BASIC GRAPH COLORING

Contents

For Rachel, Atticus and Alfred.

Graphs

A *graph* is a collection of dots we call *vertices* some of which are connected by curves we call *edges*. The relative location of the dots and the shape of the curves are not relevant, we are only concerned with whether or not a given pair of dots is connected by a curve. Initially, we forbid edges from a vertex to itself and multiple edges between two vertices. If G is a graph, then $V(G)$ is its set of vertices and $E(G)$ its set of edges. We write $|G|$ for the number of vertices in $V(G)$ and $\|G\|$ for the number of edges in $E(G)$. Two vertices are *adjacent* if they are connected by an edge. The set of vertices to which v is adjacent is its *neighborhood*, written $N(v)$. For the size of v's neighborhood $|N(v)|$, we write $d(v)$ and call this the *degree* of v. We write $E(v)$ for the set of edges containing v, these are the edges *incident* to v.

We use the shorthand $[k] := \{1, 2, \ldots, k\}$. A *path* in G is a sequence of different vertices $x_1, x_2, \ldots, x_r$ such that x_i is adjacent to x_{i+1} for all $i \in [r-1]$. We say this is a path from x_1 to x_r. If x_r is adjacent to x_1 as well, then we have a *cycle*. A graph G is *connected* if for all $x, y \in V(G)$, there is a path from x to y. Figure 1 shows all the connected graphs with at most five vertices.

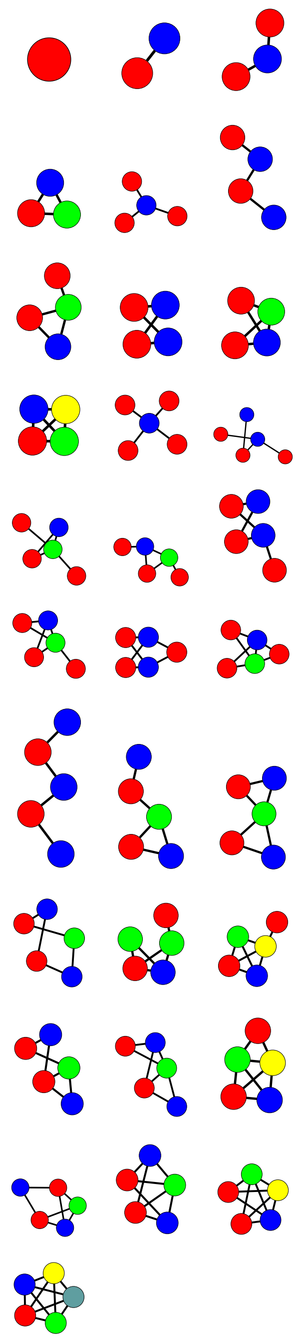

Figure 1: The connected graphs with at most five vertices.

Coloring vertices

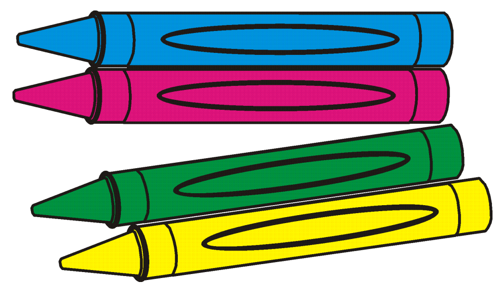

Figure 2: These are crayons.

The entire book concerns one simple task: we want to color the vertices of a given graph so that adjacent vertices receive different colors. With sufficiently many crayons and no preferences about what the coloring should look like, this is easy, we just use a different crayon for each vertex. Things get interesting when we ask how few different crayons we can use. We are definitely going to need an empty box of crayons and that will only do for the graph with no vertices at all. Given one crayon, we can handle all graphs with no edges. With two crayons, we can do any path and any cycle with an even number of vertices. But, we can't handle a triangle or any other cycle with an odd number of vertices.

Figure 3: A graph with no vertices needs no crayons at all.

In fact, odd cycles are really the only thing that will prevent us from using just two crayons. A graph H is a *subgraph* of a graph G, written $H \subseteq G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. When $H \subseteq G$, we say that G *contains* H. If $v \in V(G)$, then $G - v$ is the graph we get by removing v and all edges incident to v from G. A graph is k-colorable if we can color its vertices with (at most) k colors such that adjacent vertices receive different colors. A 0-colorable graph is *empty*, a 1-colorable graph is *edgeless* and a 2-colorable graph is *bipartite*.

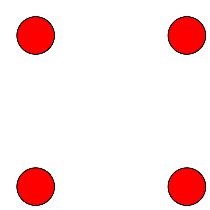

Figure 4: An edgeless graphs needs only one crayon.

Theorem 1. *A graph is* 2*-colorable just in case it contains no odd cycle.*

Proof. A graph containing an odd cycle clearly can't be 2-colored. For the other implication, suppose there is a graph that is not 2-colorable and doesn't contain an odd cycle. Then we may pick such a graph G with $|G|$ as small as possible. Surely, $|G| > 0$ and there is $v \in V(G)$ with $d(v) \geq 2$. If $x, y \in N(v)$, then x is not adjacent to y since then xyz would be an odd cycle. So we can construct a graph H from G by removing v and identifying all of $N(v)$ to a new vertex x_v. Any odd cycle in H would contain x_v and hence give rise to an odd cycle in G passing through v. So H contains no odd cycle. Since $|H| < |G|$, applying the theorem to H gives a 2-coloring of H, say with red and blue where x_v gets colored red. But this gives a 2-coloring of G by coloring all vertices in $N(v)$ red and v blue, a contradiction. $\square$

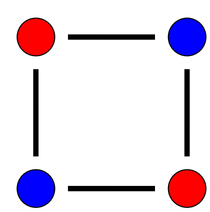

Figure 5: An even cycle needs two crayons.

Since detecting odd cycles is easy, this means 2-coloring is easy.

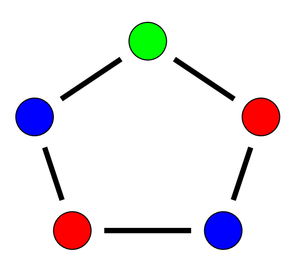

Figure 6: An odd cycle needs three crayons.

Things get more interesting when we move up to three colors.

Theorem 2. *For $k \geq 3$, determining whether or not a graph has a k-coloring is a hard problem (supposing other problems we think are hard are, in fact, hard).*

Basic estimates

Even though finding the minimum number of colors needed to color a graph is hard in general (supposing it is), we can still look for lower and upper bounds on this value. The *chromatic number* $\chi(G)$ of a graph G is the smallest k for which G is k-colorable. The simplest thing we can do is give each vertex a different color.

Theorem 3. *If G is a graph, then $\chi(G) \leq |G|$.*

The only graphs that attain the upper bound in Theorem 3 are the *complete* graphs; those in which any two vertices are adjacent. We write K_r for the complete graph with r vertices. We can usually do much better by just arbitrarily coloring vertices, reusing colors when we can. The *maximum degree* $\Delta(G)$ of a graph G is the largest degree of any vertex in G; that is

$$\Delta(G) := \max_{v \in V(G)} d(v).$$

Theorem 4. *If G is a graph, then $\chi(G) \leq \Delta(G) + 1$.*

Proof. Suppose there is a graph G that is not $(\Delta(G)+1)$-colorable. Then we may pick such a graph G with $|G|$ as small as possible. Surely, $|G| > 0$, so we may pick $v \in V(G)$. Then $|G - v| < |G|$ and $\Delta(G - v) \leq \Delta(G)$, so applying the theorem to $G - v$ gives a $(\Delta(G - v) + 1)$-coloring of $G - v$. But v has at most $\Delta(G)$ neighbors, so there is some color, say red, not used on $N(v)$, coloring v red gives a $(\Delta(G) + 1)$-coloring of G, a contradiction. $\square$

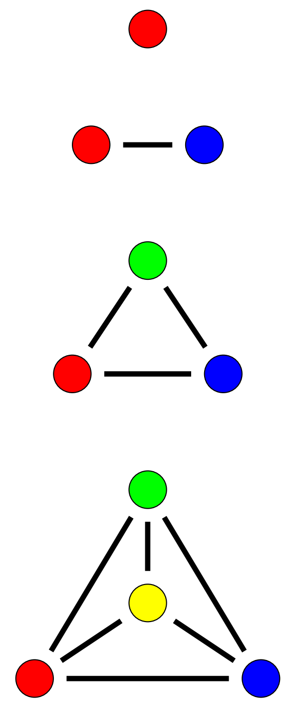

Figure 7: All vertices must get different colors in a complete graph.

Both complete graphs and odd cycles attain the upper bound in Theorem 4. Theorem 1 says we can do better for graphs that don't contain odd cycles. A *complete bipartite graph* consists of two disjoint independent sets (which we call *parts*) and all edges between them, we write $K_{a,b}$ for the complete bipartite graph with parts of size a and b. Theorem 4 gives a poor upper bound for complete bipartite graphs.

We can also do better for graphs that don't contain large complete subgraphs. A set of vertices S in a graph G is a *clique* if the vertices in S are pairwise adjacent. The *clique number* of a graph G, written $\omega(G)$, is the number of vertices in a largest clique in G.

