graph coloring tools

a wee keep company collaborative creation

Contents

preface

this comes prior to the face.

graphs

A *graph* is a collection of dots we call *vertices* some of which are connected by curves we call *edges*. The relative location of the dots and the shape of the curves are not relevant, we are only concerned with whether or not a given pair of dots is connected by a curve. Initially, we forbid edges from a vertex to itself and multiple edges between two vertices. If G is a graph, then $V(G)$ is its set of vertices and $E(G)$ its set of edges. We write $|G|$ for the number of vertices in $V(G)$ and $\|G\|$ for the number of edges in $E(G)$. Two vertices are *adjacent* if they are connected by an edge. The set of vertices to which v is adjacent is its *neighborhood*, written $N(v)$. For the size of v's neighborhood $|N(v)|$, we write $d(v)$ and call this the *degree* of v.

vertices

edges

$V(G)$, $E(G)$

$|G|$, $\|G\|$

adjacent

neighborhood

$N(v)$

[ADD PICTURES]

$d(v)$, degree

coloring vertices

The entire book concerns one simple task: we want to color the vertices of a given graph so that adjacent vertices receive different colors. With sufficiently many crayons and no preferences about what the coloring should look like, this is easy, we just use a different crayon for each vertex. Things get interesting when we ask how few different crayons we can use. We are definitely going to need an empty box of crayons and that will only do for the graph with no vertices at all. Given one crayon, we can handle all graphs with no edges. With two crayons, we can do any path and any cycle with an even number of vertices [PICTURE]. But, we can't handle a triangle or any other cycle with an odd number of vertices [PICTURE]. In fact, odd cycles are really the only thing that will prevent us from using just two crayons. A graph H is a *subgraph* of a graph G, written $H \subseteq G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. When $H \subseteq G$, we say that G *contains* H. If $v \in V(G)$, then $G - v$ is the graph we get by removing v and all edges incident to v from G. A graph is k-colorable if we can color its vertices with (at most) k colors such that adjacent vertices receive different colors. A 0-colorable graph is *empty*, a 1-colorable graph is *edgeless* and a 2-colorable graph is *bipartite*.

subgraph, $\subseteq$
contains
$G - v$

k-colorable

empty
edgeless
bipartite

Theorem 1. *A graph is 2-colorable just in case it contains no odd cycle.*

Proof. A graph containing an odd cycle clearly can't be 2-colored. For the other implication, suppose there is a graph that is not 2-colorable and doesn't contain an odd cycle. Then we may pick such a graph G with $|G|$ as small as possible. Surely, $|G| > 0$, so we may pick $v \in V(G)$. If $x, y \in N(v)$, then x is not adjacent to y since then xyz would be an odd cycle. So we can construct a graph H from G by removing v and identifying all of $N(v)$ to a new vertex x_v. Any odd cycle in H would contain x_v and hence give rise to an odd cycle in G passing through v. So H contains no odd cycle. Since $|H| < |G|$, appplying the theorem to H gives a 2-coloring of H, say with red and blue where x_v gets colored red. But this gives a 2-coloring of G by coloring all vertices in $N(v)$ red and v blue, a contradiction. $\square$

Well, this is embarrassing, coloring appears to be easy. Fortunately, things get more interesting when we move up to three colors.

Theorem 2. *3-coloring is hard supposing other things we think are hard are actually hard.*

Proof. We need a concise proof of this without having to introduce too much background. Please submit a pull request on GitHub. $\square$

basic estimates

Even though finding the minimum number of colors needed to color a graph is hard in general (supposing it is), we can still look for lower and upper bounds on this value. The *chromatic number* $\chi(G)$ of a graph G is the smallest k for which G is k-colorable. The simplest thing we can do is give each vertex a different color.

