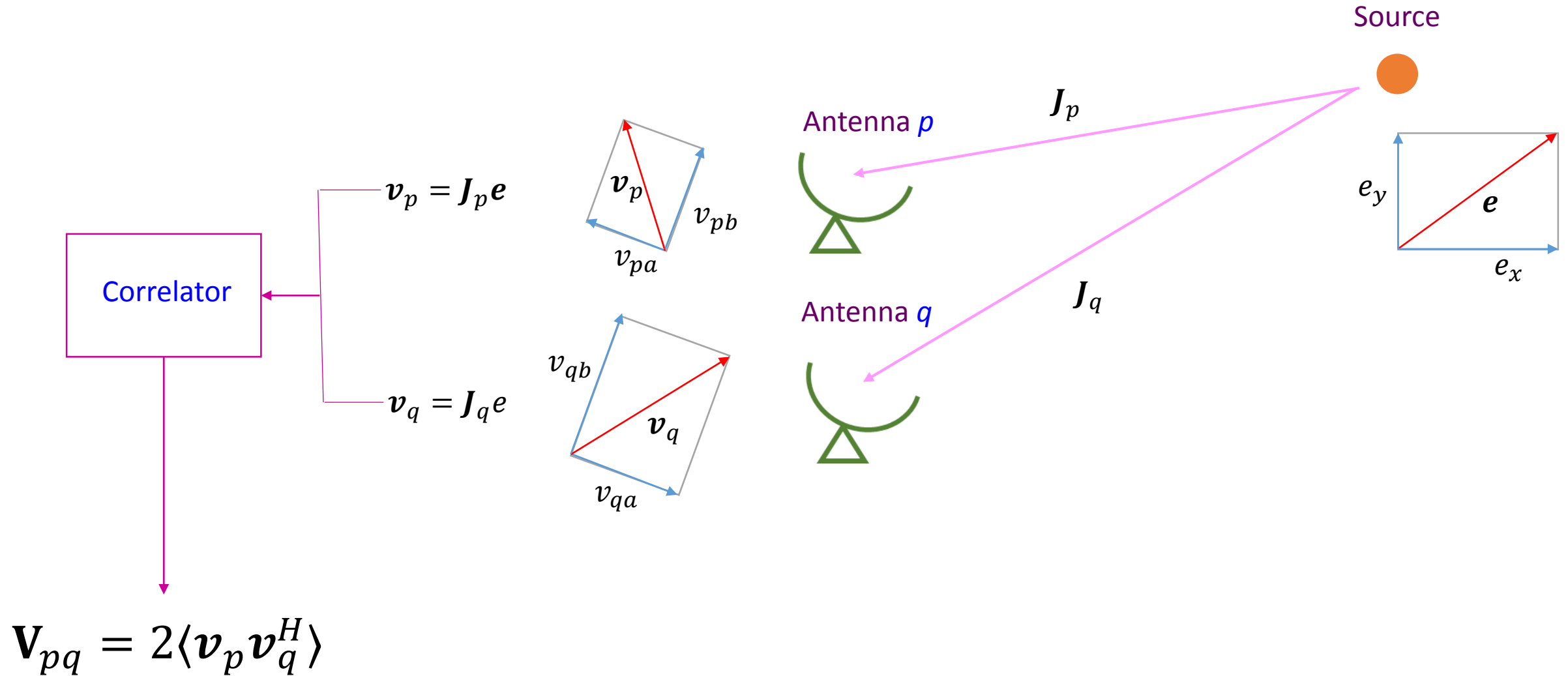
$$\mathbf{V}_{pq} = 2\langle \boldsymbol{v}_p \boldsymbol{v}_q^H \rangle$$

Visibility

- The correlator computes the visibility, $\mathbf{V}_{pq}$, on the baseline pq:

Hermitian conjugate

Outer product

$$\mathbf{V}_{pq} = 2\langle \boldsymbol{v}_p \boldsymbol{v}_q^H \rangle$$

Average

These 4 quantities are the outputs from the correlator

$$= 2\langle \begin{pmatrix} v_{pa} \\ v_{pb} \end{pmatrix} \begin{pmatrix} v_{qa}^* & v_{qb}^* \end{pmatrix} \rangle$$

$$= 2\begin{pmatrix} \langle v_{pa}v_{qa}^* \rangle & \langle v_{pa}v_{qb}^* \rangle \\ \langle v_{pb}v_{qa}^* \rangle & \langle v_{pb}v_{qb}^* \rangle \end{pmatrix}$$