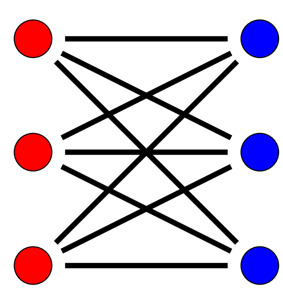

Figure 8: The graph $K_{3,3}$.

Theorem 5. *If G is a graph, then $\chi(G) \geq \omega(G)$.*

A set of vertices S in a graph G is *independent* if the vertices in S are pairwise non-adjacent. The *independence number* of a graph G, written $\alpha(G)$, is the number of vertices in a largest independent set in G.

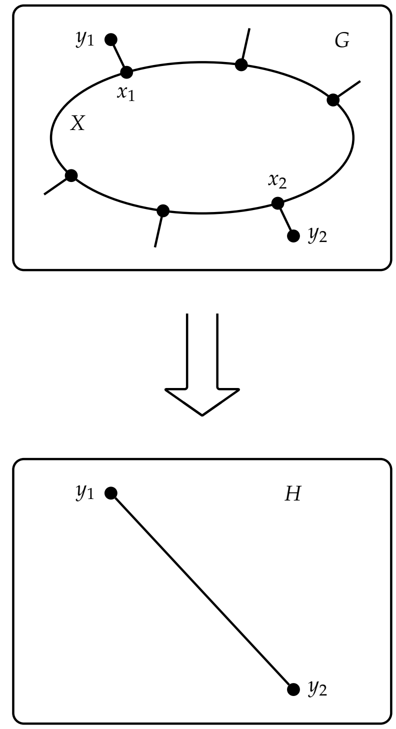

Figure 9: Removing X and adding y_1y_2 to get H.

Brooks' theorem

Theorem 6. *If G is a graph with $\Delta(G) \geq 3$ and $\omega(G) \leq \Delta(G)$, then $\chi(G) \leq \Delta(G)$.*

Proof. Suppose there is a graph G with $\Delta(G) \geq 3$ and $\omega(G) \leq \Delta(G)$ that is not $\Delta(G)$-colorable. Then we may pick such a graph G with $|G|$ as small as possible. Let S be a maximal independent set in G. Since S is maximal, every vertex in $G - S$ has a neighbor in S, so $\Delta(G) > \Delta(G - S)$. If red is an unused color in a $\chi(G - S)$-coloring of $G - S$, then by coloring all vertices in S red we get a $(\chi(G - S) + 1)$-coloring of G. So, $\Delta(G) + 1 \leq \chi(G) \leq \chi(G - S) + 1$. We conclude $\chi(G - S) > \Delta(G - S)$ and thus $\Delta(G) = \chi(G - S) = \Delta(G - S) + 1$ by Theorem 4. Since $|G - S| < |G|$, applying the theorem to $G - S$ shows that $\Delta(G - S) < 3$ or $\Delta(G - S) < \omega(G - S)$. So, either $\chi(G - S) = \Delta(G) = 3$ or $\omega(G - S) \geq \Delta(G)$. In the former case, let X be the vertex set of an odd cycle in $G - S$ guaranteed by Theorem 1. In the latter case, let X be a $\Delta(G)$-clique in $G - S$.

Since S is maximal and $\omega(G) \leq \Delta(G)$, there are $x_1, x_2 \in X$ and $y_1, y_2 \in S$ such that x_1 is adjacent to y_1 and x_2 is adjacent to y_2. Construct a graph H from $G - X$ by adding the edge y_1y_2. Since $|H| < |G|$, applying the theorem to H shows that $\omega(H) > \Delta(G)$ or $\chi(H) \leq \Delta(G)$.

Suppose $\chi(H) \leq \Delta(G)$. Then there is a $\Delta(G)$-coloring of $G - X$ where y_1 and y_2 receive different colors, say red and blue respectively. Pick the first vertex z in a shortest path P from x_1 to x_2 in X that has a blue colored neighbor in $V(H)$. Each vertex in X has $\Delta(G) - 1$ neighbors in X and hence at most one neighbor in $V(H)$. So, $z \neq x_1$ since x_1 already has a red colored neighbor in $V(H)$. Let w be be the vertex preceding z on P (it could be that $w = x_1$). Then w has no blue colored neighbor. Since X is the vertex set of a cycle or a complete graph, there is a path Q from w to z passing through every vertex of X. Color w blue and then proceed along Q, coloring one vertex at a time. Since each vertex we encounter before we get to z has at most $\Delta(G) - 1$ colored neighbors, we always have an available color to use. But, z is adjacent to both w and another blue colored vertex in $V(H)$, so there is an available color for z as well. This gives a $\Delta(G)$-coloring of G, a contradiction.

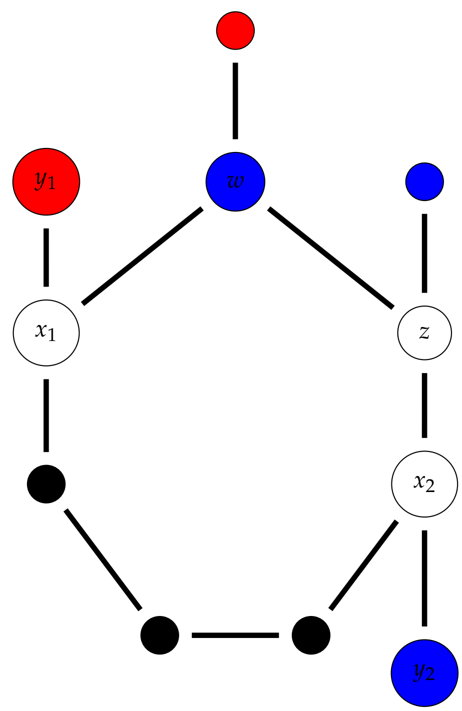

Figure 10: Right after coloring w, note that z has two blue neighbors.

So, $\omega(H) > \Delta(G)$. In particular, y_1 and y_2 each have exactly one neighbor in X and $\Delta(G) - 1$ neighbors in the same $\Delta(G) - 1$ clique A in $G - X$. Since S is maximal and $|X| \geq 3$, there must be adjacent $x_3 \in X \setminus \{x_1, x_2\}$ and $y_3 \in S \setminus \{y_1, y_2\}$. Applying the same argument with x_3, y_3 in place of x_2, y_2 shows that y_1 and y_3 each have exactly one neighbor in X and $\Delta(G) - 1$ neighbors in the same $\Delta(G) - 1$ clique B in $G - X$. Now $|A \cap B| = |A| + |B| - |A \cup B| \geq 2(\Delta(G) - 1) - d(y_1) \geq \Delta(G) - 2 > 0$. But there can't be a vertex in $A \cap B$ since it would be adjacent to y_1, y_2, y_3 as well as $\Delta(G) - 2$ vertices in A and thus have degree greater than $\Delta(G)$, a contradiction. □

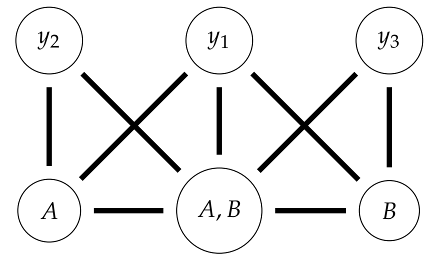

Figure 11: The final contradiction when $\Delta = 3$.

Exercise 1. *Show that if each vertex of a cycle is assigned a list of two colors, then the vertices can be colored from their lists so that adjacent vertices get different colors, unless the lists are all the same.*

Exercise 2. *Show that if each vertex of K_r is assigned a list of $r - 1$ colors, then the vertices can be colored from their lists so that adjacent vertices get different colors, unless the lists are all the same.*

Exercise 3. *Use the previous two exercises to replace the penultimate paragraph in the proof Theorem 6.*

List coloring

When attempting to k-color a graph G, it will often be convenient to first k-color $G[S]$ for some $S \subset V(G)$ and then try to k-color $G - S$ in a compatible manner. To make this precise, think of each vertex in G starting with a list of k permissible colors, say $[k]$. When we k-color $G[S]$, the colors used on $N(v) \cap S$ are no longer permissible for each $v \in V(G - S)$. For $v \in V(G - S)$, let $L(v)$ be the permissible colors for v after k-coloring $G[S]$. Now our problem is to pick $c_v \in L(v)$ for each $v \in V(G - S)$ such that $c_x \neq c_y$ whenever $xy \in E(G - S)$. This is the *list coloring* problem.

The list coloring problem arose as a subproblem in our attempt to k-color a graph. By taking it out of this context and viewing list coloring as a first-class problem in its own right, we will be able to prove more general theorems while also simplifying proofs. A *list assignment* on a graph G gives a set of colors $L(v)$ to each $v \in V(G)$. If there is $c_v \in L(v)$ for each $v \in V(G)$ such that $c_x \neq c_y$ whenever $xy \in E(G)$, then G is *L-colorable*.