chromatic number $\chi(G)$

THEOREM 3. *If G is a graph, then $\chi(G) \leq |G|$.*

The only graphs that attain the upper bound in Theorem 3 are the *complete* graphs; those in which any two vertices are adjacent. We can usually do much better by just arbitrarily coloring vertices, reusing colors when we can. The *maximum degree* $\Delta(G)$ of a graph G is the largest degree of any vertex in G; that is

complete

maximum degree $\Delta(G)$

$$\Delta(G) := \max_{v \in V(G)} d(v).$$

THEOREM 4. *If G is a graph, then $\chi(G) \leq \Delta(G) + 1$.*

PROOF. Suppose there is a graph G that is not $(\Delta(G)+1)$-colorable. Then we may pick such a graph G with $|G|$ as small as possible. Surely, $|G| > 0$, so we may pick $v \in V(G)$. Then $|G - v| < |G|$ and $\Delta(G - v) \leq \Delta(G)$, so applying the theorem to $G - v$ gives a $(\Delta(G - v) + 1)$-coloring of $G - v$. But v has at most $\Delta(G)$ neighbors, so there is some color, say red, not used on $N(v)$, coloring v red gives a $(\Delta(G) + 1)$-coloring of G, a contradiction. □

Both complete graphs and odd cycles attain the upper bound in Theorem 4. Theorem 1 says we can do better for graphs that don't contain odd cycles. We can also do better for graphs that don't contain large complete subgraphs. A set of vertices S in a graph G is a *clique* if the vertices in S are pairwise adjacent. The *clique number* of a graph G, written $\omega(G)$, is the number of vertices in a largest clique in G.

clique

$\omega(G)$

THEOREM 5. *If G is a graph, then $\chi(G) \geq \omega(G)$.*

A set of vertices S in a graph G is *independent* if the vertices in S are pairwise non-adjacent. The *independence number* of a graph G, written $\alpha(G)$, is the number of vertices in a largest independent set in G.

independent

$\alpha(G)$

THEOREM 6. *If G is a graph with $\Delta(G) \geq 3$ and $\omega(G) \leq \Delta(G)$, then $\chi(G) \leq \Delta(G)$.*

PROOF. Suppose there is a graph G with $\Delta(G) \geq 3$ and $\omega(G) \leq \Delta(G)$ that is not $\Delta(G)$-colorable. Then we may pick such a graph G with $|G|$ as small as possible. Let S be a maximal independent set in G. Since S is maximal, every vertex in $G - S$ has a neighbor in S, so $\Delta(G) > \Delta(G - S)$. If red is an unused color in a $\chi(G-S)$-coloring of $G-S$, then by coloring all vertices in S red we get a $(\chi(G - S) + 1)$-coloring of G. So, $\Delta(G) + 1 \leq \chi(G) \leq \chi(G - S) + 1$. We conclude $\chi(G - S) > \Delta(G - S)$ and thus $\Delta(G) = \chi(G - S) = \Delta(G - S) + 1$ by Theorem 4. Since $|G - S| < |G|$, applying the theorem to $G - S$ shows that $\Delta(G - S) < 3$ or $\Delta(G - S) < \omega(G - S)$. So, either $\chi(G - S) = \Delta(G) = 3$ or $\omega(G - S) \geq \Delta(G)$. In the former case, let X be the vertex set of an odd cycle in $G - S$ guaranteed by Theorem 1. In the latter case, let X be a $\Delta(G)$-clique in $G - S$.

Since S is maximal and $\omega(G) \leq \Delta(G)$, there are $x_1, x_2 \in X$ and $y_1, y_2 \in S$ such that x_1 is adjacent to y_1 and x_2 is adjacent to y_2. Construct a graph H from $G - X$ by adding the edge y_1y_2. Since $|H| < |G|$, applying the theorem to H shows that $\omega(H) > \Delta(G)$ or $\chi(H) \leq \Delta(G)$. Suppose $\chi(H) \leq \Delta(G)$. Then there is a $\Delta(G)$-coloring of $G - X$ where y_1 and y_2 receive different colors, say red and blue respectively. Pick the first vertex z in a shortest path P from x_1 to x_2 in X that has a blue colored neighbor in $V(H)$. Each vertex in X has $\Delta(G) - 1$ neighbors in X and hence at most one neighbor in $V(H)$. So, $z \neq x_1$ since x_1 already has a red colored neighbor in $V(H)$. Let w be be the vertex preceding z on P. Then w has no blue colored neighbor. Since X is the vertex set of a cycle or a complete graph, there is a path Q from w to z passing through every vertex of X. Color w blue and then proceed along Q, coloring one vertex at a time. Since each vertex we encounter before we get to z has at most $\Delta(G) - 1$ colored neighbors, we always have an available color to use. But, z is adjacent to both w and another blue colored vertex in $V(H)$, so there is an available color for z as well. This gives a $\Delta(G)$-coloring of G, a contradiction.

