COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
18. COMBINATORICS, KESZTHELY (HUNGARY), 1976.

DEGREE, GIRTH AND CHROMATIC NUMBER

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§1.

The following notations will be used throughout: $\chi(G)$ is the chromatic number of a graph G; $\mathscr{L}_\sigma$ is the class of graphs with the maximal degree of vertices not exceeding σ; $\mathscr{L}^g$ is the class of graphs whose girth is at least g; $\mathscr{L}^g_\sigma = \mathscr{L}^g \cap \mathscr{L}_\sigma$.

$\lfloor x \rfloor$ and $\lceil x \rceil$ denote respectively the lower and upper integers of x (i.e. $x - 1 < \lfloor x \rfloor \leqslant x$ and $x \leqslant \lceil x \rceil < x + 1$).

It is evident that for any σ and g

$$\max_{G \in \mathscr{L}^g_\sigma} \chi(G) \geqslant \max_{G \in \mathscr{L}^{g+1}_\sigma} \chi(G),$$

hence the sequence of these maxima becomes constant (depending on σ) $\Psi(\sigma) = \min\limits_g \max\limits_{G \in \mathscr{L}^g_\sigma} \chi(G)$ after a finite number of steps.

In 1968 Vizing [7] set up the problem: *Determine the maximal chromatic number of the graphs, contained in* $\mathscr{L}^4_\sigma$.

The following question is also of interest: *How large is the number* $\Psi(\sigma)$?

Grünbaum [5] has formulated the conjecture, suggested by the papers [9], [3], [4], that $\Psi(\sigma) = \sigma$, if $\sigma \geqslant 3$.

In [1] and [2] it has been independently shown that for any $G \in \mathscr{L}^4_\sigma$ $(\sigma \leqslant 4)$

$$\chi(G) \leqslant \left\lfloor \frac{3(\sigma + 2)}{4} \right\rfloor,$$

and, consequently

$$\Psi(\sigma) \leqslant \left\lfloor \frac{3(\sigma + 2)}{4} \right\rfloor.$$

Thus, Grünbaum's conjecture does not hold for $\sigma \geqslant 7$. The present paper is devoted to proving the following fact.

Theorem. *Let* $\sigma \geqslant 5$, $g \geqslant 7$, $G \in \mathscr{L}^g_\sigma$. *Let, further,* $q = \left\lceil \frac{g}{2} \right\rceil$. *If some natural number* χ *satisfies the inequalities*

(1) $\chi \geqslant \left\lfloor \frac{\sigma}{2} \right\rfloor + 2,$

(2) $\left(\frac{\chi - 1}{\sigma - \chi + 1}\right)^{q-1} \geqslant \frac{e}{2} q\sigma(\sigma + \chi - 2),$

then $\chi(G) \leqslant \chi$.

The theorem will be proved in §§2-5.

Corollary 1. *If* $\sigma \geqslant 5$ *then* $\Psi(\sigma) \leqslant \left\lfloor \frac{\sigma}{2} \right\rfloor + 2$.

Proof. It is easy to see that for any $\sigma \geqslant 5$, and for any natural number $g \geqslant 4(\sigma + 2)\ln \sigma$, $\chi \doteq \left\lfloor \frac{\sigma}{2} \right\rfloor + 2$ satisfies all the conditions of the theorem.

Corollary 2. $\max\limits_{G \in \mathscr{L}^{35}_5} \chi(G) \leqslant 4.$

For the proof it suffices to verify that the conditions of the theorem are satisfied for $\sigma = 5$, $g = 35$, $\chi = 4$.

According to Corollary 1, Grünbaum's conjecture is not true for $\sigma \geqslant 5$.

§ 2.

We only consider $\chi \leqslant \sigma - 1$, for at $\chi \geqslant \sigma$ the statement of the theorem is the weakening of Brooks' theorem which asserts $\chi(G) \leqslant \sigma$ for any graph $G\ (\in \mathscr{L}_\sigma)$ not containing a complete subgraph with $\sigma + 1$ vertices.

Assume that the statement of the theorem is not true. Then there exists a $(\chi + 1)$-critical graph $G \in \mathscr{L}_\sigma^g$. Let $v_0 \in V(G)$ be chosen and some colouring f of the vertices of $G \setminus \{v_0\}$ with χ colours be given.

Let $f(A)$, where $A \subseteq V(G)$, denote the set $\{f(v) \mid v \in A \setminus \{v_0\}\}$, and let $I(v)$ denote the set $\{w \in V(G) \mid (v, w) \in E(G)\}$. We shall call $w \in \in I(v_0)$ an $O_1(v_0)$-vertex, if $f(w) \notin f(I(v_0) \setminus \{w\})$. The set of all $O_1(v_0)$-vertices is denoted by $O_1(v_0)$.

As G is critical, $|O_1(v_0)| \geqslant 2\chi - \sigma \geqslant 3$.

