Improved lower bounds on the number of edges in list critical and online list critical graphs

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Abstract

We prove that every k-list-critical graph ($k \geq 8$) on $n \geq k+2$ vertices has at least $\frac{1}{2}\left(k-1+\frac{k-3}{(k-c)(k-1)+k-3}\right)n$ edges where $c=(k-3)\left(\frac{1}{2}-\frac{1}{(k-1)(k-2)}\right)$. This improves the bound established by Kostochka and Stiebitz [13]. The same bound holds for online list-critical graphs, improving the bound established by Riasat and Schauz [17]. Both bounds follow from a more general result stating that either a graph has many edges or it has an Alon-Tarsi orientable induced subgraph satisfying a certain degree condition. Finally, we use these lower bounds to prove an Ore-degree version of Brooks' theorem for online list-coloring: every graph with $\theta \geq 18$ and $\omega \leq \frac{\theta}{2}$ is online $\lfloor\frac{\theta}{2}\rfloor$-choosable. Here θ is the Ore-degree.

1 Introduction

A k-*coloring* of a graph G is a function $\pi\colon V(G) \to [k]$ such that $\pi(x) \neq \pi(y)$ for each $xy \in E(G)$. The least k for which G has a k-coloring is the *chromatic number* $\chi(G)$ of G. We say that G is k-*chromatic* when $\chi(G) = k$. A graph is k-*critical* if $\chi(G) = k$ and $\chi(H) < k$ for every proper subgraph H of G. If G is k-chromatic, then any minimal k-chromatic subgraph of G is k-critical. In this way, many questions about k-chromatic graphs can be reduced to questions about k-critical graphs which have more structure. The study of critical graphs was initiated by Dirac [4] in 1951. It is easy to see that a k-critical graph G must have minimum degree at least $k-1$ and hence $2\|G\| \geq (k-1)|G|$. The problem of determining the minimum number of edges in a k-critical graph has a long history. First, in 1957, Dirac [5] generalized Brooks' theorem [3] by showing that any k-critical graph G with $k \geq 4$ and $|G| \geq k+2$ must satisfy

$$2\|G\| \geq (k-1)|G| + k - 3.$$

In 1963, this bound was improved for large $|G|$ by Gallai [7]. Put

$$g_k(n,c) := \left(k-1+\frac{k-3}{(k-c)(k-1)+k-3}\right)n.$$

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Gallai showed that every k-critical graph G with $k \geq 4$ and $|G| \geq k+2$ satisfies $2\|G\| \geq g_k(|G|, 0)$. In 1997, Krivelevich [14] improved Gallai's bound by replacing $g_k(|G|, 0)$ with $g_k(|G|, 2)$. Then, in 2003, Kostochka and Stiebitz [13] improved this by showing that a k-critical graph with $k \geq 6$ and $|G| \geq k+2$ must satisfy $2\|G\| \geq g_k(|G|, (k-5)\alpha_k)$ where

$$\alpha_k := \frac{1}{2} - \frac{1}{(k-1)(k-2)}.$$

In 2012, Kostochka and Yancey [11] drastically improved these bounds by showing that every k-critical graph G with $k \geq 4$ must satisfy

$$\|G\| \geq \left\lceil \frac{(k+1)(k-2)|G| - k(k-3)}{2(k-1)} \right\rceil.$$

Moreover, they show that this bound is tight for $k = 4$ and $n \geq 6$ as well as for infinitely many values of $|G|$ for any $k \geq 5$. This bound has many interesting coloring applications such as a very short proof of Grötsch's theorem on the 3-colorability of triangle-free planar graphs [10] and short proofs of the results on coloring with respect to Ore degree in [9, 15, 12]. Given the applications to coloring theory, it makes sense to investigate the same problem for more general types of coloring. In this article, we obtain improved lower bounds on the number of edges for both the list coloring and online list coloring problems. To state our results we need some definitions.