Theorem 7. *If L is a list assignment on a graph G such that $|L(v)| > d(v)$ for all $v \in V(G)$, then G is L-colorable.*

Proof. Color each vertex in turn using a color in its list not appearing on any of its colored neighbors. This succeeds since each vertex has more permissible colors than neighbors. □

The requirement $|L(v)| > d(v)$ in Theorem 7 is quite strong, in the algorithm we really only need $L(v)$ to have more colors than *colored* neighbors rather than more colors than neighbors. We can encode this extra information by *orienting* the edges of G; that is, turning each edge into an arrow pointed one way or the other. If G is an oriented graph, the *out-degree* of a vertex $v \in V(G)$, written $d^+(v)$, is the number of arrows pointing away from v. An oriented graph is *acyclic* if there is no sequence of arrows that ends where it starts. An oriented graph is L-colorable just in case its underlying undirected graph is L-colorable.

Theorem 8. *If L is a list assignment on an acyclic oriented graph G such that $|L(v)| > d^+(v)$ for all $v \in V(G)$, then G is L-colorable.*

Proof. Suppose there is a list assignment L on an acyclic oriented graph G such that $|L(v)| > d^+(v)$ for all $v \in V(G)$, but G is not L-colorable. Then we may pick such an L and G with $|G|$ as small as possible. Plainly, $|G| \geq 2$. Since $|G|$ is finite and G is acyclic, there must be $w \in V(G)$ with $d^+(w) = 0$. Since $|L(w)| > d^+(w)$, we may choose $c \in L(w)$ and color w with c. Now let L' be the list assignment on $G - w$ where $L'(v) = L(v) \setminus \{c\}$ if v is adjacent to w and $L'(v) = L(v)$ otherwise. Since $d^+(w) = 0$, for any vertex v of $G - w$ that lost c from its list, we have $d^+_{G-w}(v) = d^+(v) - 1$, so $|L'(v)| > d^+_{G-w}(v)$ for all $v \in V(G - w)$. Since $|G - w| < |G|$ and $G - w$ is also acyclic, applying the theorem shows that $G - w$ is L'-colorable, but then we have an L-coloring of G, a contradiction. □

Theorem 8 is no longer true if we drop "acyclic" from the hypotheses; take a cyclically directed triangle for example. But there are ways to replace "acyclic" with weaker hypotheses and still get a true theorem. In outline-form, the proof of Theorem 8 went like this: find a vertex w we can color with some color c such that $G - w$ is still acyclic and any vertex in $G - w$ that loses c from its list also has its out-degree go down. This can be generalized in a couple natural ways. First, we could color an independent set I of vertices with c instead of just a single vertex. Second, we could replace "acyclic" with some other property of oriented graphs, say a made-up property "agliplic", and require that $G - I$ remain agliplic.

It will be convenient to work with an equivalent dual version of list assignments. Instead of assigning a list of colors to each vertex, we assign a set of vertices to each color. Given a set of colors P, a *P-assignment* on a graph G is a function from P to the subsets of $V(G)$. For a list assignment L on G and $S \subseteq V(G)$, put $L(S) := \bigcup_{v \in S} L(v)$.

Then L gives rise to the $L(V(G))$-assignment C_L given by $C_L(c) := \{v \in V(G) : c \in L(v)\}$.

Observation 1. *G is L-colorable just in case there are independent sets $I_c \subseteq C_L(c)$ for each $c \in L(V(G))$ that together cover $V(G)$.*

Viewing a list assignment in this dual fashion, there is a natural candidate for a choice of I to color with c when trying to prove Theorem 8 for agliplic oriented graphs. We want to find independent $I \subseteq C_L(c)$ such that every $v \in C_L(c) \setminus I$ has an out-neighbor in I. Such an I is a *kernel* in $G[C_L(c)]$. So, we could try taking agliplic to mean "$G[C_L(c)]$ has a kernel I_c for all $c \in L(V(G))$". That almost works, but we have no way of guaranteeing that $G - I_c$ is still agliplic. We can fix that by requiring that $G[S]$ have a kernel for *every* $S \subseteq V(G)$. Instead of agliplic, we call an oriented graph with this property *kernel-perfect*.

Theorem 9. *If L is a list assignment on a kernel-perfect oriented graph G such that $|L(v)| > d^+(v)$ for all $v \in V(G)$, then G is L-colorable.*

Proof. Suppose there is a list assignment L on a kernel-perfect oriented graph G such that $|L(v)| > d^+(v)$ for all $v \in V(G)$, but G is not L-colorable. Then we may pick such an L and G with $|G|$ as small as possible. Pick $c \in L(V(G))$ and let I be a kernel in $G[C_L(c)]$. Color all vertices in I with c and let L' be the list assignment on $G - I$ where $L'(v) = L(v) \setminus \{c\}$ if $v \in C_L(c)$ and $L'(v) = L(v)$ otherwise. Since every $v \in C_L(c)$ has an out-neighbor in I, we have $d^+_{G-I}(v) \leq d^+(v) - 1$, so $|L'(v)| > d^+_{G-I}(v)$ for all $v \in V(G - I)$. Since $|G - I| < |G|$ and $G - I$ is also kernel-perfect, applying the theorem shows that $G - I$ is L'-colorable, but then we have an L-coloring of G, a contradiction. $\square$

Given an oriented graph G that is not kernel-perfect, it is always possible to add arrows (possibly going the opposite way as a current arrow, forming a directed digon) to get what we'll call a *superoriented graph* that is kernel-perfect. One way is just to add back arrows for each arrow, then any maximal independent set is a kernel. Theorem 9 holds for superoriented graphs by a nearly identical proof. This can be useful as it gives us a way to trade in some slack in the $|L(v)| > d^+(v)$ bounds for kernel-perfection by adding some extra arrows.

Theorem 10. *If L is a list assignment on a kernel-perfect superoriented graph G such that $|L(v)| > d^+(v)$ for all $v \in V(G)$, then G is L-colorable.*

Lemma 1. *If G is a superoriented graph with an independent set I such that all edges in $G - I$ have back arrows, then G is kernel-perfect.*

Proof. Suppose there is a non-kernel-perfect superoriented graph G with independent set I such that all edges in $G - I$ have back arrows.

Then we may pick such a G with $|G|$ as small as possible. Since G is not kernel-perfect, there is $X \subseteq V(G)$ such that $G[X]$ has no kernel. If $|X| < |G|$, then we could apply the theorem to $G[X]$ to get a kernel, so we must have $X = V(G)$. So I is not a kernel in G and hence there is $v \in V(G - I)$ with none of its incident arrows pointing into I. Remove v and all its neighbors from G to get a superoriented graph H. Since $|H| < |G|$, we may apply the theorem to H to get a kernel S in H. But then $S \cup \{v\}$ is a kernel in G since any vertex other than v in $G - H$ is either in I and hence has an arrow to v or is in $G - I$ and hence has a back arrow to v, a contradiction. □

An acylic oriented graph is plainly kernel-perfect, so Theorem 9 generalizes Theorem 8. But this is not the only possible generalization of acylic that works, in the Combinatorial nullstellensatz chapter we'll present another based on polynomials. This second generalization of acyclic does not generalize kernel-perfection and kernel-perfection does not generalize it.

Question 1. *Lemma 1 says that if we take a graph G with an independent set I and direct all the edges of $G - I$ both ways and the edges between I and $V(G - I)$ arbitrarily, we get a kernel-perfect superoriented graph. Can we classify the pairs (G, F) where G is a graph and $F \subseteq E(G)$ such that every superorientation of G in which all edges in F are bidirected are kernel-perfect?*

Exercise 4. *Show that if G is a graph and $F \subseteq E(G)$, then every superorientation of G in which $G - F$ is acyclic and all edges in F are bidirected is kernel-perfect.*

Exercise 5. *Show that if G is a graph and each odd cycle of G contains at least two edges in $F \subseteq E(G)$, then every superorientation of G in which all edges in F are bidirected is kernel-perfect.*

Coloring with prescribed list sizes

Suppose we are attempting to k-color a graph G and have already k-colored $G[S]$ for some $S \subset V(G)$. If L is the list assignment of remaining permissible colors on $G - S$, then we don't really know much about L, except that $|L(v)| \geq k - |N(v) \cap S|$ for all $S \subseteq V(G)$. So, we really want to know that $G - S$ is colorable from *every* list assignment of that size.

Definition 1. *For a graph G, a G-*numbering *is a function $f : V(G) \to \mathbb{Z}$ such that $f(v) \leq d_G(v) + 1$ for all $v \in V(G)$.*

Lemma 2. *If f is a G-numbering and H is an induced subgraph of G, then $f_H : V(H) \to \mathbb{Z}$ given by $f_H(v) := f(v) - \|v, G - H\|$ is an H-numbering.*

Proof. For $v \in V(H)$,

$$\begin{aligned} f_H(v) &= f(v) - \|v, G - H\| \\ &\leq d_G(v) + 1 - \|v, G - H\| \\ &= d_G(v) + 1 - (d_G(v) - d_H(v)) \\ &= d_H(v) + 1. \end{aligned}$$

□

Definition 2. *Let G be a graph and f a G-numbering. For a nonempty induced subgraph H of G, we say that G* is f-reducible to H *if H is f_H-choosable. If G is not f-reducible to any nonempty induced subgraph, then G is f-irreducible.*

Lemma 3. *If G is f-irreducible, then $f(v) \leq d_G(v)$ for all $v \in V(G)$. In particular, $2\,\|G\| \geq f(V(G))$.*

Proof. If there is $v \in V(G)$ with $f(v) > d_G(v)$, then G is f-reducible to $G[\{v\}]$. □

A natural next question is what happens when $f(V(G)) = 2\,\|G\|$, can G be f-irreducible in this case? The examples in Figure [insert me] show that the answer is yes. We will understand the case of equality completely in the Kernel magic chapter using a tool that improves Lemma 3 in terms of the "maximum independent cover number".

