Non-Invertibility of Electromagnetic Observation Operators: Formal Structure and Physical Example

Methodological Section Draft

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Abstract

We formalize the structure of electromagnetic observation as a non-invertible projection operator mapping three-dimensional physical states to lower-dimensional observables. We prove that this operator possesses an infinite-dimensional kernel, rendering unique reconstruction impossible without regularization. The mass-sheet degeneracy in gravitational lensing is presented as an explicit physical example. This work establishes the mathematical foundation for understanding inference limitations in electromagnetic astrophysics.

1 Formal Setup

1.1 Physical State Space

Let Ω denote the space of three-dimensional mass-energy distributions within a spatial domain $V \subset \mathbb{R}^3$. A representative element $\omega \in \Omega$ is characterized by the mass density field:

$$\omega = \rho(\vec{x}), \quad \vec{x} = (x, y, z) \in V \tag{1}$$

where $\rho : V \to \mathbb{R}_{\geq 0}$ is square-integrable: $\rho \in L^2(V)$.

For galactic or cosmological applications, V may represent a galaxy halo, galaxy cluster, or cosmological volume. The dimension of Ω as a function space is infinite: $\dim(\Omega) = \infty$ (continuum of values $\rho(\vec{x})$ for each $\vec{x}$).

1.2 Electromagnetic Observable Space

Electromagnetic observations are mediated by photon detection. Relevant observables include:

$$\mathcal{D}_{\mathrm{EM}} = \{\Sigma(\vec{R}), I(\vec{\theta}), \gamma(\vec{\theta}), v_c(R), \ldots\} \tag{2}$$

where:

- $\Sigma(\vec{R})$: projected surface density at transverse coordinate $\vec{R} = (x, y)$
- $I(\vec{\theta})$: surface brightness at angular position $\vec{\theta}$
- $\gamma(\vec{\theta})$: gravitational lensing shear
- $v_c(R)$: circular velocity (rotation curve)

These observables are functions of two spatial dimensions (transverse to line of sight) or one radial dimension (for spherically symmetric systems).

1.3 Forward Observation Operator

The forward operator maps physical states to observables:

$$F_{\mathrm{EM}} : \Omega \to \mathcal{D}_{\mathrm{EM}} \tag{3}$$

Example 1 (Line-of-sight projection):

$$\Sigma(\vec{R}) = \int_{-\infty}^{\infty} \rho(\vec{R}, z)\, dz = \int_{-\infty}^{\infty} \rho(x, y, z)\, dz \tag{4}$$

This is an integral projection along the z-axis (line of sight).

Example 2 (Spherically symmetric systems):

For rotation curves in disk galaxies, assuming spherical symmetry:

$$v_c^2(R) = \frac{GM(< R)}{R} = \frac{G}{R} \int_0^R 4\pi r^2 \rho(r)\, dr \tag{5}$$

This maps 3D radial profile $\rho(r)$ to 1D observable $v_c(R)$.

Example 3 (Weak gravitational lensing):

The convergence $\kappa(\vec{\theta})$ is the projected dimensionless surface density:

$$\kappa(\vec{\theta}) = \frac{\Sigma(\vec{\theta})}{\Sigma_{\mathrm{crit}}} \tag{6}$$

where $\Sigma(\vec{\theta}) = \int \rho(D_d\vec{\theta}, z)\, dz$ and $\Sigma_{\mathrm{crit}} = \frac{c^2}{4\pi G} \frac{D_s}{D_d D_{ds}}$ (critical surface density depending on lens-source geometry).

2 Non-Invertibility Result

2.1 The Inverse Problem

Inverse problem statement: Given observation $d \in \mathcal{D}_{\mathrm{EM}}$, determine $\omega \in \Omega$ such that:

$$F_{\mathrm{EM}}[\omega] = d \tag{7}$$

The inverse operator $F_{\mathrm{EM}}^{-1} : \mathcal{D}_{\mathrm{EM}} \to \Omega$ (if it exists) would map observations back to physical states.