List coloring was introduced by Vizing [20] and independently Erdős, Rubin and Taylor [6]. Let G be a graph. A list assignment on G is a function L from $V(G)$ to the subsets of $\mathbb{N}$. A graph G is *L-colorable* if there is $\pi\colon V(G) \to \mathbb{N}$ such that $\pi(v) \in L(v)$ for each $v \in V(G)$ and $\pi(x) \neq \pi(y)$ for each $xy \in E(G)$. A graph G is *L-critical* if G is not L-colorable, but every proper subgraph H of G is $L\big|_{V(H)}$-colorable. For $f\colon V(G) \to \mathbb{N}$, a list assignment L is an *f-assignment* if $|L(v)| = f(v)$ for each $v \in V(G)$. If $f(v) = k$ for all $v \in V(G)$, then we also call an f-assignment a k-assignment. We say that G is *f-choosable* if G is L-colorable for every f-assignment L. The best known lower bound on the number of edges in an L-critical graph where L is a $(k-1)$-assignment was given by Kostochka and Stiebitz [13] in 2003 and shows that for $k \geq 9$ and $G \neq K_k$ an L-critical graph where L is a $(k-1)$-assignment on G, we have $2\|G\| \geq g_k(|G|, \frac{1}{3}(k-4)\alpha_k)$. We improve the bound to $2\|G\| \geq g_k(|G|, (k-3)\alpha_k)$.

Online list coloring was independently introduced by Zhu [21] and Schauz [18] (Schauz called it *paintability*). Let G be a graph and $f\colon V(G) \to \mathbb{N}$. We say that G is *online f-choosable* if $f(v) \geq 1$ for all $v \in V(G)$ and for every $S \subseteq V(G)$ there is an independent set $I \subseteq S$ such that $G - I$ is online f'-choosable where $f'(v) := f(v)$ for $v \in V(G) - S$ and $f'(v) := f(v) - 1$ for $v \in S - I$. We say that G is *online f-critical* if G is not online f-choosable, but every proper subgraph H of G is online $f\big|_{V(H)}$-choosable. In 2012, Riasat and Schauz [17] showed that Gallai's bound $2\|G\| \geq g_k(|G|, 0)$ holds for online f-critical graphs where $f(v) := k-1$ for all $v \in V(G)$. We improve this for $k \geq 8$ by proving the same bound as we have for list coloring: $2\|G\| \geq g_k(|G|, (k-3)\alpha_k)$.

Our main theorem shows that a graph either has many edges or an induced subgraph which has a certain kind of good orientation. To describe these good orientations we need a few definitions. A subdigraph H of a directed multigraph D is called *eulerian* if $d_H^-(v) = d_H^+(v)$ for every $v \in V(H)$. We call H *even* if $\|H\|$ is even and *odd* otherwise. We write

$EE(D)$ (resp. $EO(D)$) for the number of even (resp. odd) spanning subdigraphs of D. Note that the edgeless subgraph of D is even and hence we always have $EE(D) > 0$.

Let G be a graph and $f\colon V(G) \to \mathbb{N}$. We say that G is *f-Alon-Tarsi* (for brevity, *f-AT*) if G has an orientation D where $f(v) \geq d_D^+(v) + 1$ for all $v \in V(D)$ and $EE(D) \neq EO(D)$. Alon and Tarsi [1] showed that such orientations are very useful for list coloring; they proved the following.

Lemma 1.1. *If a graph G is f-AT for $f\colon V(G) \to \mathbb{N}$, then G is f-choosable.*

Schauz [19] extended this result to online f-choosability.

Lemma 1.2. *If a graph G is f-AT for $f\colon V(G) \to \mathbb{N}$, then G is online f-choosable.*

For a graph G, we define $d_0\colon V(G) \to \mathbb{N}$ by $d_0(v) := d_G(v)$. The d_0-choosable graphs were first characterized by Borodin [2] and independently by Erdős, Rubin and Taylor [6]. The connected graphs which are not d_0-choosable are precisely the Gallai trees (connected graphs in which every block is complete or an odd cycle). The generalization to a characterization of d_0-AT graphs was first given in [8] by Hladkỳ, Král and Schauz.

We prove the following general lemma saying that either a graph has many edges or has an induced f_H-AT subgraph H where f_H basically gives the number of colors we would expect the vertices to have left in their lists after $\delta(G)$-coloring $G - H$.

Theorem 2.13. *Let G be a graph with $\delta := \delta(G) \geq 5$ and $K_{\delta+1} \not\subseteq G$. If G does not have a nonempty induced subgraph H which is f_H-AT where $f_H(v) := \delta + d_H(v) - d_G(v)$ for all $v \in V(H)$, then $2\left\|G\right\| \geq g_{\delta+1}(\left|G\right|, c)$ where $c := (\delta-2)\alpha_{\delta+1}$ when $\delta \geq 7$ and $c := (\delta-3)\alpha_{\delta+1}$ when $\delta \in \{5, 6\}$.*

The *Alon-Tarsi number* of a graph $AT(G)$ is the least k such that G is f-AT where $f(v) := k$ for all $v \in V(G)$. We have $\chi(G) \leq \mathrm{ch}(G) \leq \mathrm{ch}_{OL}(G) \leq AT(G) \leq \mathrm{col}(G)$. We say that G is k-AT-critical if $\mathrm{AT}(G) = k$ and $AT(H) < k$ for all proper induced subgraphs H of G. From Theorem 2.13 we can conclude the following.