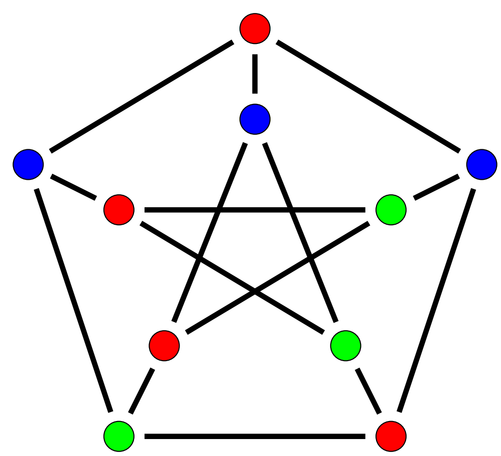

Figure 12: The Petersen graph.

Historical notes

Flows

Using Theorem 10 and Lemma 1, we will prove an easily-applicable sufficient condition for a graph to be L-colorable. But first, we need a general lemma that gives graph orientations with specified constraints on their out-degrees. This lemma can be proved directly, but a detour into the theory of flows will give us the lemma and other results for free (modulo computation).

Let V be a finite set. A *network* on V is a function $f: V \times V \to \mathbb{Q}_{\geq 0}$. For a network f and $X, Y \subseteq V$, put

$$|X, Y|_f := \sum_{(x,y) \in X \times Y} f(x, y).$$

If f and g are networks on V, then $f \leq g$ just in case $f(u, v) \leq g(u, v)$ for all $(u, v) \in V \times V$. Let $s, t \in V$ be two distinguished vertices. The *capacity* of a network f, written $|f|$, is the minimum of $|X, V \setminus X|_f$ over all $X \subseteq V \setminus \{t\}$ with $s \in X$.

We call a network f a *flow* if $|\{v\}, V|_f = |V, \{v\}|_f$ for all $v \in V \setminus \{s, t\}$. The *value* of a flow f is $\|f\| := |\{s\}, V|_f - |V, \{s\}|_f$.

Lemma 4. *If f is a flow, then $\|f\| = |X, V|_f - |V, X|_f$ for every $X \subseteq V \setminus \{t\}$ with $s \in X$.*

Proof.

$$\begin{aligned} |X, V|_f - |V, X|_f &= |\{s\}, V|_f + |X \setminus \{s\}, V|_f - |V, \{s\}|_f - |V, X \setminus \{s\}|_f \\ &= \|f\| + |X \setminus \{s\}, V|_f - |V, X \setminus \{s\}|_f \\ &= \|f\|. \end{aligned}$$

□

Lemma 5. *If $f \leq g$ where f is a flow on V and g is a network on V, then $\|f\| \leq |g|$.*

Proof. Pick $X \subseteq V \setminus \{t\}$ with $s \in X$ such that $|X, V \setminus X|_g = |g|$. Then, by Lemma 4,

$$\begin{aligned}\|f\| &= |X,V|_f - |V,X|_f \\ &= |X,V\setminus X|_f + |X,X|_f - |V\setminus X,X|_f - |X,X|_f \\ &= |X,V\setminus X|_f - |V\setminus X,X|_f \\ &\leq |X,V\setminus X|_g \\ &= |g|\,.\end{aligned}$$

□

Theorem 11. *If g is a network on V, then there exists a flow f on V with $f \leq g$ such that $\|f\| = |g|$. Moreover, if g takes only integer values, then the flow f can be chosen to as well.*

Proof. Let g be a network on V and choose a flow $f \leq g$ maximizing $\|f\|$. By Lemma 5, $\|f\| \leq |g|$. Let $X \subseteq V$ be the set of all $v \in V$ for which there is a sequence of distinct vertices $s = x_0, x_1, x_2, \ldots, x_n = v$ such that for each $0 \leq i < n$, either $f(x_i, x_{i+1}) < g(x_i, x_{i+1})$ or $f(x_{i+1}, x_i) > 0$. Plainly $s \in X$.

Suppose $t \in X$ and let $s = x_0, x_1, x_2, \ldots, x_n = t$ be a sequence witnessing this fact. Choose $\epsilon > 0$ such that for each $0 \leq i < n$, either $f(x_i, x_{i+1}) \leq g(x_i, x_{i+1}) - \epsilon$ or $f(x_{i+1}, x_i) \geq \epsilon$. Modify f to get f' by setting $f'(x_i, x_{i+1}) = f(x_i, x_{i+1}) + \epsilon$ if $f(x_i, x_{i+1}) \leq g(x_i, x_{i+1}) - \epsilon$ and $f'(x_{i+1}, x_i) = f(x_{i+1}, x_i) - \epsilon$ otherwise. Then f' is a flow with $f' \leq g$ and $\|f'\| > \|f\|$, violating maximality of $\|f\|$.

So, $t \notin X$. Since $v \in V \setminus X$ is not in X, we must have $f(x, v) = g(x, v)$ and $f(v, x) = 0$ for all $x \in X$. Hence

$$\begin{aligned}\|f\| &= |X,V|_f - |V,X|_f \\ &= |X,X|_f + |X,V\setminus X|_f - |X,X|_f - |V\setminus X,X|_f \\ &= |X,V\setminus X|_f - |V\setminus X,X|_f \\ &= |X,V\setminus X|_g \\ &\geq |g|\,.\end{aligned}$$

□

Hall's theorem

As our first application of Theorem 11, we give a sufficient condition for a graph G to be L-colorable. This condition is also necessary when G is complete.

Lemma 6. *If L is a list assignment on a graph G such that $|L(S)| \geq |S|$ for all $S \subseteq V(G)$, then G is L-colorable.*

Proof. Let L be a list assignment on a graph G such that $|L(S)| \geq |S|$ for all $S \subseteq V(G)$. We define a network g on the set $Z := \{s,t\} \cup V(G) \cup L(V(G))$ by

$$g(a,b) := \begin{cases} 1 & a = s \text{ and } b \in V(G) \\ 1 & a \in V(G) \text{ and } b \in L(a), \\ 1 & a \in L(V(G)) \text{ and } b = t, \\ 0 & \text{otherwise.} \end{cases}$$

First, we compute the capacity of g. Let $X \subseteq Z \setminus \{t\}$ with $s \in X$. If $P = X \cap L(V(G))$ and $Q = X \cap V(G)$, then

$$\begin{aligned} |X, Z \setminus X|_g &= |P, Z \setminus X|_g + |Q, Z \setminus X|_g + |\{s\}, Z \setminus X|_g \\ &= |P| + |Q, Z \setminus X|_g + |V(G) \setminus Q| \\ &\geq |P| + |L(Q) \setminus P| + |V(G) \setminus Q| \\ &\geq |L(Q)| + |G| - |Q| \\ &\geq |G|. \end{aligned}$$

The final inequality holds due to our assumption $|L(S)| \geq |S|$ for all $S \subseteq V(G)$. Hence $|g| \geq |G|$. By Theorem 11, there is a flow f on Z with $f \leq g$ such that $\|f\| = |g|$. But then

$$\begin{aligned} |\{s\}, V(G)|_f &= |\{s\}, Z|_f \\ &= |\{s\}, Z|_f - |Z, \{s\}|_f \\ &= \|f\| \\ &= |g| \\ &\geq |G|. \end{aligned}$$

Hence $f(s,v) = 1$ for all $v \in V(G)$. Since f is a flow, for each $v \in V(G)$, there is $c_v \in L(V(G))$ such that $g(v, c_v) \geq f(v, c_v) = 1$ and hence $c_v \in L(v)$. Coloring each $v \in V(G)$ with c_v gives an L-coloring of G since if $c_v = c_w$, the fact that f is a flow implies $|Z, \{c_v\}|_f = |\{c_v\}, Z|_f = f(c_v, t) = 1$ and hence $v = w$. □

Hall's theorem can be expressed in many forms, here is a more standard way that is readily seen to be equivalent to Lemma 6. A *transversal* in a collection of finite sets $\{A_i\}_{i \in [k]}$ is a set $X \subseteq \bigcup_{i \in [k]} A_i$ with $|X| = k$ such that $|X \cap A_i| = 1$ for all $i \in [k]$.

Hall's theorem. *$\{A_i\}_{i \in [k]}$ has a transversal just in case $|\bigcup_{i \in I} A_i| \geq |I|$ for all $I \subseteq [k]$.*

Orientations with prescribed degrees

If G is an oriented graph, the *in-degree* of a vertex $v \in V(G)$, written $d^-(v)$, is the number of arrows pointing into v. For $h \colon A \to \mathbb{N}$ and

$S \subseteq A$, put $h(S) := \sum_{x \in S} h(x)$.

