Generalized orientations for list coloring

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1 Basic notions

For a set S, let $\mathbb{N}^S$ be all functions from S to $\mathbb{N}$. Let $T \subseteq S$. For $g \in \mathbb{N}^S$ and $h \in \mathbb{N}^T$, define $g + h \in \mathbb{N}^S$ by $(g + h)(x) = g(x) + h(x)$ for $x \in T$ and $(g + h)(x) = g(x)$ for $x \in S \setminus T$. Define $g - h \in \mathbb{N}^S$ by $(g - h)(x) = \max\{0, g(x) + h(x)\}$ for $x \in T$ and $(g + h)(x) = g(x)$ for $x \in S \setminus T$. For $g \in \mathbb{N}^S$ let $g + \mathbb{N}^T$ be all functions of the form $g + h$ where $h \in \mathbb{N}^T$. Let $1_T \in \mathbb{N}^T$ be given by $1_T(x) = 1$ for $x \in T$. When $T = \{u\}$, we write 1_u in place of $1_{\{u\}}$. If $a : S \to \mathbb{N}$ and $b : S \to \mathbb{N}$, then $a > b$ just in case $a(v) > b(v)$ for all $v \in S$. Extend all these definitions to functions with codomain $\mathbb{Q}$ as well with the proviso that a sum or difference operation on functions with different codomains results in a function whose codomain in the union of the input codomains. If functions with different domains are compared, the comparison is on the intersection of the domains.

If G is a graph and $f : V(G) \to \mathbb{N}$, then G is *online f-list-colorable* just in case for every nonempty $S \subseteq V(G)$ there is a nonempty independent set $F \subseteq S$ such that $G - F$ is online $(f - 1_{S \setminus F})$-list-colorable.

2 Generalized orientations

Let G be a graph. For $v \in V(G)$, let $E(v)$ be the edges incident to v. For a finite set $W \subset \mathbb{Q}$, a W-orientation of G is an assignment $\pi(v, e) \in W$ to each pair (v, e) where $e \in E(v)$ such that

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(1) $\pi(v, vw) + \pi(w, wv) = 1$ for each $vw \in E(G)$; and

(2) $\sum_{e \in E(v)} \pi(v, e) \geq 0$ for all $v \in V(G)$.

When $W = \{0, 1\}$, W-orientations are equivalent to the usual notion of orienting a graph by directing its edges. Each W-orientation π of G induces a *score function* $h_\pi \colon V(G) \to \mathbb{Q}_{\geq 0}$ given by $h_\pi(v) := \sum_{e \in E(v)} \pi(v, e)$. By (1), $\sum_{v \in V(G)} h_\pi(v) = \|G\|$. Let $\eta(W, G, D)$ be shorthand for "the number of W-orientations π of G with $h_\pi \in D$". Let $\kappa(W)$ be the number of $(a, b) \in W \times W$ with $a + b = 1$.

Theorem 1 *Let G be a graph, $f \colon V(G) \to \mathbb{N}$ and $g \colon V(G) \to \mathbb{Q}$ with $f > g$. If $W \subset \mathbb{Q}$ is finite and the number of W-orientations π of G with $h_\pi = g$ is not a multiple of $\kappa(W)$, then G is online f-list-colorable.*

By induction on $|G|$, it will suffice to find, for each nonempty $S \subseteq V(G)$, a nonempty independent set $F \subseteq S$ and $g' \colon V(G - F) \to \mathbb{Q}$ such that $f - 1_{S \setminus F} > g'$ and the number of W-orientations π of $G - F$ with $h_\pi = g'$ is not a multiple of $\kappa(W)$.

Fix nonempty $S \subseteq V(G)$. Since $\sum_{v \in V(G)} h_\pi(v) = \|G\|$ for every W-orientation π of G, it follows that $\eta(W, G, \{g\}) = \eta(W, G, g + \mathbb{N}^S)$. In particular, $\eta(W, G, g - 1_\emptyset + \mathbb{N}^{S \setminus \emptyset})$ is not a multiple of $\kappa(W)$. This allows picking $X \subseteq S$ maximal such that there is $Y \subseteq X$ where $\eta\left(W, G, (g - 1_X) + \mathbb{N}^{S \setminus Y}\right)$ is not a multiple of $\kappa(W)$. Suppose $X \neq S$. Pick $v \in S \setminus X$. Put $X' := X \cup \{v\}$ and $Y' := Y \cup \{v\}$. Then $(g - 1_{X'}) + \mathbb{N}^{S \setminus Y}$ is the disjoint union of $(g - 1_{X'}) + \mathbb{N}^{S \setminus Y'}$ and $(g - 1_X) + \mathbb{N}^{S \setminus Y}$. By maximality of X, $\eta\left(W, G, (g - 1_{X'}) + \mathbb{N}^{S \setminus Y}\right)$ is a multiple of $\kappa(W)$. But the same holds for $\eta\left(W, G, (g - 1_{X'}) + \mathbb{N}^{S \setminus Y'}\right)$, so $\eta\left(W, G, (g - 1_X) + \mathbb{N}^{S \setminus Y}\right)$ must be a multiple of $\kappa(W)$, a contradiction. Hence $X = S$.

So there is $Y \subseteq S$ such that $\eta\left(W, G, (g - 1_S) + \mathbb{N}^{S \setminus Y}\right)$ is not a multiple of $\kappa(W)$. Suppose $G[S \setminus Y]$ contains an edge vw.

3 Applications