Correlation

$$\boldsymbol{v}_p = \boldsymbol{J}_p \boldsymbol{e} \quad , \quad \boldsymbol{v}_q = \boldsymbol{J}_q \boldsymbol{e}$$

$$\mathbf{V}_{pq} = 2\langle \boldsymbol{v}_p \boldsymbol{v}_q^H \rangle$$

$$= 2\langle (\boldsymbol{J}_p \boldsymbol{e})(\boldsymbol{J}_q \boldsymbol{e})^H \rangle$$

$$= 2\langle \boldsymbol{J}_p (\boldsymbol{e}\boldsymbol{e}^H) \boldsymbol{J}_q^H \rangle$$

$$= \boldsymbol{J}_p \langle 2\boldsymbol{e}\boldsymbol{e}^H \rangle \boldsymbol{J}_q^H$$

Coherency, or Brightness

$$\mathbf{V}_{pq} = \boldsymbol{J}_p \langle 2\boldsymbol{e}\boldsymbol{e}^H \rangle \boldsymbol{J}_q^H$$

By definition, the coherency, or brightness, B, is given by:

$$\mathbf{B} = \langle 2\boldsymbol{e}\boldsymbol{e}^H \rangle = \begin{pmatrix} I + Q & U + iV \\ U - iV & I - Q \end{pmatrix}$$

$\langle \boldsymbol{e}\boldsymbol{e}^H \rangle$ is the coherence of the electromagnetic field with itself, and is described by the Stokes parameters I, Q, U, V

$$\mathbf{V}_{pq} = \boldsymbol{J}_p \, \mathbf{B} \, \boldsymbol{J}_q^H$$

Radio-Interferometric Measurement Equation (RIME)

Visibility

Brightness

$$\mathbf{V}_{pq} = \boldsymbol{J}_p \, \mathbf{B} \, \boldsymbol{J}_q^H$$

Jones matrices

$$\begin{pmatrix} v_{aa} & v_{ab} \\ v_{ba} & v_{bb} \end{pmatrix} = \begin{pmatrix} j_{11a} & j_{12a} \\ j_{21a} & j_{22a} \end{pmatrix} \begin{pmatrix} I+Q & U+iV \\ U-iV & I-Q \end{pmatrix} \begin{pmatrix} j_{11b} & j_{12b} \\ j_{21b} & j_{22b} \end{pmatrix}^H$$

Component Jones matrices

The Jones matrix for an antenna is a product of several component Jones matrices, corresponding to different corrupting effects along the signal path

Example:

$$J = B \ G \ D \ E \ P \ T$$

Bandpass gain

Instrumental gain

Polarization leakage

Primary beam

Parallactic angle feed rotation

Ionospheric and tropospheric effects

Component Jones matrices

Jones chain:

$$J = J_n \, J_{n-1} \, \cdots \, J_2 \, J_1$$

Later
in signal path

Earlier
in signal path

Component Jones matrices

Source

Propagation effects

Jones matrix, J

$\dots$ J_2 J_1

Earlier
in signal path

J_n J_{n-1}

Later
in signal path

Antenna

Component Jones matrices

Antenna p : $J_p = J_{pn} \, J_{p(n-1)} \cdots J_{p2} \, J_{p1}$

Antenna q : $J_q = J_{qn} \, J_{q(n-1)} \cdots J_{q2} \, J_{q1}$

Visibility   Brightness

$$\mathbf{V}_{pq} = J_p \, \mathbf{B} \, J_q^H$$

Jones matrices

$$\mathbf{V}_{pq} = J_{pn} J_{p(n-1)} \cdots J_{p2} J_{p1} \, \mathbf{B} \, J_{q1}^H J_{q2}^H \cdots J_{q(n-1)}^H \, J_{qn}^H$$

$$\mathbf{V}_{pq} = J_{pn}\big(J_{p(n-1)}\big(\cdots \big(J_{p2}\big(J_{p1}\,\mathbf{B}\,J_{q1}^H\big)J_{q2}^H\big)\cdots\big)J_{q(n-1)}^H\big)J_{qn}^H$$

Direction-independent and direction-dependent effects

Propagation effects can be of two kinds:

- Direction-independent effects
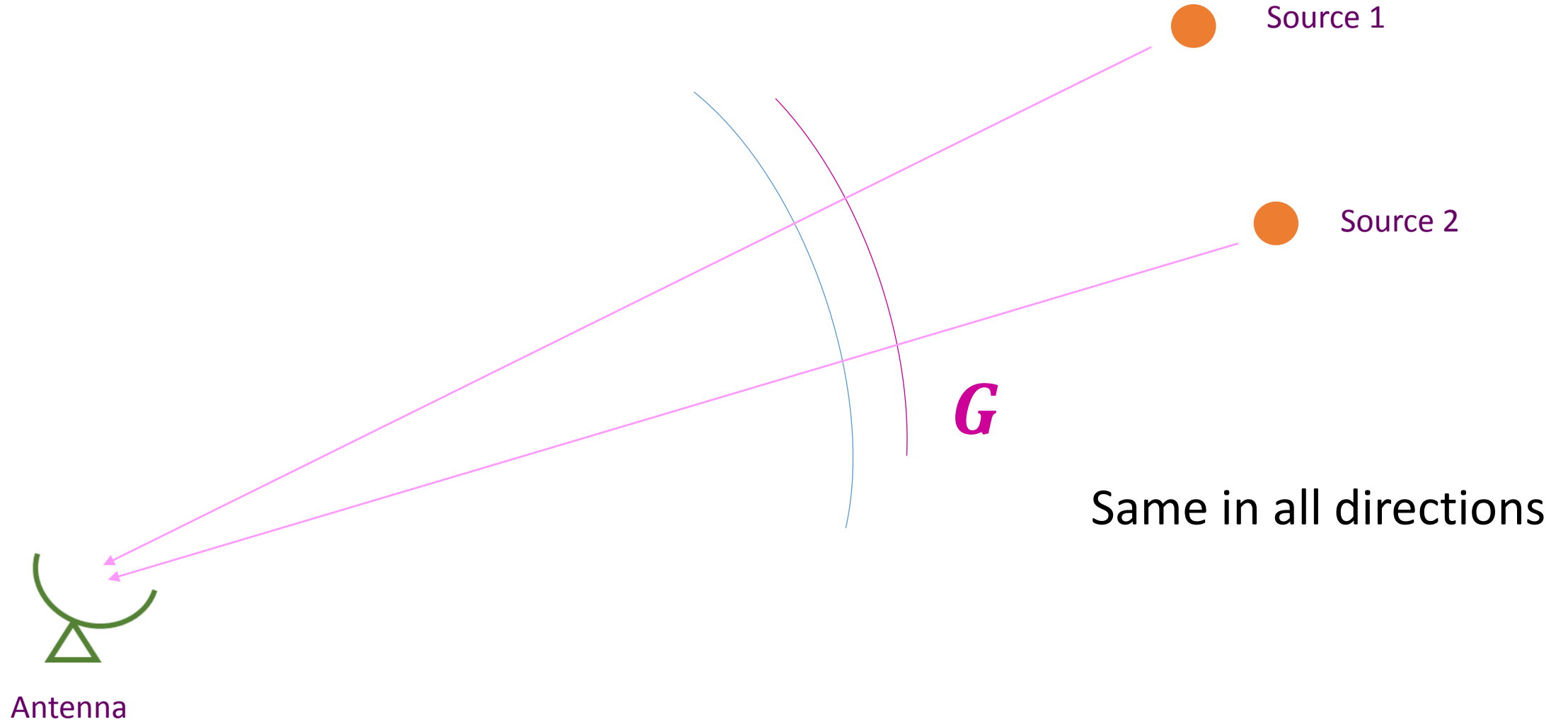- Direction-dependent effects

These effects can be represented by different Jones matrices:

$$J = G\,E$$

Final Jones matrix

Direction-independent effects

Direction-dependent effects

Direction-independent effects

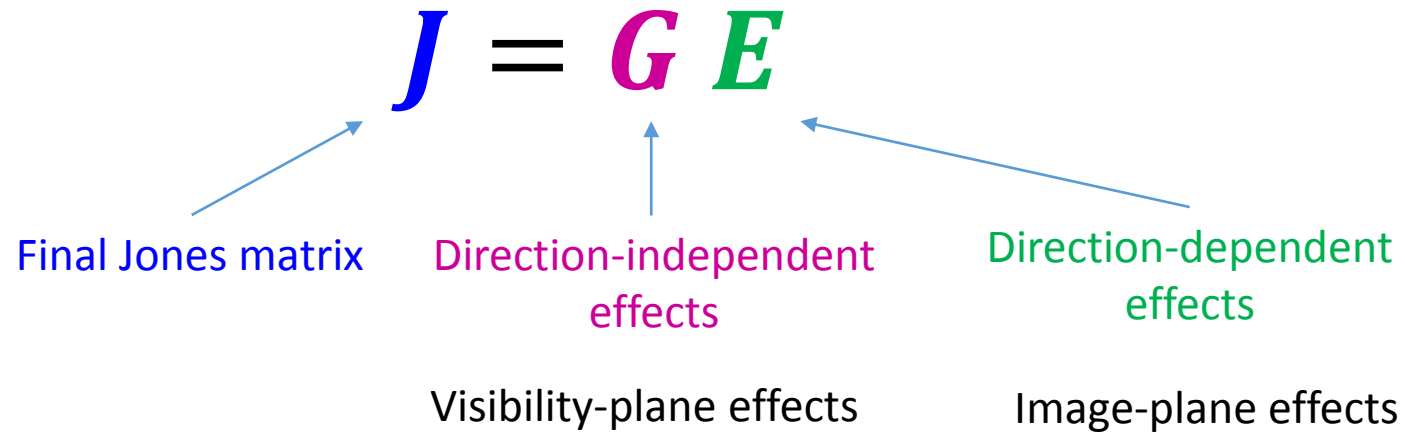
Source 1

Source 2

G

Same in all directions

Antenna

Direction-dependent effects

Source 1

Source 2

E

Varies with direction

Antenna

Direction-independent and direction-dependent effects

$$J = G\ E$$

Final Jones matrix

Direction-independent effects

Direction-dependent effects
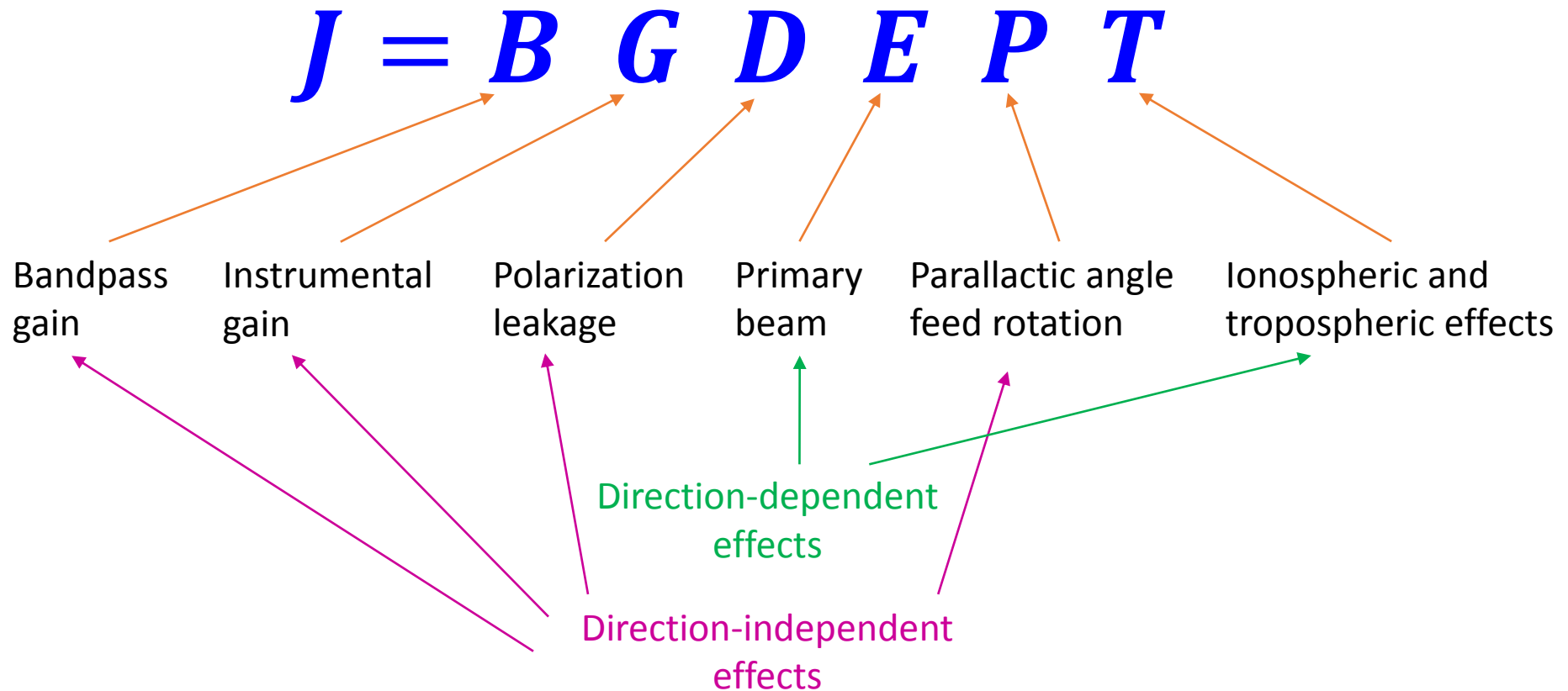
Visibility-plane effects
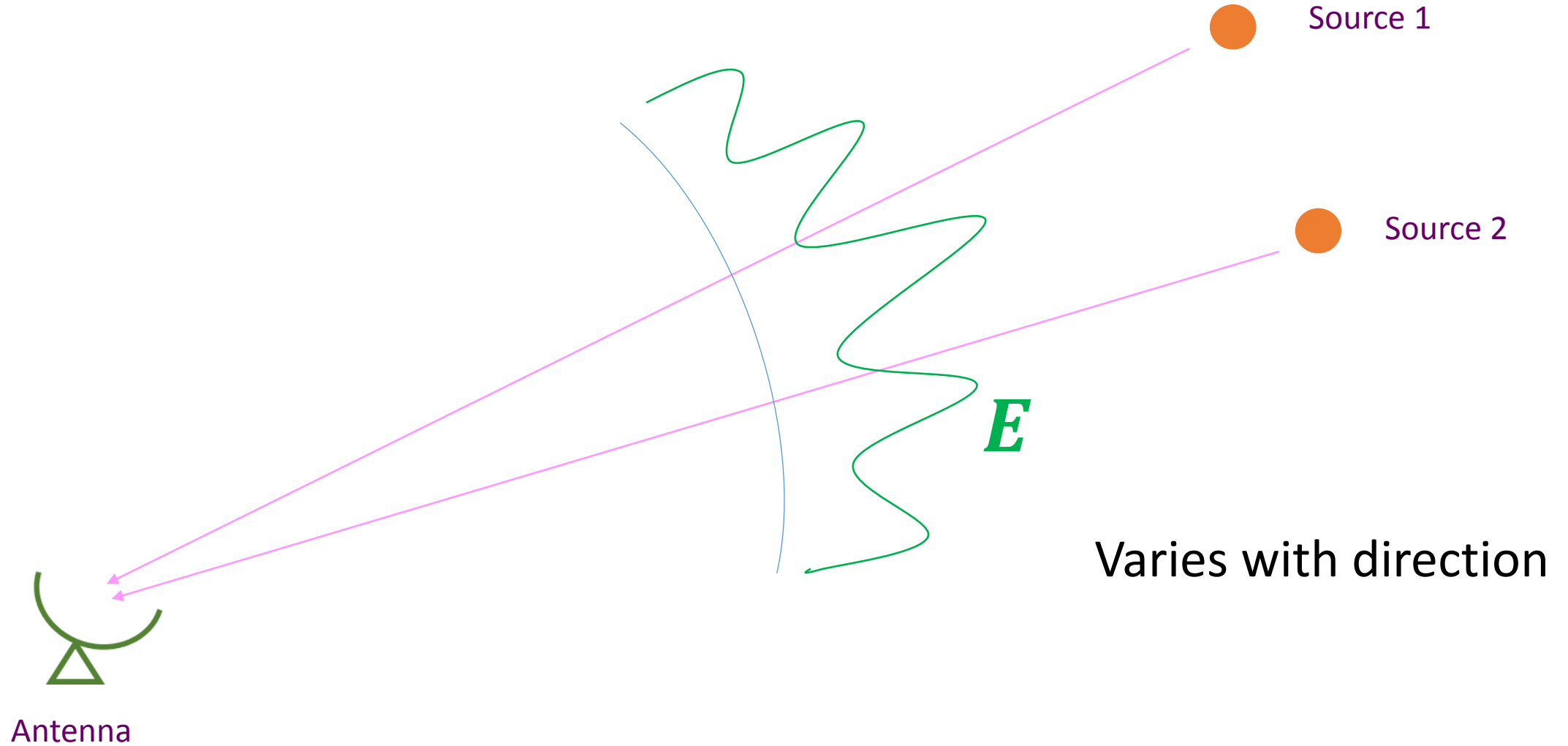
Image-plane effects

$$\mathbf{V}_{pq} = J_p\ \mathbf{B}\ J_q^H$$

$$\mathbf{V}_{pq} = G_p\left(E_p\mathbf{B}E_q^H\right)G_q^H$$

Direction-independent and direction-dependent effects

Example:

$$J = B \ G \ D \ E \ P \ T$$

Bandpass gain

Instrumental gain

Polarization leakage

Primary beam

Parallactic angle feed rotation

Ionospheric and tropospheric effects

Direction-dependent effects

Direction-independent effects

Explicit RIME and phenomenological RIME

- Explicit RIME:

$$J = B\ G\ D\ E\ P\ T$$ → Useful for understanding the component corrupting effects along the propagation path