Lemma 7. *If G is a graph and $h\colon V(G) \to \mathbb{N}$, then G has an orientation such that $d^-(v) \geq h(v)$ for all $v \in V(G)$ just in case*

$$\|S\| + \|S, V(G) \setminus S\| \geq h(S), \tag{1}$$

for every $S \subseteq V(G)$.

Proof. An easy computation shows that if G has an orientation with $d^-(v) \geq h(v)$ for all $v \in V(G)$, then (1) holds.

For the other implication, we define a network g on the set $Z := \{s, t\} \cup V(G) \cup E(G)$ by

$$g(a,b) := \begin{cases} h(b) & a = s \text{ and } b \in V(G) \\ 1 & a \in V(G) \text{ and } b \in E(a), \\ 1 & a \in E(G) \text{ and } b = t, \\ 0 & \text{otherwise.} \end{cases}$$

First, we compute the capacity of g. Let $X \subseteq Z \setminus \{t\}$ with $s \in X$. If $P = X \cap E(G)$ and $Q = X \cap V(G)$, then

$$\begin{aligned} |X, Z \setminus X|_g &= |P, Z \setminus X|_g + |Q, Z \setminus X|_g + |\{s\}, Z \setminus X|_g \\ &= |P| + |Q, Z \setminus X|_g + h\left(V(G) \setminus Q\right) \\ &\geq |P| + 2\|Q\| + \|Q, V(G) \setminus Q\| - 2\left|E(Q) \cap P\right| - \left|E(Q, V(G) \setminus Q) \cap P\right| + h\left(V(G) \setminus Q\right) \\ &\geq |P| + 2\|Q\| + \|Q, V(G) \setminus Q\| - \|Q\| - |P| + h\left(V(G) \setminus Q\right) \\ &= \|Q\| + \|Q, V(G) \setminus Q\| + h\left(V(G) \setminus Q\right) \\ &\geq h(Q) + h\left(V(G) \setminus Q\right) \\ &= h(V(G)). \end{aligned}$$

The penultimate inequality holds due to (1). Hence $|g| \geq h(V(G))$. By Theorem 11, there is a flow f on Z with $f \leq g$ such that $\|f\| = |g|$. But then

$$\begin{aligned} |\{s\}, V(G)|_f &= |\{s\}, Z|_f \\ &= |\{s\}, Z|_f - |Z, \{s\}|_f \\ &= \|f\| \\ &= |g| \\ &\geq h(V(G)). \end{aligned}$$

Hence $f(s, v) = h(v)$ for all $v \in V(G)$. Since f is a flow, for each $v \in V(G)$, there is $I(v) \subseteq E(G)$ with $|I(v)| = h(v)$ such that $f(v, e) = 1$ for all $e \in I(v)$. Now $I(v) \cap I(w) = \emptyset$ for different $v, w \in V(G)$ since otherwise, $e \in I(v) \cap I(w)$ would have $f(e, t) \geq 2 > g(e, t)$. So, we can define our desired orientation of G by directing all edges in $I(v)$ into v for all $v \in V(G)$ and then directing any unoriented edges arbitrarily. $\square$

Exercise 6. *Carry out the computation mentioned at the start of the proof of Lemma 7.*

The proofs of Lemma 6 and Lemma 7 look so similar we might suspect there is some relation between the two lemmas.

Exercise 7. *Show that Lemma 7 follows from Lemma 6. Hint: split each $b \in V(G)$ into $h(b)$ copies of itself.*

Exercise 8. *Prove Lemma 6 without using flows.*

Exercise 9. *Give another proof of Lemma 7 by starting with an arbitrary orientation and repeatedly reversing paths that "improve" the orientation. Note the similarity to the proof of Theorem 11.*

Historical notes

Kernel magic

Definition 3. *The* maximum independent cover number *of a graph G is the maximum* $\text{mic}(G)$ *of* $\|I, V(G) \setminus I\|$ *over all independent sets I of G.*

Lemma 8. *If G is f-irreducible, then* $2\|G\| > f(V(G)) + \text{mic}(G) - |G|$.

Brooks' theorem for list coloring

Triangle-free graphs

Historical notes

Combinatorial nullstellensatz

Figure 13: Hilbert had good taste in hats.

Fix an arbitrary field $\mathbb{F}$. We write $f_{k_1,\ldots,k_n}$ for the coefficient of $x_1^{k_1} \cdots x_n^{k_n}$ in the polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$.

Lemma 9. *Suppose $f \in \mathbb{F}[x_1, \ldots, x_n]$ and $k_1, \ldots, k_n \in \mathbb{N}$ with $\sum_{i\in[n]} k_i = \deg(f)$. If $f_{k_1,\ldots,k_n} \neq 0$, then for any $A_1, \ldots, A_n \subseteq \mathbb{F}$ with $|A_i| \geq k_i + 1$, there exists $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$ with $f(a_1, \ldots, a_n) \neq 0$.*

Proof. Suppose the result is false and choose $f \in \mathbb{F}[x_1, \ldots, x_n]$ for which it fails minimizing $\deg(f)$. Then $\deg(f) \geq 2$ and we have $k_1, \ldots, k_n \in \mathbb{N}$ with $\sum_{i\in[n]} k_i = \deg(f)$ and $A_1, \ldots, A_n \subseteq \mathbb{F}$ with $|A_i| \geq k_i + 1$ such that $f(a_1, \ldots, a_n) = 0$ for all $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$. By symmetry, we may assume that $k_1 > 0$. Fix $a \in A_1$ and divide f by $x_1 - a$ to get $f = (x_1 - a)Q + R$ where the degree of x_1 in R is zero. Then the coefficient of $x_1^{k_1-1} x_2^{k_2} \cdots x_n^{k_n}$ in Q must be non-zero and $\deg(Q) < \deg(f)$. So, by minimality of $\deg(f)$ there is $(a_1, \ldots, a_n) \in (A_1 \setminus \{a\}) \times \cdots \times A_n$ such that $Q(a_1, \ldots, a_n) \neq 0$. Since $0 = f(a_1, \ldots, a_n) = (a_1 - a)Q(a_1, \ldots, a_n) + R(a_1, \ldots, a_n)$ we must have $R(a_1, \ldots, a_n) \neq 0$. But x_1 has degree zero in R, so $R(a, \ldots, a_n) = R(a_1, \ldots, a_n) \neq 0$. Finally, this means that $f(a, \ldots, a_n) = (a - a)Q(a, \ldots, a_n) + R(a, \ldots, a_n) \neq 0$, a contradiction. $\square$

The graph polynomial

Let G be a loopless multigraph with vertex set $V := \{x_1, \ldots, x_n\}$ and edge multiset E. The *graph polynomial* of G is

$$p_G(x_1, \ldots, x_n) := \prod_{\substack{\{x_i, x_j\} \in E \\ i<j}} (x_i - x_j).$$

To each orientation $\vec{G}$ of G, there is a corresponding monomial $m_{\vec{G}}(x_1, \ldots, x_n)$ given by choosing either x_i or $-x_j$ from each factor $(x_i - x_j)$ according to $\vec{G}$. Precisely, given an orientation $\vec{G}$ of G

with edge multiset $\vec{E}$, put

$$m_{\vec{G}}(x_1, \ldots, x_n) := \left(\prod_{\substack{(x_i,x_j)\in\vec{E} \\ i<j}} x_i \right) \left(\prod_{\substack{(x_j,x_i)\in\vec{E} \\ i<j}} -x_j \right).$$

Then $p_G(x_1, \ldots, x_n) = \sum_{\vec{G}} m_{\vec{G}}(x_1, \ldots, x_n)$, where the sum is over all orientations $\vec{G}$ of G.

Each $m_{\vec{G}}(x_1, \ldots, x_n)$ has coefficient either 1 or -1. We are interested in collecting up all monomials of the form $x_1^{k_1} \cdots x_n^{k_n}$. Let $DE_{k_1,\ldots,k_n}(G)$ be the orientations of G where $m_{\vec{G}}(x_1, \ldots, x_n) = x_1^{k_1} \cdots x_n^{k_n}$ and $DO_{k_1,\ldots,k_n}(G)$ the orientations of G where $m_{\vec{G}}(x_1, \ldots, x_n) = -x_1^{k_1} \cdots x_n^{k_n}$. Write $p_{k_1,\ldots,k_n}(G)$ for the coefficient of $x_1^{k_1} \cdots x_n^{k_n}$ in $p_G(x_1, \ldots, x_n)$. Then we have

$$p_{k_1,\ldots,k_n}(G) = |DE_{k_1,\ldots,k_n}(G)| - |DO_{k_1,\ldots,k_n}(G)|.$$

This gives a combinatorial interpretation of the coefficients of p_G, but unfortunately it quantifies over all orientations of G. For applying the Combinatorial Nullstellensatz, it is useful to have a single orientation of G as a certificate that $p_{k_1,\ldots,k_n}(G) \neq 0$. This can be achieved in terms of Eulerian subgraphs. A digraph is Eulerian if the in-degree and out-degree are equal at every vertex. Let $EE(\vec{G})$ be the spanning Eulerian subgraphs of $\vec{G}$ with an even number of edges and let $EO(\vec{G})$ be the spanning Eulerian subgraphs of $\vec{G}$ with an odd number of edges.