So, $\omega(H) > \Delta(G)$. In particular, y_1 and y_2 each have exactly one neighbor in X and $\Delta(G) - 1$ neighbors in the same $\Delta(G) - 1$ clique A in $G - X$. Since S is maximal and $|X| \geq 3$, there must be adjacent $x_3 \in X \backslash \{x_1, x_2\}$ and $y_3 \in S \backslash \{y_1, y_2\}$. Applying the same argument with x_3, y_3 in place of x_2, y_2 shows that y_1 and y_3 each have exactly one neighbor in X and $\Delta(G) - 1$ neighbors in the same $\Delta(G) - 1$ clique B in $G - X$. Now $|A \cap B| = |A| + |B| - |A \cup B| \geq 2(\Delta(G) - 1) - d(y_1) \geq \Delta(G) - 2 > 0$. But there can't be a vertex in $A \cap B$ since it would be adjacent to y_1, y_2, y_3 as well as $\Delta(G) - 2$ vertices in A and thus have degree greater than $\Delta(G)$, a contradiction.

list coloring

When attempting to k-color a graph G, it will often be convenient to first k-color $G[S]$ for some $S \subset V(G)$ and then try to k-color $G - S$ in a compatible manner. To make this precise, think of each vertex in G starting with a list of k permissible colors, say $[k]$. When we k-color $G[S]$, the colors used on $N(v) \cap S$ are no longer permissible for each $v \in V(G - S)$. For $v \in V(G - S)$, let $L(v)$ be the permissible colors for v after k-coloring $G[S]$. Now our problem is to pick $c_v \in L(v)$ for each $v \in V(G - S)$ such that $c_x \neq c_y$ whenever $xy \in E(G - S)$. This is the *list coloring* problem.

The list coloring problem arose as a subproblem in our attempt to k-color a graph. By taking it out of this context and viewing list coloring as a first-class problem in its own right, we will be able to prove more general theorems while also simplifying proofs. A *list assignment* on a graph G gives a set of colors $L(v)$ to each $v \in V(G)$. If there is $c_v \in L(v)$ for each $v \in V(G)$ such that $c_x \neq c_y$ whenever $xy \in E(G)$, then G is *L-colorable.*

Theorem 7. *If L is a list assignment on a graph G such that $|L(v)| > d(v)$ for all $v \in V(G)$, then G is L-colorable.*

Proof. Color each vertex in turn using a color in its list not appearing on any of its colored neighbors. This succeeds since each vertex has more permissible colors than neighbors. □

The requirement $|L(v)| > d(v)$ in Theorem 7 is quite strong, in the algorithm we really only need $L(v)$ to have more colors than *colored* neighbors rather than more colors than neighbors. We can encode this extra information by *orienting* the edges of G; that is, turning each edge into an arrow pointed one way or the other. If G is an oriented graph, the *out-degree* of a vertex $v \in V(G)$, written $d^+(v)$, is the number of arrows pointing away from v. An oriented graph is *acyclic* if there is no sequence of arrows that ends where it starts. An oriented graph is L-colorable just in case its underlying undirected graph is L-colorable.

