Radio-Interferometric Measurement Equation

Introductory Radio Interferometry Course

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Radio-Interferometric Measurement Equation (RIME)

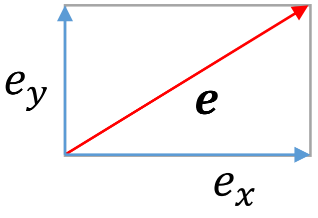
- Compact, intuitive, matrix-based way of representing propagation effects in radio interferometry.
- Useful for calibration (solving for and correcting these propagation effects).

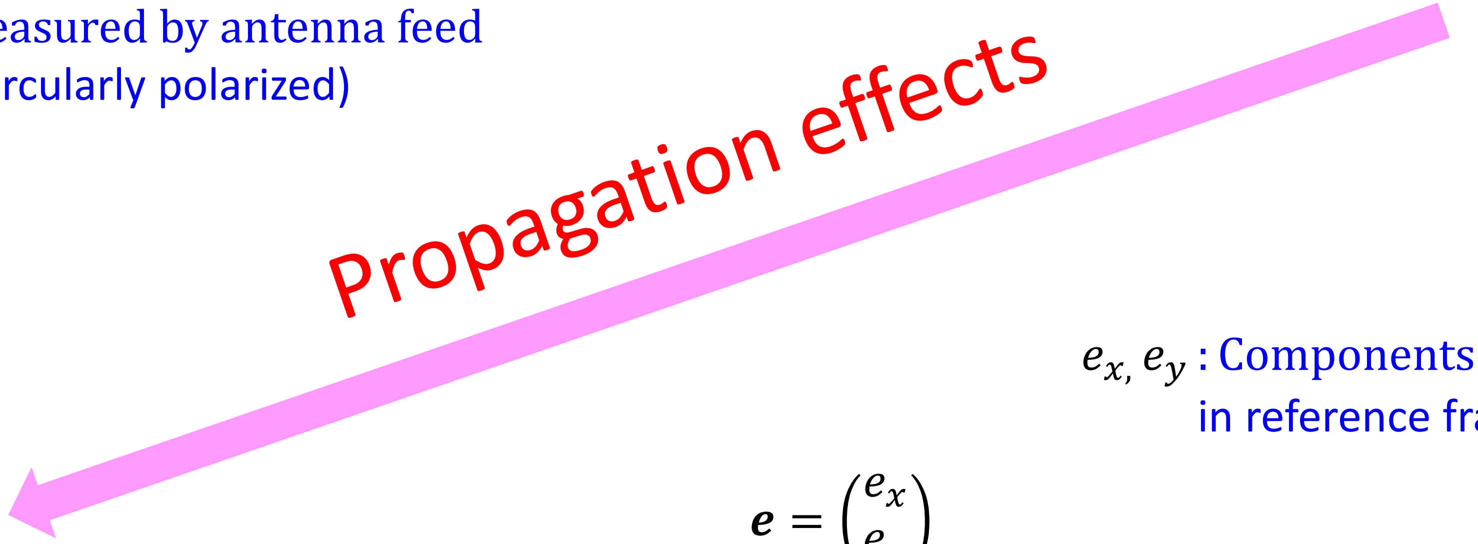
Introduction

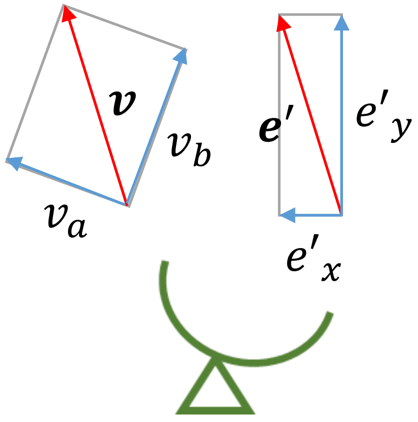
e'_x, e'_y : Components of electric field vector
in reference frame of sky, at the observer

v_a, v_b : Voltages measured by antenna feed
(linearly or circularly polarized)

Source







Antenna

e_x, e_y : Components of electric field vector
in reference frame of sky, at the source

Can be represented
as vectors:

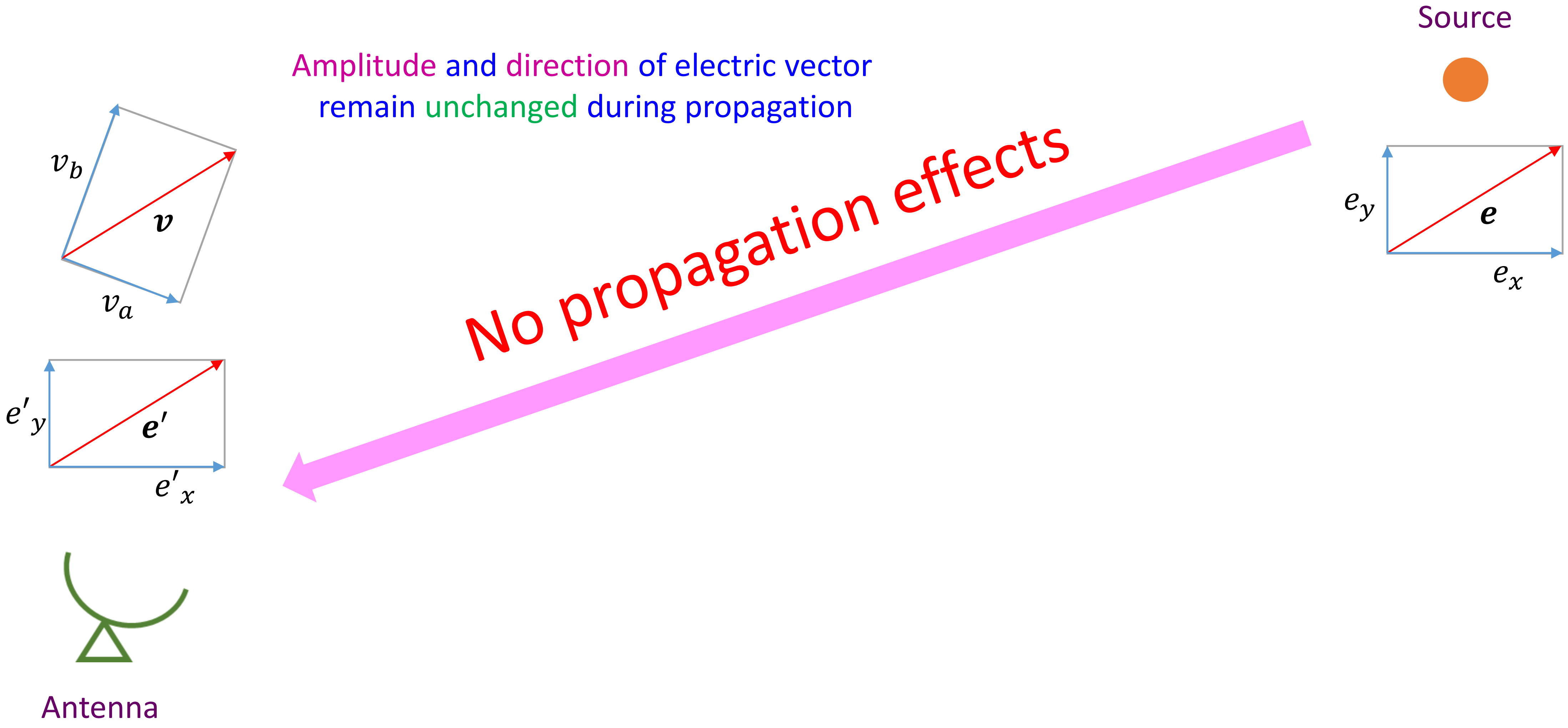
$$\boldsymbol{e} = \begin{pmatrix} e_x \\ e_y \end{pmatrix}$$