If $\vec{G}$ is an orientation of G, then we have

$$\left| |EE(\vec{G})| - |EO(\vec{G})| \right| = \left| |DE_{d^+_{\vec{G}}(x_1),\ldots,d^+_{\vec{G}}(x_n)}(G)| - |DO_{d^+_{\vec{G}}(x_1),\ldots,d^+_{\vec{G}}(x_n)}(G)| \right|.$$

Proof. For $D \in DE_{d^+_{\vec{G}}(x_1),\ldots,d^+_{\vec{G}}(x_n)}(G) \cup DO_{d^+_{\vec{G}}(x_1),\ldots,d^+_{\vec{G}}(x_n)}(G)$, let $\vec{G} \oplus D$ be the spanning subgraph of $\vec{G}$ with edge set

$$\left\{ (x_i, x_j) \in E(\vec{G}) : (x_j, x_i) \in E(D) \right\}.$$

Then $\vec{G} \oplus D$ is Eulerian since all vertices have the same out-degree in $\vec{G}$ and D. If $\vec{G}$ is even, this gives bijections between $DE_{d^+_{\vec{G}}(x_1),\ldots,d^+_{\vec{G}}(x_n)}(G)$ and $EE(\vec{G})$ as well as between $DO_{d^+_{\vec{G}}(x_1),\ldots,d^+_{\vec{G}}(x_n)}(G)$ and $EO(\vec{G})$. When $\vec{G}$ is odd, the bijections are between $DE_{d^+_{\vec{G}}(x_1),\ldots,d^+_{\vec{G}}(x_n)}(G)$ and $EO(\vec{G})$ as well as between $DO_{d^+_{\vec{G}}(x_1),\ldots,d^+_{\vec{G}}(x_n)}(G)$ and $EE(\vec{G})$. In either case, we have

$$\left| |EE(\vec{G})| - |EO(\vec{G})| \right| = \left| |DE_{d^+_{\vec{G}}(x_1),\ldots,d^+_{\vec{G}}(x_n)}(G)| - |DO_{d^+_{\vec{G}}(x_1),\ldots,d^+_{\vec{G}}(x_n)}(G)| \right|.$$

$\square$

Therefore, if $\vec{G}$ is an orientation of G with $d^+_{\vec{G}}(x_i) = k_i$ for all $i \in [n]$, then

$$|p_{k_1,\ldots,k_n}(G)| = \Big| |EE(\vec{G})| - |EO(\vec{G})| \Big|.$$

A coefficient formula

Lemma 10. *Suppose $f \in \mathbb{F}[x_1, \ldots, x_n]$ and $k_1, \ldots, k_n \in \mathbb{N}$ with $\sum_{i\in[n]} k_i = \deg(f)$. For any $A_1, \ldots, A_n \subseteq \mathbb{F}$ with $|A_i| = k_i + 1$, we have*

$$f_{k_1,\ldots,k_n} = \sum_{(a_1,\ldots,a_n)\in A_1\times\cdots\times A_n} \frac{f(a_1,\ldots,a_n)}{N(a_1,\ldots,a_n)},$$

where

$$N(a_1,\ldots,a_n) := \prod_{i\in[n]} \prod_{b\in A_i\setminus\{a_i\}} (a_i - b).$$

Proof. Let $f \in \mathbb{F}[x_1, \ldots, x_n]$ and $k_1, \ldots, k_n \in \mathbb{N}$ with $\sum_{i\in[n]} k_i = \deg(f)$. Also, let $A_1, \ldots, A_n \subseteq \mathbb{F}$ with $|A_i| = k_i + 1$. For each $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$, let $\chi_{(a_1,\ldots,a_n)}$ be the characteristic function of the set $\{(a_1, \ldots, a_n)\}$; that is $\chi_{(a_1,\ldots,a_n)} \colon A_1 \times \cdots \times A_n \to \mathbb{F}$ with $\chi_{(a_1,\ldots,a_n)}(x_1, \ldots, x_n) = 1$ when $(x_1, \ldots, x_n) = (a_1, \ldots, a_n)$ and $\chi_{(a_1,\ldots,a_n)}(x_1, \ldots, x_n) = 0$ otherwise. Consider the function

$$F = \sum_{(a_1,\ldots,a_n)\in A_1\times\cdots\times A_n} f(a_1,\ldots,a_n)\chi_{(a_1,\ldots,a_n)}.$$

Then F agrees with f on all of $A_1 \times \cdots \times A_n$ and hence $f - F$ is zero on $A_1 \times \cdots \times A_n$. We will apply the Combinatorial Nullstellensatz to $f - F$ to conclude that $(f - F)_{k_1,\ldots,k_n} = 0$ and hence $f_{k_1,\ldots,k_n} = F_{k_1,\ldots,k_n}$ where $F_{k_1,\ldots,k_n}$ will turn out to be our desired sum. To apply the Combinatorial Nullstellensatz, we need to represent F as a polynomial, we can do so by representing each $\chi_{(a_1,\ldots,a_n)}$ as a polynomial as follows. For $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$, let

$$N(a_1,\ldots,a_n) := \prod_{i\in[n]} \prod_{b\in A_i\setminus\{a_i\}} (a_i - b).$$

Then it is readily verified that

$$\chi_{(a_1,\ldots,a_n)}(x_1,\ldots,x_n) = \frac{\prod_{i\in[n]} \prod_{b\in A_i\setminus\{a_i\}} (x_i - b)}{N(a_1,\ldots,a_n)}.$$

Using this to define F we get

$$F(x_1,\ldots,x_n) = \sum_{(a_1,\ldots,a_n)\in A_1\times\cdots\times A_n} f(a_1,\ldots,a_n)\frac{\prod_{i\in[n]} \prod_{b\in A_i\setminus\{a_i\}} (x_i - b)}{N(a_1,\ldots,a_n)}.$$

Now $\deg(F) = \sum_{i\in[n]}(|A_i| - 1) = \sum_{i\in[n]} k_i = \deg(f)$. Since $f - F$ is zero on $A_1 \times \cdots \times A_n$, applying the Combinatorial Nullstellensatz to

$f - F$ with $k_1, \ldots, k_n$ and sets $A_1, \ldots, A_n$ gives $(f - F)_{k_1,\ldots,k_n} = 0$ and hence

$$f_{k_1,\ldots,k_n} = F_{k_1,\ldots,k_n} = \sum_{(a_1,\ldots,a_n)\in A_1\times\cdots\times A_n} \frac{f(a_1, \ldots, a_n)}{N(a_1, \ldots, a_n)}.$$

□

Uniquely colorable graphs

Adding edges to simplify coefficient computation

AT preserving operations

Historical notes

Independent transversals

Let H be a graph and $V_1, \ldots, V_r$ a partition of $V(H)$. An *independent transversal* of $V_1, \ldots, V_r$ is an independent set I in H with $I \cap V_i \neq \emptyset$ for all $i \in [r]$.

Corollary 1. *Let H be a graph and $V_1, \ldots, V_r$ a partition of $V(H)$. If $|V_i| \geq 2\Delta(H)$ for each $i \in [r]$, then $V_1, \ldots, V_r$ has an independent transversal.*

With a more technical statement, we can account for non-uniform $|V_i|$. Corollary 1 follows from the case when $t = 0$ and $S = \emptyset$.

Theorem 12. *Let H be a graph and $V_1 \cup \cdots \cup V_r$ a partition of $V(H)$. Suppose there exists $t \geq 1$ such that for each $i \in [r]$ and each $v \in V_i$ we have $d(v) \leq \min\{t, |V_i| - t\}$. For any $S \subseteq V(H)$ with $|S| < \min\{|V_1|, \ldots, |V_r|\}$, there is an independent transversal I of $V_1, \ldots, V_r$ with $I \cap S = \emptyset$.*

In fact, an even more general statement holds and the strengthened induction hypothesis simplifies the proof. First we need some definitions and notation. Write $f\colon A \twoheadrightarrow B$ for a surjective (onto) function from A to B. Let G be a graph. A set of vertices $S \subseteq V(G)$ *dominates* G if each $v \in V(G) \setminus S$ has a neighbor in S. A set of vertices $S \subseteq V(G)$ *totally dominates* G if each $v \in V(G)$ has a neighbor in S. A set of edges $M \subseteq E(G)$ is a *matching* of G if $e_1 \cap e_2 = \emptyset$ for all $e_1, e_2 \in M$. A matching is *induced* if $\|G[e_1 \cup e_2]\| = 2$ for all $e_1, e_2 \in M$. For a k-coloring $\pi\colon V(G) \twoheadrightarrow [k]$ of G and a subgraph H of G we say that $I := \{x_1, \ldots, x_k\} \subseteq V(H)$ is an H-independent transversal of π if I is an independent set in H and $\pi(x_i) = i$ for all $i \in [k]$.