THEOREM 8. *If L is a list assignment on an acyclic oriented graph G such that $|L(v)| > d^+(v)$ for all $v \in V(G)$, then G is L-colorable.*

PROOF. Suppose there is a list assignment L on an acyclic oriented graph G such that $|L(v)| > d^+(v)$ for all $v \in V(G)$, but G is not L-colorable. Then we may pick such an L and G with $|G|$ as small as possible. Plainly, $|G| \geq 2$. Since $|G|$ is finite and G is acyclic, there must be $w \in V(G)$ with $d^+(w) = 0$. Since $|L(w)| > d^+(w)$, we may choose $c \in L(w)$ and color w with c. Now let L' be the list assignment on $G-w$ where $L'(v) = L(v) \setminus \{c\}$ if v is adjacent to w and $L'(v) = L(v)$ otherwise. Since $d^+(w) = 0$, for any vertex v of $G - w$ that lost c from its list, we have $d^+_{G-w}(v) = d^+(v) - 1$, so $|L'(v)| > d^+_{G-w}(v)$ for all $v \in V(G - w)$. Since $|G - w| < |G|$ and $G - w$ is also acyclic, applying the theorem shows that $G - w$ is L'-colorable, but then we have an L-coloring of G, a contradiction. $\square$

Theorem 8 is no longer true if we drop "acyclic" from the hypotheses; take a cyclically directed triangle for example. But there are ways to replace "acyclic" with weaker hypotheses and still get a true theorem. In outline-form, the proof of Theorem 8 went like this: find a vertex w we can color with some color c such that $G - w$ is still acyclic and any vertex in $G - w$ that loses c from its list also has its out-degree go down. This can be generalized in a couple natural ways. First, we could color an independent set I of vertices with c instead of just a single vertex. Second, we could replace "acyclic" with some other property of oriented graphs, say a made-up property "agliplic", and require that $G - I$ remain agliplic.

It will be convenient to work with an equivalent dual version of list assignments. Instead of assigning a list of colors to each vertex, we assign a set of vertices to each color. Given a set of colors P, a *P-assignment* on a graph G is a function from P to the subsets of $V(G)$. For a list assigment L, let the *pot of L* be the set of all colors appearing in at least one $L(v)$; that is, $\mathrm{pot}(L) := \bigcup_{v\in V(G)} L(v)$. Then L gives rise to the $\mathrm{pot}(L)$-assignment C_L given by $C_L(c) := \{v \in V(G) : c \in L(v)\}$.

OBSERVATION 1. *G is L-colorable just in case there are independent sets $I_c \subseteq C_L(c)$ for each $c \in \mathrm{pot}(L)$ that together cover $V(G)$.*

Viewing a list assignment in this dual fashion, there is a natural candidate for a choice of I to color with c when trying to prove Theorem 8 for agliplic oriented graphs. We want to find independent $I \subseteq C_L(c)$ such that every $v \in C_L(c) \setminus I$ has an out-neighbor in I. Such an I is a *kernel* in $G\,[C_L(c)]$. So, we could try taking agliplic to mean "$G\,[C_L(c)]$ has a kernel I_c for all $c \in \mathrm{pot}(L)$". That almost works, but we have no way of guaranteeing that $G - I_c$ is still agliplic. We can fix that by requiring that $G[S]$ have a kernel for *every* $S \subseteq V(G)$. Instead of agliplic, we call an oriented graph with this property *kernel-perfect*.

THEOREM 9. *If L is a list assignment on a kernel-perfect oriented graph G such that $|L(v)| > d^+(v)$ for all $v \in V(G)$, then G is L-colorable.*

PROOF. Suppose there is a list assignment L on a kernel-perfect oriented graph G such that $|L(v)| > d^+(v)$ for all $v \in V(G)$, but G is not L-colorable. Then we may pick such an L and G with $|G|$ as small as possible. Pick $c \in \mathrm{pot}(L)$ and let I be a kernel in $G\,[C_L(c)]$. Color all vertices in I with c and let L' be the list assignment on $G-I$ where $L'(v) = L(v) \setminus \{c\}$ if $v \in C_L(c)$ and $L'(v) = L(v)$ otherwise. Since every $v \in C_L(c)$ has an out-neighbor in I, we have $d^+_{G-I}(v) \leq d^+(v) - 1$, so $|L'(v)| > d^+_{G-I}(v)$ for all $v \in V(G-I)$. Since $|G-I| < |G|$ and $G-I$ is also kernel-perfect, applying the theorem shows that $G-I$ is L'-colorable, but then we have an L-coloring of G, a contradiction. □