$$\boldsymbol{e}' = \begin{pmatrix} e'_x \\ e'_y \end{pmatrix}$$

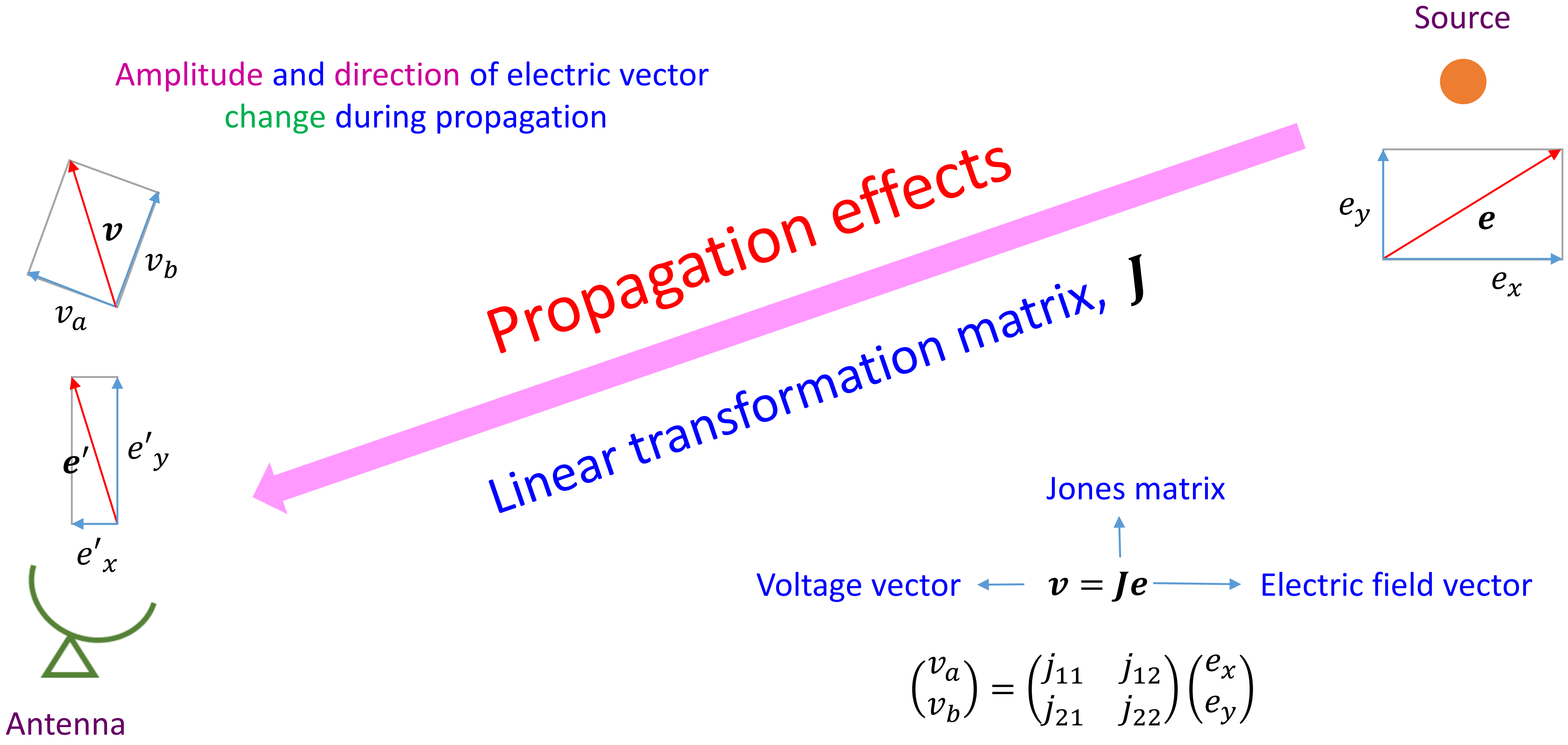
$$\boldsymbol{v} = \begin{pmatrix} v_a \\ v_b \end{pmatrix}$$