Theorem 3.5. *For $k \geq 6$ and $G \neq K_k$ a k-AT-critical graph, we have $2\left\|G\right\| \geq g_k(\left|G\right|, c)$ where $c := (k-3)\alpha_k$ when $k \geq 8$ and $c := (k-4)\alpha_k$ when $k \in \{6, 7\}$.*

Similarly, applying Lemma 1.1 gives the following.

Theorem 3.1. *For $k \geq 6$ and $G \neq K_k$ an L-critical graph where L is a $(k-1)$-assignment on G, we have $2\left\|G\right\| \geq g_k(\left|G\right|, c)$ where $c := (k-3)\alpha_k$ when $k \geq 8$ and $c := (k-4)\alpha_k$ when $k \in \{6, 7\}$.*

This improves the bound given by Kostochka and Stiebitz in [13]; for L-critical graphs, they have $2\left\|G\right\| \geq g_k(\left|G\right|, \frac{1}{3}(k-4)\alpha_k)$ for $k \geq 9$. Now, applying Lemma 1.2 gives the following.

Theorem 3.3. *For $k \geq 6$ and $G \neq K_k$ an online f-critical graph where $f(v) := k-1$ for all $v \in V(G)$, we have $2\left\|G\right\| \geq g_k(\left|G\right|, c)$ where $c := (k-3)\alpha_k$ when $k \geq 8$ and $c := (k-4)\alpha_k$ when $k \in \{6, 7\}$.*

Definition 1. The *Ore-degree* of an edge xy in a graph G is $\theta(xy) := d(x) + d(y)$. The *Ore-degree* of a graph G is $\theta(G) := \max_{xy \in E(G)} \theta(xy)$.

A bound like Brooks' theorem in terms of the Ore-degree was given by Kierstead and Kostochka [9] and subsequently the required lower bound on Δ was improved in [15, 12, 16]. For example, we have the following.

Theorem 1.3. *Every graph with $\theta \geq 10$ and $\omega \leq \frac{\theta}{2}$ is $\left\lfloor \frac{\theta}{2} \right\rfloor$-colorable.*

Another method for achieving the tightest of these results on Ore-degree was given by Kostochka and Yancey [11]. Their proof combined their new lower bound on the number of edges in a color critical graph together with a list coloring lemma derived via the kernel lemma. In Section 4 we improve this latter lemma and, in a similar way, use it in combination with Theorem 3.1 to prove an Ore-degree version of Brooks' theorem for list coloring. The improved lemma can be seen to be giving another lower bound on the number of edges in G. Let $\mathrm{mic}(G)$ be the maximum of $\sum_{v \in I} d_G(v)$ over all independent sets I of G.

Theorem 4.5. *For any graph G we have either:*

1. *G has a nonempty induced subgraph H which is online f_H-choosable where $f_H(v) := \delta(G) + d_H(v) - d_G(v)$ for all $v \in V(H)$; or*

2. $2\left\|G\right\| \geq (\delta(G) - 1)\left|G\right| + \mathrm{mic}(G) + 1$.

This quickly gives the aforementioned Ore degree version of Brooks' theorem for list coloring.

Theorem 5.9. *Every graph with $\theta \geq 18$ and $\omega \leq \frac{\theta}{2}$ is $\left\lfloor \frac{\theta}{2} \right\rfloor$-choosable.*

Note that using Kostochka and Stiebitz's above lower bound on the number of edges in a list critical graph gives Theorem 5.9 with $\theta \geq 54$. Similarly, we get the online version.

Theorem 5.8. *Every graph with $\theta \geq 18$ and $\omega \leq \frac{\theta}{2}$ is online $\left\lfloor \frac{\theta}{2} \right\rfloor$-choosable.*

We expect that Theorems 5.8 and 5.9 actually hold for $\theta \geq 10$. In the regular coloring case, it was shown in [12] that the only exception when $\theta \geq 8$ is the graph O_5; again, the expectation is that the same result will hold for Theorems 5.8 and 5.9.