Given an oriented graph G that is not kernel-perfect, it is always possible to add arrows (possibly going the opposite way as a current arrow, forming a directed digon) to get what we'll call a *superoriented graph* that is kernel-perfect. One way is just to add back arrows for each arrow, then any maximal independent set is a kernel. Theorem 9 holds for superoriented graphs by a nearly identical proof. This can be useful as it gives us a way to trade in some slack in the $|L(v)| > d^+(v)$ bounds for kernel-perfection by adding some extra arrows. example, picture

Theorem 10. *If L is a list assignment on a kernel-perfect superoriented graph G such that $|L(v)| > d^+(v)$ for all $v \in V(G)$, then G is L-colorable.*

An acylic oriented graph is plainly kernel-perfect, so Theorem 9 generalizes Theorem 8. But this is not the only possible generalization of acylic that works, later we will see another based on counting certain substructures of the oriented graph. This second generalization of acyclic does not generalize kernel-pefection and kernel-perfection does not generalize it. when?

flows

Using Theorem 10, we will prove an easily-applicable sufficient condition for a graph to be L-colorable. But first, we need a general lemma that gives graph orientations with specified constraints on their out-degrees. This lemma can be proved directly, but a detour into the theory of flows will give us the lemma and many other results for free.

Let V be a finite set. A *network* on V is a function $f\colon V \times V \to \mathbb{Q}_{\geq 0}$. For a network f and $X, Y \subseteq V$, put

$$|X, Y| := \sum_{(x,y) \in X \times Y} f(x, y).$$

If f and g are networks on V, then $f \leq g$ just in case $f(u, v) \leq g(u, v)$ for all $(u, v) \in V \times V$. Let $s, t \in V$ be two distinguished vertices. The *capacity* of a network f, written $|f|$, is the minimum of $|X, V|$ over all $X \subseteq V \setminus \{t\}$ with $s \in X$.

We call a network f a *flow* if $|\{v\}, V| = |V, \{v\}|$ for all $v \in V \setminus \{s, t\}$. The *value* of a flow f is $\|f\| := |\{s\}, V| - |V, \{s\}|$.

Lemma 1. *If f is a flow, then $\|f\| = |X, V| - |V, X|$ for every $X \subseteq V \setminus \{t\}$ with $s \in X$.*

Proof.

$$\begin{aligned} |X, V| - |V, X| &= |\{s\}, V| + |X \setminus \{s\}, V| - |V, \{s\}| - |V, X \setminus \{s\}| \\ &= \|f\| + |X \setminus \{s\}, V| - |V, X \setminus \{s\}| \\ &= \|f\|. \end{aligned}$$

□

Lemma 2. *If $f \leq g$ where f is a flow on V and g is a network on V, then $\|f\| \leq |g|$.*

Proof. Pick $X \subseteq V \setminus \{t\}$ with $s \in X$ such that $|X, V| = |g|$. Then, by Lemma 1, $\|f\| = |X, V| - |V, X| \leq |g|$. □

Theorem 11. *If g is a network on V, then there exists a flow f on V with $f \leq g$ such that $\|f\| = |g|$.*

Proof. Let g be a network on V and choose a flow $f \leq g$ maximizing $\|f\|$. By Lemma 2, $\|f\| \leq |g|$. Let $X \subseteq V$ be the set of all $v \in V$ for which there is a sequence of distinct vertices $s = x_0, x_1, x_2, \ldots, x_n = v$ such that for each $0 \leq i < n$, either $f(x_i, x_{i+1}) < g(x_i, x_{i+1})$ or $f(x_{i+1}, x_i) > 0$.