Propagation effects absent

Amplitude and direction of electric vector remain unchanged during propagation



Propagation effects present

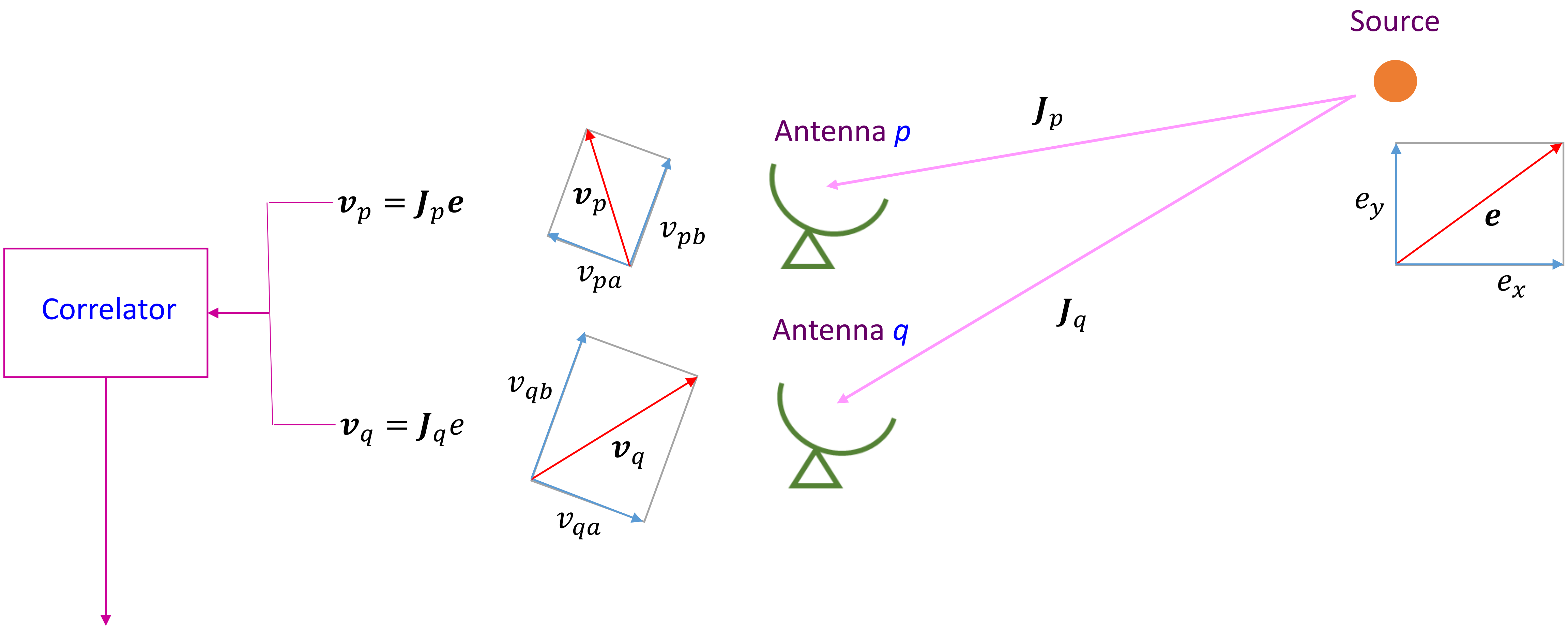


Jones matrix

Voltage vector ← $\boldsymbol{v} = \boldsymbol{J}\boldsymbol{e}$ → Electric field vector

$$\begin{pmatrix} v_a \\ v_b \end{pmatrix} = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix} \begin{pmatrix} e_x \\ e_y \end{pmatrix}$$

Correlation



$$\mathbf{V}_{pq} = 2\langle \boldsymbol{v}_p \boldsymbol{v}_q^H \rangle$$

Visibility

- The correlator computes the visibility, $\mathbf{V}_{pq}$, on the baseline pq:

Hermitian conjugate

$$\mathbf{V}_{pq} = 2\underbrace{\langle \overbrace{\boldsymbol{v}_p \boldsymbol{v}_q^H}^{\text{Outer product}} \rangle}_{\text{Average}}$$

These 4 quantities are the outputs from the correlator

$$= 2\left\langle \begin{pmatrix} v_{pa} \\ v_{pb} \end{pmatrix} \begin{pmatrix} v_{qa}^* & v_{qb}^* \end{pmatrix} \right\rangle$$

$$= 2 \begin{pmatrix} \overbrace{\langle v_{pa} v_{qa}^* \rangle \quad \langle v_{pa} v_{qb}^* \rangle}^{} \\ \langle v_{pb} v_{qa}^* \rangle \quad \langle v_{pb} v_{qb}^* \rangle \end{pmatrix}$$

Correlation

$$\boldsymbol{v}_p = \boldsymbol{J}_p \boldsymbol{e} \quad , \quad \boldsymbol{v}_q = \boldsymbol{J}_q \boldsymbol{e}$$

$$\mathbf{V}_{pq} = 2\langle \boldsymbol{v}_p \boldsymbol{v}_q^H \rangle$$

$$= 2\langle (\boldsymbol{J}_p \boldsymbol{e})(\boldsymbol{J}_q \boldsymbol{e})^H \rangle$$

$$= 2\langle \boldsymbol{J}_p (\boldsymbol{e}\boldsymbol{e}^H) \boldsymbol{J}_q^H \rangle$$

$$= \boldsymbol{J}_p \langle 2\boldsymbol{e}\boldsymbol{e}^H \rangle \boldsymbol{J}_q^H$$

Coherency, or Brightness

$$\mathbf{V}_{pq} = \boldsymbol{J}_p \langle 2\boldsymbol{e}\boldsymbol{e}^H \rangle \boldsymbol{J}_q^H$$

By definition, the coherency, or brightness, B, is given by:

$$\mathbf{B} = \langle 2\boldsymbol{e}\boldsymbol{e}^H \rangle = \begin{pmatrix} I + Q & U + iV \\ U - iV & I - Q \end{pmatrix}$$

$\langle \boldsymbol{e}\boldsymbol{e}^H \rangle$ is the coherence of the electromagnetic field with itself, and is described by the Stokes parameters I, Q, U, V

$$\mathbf{V}_{pq} = \boldsymbol{J}_p \, \mathbf{B} \, \boldsymbol{J}_q^H$$

Radio-Interferometric Measurement Equation (RIME)