We propose an algorithm for determining a subset Γ of the set $V(G) \cup E(G)$ in G. The set Γ will play the main role in the proof of the theorem. The edges of Γ will be oriented, some of them in both directions. While constructing Γ, the edges and vertices, belonging to Γ, will be called Γ-edges and Γ-vertices respectively. Further, the edges in Γ will be divided into Γ_1-edges and Γ_2-edges. The algorithm will work in not more than $\sigma |V(G)|$ steps. At the i-th step Γ-edges and Γ-vertices of i-th level will be defined. Simultaneously, with the construction of Γ we shall construct a mapping P, defined on the set of Γ_2-edges with the values in the set of Γ-vertices.

THE ALGORITHM OF CONSTRUCTING Γ

Step 0. v_0 is called a Γ-vertex of level 0.

Step 1. Direct each edge (v_0, w), where w is a $O_1(v_0)$-vertex, towards w. We call these directed edges Γ_1-edges of level 1, and we refer to the $O_1(v_0)$-vertices as Γ-vertices of level 1. There are no Γ_2-edges of level 1. Go to Step 2.

Definition. For any natural number i and for each Γ-vertex $v \neq v_0$ we denote by $T_i(v)$ the set of those Γ-vertices, which belong to $I(v)$, and from which Γ_1-edges of level $\leqslant i$ go to v.

Example. If v is an $O_1(v_0)$-vertex then $T_1(v) = \{v_0\}$.

Definition. For any natural number $i \geqslant 2$ and each Γ-vertex $v \neq v_0$ we denote

$$O_i(v) = \Big\{ w \in I(v) \setminus (T_{i-1}(v) \cup \{v_0\}) \mid f(w) \notin \\ \notin f(I(v) \setminus (T_{i-1}(v) \cup \{w\})) \;\&\; w \notin \bigcup_{j=2}^{i-1} O_j(v) \Big\}.$$

Example. If $v \in O_1(v)$, then

$$O_2(v) = \{w \in I(v) \setminus \{v_0\} \mid f(w) \notin f(I(v) \setminus \{w, v_0\})\}.$$

Definition. Let $v \neq v_0$ be a Γ-vertex. We shall say that the $D_i(v)$-situation takes place in G, if $|f(I(v) \setminus T_{i-1}(v))| < \chi - 1$.

Step k $(k \geqslant 2)$.

(a) If for at least one Γ-vertex $v \neq v_0$ the $D_{k-1}(v)$-situation takes place, then the algorithm terminates. Otherwise, the algorithm terminates if no Γ-edge of the $(k-1)$-th level has been constructed in the $(k-1)$-th step. In all other cases go to item (b).

(b) For each ordered pair of vertices $\{v, w\}$, where $v \neq v_0$ is a Γ-vertex of level 1 or 2 or 3 ... or $k-1$), and $w \in O_k(v)$, we direct the edge (v, w) towards w. We call all such edges the Γ-edges of the k-th level. Go to item (c).

Remark 1. It may happen that some edge is directed in both senses.

(c) Let us consider an arbitrary Γ-edge $\overrightarrow{(v, w)}$ of the k-th level. $f(v) = \alpha$, $f(w) = \beta$. If there is a Γ-vertex u $(\in I(v) \cup I(w))$ of level not exceeding $k - q$, then we call $\overrightarrow{(v, w)}$ a Γ_2-edge of the k-th level. Besides, we call $\overrightarrow{(v, w)}$ a Γ_2-edge of the k-th level if for some $s\ (\geqslant 2)$ there exists in G a directed chain

$$\overrightarrow{(v_1, v_2)}, \overrightarrow{(v_2, v_3)}, \ldots, \overrightarrow{(v_{s-1}, v_s)}$$

of Γ_1-edges such that $v_s = v$, $f(v_j) \in \{\alpha, \beta\}$ (where $1 \leqslant j \leqslant s$) and at least one of the vertices $v_1, v_2, \ldots, v_s$ is adjacent to a Γ-vertex u' the level of which does not exceed $k - q$. The vertex u (or u'), because of which $\overrightarrow{(v, w)}$ became a Γ_2-edge, will be called the image of $\overrightarrow{(v, w)}$ in the mapping F. (If there exist more than one such vertices u or u', then we choose $F(\overrightarrow{(v, w)})$ arbitrarily from among them.) We check each Γ-edge of level k whether or not it is a Γ_2-edge; if it is not, we call it a Γ_1-edge of level k. Go to item (d).

(d) A vertex $v \in V(G)$ will be called a Γ-vertex of the k-th level, if at least one Γ_1-edge of the k-th level enters it, but no Γ_1-edge of lower level. Go to Step $k + 1$.

Remark 2. If an edge of G is directed in both directions, it may be a Γ_1-edge in one direction and a Γ_2-edge in the other.

If no Γ_1-edge of the k-th level appears in Step k of the algorithm, then for any Γ-vertex v

$$T_{k-1}(v) = T_k(v), \qquad O_{k+1}(v) = \phi,$$

and at Step $(k + 1)$ there will not appear any Γ-edge of the $(k + 1)$-th level. That is, the algorithm terminates not later than at Step $(k + 1)$. Consequently, the algorithm works in at most $2 \cdot |E(G)|$ steps.

Later we shall denote the level of the Γ-vertex v or that of the Γ-edge $\vec{e}$ by $Y(v)$ or $Y(\vec{e})$ respectively. It is clear that $Y(\overrightarrow{(v, w)})$ and $Y(\overrightarrow{(w, v)})$ may be different.