Suppose $t \in X$ and let $s = x_0, x_1, x_2, \ldots, x_n = t$ be a sequence witnessing this fact. Choose $\epsilon > 0$ such that for each $0 \leq i < n$, either $f(x_i, x_{i+1}) \leq g(x_i, x_{i+1}) - \epsilon$ or $f(x_{i+1}, x_i) \geq \epsilon$. Modify f to get f' by setting $f'(x_i, x_{i+1}) = f(x_i, x_{i+1}) + \epsilon$ if $f(x_i, x_{i+1}) \leq g(x_i, x_{i+1}) - \epsilon$ and $f'(x_{i+1}, x_i) = f(x_{i+1}, x_i) - \epsilon$ otherwise. Then f' is a flow with $f' \leq g$ and $\|f'\| > \|f\|$, violating maximality of $\|f\|$.

So, $t \notin X$. □

degree again.

maximum independent covers.

polynomial tools

combinatorial nullstellensatz.

coefficient formulae.

a combinatorial interpretation.

coloring edges

It is also useful to consider coloring the edges of a graph so that incident edges receive different colors. This appears to be at odds with our previous claim that this book was only about coloring vertices of graph; fortunately, edge coloring is just a special case of vertex coloring. If G is a graph, the *line graph* of G, written $L(G)$ is the graph with vertex set $E(G)$ where two edges of G are adjacent in $L(G)$ if they are incident in G. Coloring the edges of G is equivalent to coloring the vertices of $L(G)$.

For graphs with maximum degree zero (that is, no edges at all), we can get by with zero colors. With just one color we can edge color any graph with maximum degree at most one. We will definitely always need at least $\Delta(G)$ colors to edge color a graph G. Could we be so fortunate that the pattern continues and we can edge color any graph G with only $\Delta(G)$-colors? Not quite, but we can do so for bipartite (2-colorable) graphs. A graph is k-edge-colorable if we can color its edges with (at most) k colors such that incident edges receive different colors. A color c us *used* at a vertex v of G if an edge incident to v in G is colored with c. Otherwise, c is *available* at v. A *path* in G is a sequence of pairwise distinct vertices $x_1x_2\cdots x_k$ such that x_i is adjacent to x_{i+1} for $i \in [k-1]$.

used

available

path

THEOREM 12. *If G is a bipartite graph, then G is $\Delta(G)$-edge-colorable.*

PROOF. Suppose there is a graph G that is not $\Delta(G)$-edge-colorable. Then we may pick such a graph G with $\|G\|$ as small as possible. Now $\|G\| > 0$, since we can surely edge color a graph with zero edges using zero colors. Let xy be an edge in G. Since $\|G - xy\| < \|G\|$, applying the theorem to $G - xy$ gives an edge coloring of $G - xy$ using at most $\Delta(G)$ colors. Now each of x and y are incident to at most $\Delta(G) - 1$ edges in $G - xy$ and G has no $\Delta(G)$-edge-coloring, so there is a color red available at x and a different color blue available at y. There is a unique maximal path P starting at x with edges alternately colored blue and red. If P does not end at y, then we get a $\Delta(G)$-edge-coloring of G by swapping the colors red and blue along P and coloring xy blue, a contradiction. Since P ends at y and alternates between red and blue, it has even length. But then $P + xy$ is an odd cycle in G, violating Theorem 1. □

It may come as a surpise that even though we might need more than $\Delta(G)$ colors to edge color a graph G, we will only ever need at most one extra color. For bipartite graphs we were able to repair an almost correct coloring by swapping colors along a path because we had control over where this path ended. In the general case we don't have the same control over a path between two vertices, but we can exert some measure of control over paths leaving and entering a larger structure. The larger structure we use here is the whole neighborhood of a vertex.

This is a good time to introduce a lemma that will be useful throughout the book. A *transversal* in a collection of sets $\{A_i\}_{i\in[k]}$ is a set $X \subseteq \bigcup_{i\in[k]} A_i$ with $|X| = k$ such that $|X \cap A_i| = 1$ for all $i \in [k]$.