Visibility

Brightness

$$\mathbf{V}_{pq} = \boldsymbol{J}_p \, \mathbf{B} \, \boldsymbol{J}_q^H$$

Jones matrices

$$\begin{pmatrix} v_{aa} & v_{ab} \\ v_{ba} & v_{bb} \end{pmatrix} = \begin{pmatrix} j_{11a} & j_{12a} \\ j_{21a} & j_{22a} \end{pmatrix} \begin{pmatrix} I+Q & U+iV \\ U-iV & I-Q \end{pmatrix} \begin{pmatrix} j_{11b} & j_{12b} \\ j_{21b} & j_{22b} \end{pmatrix}^H$$

Component Jones matrices

The Jones matrix for an antenna is a product of several component Jones matrices, corresponding to different corrupting effects along the signal path

Example:

$$\boldsymbol{J} = \underbrace{\boldsymbol{B}}_{\text{Bandpass gain}} \; \underbrace{\boldsymbol{G}}_{\text{Instrumental gain}} \; \underbrace{\boldsymbol{D}}_{\text{Polarization leakage}} \; \underbrace{\boldsymbol{E}}_{\text{Primary beam}} \; \underbrace{\boldsymbol{P}}_{\text{Parallactic angle feed rotation}} \; \underbrace{\boldsymbol{T}}_{\text{Ionospheric and tropospheric effects}}$$

Component Jones matrices

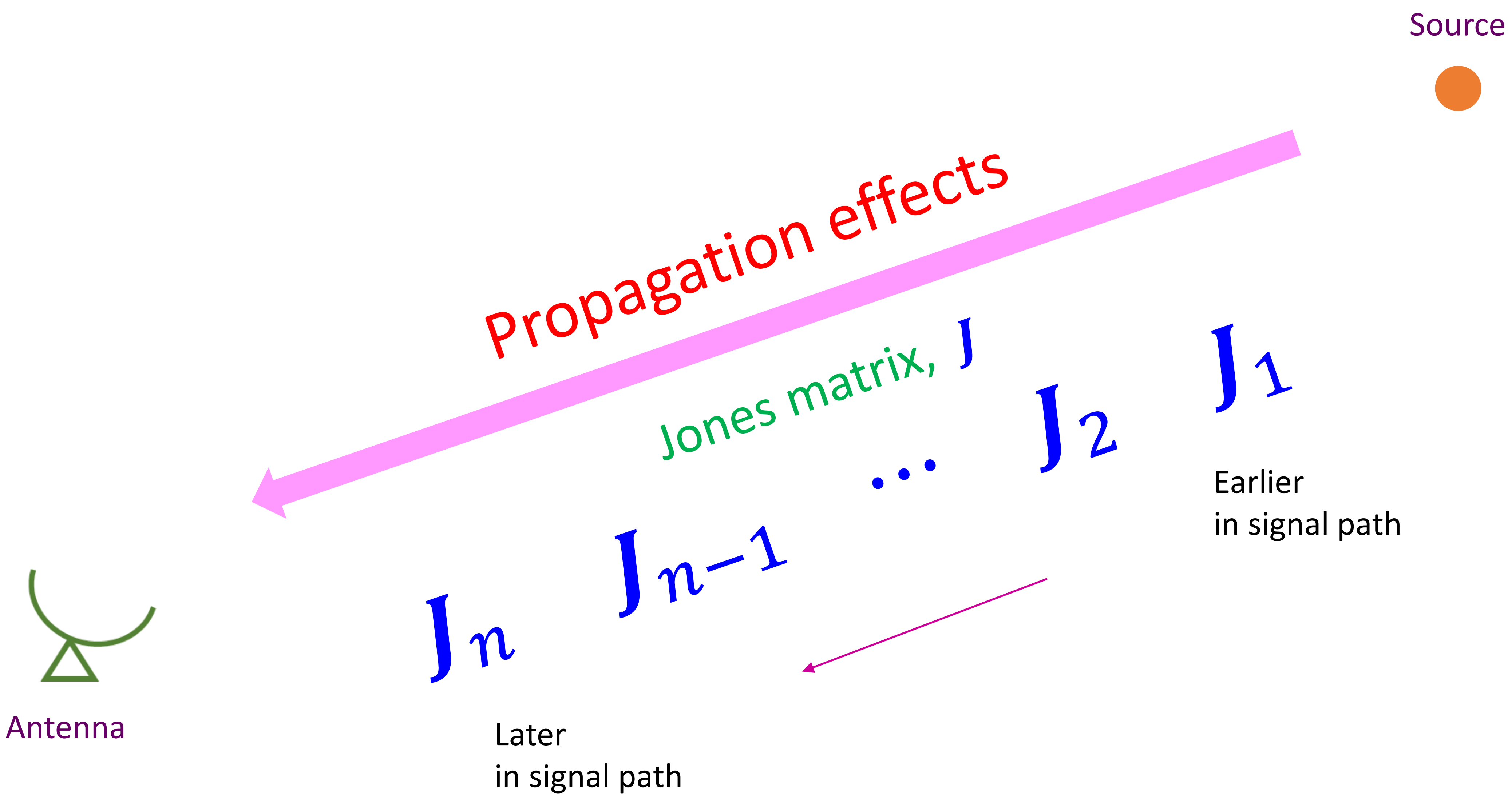
Jones chain:

$$\boldsymbol{J} = \boldsymbol{J}_n \, \boldsymbol{J}_{n-1} \cdots \boldsymbol{J}_2 \, \boldsymbol{J}_1$$

Later
in signal path

Earlier
in signal path

Component Jones matrices



Component Jones matrices

Antenna p : $$\boldsymbol{J}_p = \boldsymbol{J}_{pn}\, \boldsymbol{J}_{p(n-1)} \cdots \boldsymbol{J}_{p2}\, \boldsymbol{J}_{p1}$$