§ 3.

In this section we consider some properties of Γ.

(I). *If $\overrightarrow{(v, w)}$ is a Γ-edge, then $Y(\overrightarrow{(v, w)}) \geqslant Y(v) + 1$. If $\overrightarrow{(v, w)}$ is a Γ_1-edge, then $Y(\overrightarrow{(v, w)}) \geqslant Y(w)$.*

Proof. The first inequality follows from the definition of Γ-edges of the k-th level. By the definition of Γ-vertices of level k, the level of any Γ-vertex v is equal to the minimum of levels of Γ_1-edges entering into v. This implies the second inequality.

(II). *If* $(\overrightarrow{v, w})$ *and* $(\overrightarrow{w, u})$ *are* Γ*-edges,* $v \neq u$ *and* $f(v) = f(u)$*, then* $(\overrightarrow{v, w})$ *is a* Γ_1*-edge, and* $Y((\overrightarrow{v, w})) \leqslant Y((\overrightarrow{w, u})) - 1$.

Proof. Let $Y((\overrightarrow{w, u})) = i$. If $v \notin T_{i-1}(v)$, then $u \notin O_i(v)$, and $(\overrightarrow{w, u})$ would not be a Γ-edge of the i-th level. Consequently, $v \in T_{i-1}(v)$. That is, $(\overrightarrow{v, w})$ is a Γ_i-edge, and $Y((\overrightarrow{v, w})) \leqslant i - 1$.

The next statement is obvious.

(III). *For any* Γ*-vertex* $v \neq v_0$ *there exists a* Γ_1*-edge* $(\overrightarrow{w, v})$ *such that* $Y(v) = Y((\overrightarrow{w, v})) \geqslant Y(w) + 1$.

As an immediate corollary of (I) and (III) we state:

(IV). *For any* Γ*-vertex* v *the length of the shortest directed chain, consisting of* Γ_1*-edges and leading from* v_0 *to* v*, does not exceed* $Y(v)$.

(V). *There is no* Γ_1*-edge of level* > 1*, which terminates at a vertex adjacent to* v_0.

Proof. Each edge, whose level exceeds 1, and whose end vertex belongs to $I(v_0)$ is a Γ_2-edge. By (IV) and since $G \in \mathscr{L}^g$, its level is at least $g - 2$.

(VI). *For any directed two-coloured chain* $(\overrightarrow{v_1, v_2}), (\overrightarrow{v_2, v_3}), \ldots$ $\ldots, (\overrightarrow{v_{s-1}, v_s})$*, consisting of* Γ_1*-edges, the following is true:*

$$Y((\overrightarrow{v_{s-1}, v_s})) \leqslant Y(v_1) + q - 2.$$

Proof. Since the chain is two-coloured, v_1 has a colour, and $v_1 \neq v_0$. Then, according to (III), there exists a $w \in I(v_1)$ with $Y(w) \leqslant Y(v_1) - 1$. But, taking into account the definition of Γ_2-edges, if $Y((\overrightarrow{v_{s-1}, v_s})) \geqslant Y(w) + q$ then $(\overrightarrow{v_{s-1}, v_s})$ will be a Γ_2-edge.

The following statement results from the definition of the sets $O_i(\nu)$ and the Γ-edges of the k-th level.

(VII). *Let* $\nu \in \Gamma \setminus \{\nu_0\}$, $f(\nu) = \alpha$. *Then for any* $\beta \neq \alpha$ *there exists at most one* Γ*-edge, going from* ν *to a vertex of colour* β. *Moreover, if there exists a vertex* $w \in I(\nu)$ *such that* $f(w) = \beta$, $\overrightarrow{(\nu, w)}$ *is a* Γ*-edge and* $Y(\overrightarrow{(\nu, w)}) = k$, *then any vertex* $u \in I(\nu) \setminus \{w\}$ *with* $f(u) = \beta$ *belongs to* $T_{k-1}(\nu)$.

Definition. Let α and β be arbitrary colours. We denote by $G_{\alpha\beta}$ the subgraph of the graph G, spanned by the vertices whose colour is α and β.

Definition. Let $\overrightarrow{(u, \nu)}$ be Γ_1-edge, $f(u) = \alpha$, $f(\nu) = \beta$. We denote by $G_{\alpha\beta}(\overrightarrow{(u, \nu)})$ the connected component of the graph $G_{\alpha\beta} \setminus \{(u, \nu)\}$, containing the vertex u.

From (II), (VI) and (VII) we obtain:

(VIII). *Each component* $G_{\alpha\beta}(\overrightarrow{(u, \nu)})$ *is a rooted tree* with the root* u, *and its height does not exceed* $q - 3$; *furthermore, any edge of the tree is a* Γ_1*-edge, and is directed towards* u.