- Phenomenological RIME:

$$J = G\ E$$ → Useful for calibration, as these matrices are easier to solve for

Polarization

- Polarization of an electromagnetic wave describes the direction of oscillation of the electric field.
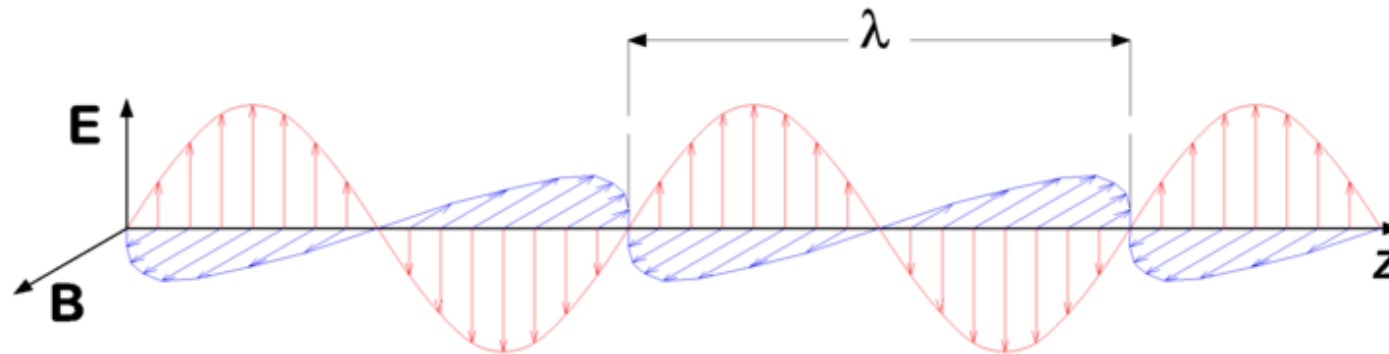
(From https://commons.wikimedia.org/wiki/File:Electromagnetic_wave.png, author: User:P.wormer)

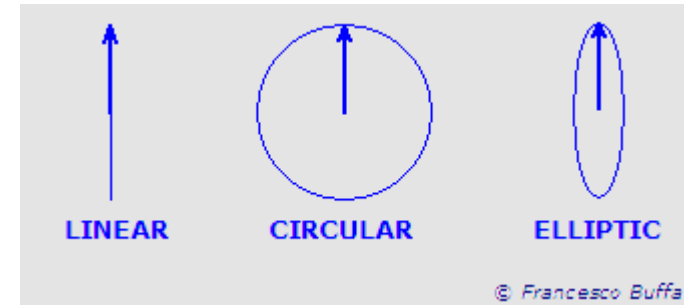- The plane of polarization is perpendicular to the direction of propagation of the wave.

Polarization

- Electromagnetic waves can be:

  - Linearly polarized

  - Circularly polarized

  - Elliptically polarized

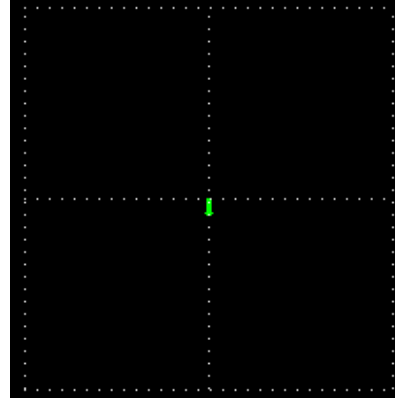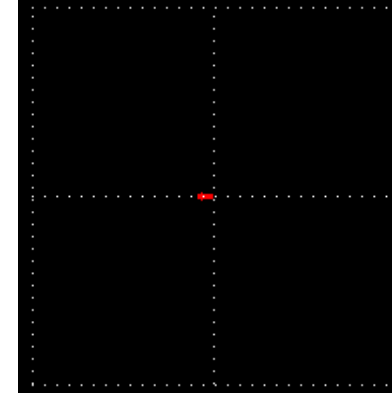LINEAR   CIRCULAR   ELLIPTIC

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Antenna feeds

An antenna contains feeds, which detect and measure specific polarized components of an electromagnetic wave:

- Antenna with orthogonal linearly polarized feeds:
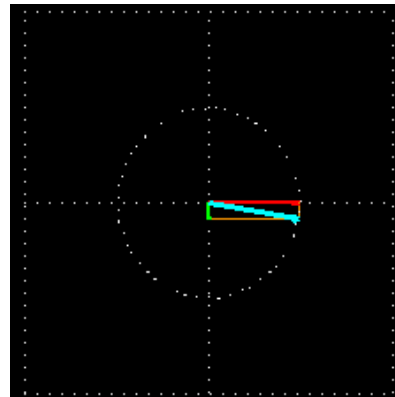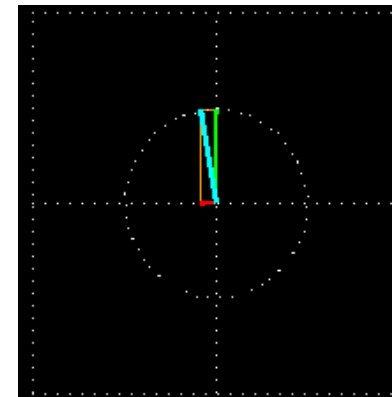
x-polarized feed

y-polarized feed

- Antenna with orthogonal circularly polarized feeds:

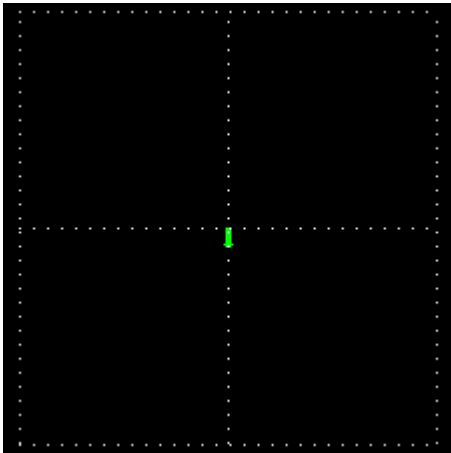Right-circularly polarized (RCP) feed

Left-circularly polarized (LCP) feed
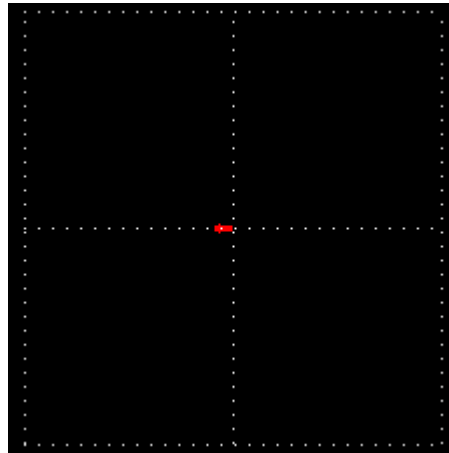
(From http://cddemo.szialab.org/)
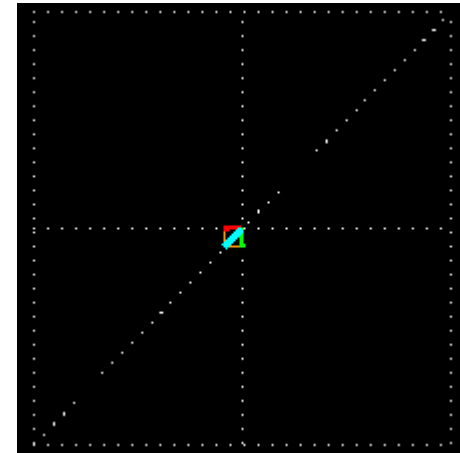
Antenna feeds

- Linearly polarized wave, measured by linearly polarized feeds:

x component

y component

Combination of
x and y components

Antenna feeds

- Linearly polarized wave, measured by circularly polarized feeds:
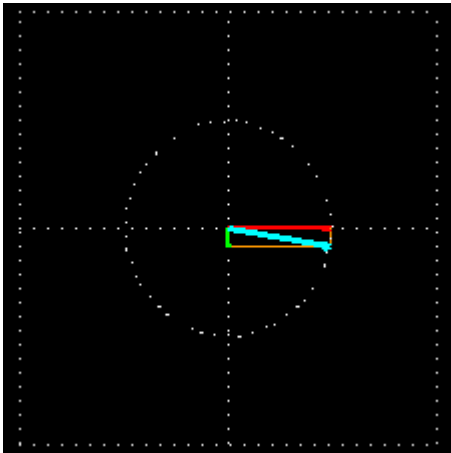
RCP component
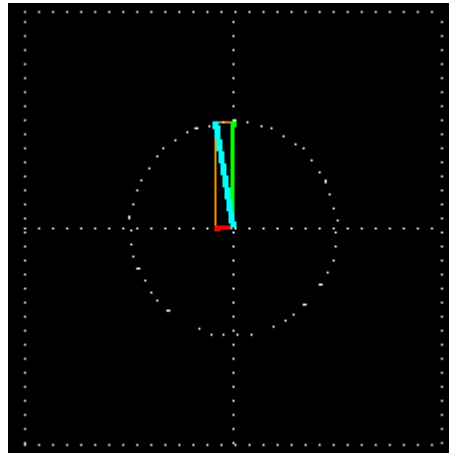
LCP component
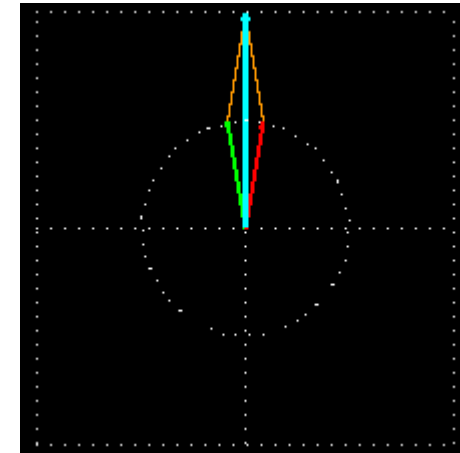
Combination of
RCP and LCP components