2.2 Demonstration of Non-Uniqueness

Theorem 2.1 (Non-uniqueness of line-of-sight projection)**.** *Let $\Sigma_0(\vec{R})$ be a given projected surface density. Then there exist infinitely many distinct 3D density distributions $\rho(\vec{x})$ satisfying:*

$$\Sigma_0(\vec{R}) = \int_{-\infty}^{\infty} \rho(\vec{R}, z)\, dz \tag{8}$$

Proof. Consider two density distributions:

$$\rho_1(\vec{R}, z) = \Sigma_0(\vec{R})\delta(z) \tag{9}$$

$$\rho_2(\vec{R}, z) = \frac{\Sigma_0(\vec{R})}{2h}\Theta(h - |z|) \tag{10}$$

where $\delta(z)$ is the Dirac delta, Θ is the Heaviside function, and $h > 0$ is arbitrary.

Both satisfy:

$$\int_{-\infty}^{\infty} \rho_1(\vec{R}, z)\, dz = \Sigma_0(\vec{R}) \tag{11}$$

$$\int_{-\infty}^{\infty} \rho_2(\vec{R}, z)\, dz = \Sigma_0(\vec{R}) \tag{12}$$

Yet $\rho_1 \neq \rho_2$ (one is a delta function, the other is a uniform slab of arbitrary thickness $2h$).

Since h is a free parameter, there exist infinitely many such distributions. □

Corollary 2.2. *The preimage $F_{EM}^{-1}(\Sigma_0)$ is infinite-dimensional.*

2.3 Characterization of the Kernel

Definition 2.3 (Kernel)**.** The kernel of F_{EM} consists of all perturbations $\delta\rho \in \Omega$ that produce no change in observables:

$$\ker(F_{\mathrm{EM}}) = \{\delta\rho \in \Omega : F_{\mathrm{EM}}[\rho + \delta\rho] = F_{\mathrm{EM}}[\rho]\ \forall \rho\} \tag{13}$$

Theorem 2.4 (Kernel of line-of-sight projection)**.** *For line-of-sight projection $\Sigma(\vec{R}) = \int \rho(\vec{R}, z)\, dz$, the kernel consists of all $\delta\rho(\vec{R}, z)$ satisfying:*

$$\int_{-\infty}^{\infty} \delta\rho(\vec{R}, z)\, dz = 0 \quad \forall \vec{R} \tag{14}$$

Proof. If $F[\rho + \delta\rho] = F[\rho]$, then:

$$\int [\rho(\vec{R}, z) + \delta\rho(\vec{R}, z)] dz = \int \rho(\vec{R}, z)\, dz \tag{15}$$

This requires:

$$\int \delta\rho(\vec{R}, z)\, dz = 0 \tag{16}$$

Conversely, any $\delta\rho$ satisfying this condition leaves $\Sigma(\vec{R})$ unchanged. □

Dimension of kernel:

The space of functions satisfying $\int \delta\rho(\vec{R}, z)\, dz = 0$ has dimension:

$$\dim(\ker F) = \infty \tag{17}$$

This is an infinite-dimensional subspace of $L^2(V)$ (uncountably many degrees of freedom in z-structure are unobservable).

3 Physical Example: Gravitational Lensing Mass-Sheet Degeneracy

3.1 Setup

Consider a lens at redshift z_d and source at z_s. Weak lensing observables are the convergence $\kappa(\vec{\theta})$ and shear $\gamma(\vec{\theta})$.

The convergence is related to the projected surface density via:

$$\kappa(\vec{\theta}) = \frac{\Sigma(\vec{\theta})}{\Sigma_{\text{crit}}} \tag{18}$$

where:

$$\Sigma(\vec{\theta}) = \int \rho(D_d\vec{\theta}, z)\, dz \tag{19}$$

is the projection of 3D density $\rho(\vec{x})$ along the line of sight.

3.2 Mass-Sheet Degeneracy

Theorem 3.1 (Mass-sheet transformation; Falco et al. (1985); Schneider & Seitz (1995))**.** *The reduced shear:*

$$g(\vec{\theta}) = \frac{\gamma(\vec{\theta})}{1 - \kappa(\vec{\theta})} \tag{20}$$

is invariant under the transformation:

$$\kappa(\vec{\theta}) \to \kappa_\lambda(\vec{\theta}) = \lambda\kappa(\vec{\theta}) + (1 - \lambda) \tag{21}$$

for any constant $\lambda \in \mathbb{R}$.