Definition. For any Γ-vertex $\nu \neq \nu_0$ we define the notion of the *ν-tree* by induction with respect to the level of the vertices:

1. If $Y(\nu) = 1$, then a ν-tree consists of the vertices ν_0, ν and of the Γ_1-edge $\overrightarrow{(\nu_0, \nu)}$.

2. Let the u-tree be defined for each Γ-vertex $u \neq \nu_0$ with $Y(w) < k$. We consider a Γ-vertex ν with $Y(\nu) = k$ *and* $f(\nu) = \alpha$. According to (III), there exists a $u_0 \in T_k(\nu)$. Let $f(u_0) = \beta$. We choose among the initial vertices of the graph $G_{\beta\alpha}(\overrightarrow{(u_0, \nu)})$ a vertex w in such a way, that the directed chain, consisting of Γ_1-edges leading in the graph $G_{\beta\alpha}((u_0, \nu))$ from w to u_0, would end in a Γ_1-edge $\overrightarrow{(w', u_0)}$,

*The root is an arbitrary distinguished vertex of the tree. The height of a vertex of a rooted tree is its distance from the root. By (VII), each Γ_1-edge of the tree in question is directed towards the root u.

having the least level among the edges of $G_{\beta\alpha}((\overrightarrow{u_0, v}))$, entering u_0. If $G_{\beta\alpha}((\overrightarrow{u_0, v})) = \{u_0\}$, then we take $w \doteq u_0$. From (I) and (II) it follows that $Y(w) < Y(v)$. Then all the vertices and Γ_1-edges of the w-tree, all the vertices and Γ_1-edges of $G_{\beta\alpha}((\overrightarrow{u_0, v}))$, the Γ_1-edge $(\overrightarrow{u_0, v})$ and the Γ-vertex v belong to the v-tree (and the v-tree consists of these elements only).

Remark 3. Generally speaking, the v-tree is not unique (since it depends on the choice of the vertices w).

(IX). *Let* $v \neq v_0$ *be some* Γ*-vertex, and* $F(v)$ *some arbitrary* v*-tree. Then*

(a) *if* $(\overrightarrow{u, w}) \in F(v)$, $(\overrightarrow{w, y}) \in F(v)$, $u \neq y$, *then* $Y((\overrightarrow{u, w})) < Y((\overrightarrow{w, y}))$.

(b) $F(v)$ *is the directed tree with root* v; *each edge in* $F(v)$ *is a* Γ_1*-edge, its height does not exceed* $Y(v)$. *Its edges are directed towards* v. *The vertex* v_0 *is one of the initial vertices of this tree. Only one* Γ_1*-edge, belonging to* $F(v)$, *goes from any of the vertices* $w \in F(v) \setminus \{v\}$.

Proof. Let us prove this statement by the induction on level v. If $Y(v) = 1$, then this statement is obvious.

Suppose that this statement is true for all Γ-vertices of the level not exceeding $k - 1$ and $Y(v) = k$. According to the definition of the v-tree, there exist such Γ-vertices u_0 and w, that $F(v)$ consists of vertices and Γ_1-edges of a w-tree and $G_{\beta\alpha}((\overrightarrow{u_0, v}))$, and of the Γ_1-edge $(\overrightarrow{u_0, v})$ and the vertex v. Since $Y(w) < k$, for the w-tree (IX) is valid. We show that no vertex of $G_{\beta\alpha}((\overrightarrow{u_0, v}))$, except w, belongs to the w-tree. Suppose that some vertex $u \neq w$ lies simultaneously in the w-tree and in $G_{\beta\alpha}((\overrightarrow{u_0, v}))$. Then a directed chain $(\overrightarrow{u, u_1}), (\overrightarrow{u_1, u_2}), \ldots, (\overrightarrow{u_{s-1}, u_s})$ leads from u to u_0 such that $u_s = u_0$ and $f(u_j) \in \{\alpha, \beta\}$ $(1 \leqslant j \leqslant s)$. Besides, another directed chain

$$(\overrightarrow{u, y_1}), (\overrightarrow{y_1, y_2}), \ldots, (\overrightarrow{y_{r-1}, w}), (\overrightarrow{w, y_{r+1}}), \ldots, (\overrightarrow{y_{l-1}, u_0})$$

leads from u to u_0. According to (II) and (VI)

(3) $$Y((\overrightarrow{u_0, v})) \leqslant Y(u) + q - 2;$$

$$Y((\overrightarrow{u_i, u_{i+1}})) < Y((\overrightarrow{u_{i+1}, u_{i+2}})), \qquad i = 1, 2, \ldots, s-2;$$

$$Y((\overrightarrow{y_i, y_{i+1}})) < Y((\overrightarrow{y_{i+1}, y_{i+2}})), \qquad i = r, r+1, \ldots, l-2.$$

Besides, according to the definition of the v-tree $Y((\overrightarrow{y_{r-1}, w})) <$ $< Y((\overrightarrow{w, y_{r+1}}))$ and in accordance with the induction hypothesis

$$Y((\overrightarrow{y_i, y_{i+1}})) \leqslant Y((\overrightarrow{y_{i+1}, y_{i+2}})), \qquad i = 1, 2, \ldots, r-2;$$

$$Y((\overrightarrow{u, y_1})) \leqslant Y((\overrightarrow{y_1, y_2})),$$

However, by (I),

$$Y((\overrightarrow{u, u_1})) \geqslant Y(u) + 1, \qquad Y((\overrightarrow{u, y_1})) \geqslant Y(u) + 1.$$

Consequently,

$$Y((\overrightarrow{u_{s-1}, u_0})) \geqslant Y(u) + s, \qquad Y((\overrightarrow{y_{l-1}, u_0})) \geqslant Y(u) + l.$$

Since $Y((\overrightarrow{u_0, v})) \geqslant \max\{Y((\overrightarrow{y_{l-1}, u_0})) + 1,\ Y((\overrightarrow{u_{s-1}, u_0})) + 1\}$, it follows from (3) that $\max\{s, l\} \leqslant q - 3$.