Structure of Jones matrices

Most Jones matrices have a simple form:

- $J_{\text{rotation}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

- $J_{\text{leakage}} = \begin{bmatrix} 1 & D_X \\ D_Y & 1 \end{bmatrix}$

- $J_{\text{gain}} = \begin{bmatrix} G_X & 0 \\ 0 & G_Y \end{bmatrix}$

(in a linearly polarized basis)

Structure of Jones matrices

- The structure of individual Jones matrices depends on the antenna feed polarization basis, but the RIME is valid independent of the polarization basis.

- For example,

  - $$J_{\text{rotation}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
    in a linearly polarized basis

  - $$J_{\text{rotation}} = \begin{bmatrix} e^{j\theta} & 0 \\ 0 & e^{-j\theta} \end{bmatrix}$$
    in a circularly polarized basis

Rotation matrices

- $\boldsymbol{J}_{\text{rotation}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Examples:

Parallactic angle feed rotation:

Parallactic angle

$\boldsymbol{P} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

Ionospheric Faraday rotation:

$\boldsymbol{T} = \begin{bmatrix} \cos\chi & -\sin\chi \\ \sin\chi & \cos\chi \end{bmatrix}$
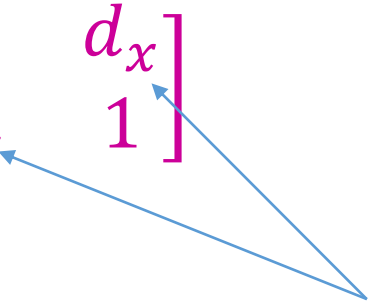
Faraday rotation angle

Leakage matrices

- $\boldsymbol{J}_{\text{leakage}} = \begin{bmatrix} 1 & D_X \\ D_Y & 1 \end{bmatrix}$

Examples:

Polarization leakage: $\boldsymbol{D} = \begin{bmatrix} 1 & d_x \\ d_y & 1 \end{bmatrix}$

Polarization leakage terms

Gain matrices

- $\boldsymbol{J}_{\text{gain}} = \begin{bmatrix} G_X & 0 \\ 0 & G_Y \end{bmatrix}$

Examples:

Instrumental gain: $\boldsymbol{G} = \begin{bmatrix} g_x & 0 \\ 0 & g_y \end{bmatrix} = \begin{bmatrix} a_x e^{j\phi_x} & 0 \\ 0 & a_y e^{j\phi_y} \end{bmatrix}$

Bandpass gain: $\boldsymbol{B} = \begin{bmatrix} B_x & 0 \\ 0 & B_y \end{bmatrix} = \begin{bmatrix} b_x(\nu) e^{j\psi_x(\nu)} & 0 \\ 0 & b_y(\nu) e^{j\psi_y(\nu)} \end{bmatrix}$

Order of Jones matrices in Jones chain

$$J = J_n \, J_{n-1} \, \cdots \, J_2 \, J_1$$

- Order of matrices in the Jones chain is important, because matrix multiplication is not commutative, in general.

- However, specific kinds of matrices do commute – scalar matrices commute with all kinds of matrices, rotation matrices with each other, diagonal matrices with each other.

References

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- G. B. Taylor, C. L. Carilli, & R. A. Perley, editors (1999), *Synthesis Imaging in Radio Astronomy II*, volume 180 of *Astronomical Society of the Pacific Conference Series*

- *14th Synthesis Imaging Workshop* lecture slides (2014), National Radio Astronomy Observatory, Socorro, New Mexico, USA

- Oleg Smirnov's RIME lecture from *3GC3 Workshop and Interferometry School* (2013), Port Alfred, South Africa

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