Lemma 11. *Let G be a graph and $\pi\colon V(G) \twoheadrightarrow [k]$ a proper k-coloring of G. Suppose that π has no G-independent transversal, but for every $e \in E(G)$, π has a $(G - e)$-independent transversal. Then for every $xy \in E(G)$ there is $J \subseteq [k]$ with $\pi(x), \pi(y) \in J$ and an induced matching M of $G\left[\pi^{-1}(J)\right]$ with $xy \in M$ such that:*

1. *$\bigcup M$ totally dominates $G\left[\pi^{-1}(J)\right]$,*
2. *the multigraph with vertex set J and an edge between $a, b \in J$ for each $uv \in M$ with $\pi(u) = a$ and $\pi(v) = b$ is a (simple) tree. In particular $|M| = |J| - 1$.*

Proof. Suppose the lemma is false and choose a counterexample G with $\pi\colon V(G) \twoheadrightarrow [k]$ so as to minimize k. Let $xy \in E(G)$. By assumption π has a $(G - xy)$-independent transversal T. Note that we must have $x, y \in T$ lest T be a G-independent transversal of π.

By symmetry we may assume that $\pi(x) = k - 1$ and $\pi(y) = k$. Put $X := \pi^{-1}(k-1)$, $Y := \pi^{-1}(k)$ and $H := G - N(\{x, y\}) - E(X, Y)$. Define $\zeta\colon V(H) \twoheadrightarrow [k-1]$ by $\zeta(v) := \min\{\pi(v), k-1\}$. Note that since $x, y \in T$, we have $\left|\zeta^{-1}(i)\right| \geq 1$ for each $i \in [k-2]$. Put $Z := \zeta^{-1}(k-1)$. Then $Z \neq \emptyset$ for otherwise $M := \{xy\}$ totally dominates $G[X \cup Y]$ giving a contradiction.

Suppose ζ has an H-independent transversal S. Then we have $z \in S \cap Z$ and by symmetry we may assume $z \in X$. But then $S \cup \{y\}$ is a G-independent transversal of π, a contradiction.

Let $H' \subseteq H$ be a minimal spanning subgraph such that ζ has no H'-independent transversal. Now $d(z) \geq 1$ for each $z \in Z$ for otherwise $T - \{x, y\} \cup \{z\}$ would be an H'-independent transversal of ζ. Pick $zw \in E(H')$. By minimality of k, we have $J \subseteq [k-1]$ with $\zeta(z), \zeta(w) \in J$ and an induced matching M of $H'\left[\zeta^{-1}(J)\right]$ with $zw \in M$ such that

1. $\bigcup M$ totally dominates $H'\left[\zeta^{-1}(J)\right]$,

2. the multigraph with vertex set J and an edge between $a, b \in J$ for each $uv \in M$ with $\zeta(u) = a$ and $\zeta(v) = b$ is a (simple) tree.

Put $M' := M \cup \{xy\}$ and $J' := J \cup \{k\}$. Since H' is a spanning subgraph of H, $\bigcup M$ totally dominates $H\left[\zeta^{-1}(J)\right]$ and hence $\bigcup M'$ totally dominates $G\left[\pi^{-1}(J')\right]$. Moreover, the multigraph in (2) for M' and J' is formed by splitting the vertex $k-1 \in J$ into two vertices and adding an edge between them and hence it is still a tree. This final contradiction proves the lemma. $\square$

Proof of Theorem 12. Suppose the lemma fails for such an $S \subseteq V(H)$. Put $H' := H - S$ and let $V_1', \ldots, V_r'$ be the induced partition of H'. Then there is no independent transversal of $V_1', \ldots, V_r'$ and $\left|V_i'\right| \geq 1$ for each $i \in [r]$. Create a graph Q by removing edges from H' until it is edge minimal without an independent transversal. Pick $yz \in E(Q)$ and apply Lemma 11 on yz with the induced partition to get the guaranteed $J \subseteq [r]$ and the tree T with vertex set J and an edge between $a, b \in J$ for each $uv \in M$ with $u \in V_a'$ and $v \in V_b'$. By our condition, for each $uv \in E(V_i, V_j)$, we have $|N_H(u) \cup N_H(v)| \leq \min\{|V_i|, |V_j|\}$.

Choose a root c of T. Traversing T in leaf-first order and for each leaf a with parent b picking $|V_a|$ from $\min\{|V_a|, |V_b|\}$ we get that the vertices in M together dominate at most $\sum_{i \in J \setminus \{c\}} |V_i|$ vertices in H.

Since $|S| < |V_c|$, M cannot totally dominate $\bigcup_{i \in J} V_i'$, a contradiction.

□

The condition on S can be weakened slightly. Suppose we have ordered the V_i so that $|V_1| \leq |V_2| \leq \cdots \leq |V_r|$. Then for any $S \subseteq V(H)$ with $|S| < |V_2|$ such that $V_1 \not\subseteq S$, there is an independent transversal I of $V_1, \ldots, V_r$ with $I \cap S = \emptyset$. The proof is the same except when we choose our root c, choose it so as to maximize $|V_c|$. Since $|J| \geq 2$, we get $|V_c| \geq |V_2| > |S|$ at the end.

Lists with low degree color graphs

Hitting all maximum cliques

Theorem 13. *If G is a graph with $\omega(G) > \frac{2}{3}(\Delta(G) + 1)$, then G contains an independent set I such that $\omega(G - I) < \omega(G)$.*

Let G be a graph. For a collection of cliques $\mathcal{Q}$ in G, let $X_{\mathcal{Q}}$ be the intersection graph of $\mathcal{Q}$; that is, the vertex set of $X_{\mathcal{Q}}$ is $\mathcal{Q}$ and there is an edge between $Q_1, Q_2 \in \mathcal{Q}$ just in case $Q_1 \neq Q_2$ and Q_1 and Q_2 intersect. When $\mathcal{Q}$ is a collection of maximum cliques, we get a lot of information about $X_{\mathcal{Q}}$.

Lemma 12. *If G is a graph and $\mathcal{Q}$ is a non-empty collection of maximum cliques in G, then*

$$\left|\bigcup \mathcal{Q}\right| + \left|\bigcap \mathcal{Q}\right| \geq 2\omega(G). \tag{2}$$

Proof. Suppose there is a graph G and collection $\mathcal{Q}$ of maximum cliques in G for which (2) fails. Then we may pick such a $\mathcal{Q}$ with $|\mathcal{Q}|$ as small as possible. Put $r := |\mathcal{Q}|$ and suppose $\mathcal{Q} = \{Q_1, \ldots, Q_r\}$. Consider the set

$$W := (Q_1 \cap \bigcup_{i=2}^{r} Q_i) \cup \bigcap_{i=2}^{r} Q_i.$$

Plainly, W is a clique. Thus

$$\begin{aligned}
\omega(G) &\geq |W| \\
&= \left|(Q_1 \cap \bigcup_{i=2}^{r} Q_i) \cup \bigcap_{i=2}^{r} Q_i\right| \\
&= \left|Q_1 \cap \bigcup_{i=2}^{r} Q_i\right| + \left|\bigcap_{i=2}^{r} Q_i\right| - \left|\bigcap_{i=1}^{r} Q_i \cap \bigcup_{i=2}^{r} Q_i\right| \\
&= |Q_1| + \left|\bigcup_{i=2}^{r} Q_i\right| - \left|\bigcup_{i=1}^{r} Q_i\right| + \left|\bigcap_{i=2}^{r} Q_i\right| - \left|\bigcap_{i=1}^{r} Q_i\right| \\
&= \omega(G) + \left|\bigcup_{i=2}^{r} Q_i\right| + \left|\bigcap_{i=2}^{r} Q_i\right| - \left|\bigcup_{i=1}^{r} Q_i\right| - \left|\bigcap_{i=1}^{r} Q_i\right| \\
&\geq \omega(G) + 2\omega(G) - \left(\left|\bigcup_{i=1}^{r} Q_i\right| + \left|\bigcap_{i=1}^{r} Q_i\right|\right) \\
&> \omega(G),
\end{aligned}$$

a contradiction. Here the final inequality comes from the failure of (2) and the penultimate inequality comes from minimality of $|\mathcal{Q}|$ by applying the lemma to $\{Q_2, \ldots, Q_r\}$. $\square$

Lemma 13. *If $\mathcal{Q}$ is a non-empty collection of maximum cliques in a graph G with $\omega(G) > \frac{2}{3}(\Delta(G)+1)$ such that $X_{\mathcal{Q}}$ is connected, then $\cap\mathcal{Q} \neq \emptyset$.*