So we obtained that there exists in G a cycle $(u, u_1, u_2, \ldots$ $\ldots, u_s, y_{l-1}, y_{l-2}, \ldots, y_1, u)$, the length of which is $s + l$. But

$$s + l \leqslant 2(q-3) < g.$$

Hence, $F(v)$ is a tree. Verification of the further parts of the statement does not raise any difficulties.

(X). *Let* $v \neq v_0$ *be a* Γ*-vertex. Only the* Γ_1*-edges, belonging to* $F(v)$*, are the edges of the subgraph of the graph* G*, generated by the vertices of an arbitrary* v*-tree* $F(v)$.

Proof. Suppose that the vertices x and y, belonging to $F(v)$, are connected by the edge $(x, y) \in E(G) \setminus E(F(v))$.

Case 1. Let the vertex y lie in the directed chain, leading in $F(v)$ from the vertex x to the vertex v. The necessary condition for $(\overrightarrow{w, y})$

lying in this chain, to be a Γ_1-edge, is*

$$Y((\overrightarrow{w, y})) \leqslant Y(x) + q - 1.$$

Taking (IX/a) into account we obtain that G contains a cycle of length not exceeding q.

Since the roles of the vertices x and y are symmetric, only the following case remained open.

Case 2. Let u be the first common vertex of the directed chains in $F(v)$, going from x to v, and from y to v, $u \notin \{x, y\}$. Then let $(\overrightarrow{u, u_1}) \in F(v)$, $f(u) = \alpha$, $f(u_1) = \beta$. According to the construction of $F(v)$ at least one of the vertices x and y belongs to $G_{\alpha\beta}((\overrightarrow{u, u_1}))$. (VIII) implies that at most one of the vertices x and y can lie in $G_{\alpha\beta}((\overrightarrow{u, u_1}))$. Let, for sake of definiteness, $y \in G_{\alpha\beta}((\overrightarrow{u, u_1}))$. Let $(\overrightarrow{u_x, x})$ (or $(\overrightarrow{u_y, u})$) denote the last edge of the directed chain in $F(v)$, going from x (from y,) respectively to u. Then, according to the construction of $F(v)$,

$$Y((\overrightarrow{u_y, u})) \leqslant Y((\overrightarrow{u, u_1})) - 1, \qquad Y((\overrightarrow{u_x, u})) \leqslant Y((\overrightarrow{u, u_1})) - 1.$$

Since $(x, y) \in E(G)$, $(\overrightarrow{u, u_1})$ is a Γ_1-edge and the directed chain, going from y to u_1 is two-coloured, therefore

$$Y((\overrightarrow{u, u_1})) \leqslant Y(x) + q - 1$$

and (by (VI))

$$Y((\overrightarrow{u, u_1})) \leqslant Y(y) + q - 2.$$

According to (IX/a) and (I) the length of the directed chain, going from x (or from y) to u in $F(v)$, does not exceed $q - 2$ ($q - 3$, respectively). Thus G contains a cycle with length at most

$$(q - 2) + (q - 3) + 1 = 2q - 4 < g.$$

Hence the proof is complete.

*Suppose the contrary, i.e.

$$Y((\overrightarrow{w, y})) \geqslant Y(x) + q.$$

Then $(\overrightarrow{w, y})$ is a Γ_2-edge (by definition, such that u is replaced by x).

Definition. Suppose that the $D_k(\nu)$-situation arises for some Γ-vertex $\nu \neq \nu_0$, and for some natural k in G. Let, further $f(\nu) = \alpha$. We define the *ν-trace* according to the following rules.

1. If $|f(I(\nu))| < \chi - 1$, then any ν-tree is a ν-trace.

2. Suppose that $|f(I(\nu))| = \chi - 1$. Then, according to the definition of the $D_k(\nu)$-situation, there exists such a colour β that $\beta \notin$ $\notin f(I(\nu) \setminus T_k(\nu))$. We consider the connected component $G_{\alpha\beta}(\nu)$ of the graph $G_{\alpha\beta}$, containing the vertex ν. Due to (VI) and (VII) $G_{\alpha\beta}(\nu)$ is a directed tree with root ν, whose height does not exceed $q - 2$. Each edge of this tree is a Γ_1-edge, and is directed towards ν. Let ν_1 be an initial vertex of the tree $G_{\alpha\beta}(\nu)$, with the property that the level of the last Γ_1-edge $(\overrightarrow{\nu', \nu})$ in the directed chain leading in the graph $G_{\alpha\beta}(\nu)$ from ν_1 to ν, is the least in comparison with the levels of the edges from $G_{\alpha\beta}(\nu)$, entering ν. As ν-trace we take all the edges and vertices of $G_{\alpha\beta}(\nu)$ and of arbitrary ν_1-tree.