Antenna q : $$\boldsymbol{J}_q = \boldsymbol{J}_{qn}\, \boldsymbol{J}_{q(n-1)} \cdots \boldsymbol{J}_{q2}\, \boldsymbol{J}_{q1}$$

Visibility

Brightness

$$\mathbf{V}_{pq} = \boldsymbol{J}_p\, \mathbf{B}\, \boldsymbol{J}_q^H$$

Jones matrices

$$\mathbf{V}_{pq} = \boldsymbol{J}_{pn}\, \boldsymbol{J}_{p(n-1)} \cdots \boldsymbol{J}_{p2}\, \boldsymbol{J}_{p1}\, \mathbf{B}\, \boldsymbol{J}_{q1}^H\, \boldsymbol{J}_{q2}^H \cdots \boldsymbol{J}_{q(n-1)}^H\, \boldsymbol{J}_{qn}^H$$

$$\mathbf{V}_{pq} = \boldsymbol{J}_{pn}\Big(\boldsymbol{J}_{p(n-1)}\Big(\cdots\Big(\boldsymbol{J}_{p2}\Big(\boldsymbol{J}_{p1}\, \mathbf{B}\, \boldsymbol{J}_{q1}^H\Big)\boldsymbol{J}_{q2}^H\Big)\cdots\Big)\boldsymbol{J}_{q(n-1)}^H\Big)\boldsymbol{J}_{qn}^H$$

Direction-independent and direction-dependent effects

Propagation effects can be of two kinds:

- Direction-independent effects
- Direction-dependent effects

These effects can be represented by different Jones matrices:

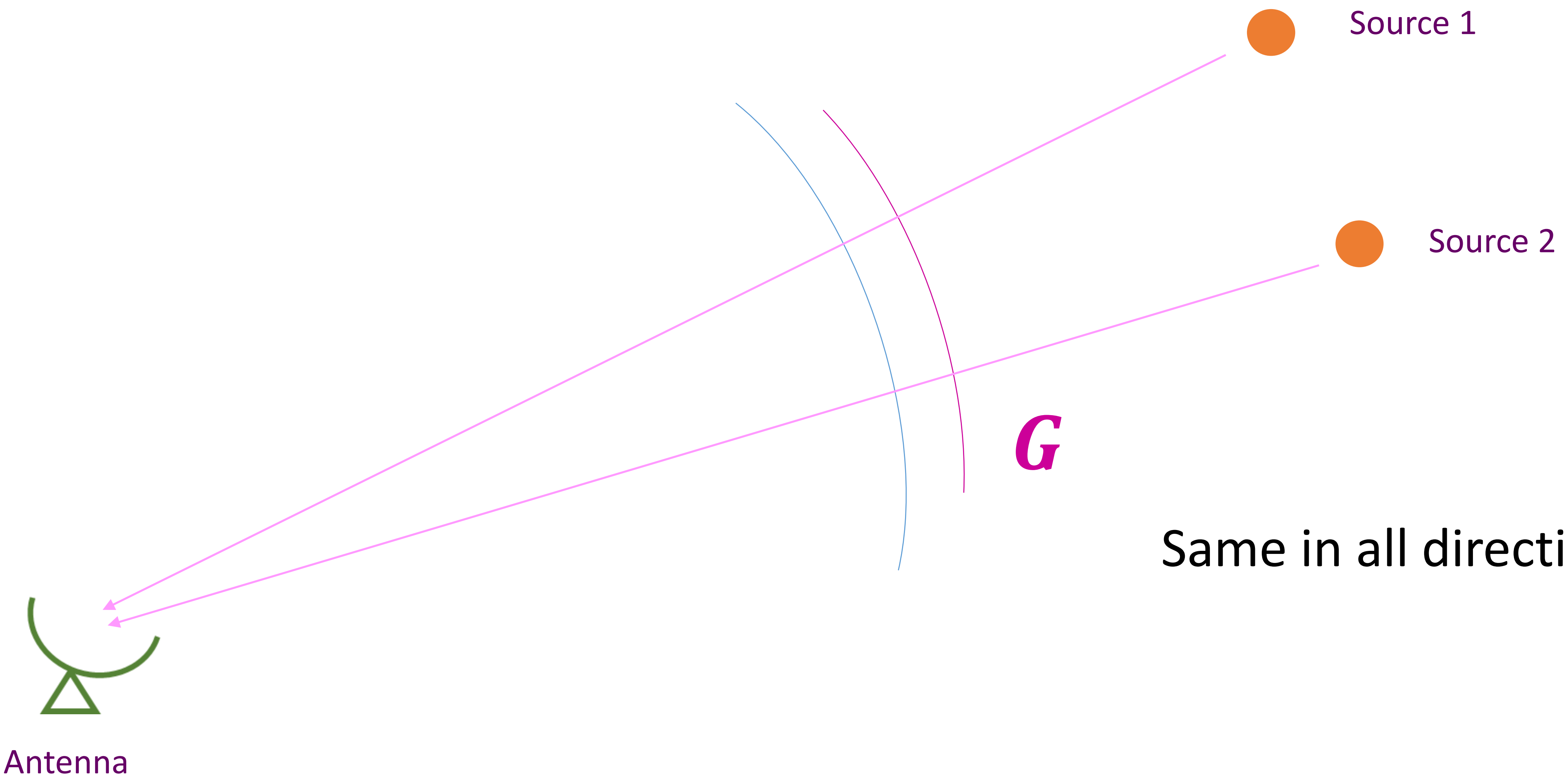
$$\boldsymbol{J} = \boldsymbol{G}\ \boldsymbol{E}$$