Proof. Under the transformation $\kappa \to \kappa_\lambda = \lambda\kappa + (1 - \lambda)$, the shear transforms as $\gamma \to \gamma_\lambda = \lambda\gamma$.

The reduced shear becomes:

$$g_\lambda = \frac{\gamma_\lambda}{1 - \kappa_\lambda} = \frac{\lambda\gamma}{1 - \lambda\kappa - (1 - \lambda)} = \frac{\lambda\gamma}{\lambda(1 - \kappa)} = \frac{\gamma}{1 - \kappa} = g \tag{22}$$

Thus $g_\lambda = g$, and the reduced shear is unchanged. □

Physical interpretation:

Weak lensing measures only the reduced shear $g(\vec{\theta})$, not κ and γ independently. The transformation:

$$\Sigma(\vec{\theta}) \to \lambda\Sigma(\vec{\theta}) + (1 - \lambda)\Sigma_{\text{crit}} \tag{23}$$

corresponds to adding a uniform mass sheet of surface density $(1-\lambda)\Sigma_{\text{crit}}$ while rescaling the lens by λ. Since λ is a free parameter, infinitely many surface densities produce identical lensing observables.

3.3 Explicit Kernel Element

For a given convergence profile $\kappa_0(\vec{\theta})$, define the one-parameter family:

$$\kappa_\lambda(\vec{\theta}) = \lambda\kappa_0(\vec{\theta}) + (1 - \lambda) \tag{24}$$

All members of this family yield the same reduced shear:

$$g(\vec{\theta}) = \frac{\gamma_0(\vec{\theta})}{1 - \kappa_0(\vec{\theta})} \tag{25}$$

The difference:

$$\delta\kappa = \kappa_{\lambda'} - \kappa_\lambda = (\lambda' - \lambda)[\kappa_0(\vec{\theta}) - 1] \tag{26}$$

lies in the kernel: $F_{\text{EM}}[\kappa_0 + \delta\kappa] = F_{\text{EM}}[\kappa_0]$ (both produce the same g).

Consequence: Weak lensing data alone cannot uniquely determine $\Sigma(\vec{\theta})$ without additional constraints.

3.4 Degeneracy in 3D Reconstruction

Even if the mass-sheet degeneracy were broken (e.g., using time delays in strong lensing; Refsdal (1964)), determining $\Sigma(\vec{\theta})$ uniquely, the 3D density $\rho(\vec{x})$ remains underdetermined because:

$$\Sigma(\vec{R}) = \int_{-\infty}^{\infty} \rho(\vec{R}, z)\, dz \tag{27}$$

Any redistribution of mass along the line of sight satisfying $\int \delta\rho(\vec{R}, z)\, dz = 0$ leaves $\Sigma(\vec{R})$ unchanged.

Example: A spherical halo $\rho(r) = \rho_0/(1 + r/r_s)^2$ and a flattened halo $\rho(R, z) = \rho_0/[(1 + \sqrt{R^2 + (qz)^2}/r_s)^2]$ (where q is axis ratio) can produce nearly identical $\Sigma(R)$ for appropriate choice of q and r_s, despite having different 3D geometries.

4 Role of Regularization

4.1 Regularization as Methodological Necessity

Given the non-invertibility of F_{EM} and the infinite-dimensionality of $\ker(F_{\text{EM}})$, selecting a unique solution from the preimage $F_{\text{EM}}^{-1}(d)$ requires additional constraints. These are introduced via **regularization**.

Definition 4.1 (Regularized inverse problem)**.** Find $\omega_{\text{reg}} \in \Omega$ that minimizes:

$$\mathcal{J}[\omega] = \|F_{\text{EM}}[\omega] - d\|^2 + \alpha\mathcal{R}[\omega] \tag{28}$$

where:

- $\|F_{\text{EM}}[\omega] - d\|^2$ is the data misfit (chi-squared)
- $\mathcal{R}[\omega]$ is a regularization functional
- $\alpha > 0$ is the regularization parameter (Lagrange multiplier)

The regularization functional $\mathcal{R}[\omega]$ imposes smoothness, symmetry, or other structural constraints.

4.2 Standard Regularization Functionals

Example 1 (Tikhonov regularization):

$$\mathcal{R}[\rho] = \int |\nabla\rho|^2 d^3x \tag{29}$$

Penalizes rapid spatial variations, enforcing smoothness.