LEMMA 3. *$\{A_i\}_{i\in[k]}$ has a transversal just in case $\left|\bigcup_{i\in I} A_i\right| \geq |I|$ for all $I \subseteq [k]$.*

PROOF. □

COROLLARY 1. *If $\left|\bigcup_{i\in[k]} A_i\right| \geq k$, then there is nonempty $I \subseteq [k]$ such that $\{A_i\}_{i\in I}$ has a transversal X where $X \cap A_i = \emptyset$ for all $i \in [k] \setminus I$.*

PROOF. □

THEOREM 13. *If G is a graph, then G is $(\Delta(G) + 1)$-edge-colorable.*

PROOF. Suppose there is a graph G that is not $(\Delta(G) + 1)$-edge-colorable. Then we may pick such a graph G with $|G|$ as small as possible. Now $|G| > 0$, since we can surely edge color a graph with zero vertices using at most one color. Let x be a vertex in G. Call $S \subseteq N(x)$ *acceptable* for an edge coloring π of $G - x$ if $\{\bar{\pi}(v)\}_{v\in S}$ has a transversal T_S such that $|\bar{\pi}(v) \setminus T_S| \geq 2$ for all $v \in N(x) \setminus S$.

Since $|G - x| < |G|$, applying the theorem to $G - x$ gives an edge coloring ζ of $G - x$ using at most $\Delta(G) + 1$ colors. Note that the empty set is acceptable for ζ. So, we may choose an edge coloring π of $G - x$ using at most $\Delta(G) + 1$ colors and $S \subseteq N(x)$ that is acceptable for π so as to maximize $|S|$ and subject to that to maximize $|C_S|$, where

$$C_S := \bigcup_{v\in N(x)\setminus S} \bar{\pi}(v) \setminus T_S.$$

Now $S \neq N(x)$ for otherwise we can extend π to all of G using T_S. Suppose $|C_S| \geq |N(x) \setminus S|$. Then, by Corollary 1, there is nonempty $A \subseteq N(x)\setminus S$ such that $\{\bar{\pi}(v) \setminus T_S\}_{v\in A}$ has a transversal X where $X \cap \bar{\pi}(v) = \emptyset$ for all $v \in N(x) \setminus (S \cup A)$. But then $S \cup A$ is acceptable for π, contradicting maximality of $|S|$.

So, $|C_S| < |N(x) \setminus S|$ and hence $|C_S \cup T_S| < |N(x)| \leq \Delta(G)$. Pick $\tau \in [\Delta(G)] \setminus (C_S \cup T_S)$. Since S is acceptable for π, $|\bar{\pi}(v) \setminus T_S| \geq 2$ for all $v \in N(x) \setminus S$. Hence there are $v_1, v_2, v_3 \in N(x) \setminus S$ and $\gamma \in C_S$ such that $\gamma \in \bar{\pi}(v_i) \setminus T_S$ for all $i \in [3]$. There is a unique maximal path P starting at v_1 with edges alternately colored τ and γ. Let ζ be the edge coloring made from π by swapping τ and γ on P. Then ζ violates the maximality of $|C_S|$ since S is acceptable for ζ and

$$\bigcup_{v\in N(x)\setminus S} \bar{\zeta}(v) \setminus T_S = \{\tau\} \cup C_S.$$

□

hardness

We now know that every graph G can be edge colored with either $\Delta(G)$ or $\Delta(G) + 1$ colors. So, edge coloring is basically trivial, right? Furtunately, no it isn't, the collection of graphs requiring $\Delta(G) + 1$ colors is very rich. Another way to say this, is that it is a hard problem to decide whether or not edge coloring a given graph G requires $\Delta(G) + 1$ colors.

THEOREM 14. *Deciding whether or not edge coloring a given graph G requires $\Delta(G) + 1$ colors is hard supposing other things we think are hard are actually hard.*

Proof. We need a concise proof of this without having to introduce too much background. Please submit a pull request on GitHub. $\square$

vertex coloring, again

edge coloring, again

fans as a greedy strategy

acceptable paths

acceptable trees

edge list coloring

more kernel method.

2-edge-coloring.

improved degree theorem.

a hint of quasiline and claw-free graphs.

shuffle tool

destroying non-complete

shuffling with more rules

looking at the entire recoloring digraph

independent transversalls

randomly

Haxell's tool

triangles within triangles are neat.

big maximum cliques intersect in simple ways

going lopsided

vertex transitive graphs

strong coloring

medium clique implies big clique

potential potential tool