Final Jones matrix

Direction-independent effects

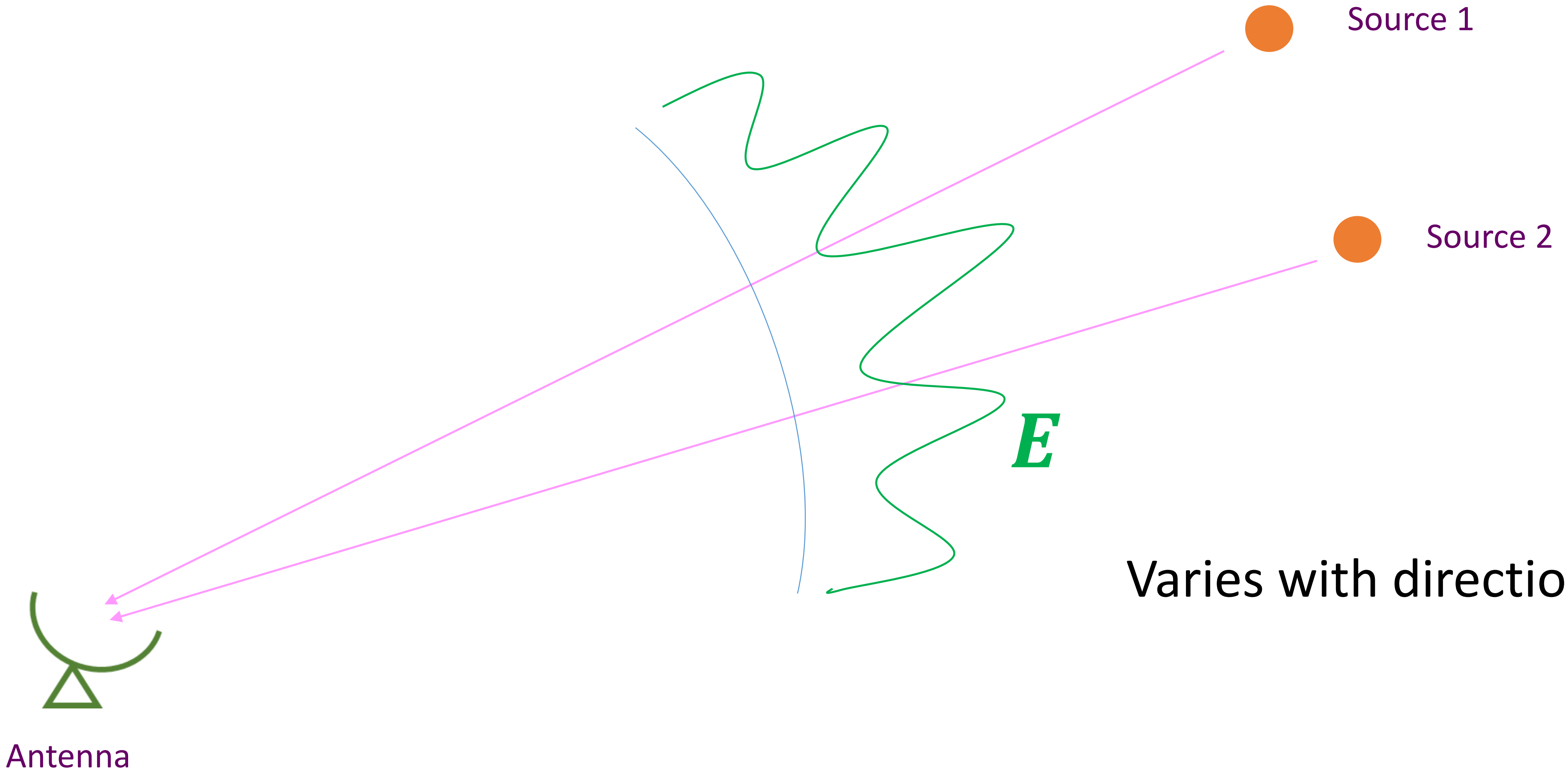
Direction-dependent effects

Direction-independent effects



Same in all directions

Direction-dependent effects



Varies with direction

Direction-independent and direction-dependent effects

$$\boldsymbol{J} = \boldsymbol{G}\ \boldsymbol{E}$$

Final Jones matrix

Direction-independent effects

Direction-dependent effects

Visibility-plane effects

Image-plane effects

$$\mathbf{V}_{pq} = \boldsymbol{J}_p\ \mathbf{B}\ \boldsymbol{J}_q^H$$

$$\mathbf{V}_{pq} = \boldsymbol{G}_p\left(\boldsymbol{E}_p \mathbf{B} \boldsymbol{E}_q^H\right)\boldsymbol{G}_q^H$$

Direction-independent and direction-dependent effects

Example:

$$\boldsymbol{J} = \boldsymbol{B}\ \boldsymbol{G}\ \boldsymbol{D}\ \boldsymbol{E}\ \boldsymbol{P}\ \boldsymbol{T}$$

Bandpass gain

Instrumental gain

Polarization leakage

Primary beam

Parallactic angle feed rotation

Ionospheric and tropospheric effects

Direction-dependent effects

Direction-independent effects

Explicit RIME and phenomenological RIME

- Explicit RIME:

$$\boldsymbol{J} = \boldsymbol{B}\ \boldsymbol{G}\ \boldsymbol{D}\ \boldsymbol{E}\ \boldsymbol{P}\ \boldsymbol{T}$$

Useful for understanding the component corrupting effects along the propagation path

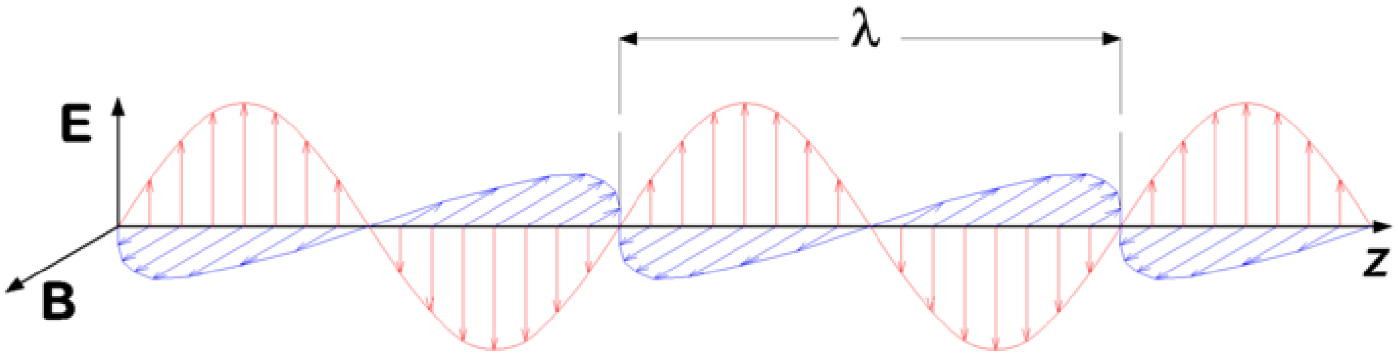
- Phenomenological RIME:

$$\boldsymbol{J} = \boldsymbol{G}\ \boldsymbol{E}$$

Useful for calibration, as these matrices are easier to solve for

Polarization

- Polarization of an electromagnetic wave describes the direction of oscillation of the electric field.