Example 2 (Maximum entropy):

$$\mathcal{R}[\rho] = -\int \rho \ln\left(\frac{\rho}{\rho_0}\right) d^3x \tag{30}$$

Selects the density distribution with maximum entropy relative to prior ρ_0.

Example 3 (Parameterization):

Restrict Ω to a finite-dimensional family:

$$\rho(\vec{x}) = \rho(\vec{x}; \theta_1, \ldots, \theta_n) \tag{31}$$

where θ_i are free parameters. Example: NFW profile for dark matter halos:

$$\rho(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2} \tag{32}$$

with parameters $\{\rho_s, r_s\}$.

Example 4 (Symmetry constraints):

Assume spherical symmetry: $\rho(\vec{x}) = \rho(r)$, reducing dimensionality from ∞ (3D field) to ∞ (1D function) but eliminating angular structure.

4.3 What Regularization Does Not Provide

Regularization supplements select a particular element from the preimage $F_{\mathrm{EM}}^{-1}(d)$ by imposing additional structure. However:

1. **Regularization does not eliminate the kernel.** The transformation $\rho \to \rho + \delta\rho$ with $\delta\rho \in \ker(F_{\mathrm{EM}})$ still satisfies $F_{\mathrm{EM}}[\rho + \delta\rho] = F_{\mathrm{EM}}[\rho]$. Regularization penalizes such transformations via $\mathcal{R}[\rho + \delta\rho]$ but does not make them physically impossible.

2. **Regularization is a methodological choice.** The functional $\mathcal{R}[\omega]$ and parameter α are selected by the investigator based on prior expectations (smoothness, symmetry, simplicity) or computational convenience. Different choices yield different ω_{reg}.

3. **Regularization does not provide microstructure.** Selecting a smooth or parameterized $\rho(\vec{x})$ does not imply the true density is smooth or has that functional form—only that the data are consistent with such a distribution and additional structure cannot be resolved.

4.4 Boundary Supplements

In practice, regularization is supplemented by:

- **Boundary conditions:** Asymptotic behavior (e.g., $\rho(r) \to 0$ as $r \to \infty$)
- **Physical priors:** Non-negativity ($\rho \geq 0$), total mass constraints ($\int \rho \, d^3x = M_{\mathrm{tot}}$)
- **Calibration data:** Independent measurements (e.g., stellar kinematics, strong lensing time delays) that constrain specific aspects of ρ

These are collectively termed **boundary supplements**—additional constraints required to close the inverse problem.

5 Conclusion

We have demonstrated that the electromagnetic observation operator $F_{\mathrm{EM}} : \Omega \to \mathcal{D}_{\mathrm{EM}}$, which maps three-dimensional mass-energy distributions to two-dimensional (or one-dimensional) photon-based observables, is non-invertible. Specifically:

1. **Non-uniqueness:** For any observation $d \in \mathcal{D}_{\mathrm{EM}}$, the preimage $F_{\mathrm{EM}}^{-1}(d)$ contains infinitely many distinct elements $\omega \in \Omega$.

2. **Kernel structure:** The kernel $\ker(F_{\text{EM}})$ is infinite-dimensional, consisting of all density perturbations $\delta\rho(\vec{x})$ that integrate to zero along the line of sight: $\int \delta\rho(\vec{R}, z)\, dz = 0$.

3. **Physical example:** Gravitational lensing exhibits the mass-sheet degeneracy, where infinitely many surface densities $\Sigma(\vec{\theta})$ produce identical reduced shear $g(\vec{\theta})$. Even if this degeneracy is broken, 3D structure along the line of sight remains underdetermined.

4. **Regularization necessity:** Selecting a unique density distribution from observational data requires regularization—imposing smoothness, symmetry, parameterization, or other constraints via a penalty functional $\mathcal{R}[\rho]$. These are methodological supplements, not physical derivations.

The formal structure established here applies to any electromagnetic observation involving line-of-sight projection or dimensional reduction: rotation curves (3D $\to$ 1D), surface brightness profiles (3D $\to$ 2D), lensing convergence (3D $\to$ 2D), and spectroscopic measurements integrated over solid angle. In each case, $\dim(\ker F_{\text{EM}}) > 0$, rendering inversion non-unique without additional constraints.

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