Proof. Suppose there is a graph G with $\omega(G) > \frac{2}{3}(\Delta(G)+1)$ and collection $\mathcal{Q}$ of maximum cliques in G for which $X_{\mathcal{Q}}$ is connected and $\cap\mathcal{Q} = \emptyset$. Then we may pick such a $\mathcal{Q}$ with $|\mathcal{Q}|$ as small as possible. Then $|\mathcal{Q}| \geq 2$ since $\cap\mathcal{Q} = \emptyset$. Let A be a noncutvertex in $X_{\mathcal{Q}}$ and B a neighbor of A. Put $\mathcal{Z} := \mathcal{Q} \setminus \{A\}$. Then $\mathcal{Z}$ is non-empty, $X_{\mathcal{Z}}$ is connected and $|\mathcal{Z}| < |\mathcal{Q}|$. Hence, by minimality of $|\mathcal{Q}|$, we may apply the lemma to $\mathcal{Z}$ which gives $\cap\mathcal{Z} \neq \emptyset$. Since a vertex in $\cap\mathcal{Z}$ is adjacent to all other vertices in $\cup\mathcal{Z}$, we conclude $|\cup\mathcal{Z}| \leq \Delta(G)+1$. By assumption, $\cap\mathcal{Q} = \emptyset$, so

$$\begin{aligned}
|\cap\mathcal{Q}| + |\cup\mathcal{Q}| &\leq 0 + (|\cup\mathcal{Z}| + |A \setminus B|) \\
&\leq (\Delta(G)+1) + (\Delta(G)+1-\omega(G)) \\
&= 2(\Delta(G)+1) - \omega(G) \\
&< 2\omega(G),
\end{aligned}$$

contradicting Lemma 12. Here the final inequality holds because $\omega(G) > \frac{2}{3}(\Delta(G)+1)$. $\square$

Proof of Theorem 13. Let $\mathcal{Q}$ be all the maximum cliques in G and $\mathcal{Q}_1, \ldots, \mathcal{Q}_k$ the vertex sets of the components of $X_{\mathcal{Q}}$. For $i \in [k]$, put $F_i := \cap\mathcal{Q}_i$ and $D_i := \cup\mathcal{Q}_i$. Since the components of $X_{\mathcal{Q}}$ satisfy the hypotheses of Lemma 13, we have $F_i \neq \emptyset$ for all $i \in [k]$.

Put $t := \frac{1}{3}(\Delta(G)+1)$. Fix $i \in [k]$. Each $v \in F_i$ has at most

$$\begin{aligned} d_G(v) + 1 - |D_i| &\leq \Delta(G) + 1 - |D_i| \\ &\leq \Delta(G) + 1 - \omega(G) \\ &< t, \end{aligned}$$

neighbors in the rest of the F_j.

Applying Lemma 12 gives $|F_i| + |D_i| \geq 2\omega(G) > \frac{4}{3}(\Delta(G)+1)$. Thus each $v \in F_i$ also has at most

$$\begin{aligned} d_G(v) + 1 - |D_i| &\leq \Delta(G) + 1 - |D_i| \\ &< \Delta(G) + 1 - \left(\frac{4}{3}(\Delta(G)+1) - |F_i|\right) \\ &= |F_i| - t, \end{aligned}$$

neighbors in the rest of the F_j.

Consider the subgraph H of G with vertex set $\bigcup_{j \in [k]} F_j$ formed by making each F_j edgeless. Applying Theorem 12 to H with partition $F_1, \ldots, F_k$ and $S = \emptyset$ gives an independent set I intersecting each F_i and hence every maximum clique in G. So, we have $\omega(G - I) < \omega(G)$ as desired. □

Historical notes

Vertex partitions and shuffling

Historical notes

Coloring edges

It is also useful to consider coloring the edges of a graph so that incident edges receive different colors. This appears to be at odds with our previous claim that this book was only about coloring vertices of graph; fortunately, edge coloring is just a special case of vertex coloring. If G is a graph, the *line graph* of G, written $L(G)$ is the graph with vertex set $E(G)$ where two edges of G are adjacent in $L(G)$ if they are incident in G. Coloring the edges of G is equivalent to coloring the vertices of $L(G)$.

For graphs with maximum degree zero (that is, no edges at all), we can get by with zero colors. With just one color we can edge color any graph with maximum degree at most one. We will definitely always need at least $\Delta(G)$ colors to edge color a graph G. Could we be so fortunate that the pattern continues and we can edge color any graph G with only $\Delta(G)$-colors? Not quite, but we can do so for bipartite (2-colorable) graphs. A graph is k-edge-colorable if we can color its edges with (at most) k colors such that incident edges receive different colors. A color c us *used* at a vertex v of G if an edge incident to v in G is colored with c. Otherwise, c is *available* at v. A *path* in G is a sequence of pairwise distinct vertices $x_1 x_2 \cdots x_k$ such that x_i is adjacent to x_{i+1} for $i \in [k-1]$.

Theorem 14. *If G is a bipartite graph, then G is $\Delta(G)$-edge-colorable.*

Proof. Suppose there is a graph G that is not $\Delta(G)$-edge-colorable. Then we may pick such a graph G with $\|G\|$ as small as possible. Now $\|G\| > 0$, since we can surely edge color a graph with zero edges using zero colors. Let xy be an edge in G. Since $\|G - xy\| < \|G\|$, applying the theorem to $G - xy$ gives an edge coloring of $G - xy$ using at most $\Delta(G)$ colors. Now each of x and y are incident to at most $\Delta(G) - 1$ edges in $G - xy$ and G has no $\Delta(G)$-edge-coloring, so there is a color red available at x and a different color blue available at y. There is a unique maximal path P starting at x with edges alternately colored blue and red. If P does not end at y, then we get a $\Delta(G)$-edge-coloring of G by swapping the colors red and blue along P and coloring xy blue, a contradiction. Since P ends at y and alternates between red and blue, it has even length. But then $P + xy$ is an odd cycle in G, violating Theorem 1. $\square$

It may come as a surpise that even though we might need more than $\Delta(G)$ colors to edge color a graph G, we will only ever need at most one extra color. For bipartite graphs we were able to repair an almost correct coloring by swapping colors along a path because we had control over where this path ended. In the general case we don't have the same control over a path between two vertices, but we can exert some measure of control over paths leaving and entering a larger structure. The larger structure we use here is the whole neighborhood of a vertex.

Corollary 2. *If $\left|\bigcup_{i\in[k]} A_i\right| \geq k$, then there is nonempty $I \subseteq [k]$ such that $\{A_i\}_{i\in I}$ has a transversal X where $X \cap A_i = \emptyset$ for all $i \in [k] \setminus I$.*

Proof. Immediate from Lemma 6. $\square$

Theorem 15. *If G is a graph, then G is $(\Delta(G)+1)$-edge-colorable.*

Proof. Suppose there is a graph G that is not $(\Delta(G)+1)$-edge-colorable. Then we may pick such a graph G with $|G|$ as small as possible. Now $|G| > 0$, since we can surely edge color a graph with zero vertices using at most one color. Let x be a vertex in G. Call $S \subseteq N(x)$ *acceptable* for an edge coloring π of $G - x$ if $\{\bar{\pi}(v)\}_{v\in S}$ has a transversal T_S such that $|\bar{\pi}(v) \setminus T_S| \geq 2$ for all $v \in N(x) \setminus S$.

Since $|G - x| < |G|$, applying the theorem to $G - x$ gives an edge coloring ζ of $G - x$ using at most $\Delta(G) + 1$ colors. Note that the empty set is acceptable for ζ. So, we may choose an edge coloring π of $G - x$ using at most $\Delta(G) + 1$ colors and $S \subseteq N(x)$ that is acceptable for π so as to maximize $|S|$ and subject to that to maximize $|C_S|$, where

$$C_S := \bigcup_{v\in N(x)\setminus S} \bar{\pi}(v) \setminus T_S.$$

Now $S \neq N(x)$ for otherwise we can extend π to all of G using T_S. Suppose $|C_S| \geq |N(x) \setminus S|$. Then, by Corollary 2, there is nonempty $A \subseteq N(x) \setminus S$ such that $\{\bar{\pi}(v) \setminus T_S\}_{v\in A}$ has a transversal X where $X \cap \bar{\pi}(v) = \emptyset$ for all $v \in N(x) \setminus (S \cup A)$. But then $S \cup A$ is acceptable for π, contradicting maximality of $|S|$.

So, $|C_S| < |N(x) \setminus S|$ and hence $|C_S \cup T_S| < |N(x)| \leq \Delta(G)$. Pick $\tau \in [\Delta(G)] \setminus (C_S \cup T_S)$. Since S is acceptable for π, $|\bar{\pi}(v) \setminus T_S| \geq 2$ for all $v \in N(x) \setminus S$. Hence there are $v_1, v_2, v_3 \in N(x) \setminus S$ and $\gamma \in C_S$ such that $\gamma \in \bar{\pi}(v_i) \setminus T_S$ for all $i \in [3]$. There is a unique maximal path P starting at v_1 with edges alternately colored τ and γ. Let ζ be the edge coloring made from π by swapping τ and γ on P. Then ζ

violates the maximality of $|C_S|$ since S is acceptable for ζ and

$$\bigcup_{v \in N(x) \setminus S} \bar{\zeta}(v) \setminus T_S = \{\tau\} \cup C_S.$$

$\square$

hardness

We now know that every graph G can be edge colored with either $\Delta(G)$ or $\Delta(G) + 1$ colors. So, edge coloring is basically trivial, right? Furtunately, no it isn't, the collection of graphs requiring $\Delta(G) + 1$ colors is very rich. Another way to say this, is that it is a hard problem to decide whether or not edge coloring a given graph G requires $\Delta(G) + 1$ colors.

Theorem 16. *Deciding whether or not edge coloring a given graph G requires* $\Delta(G) + 1$ *colors is hard supposing other things we think are hard are actually hard.*

Historical notes

This is not Vizing's proof.