Remark 4. Like the ν-tree, the ν-trace is not unique either.

(XI). *Let $\nu \neq \nu_0$ be a Γ-vertex. Then the subgraph of G, generated by the vertices of any ν-trace, coincides with this trace, and is the root-orientated tree with root ν, the height of which does not exceed $Y(\nu)$. Each edge of this tree is Γ_1-edge, and is directed in the direction of ν. Only one Γ_1-edge, belonging to the ν-trace, goes from each vertex of this trace except vertex ν. Vertex ν_0 is one of the initial vertices of this tree.*

The proof of (XI) is analogous to that of (X).

Lemma. *In the process of constructing Γ we do not get $D_k(\nu)$-situations for any pair $\nu\ (\in \Gamma)$, k.*

Proof. It suffices to show that if for some pair ν, k while constructing Γ, $D_k(\nu)$-situation arises, then G is χ-colourable.

Let $F(\nu)$ be some ν-trace. We define a function h on the vertices of $F(\nu)$ according to the following rules. If $|f(I(\nu))| < \chi - 1$, then a colour $\alpha \notin f(I(\nu) \cup \{\nu\})$ will be the image of the vertex ν for the mapping h. Let $|f(I(\nu))| = \chi - 1$. Let us recall the definitions of the $D_k(\nu)$-situation and the ν-trace. All the vertices from $F(\nu)$, whose Γ_1-edges enter to ν, are coloured with the same colour β, and, besides, this colour has not been used for the colouring of the vertices from $I(\nu) \setminus F(\nu)$. Then we assume that $h(\nu) = \beta$. As about $h(w)$ for each vertex $w \in F(\nu) \setminus \{\nu\}$, we take the colour of such a vertex $w' \in F(\nu)$, that $\overrightarrow{(w, w')} \in F(\nu)$. We define the function f': $V(G) \to \{1, 2, \ldots, \chi\}$ such that

$$f'(w) = \begin{cases} f(w), & w \in V(G) \setminus F(\nu); \\ h(w), & w \in V(G) \cap F(\nu). \end{cases}$$

It follows from (XI), that f' is a correct colouring of the vertices of G by χ colours. Hence the Lemma is proved.

§4.

Thus, in course of constructing Γ, for each pair ν $(\in \Gamma)$, k

(4) $\qquad |f(I(\nu) \setminus T_k(\nu))| = \chi - 1$

is fulfilled. Thus (in addition to (I)-(VIII)) the following statements are valid for Γ.

(XII). *For any* Γ*-vertex* $\nu \neq \nu_0$ *and for any natural* i, j

$$O_j(\nu) \cap T_i(\nu) = \phi.$$

Proof. Suppose $w \in O_j(\nu) \cap T_i(\nu)$, $f(w) = \alpha$. According to (VII) all the vertices of colour α, lying in $I(\nu) \setminus \{w\}$, belong to $T_{j-1}(\nu)$. Consequently, $\alpha \notin f(I(\nu) \setminus T_{\max\{i, j-1\}}(\nu))$ which contradicts the relation (4).

(XIII). *If* $|T_i(\nu)| = a \geqslant 1$, *then* $\left|\bigcup_{j=2}^{i+1} O_j(\nu)\right| \geqslant 2\chi - \sigma - 2 + a$.

Proof. We have $|I(\nu) \setminus T_i(\nu)| \leqslant \sigma - a$,

$$\bigcup_{j=2}^{i+1} O_j = \{w \in I(\nu) \setminus T_i(\nu) \mid f(w) \notin f(I(\nu) \setminus (T_i(\nu) \cup \{w\}))\}.$$

By (4), among the $|I(\nu) \setminus T_i(\nu)|$ vertices we must come across those of $\chi - 1$ colours. But if not more than $\sigma - a$ elements are coloured by

$\chi - 1$ colours and each colour really occurs, then the number of colours, used only once is not less than

$$(\chi - 1) - ((\sigma - a) - (\chi - 1)) = 2\chi - 2 - \sigma + a.$$

The following statement immediately follows from (4) and (XIII).

(XIV). *For any* Γ-*vertex* $v \neq v_0$ *and for any* $i \geqslant 1$, $|T_i(v)| \leqslant \leqslant \sigma - \chi + 1$ *holds and, hence,*

$$\frac{\left|\bigcup_{j=2}^{i+1} O_j(v)\right|}{|T_i(v)|} \overset{(\mathrm{XIII})}{\geqslant} \frac{2\chi - 2 - \sigma + |T_i(v)|}{|T_i(v)|} =$$
$$= 1 + \frac{2\chi - 2 - \sigma}{|T_i(v)|} \geqslant \frac{\chi - 1}{\sigma - \chi + 1}.$$

§5.

For completing the proof of the Theorem it remains to show that if the $D_k(v)$-situation never arises in course of constructing Γ, then the number of edges in Γ unboundedly increases; this contradicts the finiteness of G.