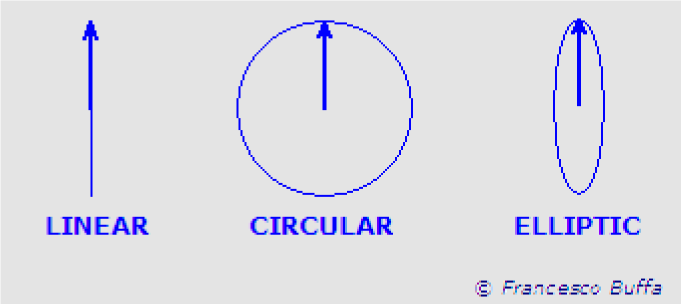


(From https://commons.wikimedia.org/wiki/File:Electromagnetic_wave.png, author: User:P.wormer)

- The plane of polarization is perpendicular to the direction of propagation of the wave.

Polarization

- Electromagnetic waves can be:
  - Linearly polarized
  - Circularly polarized
  - Elliptically polarized

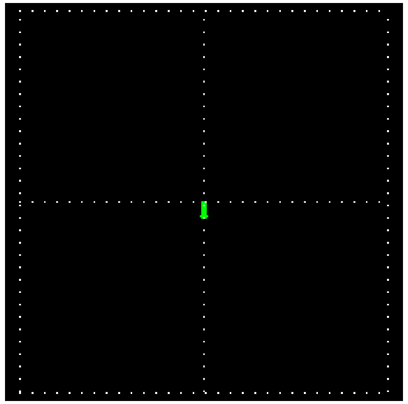


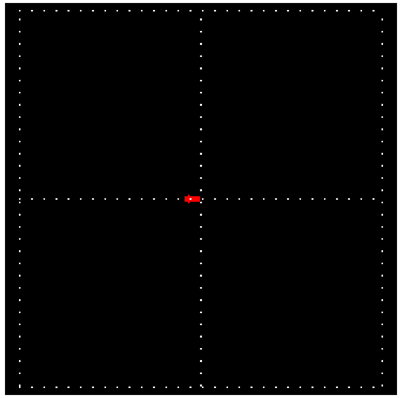
Antenna feeds

An antenna contains feeds, which detect and measure specific polarized components of an electromagnetic wave:

- Antenna with orthogonal linearly polarized feeds:

x-polarized feed

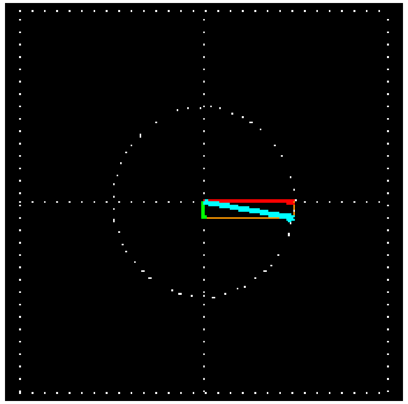


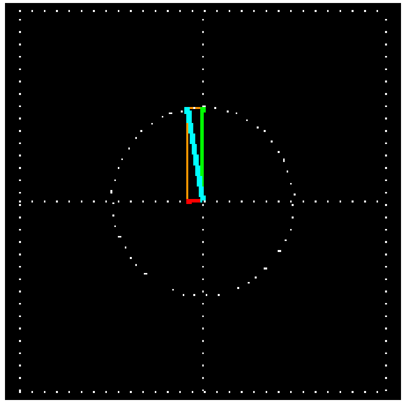


y-polarized feed

- Antenna with orthogonal circularly polarized feeds:

Right-circularly polarized (RCP) feed



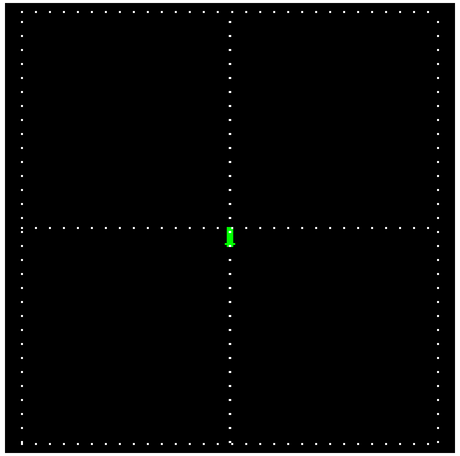


Left-circularly polarized (LCP) feed

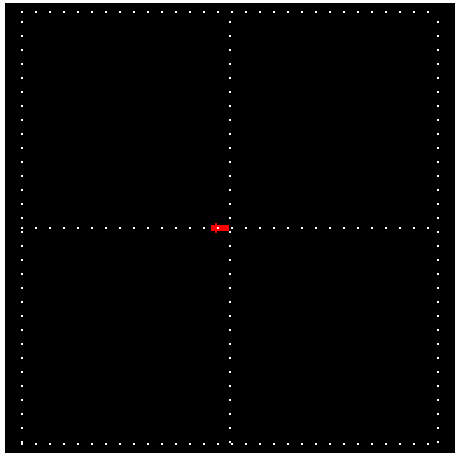
(From http://cddemo.szialab.org/)

