May 6, 2020

Lemma 1. *The homogeneous cliques in a vertex-critical counterexample to BK have at most three vertices.*

Proof. Suppose G is a vertex-critical counterexample to BK. Choose homogeneous clique $S \subseteq V(G)$ maximizing $|S|$. Put $T := N(S) \setminus S$. By homogeneity, S is joined to T. Plainly, $|S| + |T| \geq \Delta(G) + 1$.

Suppose the lemma is false. Then $|S| \geq 4$. By Lemma 4.49 in original mules $|T| \leq 3$. By Lemma 4.29 in original mules there is no induced $K_6 \vee E_3$, so we must have $|S| \leq 5$. But then $\Delta(G) + 1 \geq 10 > 8 \geq |S| + |T|$, a contradiction.

□