Let P be the mapping, defined in the course of constructing the Γ_2-edges. Now we consider an arbitrary Γ-vertex u. Let $y \in I(u)$ and $P_y^{-1}(u)$ be the set of such Γ_2-edges $\overrightarrow{(v, w)}$ from $P^{-1}(u)$, which became Γ_2-edges because of the fact that they are themselves incident to y, or because of the fact that the two-coloured chain, consisting of Γ_1-edges of colours $f(v)$ and $f(w)$, passing through y, leads to vertex v. It is obvious that $P^{-1}(u) = \bigcup_{y \in I(u)} P_y^{-1}(u)$. The number of Γ_2-edges, entering y, does not exceed σ. Besides, according to (VII), not more than $\chi - 1$ directed two-coloured chains come from y. Moreover, for at least one Γ-edge coming from y, it is necessary that $T_{2|E(G)|}(y) \neq \phi$.

Consequently, for each Γ-vertex u, and for each vertex $y \in I(u)$

$$|P_y^{-1}(u)| \leqslant \max\{\sigma, (\sigma - 1) + (\chi - 1)\} = \sigma + \chi - 2.$$

Thus, for any Γ-vertex u

$$|P^{-1}(u)| = \Big|\bigcup_{y \in I(u)} P_y^{-1}(u)\Big| \leqslant \sigma(\sigma + \chi - 2).$$

It is clear that for each Γ_2-edge $\overrightarrow{(v, w)} \in P^{-1}(u)$

$$Y(\overrightarrow{(v, w)}) \geqslant Y(u) + q.$$

Therefore, the number of Γ_2-edges, the level of which does not exceed k, is bounded from above by $\sigma(\sigma + \chi - 2) \sum_{i=0}^{k-q} |V_i|$, where V_i is the set of the Γ-vertices of the i-th level.

Let E_i' (or E_i'') be the set of Γ_1-edges (Γ_2-edges respectively) of the i-th level. The set of all Γ-edges of the i-th level is $E_i = E_i' \cup E_i''$. Then

$$(5) \qquad \sum_{i=1}^{k} |E_i''| \leqslant \sigma(\sigma + \chi - 2) \sum_{i=0}^{k-q} |V_i|.$$

Since $\chi \geqslant \lfloor \frac{\sigma}{2} \rfloor + 2$, and since for any Γ-vertex $v \in V_i$ $(i = 1, 2, \ldots)$, $|T_i(v)| \geqslant 1$, according to (XIII), $|O_{i+1}(v)| \geqslant 2$ $(v \in V_i)$. Consequently,

$$(6) \qquad |V_i| \leqslant \frac{1}{2} |E_{i+1}|, \qquad i = 0, 1, 2, \ldots$$

Thus, for every $k \geqslant 1$

$$(7) \qquad \sum_{i=1}^{k+1} |E_i| = \sum_{v \in \bigcup_{i=0}^{k} V_i} \Big|\bigcup_{j=1}^{k+1} O_j(v)\Big| \geqslant$$

$$\geqslant \sum_{v \in \bigcup_{i=1}^{k} V_i} \frac{\Big|\bigcup_{j=2}^{k+1} O_j(v)\Big|}{|T_k(v)|} |T_k(v)| \overset{(XIV)}{\geqslant}$$

$$\overset{(XIV)}{\geqslant} \frac{\chi - 1}{\sigma - \chi + 1} \sum_{v \in \bigcup_{i=1}^{k} V_i} |T_k(v)| \geqslant \frac{\chi - 1}{\sigma + 1 - \chi} \sum_{i=1}^{k} |E_i'| =$$

$$= \frac{\chi - 1}{\sigma - \chi + 1} \Big(\sum_{i=1}^{k} |E_i| - \sum_{i=1}^{k} |E_i''| \Big) \overset{(5)}{\geqslant}$$

$$\overset{(5)}{\geqslant} \frac{\chi - 1}{\sigma - \chi + 1} \Big(\sum_{i=1}^{k} |E_i| - \sigma(\sigma + \chi - 2) \sum_{i=1}^{k-q} |V_i| \Big) \overset{(6)}{\geqslant}$$

$$\overset{(6)}{\geqslant} \frac{\chi - 1}{\sigma - \chi + 1} \Big(\sum_{i=1}^{k} |E_i| - \frac{\sigma(\sigma + \chi - 2}{2} \sum_{i=1}^{k-q+1} |E_i| \Big).$$

We show that for all $k \geqslant 1$

$$(8) \qquad \sum_{i=1}^{k+1} |E_i| \geqslant \frac{(\chi - 1)(q - 1)}{(\sigma - \chi + 1)q} \sum_{i=1}^{k} |E_i|.$$

Since $E_1 \neq \phi$, and, according to (2), $\sqrt[q-1]{\frac{1}{e} \Big(\frac{\chi - 1}{\sigma - \chi + 1}\Big)^{q-1}} > 1$,

$$\frac{(\chi - 1)(q - 1)}{(\sigma - \chi + 1)q} > 1.$$

(We have used that

$$\frac{1}{e} \Big(\frac{\chi - 1}{\sigma - \chi + 1}\Big)^{q-1} < \Big(\frac{\chi - 1}{\sigma - \chi + 1}\Big)^{q-1},$$

$$\frac{1}{\Big(1 + \frac{1}{q-1}\Big)^{q-1}} = \Big(\frac{(\chi - 1)(q - 1)}{(\sigma - \chi + 1)q}\Big)^{q-1}.)$$

So (8) will imply that the number of Γ-edges increases unboundedly, which contradicts to the finiteness of G.