Antenna feeds

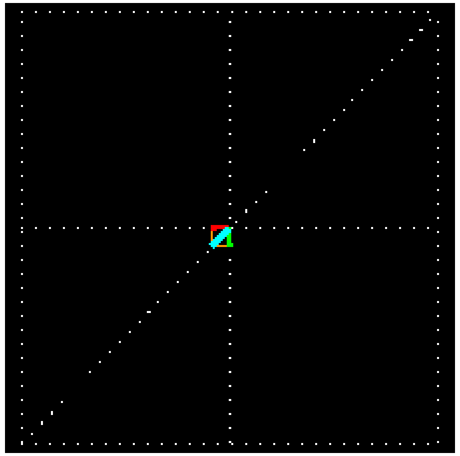
- Linearly polarized wave, measured by linearly polarized feeds:



x component



y component

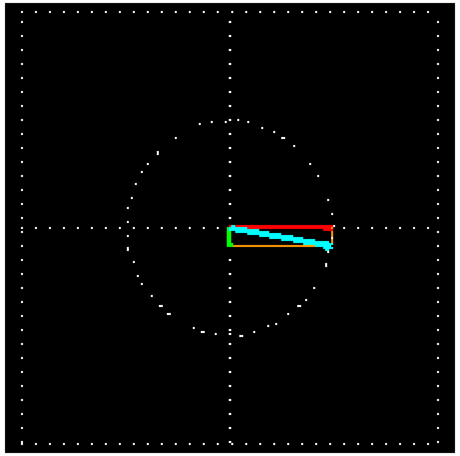


Combination of
x and y components

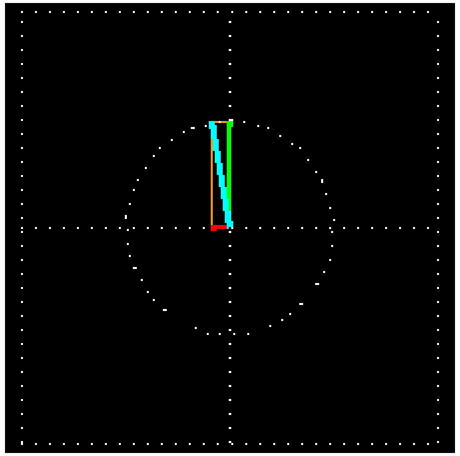
(From http://cddemo.szialab.org/)

Antenna feeds

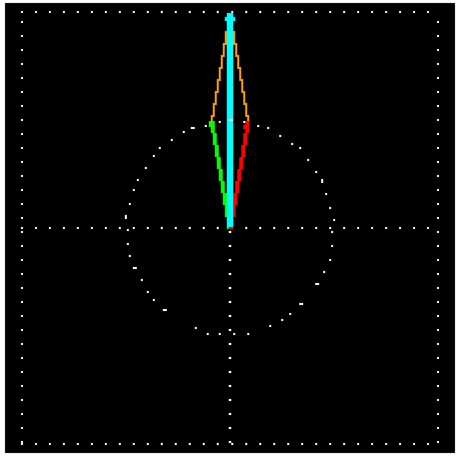
- Linearly polarized wave, measured by circularly polarized feeds:



RCP component



LCP component



Combination of
RCP and LCP components

(From http://cddemo.szialab.org/)

Structure of Jones matrices

Most Jones matrices have a simple form:

- $\boldsymbol{J}_{\text{rotation}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

- $\boldsymbol{J}_{\text{leakage}} = \begin{bmatrix} 1 & D^R \\ D^L & 1 \end{bmatrix}$

- $\boldsymbol{J}_{\text{gain}} = \begin{bmatrix} G^R & 0 \\ 0 & G^L \end{bmatrix}$

(in a linearly polarized basis)

Structure of Jones matrices

- The structure of individual Jones matrices depends on the antenna feed polarization basis, but the RIME is valid independent of the polarization basis.
- For example,
  - $J_{\text{rotation}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ in a linearly polarized basis
  - $J_{\text{rotation}} = \begin{bmatrix} e^{j\theta} & 0 \\ 0 & e^{-j\theta} \end{bmatrix}$ in a circularly polarized basis

Rotation matrices

- $J_{\text{rotation}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Examples:

Parallactic angle feed rotation:

$$\boldsymbol{P} = \begin{bmatrix} \cos\underset{\text{Parallactic angle}}{\alpha} & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

Ionospheric Faraday rotation:

$$\boldsymbol{T} = \begin{bmatrix} \cos\chi & -\sin\chi \\ \sin\chi & \cos\underset{\text{Faraday rotation angle}}{\chi} \end{bmatrix}$$

Leakage matrices

- $\boldsymbol{J}_{\text{leakage}} = \begin{bmatrix} 1 & D^R \\ D^L & 1 \end{bmatrix}$

Examples:

Polarization leakage: $\boldsymbol{D} = \begin{bmatrix} 1 & d^R \\ d^L & 1 \end{bmatrix}$

Polarization leakage terms

Gain matrices

- $\boldsymbol{J}_{\text{gain}} = \begin{bmatrix} G^R & 0 \\ 0 & G^L \end{bmatrix}$

Examples:

Instrumental gain: $\boldsymbol{G} = \begin{bmatrix} g^R & 0 \\ 0 & g^L \end{bmatrix} = \begin{bmatrix} a^R e^{j\phi^R} & 0 \\ 0 & a^L e^{j\phi^L} \end{bmatrix}$

Bandpass gain: $\boldsymbol{B} = \begin{bmatrix} B^R & 0 \\ 0 & B^L \end{bmatrix} = \begin{bmatrix} b^R(\nu) e^{j\psi^R(\nu)} & 0 \\ 0 & b^L(\nu) e^{j\psi^L(\nu)} \end{bmatrix}$

Order of Jones matrices in Jones chain

$$\boldsymbol{J} = \boldsymbol{J}_n \, \boldsymbol{J}_{n-1} \cdots \boldsymbol{J}_2 \, \boldsymbol{J}_1$$

- Order of matrices in the Jones chain is important, because matrix multiplication is not commutative, in general.
- However, specific kinds of matrices do commute – scalar matrices commute with all kinds of matrices, rotation matrices with each other, diagonal matrices with each other.

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