So, for $1 \leqslant k < q$ inequality (8) immediately follows from (7). Let now (8) hold for all $k < k_0$. Then

$$\sum_{i=1}^{k_0+1} |E_i| \overset{(7)}{\geqslant}$$

$$\overset{(7)}{\geqslant} \frac{\chi - 1}{\sigma - \chi + 1} \Big(\sum_{i=1}^{k_0} |E_i| - \frac{\sigma(\sigma + \chi - 2)}{2} \sum_{i=1}^{k_0 - q + 1} |E_i| \Big) =$$

$$= \frac{(\chi - 1)(q - 1)}{(\sigma - \chi + 1)q} \sum_{i=1}^{k_0} |E_i| +$$

$$+\frac{(\chi-1)}{(\sigma-\chi+1)}\Big(\frac{1}{q}\sum_{i=1}^{k_0}|E_i|-$$

$$-\frac{\sigma(\sigma+\chi-2)}{2}\sum_{i=1}^{k_0-q+1}|E_i|\Big).$$

By the induction hypothesis

$$\sum_{i=1}^{k_0}|E_i|\geqslant\frac{(\chi-1)(q-1)}{(\sigma-\chi+1)q}\sum_{i=1}^{k_0-1}|E_i|\geqslant\ldots$$

$$\ldots\geqslant\Big(\frac{(\chi-1)(q-1)}{(\sigma-\chi+1)q}\Big)^{q-1}\sum_{i=1}^{k_0-q+1}|E_i|.$$

Consequently,

$$\frac{1}{q}\sum_{i=1}^{k_0}|E_i|-\frac{\sigma(\sigma+\chi-2)}{2}\sum_{i=1}^{k_0-q+1}|E_i|\geqslant$$

$$\geqslant\Big(\frac{1}{q}\Big(\frac{(\chi-1)(q-1)}{(\sigma-\chi+1)q}\Big)^{q-1}-\frac{\sigma(\sigma+\chi-2)}{2}\Big)\sum_{i=1}^{k_0-q+1}|E_i|,$$

and, in accordance with (2) $\frac{1}{q}\sum_{i=1}^{k_0}|E_i|-\frac{\sigma(\sigma+\chi-2)}{2}\sum_{i=1}^{k_0-q+1}|E_i|\geqslant 0.$

Thus,

$$\sum_{i=1}^{k_0+1}|E_i|\geqslant\frac{(\chi-1)(q-1)}{(\sigma-\chi+1)q}\sum_{i=1}^{k_0}|E_i|.$$

which proves the theorem.

§6.

Remark 5. Using a theorem of Lovász [6] it is easy to prove that for any real $\alpha>\frac{1}{2}$ there exist natural numbers $g(\alpha)$ and $\sigma(\alpha)$ such that for any $g\geqslant g(\alpha)$, $\sigma\geqslant\sigma(\alpha)$ and $G\in\mathscr{L}_\sigma^g$

$$\chi(G)\leqslant\alpha\sigma.$$

The theorem of Lovász states: if $G\in\mathscr{L}_\sigma$ and $\sigma_1,\ldots,\sigma_n$ are non-negative integers such that $\sigma + 1 = \sum_{i=1}^{n} (\sigma_i + 1)$, then the vertices of G admit a covering by subgraphs $G_1, G_2, \ldots, G_n$ such that, for any $1 \leqslant i \leqslant n$,

$$G_i \in \mathscr{L}_{\sigma_i}.$$

We remind now the notion, (introduced by V.G. Vizing [8]), of the prescribed colouring and the upper chromatic number.

Definition. By a prescription for the vertices of a graph $G(V, E)$ we understand the mapping Φ of the set of vertices of G to the set of subsets of natural numbers.

Definition. We say that the colouring f of the vertices of a graph G satisfies the prescription Φ, if $f(v) \in \Phi(v)$ for each vertex $v \in V(G)$.

Definition. The smallest natural number k with the following properties is called the upper chromatic number $W(G)$ of a graph G: for each prescription Φ, satisfying

$$(\forall v)(v \in V(G) \Rightarrow |\Phi(v)| \geqslant k)$$

there exists a colouring f_Φ of vertices of G such that f_Φ satisfies Φ. It is clear that $W(G) \geqslant \chi(G)$. V.G. Vizing [8] has constructed a graph G_k for each $k \geqslant 2$ such that $\chi(G_k) = 2$, $W(G_k) \geqslant k$.

Remark 6. The proof of our Theorem can be applied, practically without any alterations for the prescribed colourings.

In conclusion I would like to call the reader's attention to the difficult and important problem:

To find the best upper estimate for the chromatic number of the graph in terms of the maximal degree and density or girth.

Acknowledgement. I am indebted to Prof. A. Ádám for his useful remarks concerning improvements in my mathematical and linguistic style.

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