Extracting colorings from large independent sets

H. A. Kierstead and Landon Rabern*

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Abstract

We take an application of the Kernel Lemma by Kostochka and Yancey [10] to its logical conclusion. The consequence is a sort of magical way to draw conclusions about list coloring (and online list coloring) just from the existence of an independent set incident to many edges. We use this to prove an Ore-degree version of Brooks' theorem for online list-coloring: every graph with $\theta \geq 18$ and $\omega \leq \frac{\theta}{2}$ is online $\lfloor \frac{\theta}{2} \rfloor$-choosable. Here θ is the Ore-degree. In addition, we prove an upper bound for online list-coloring triangle-free graphs: $\chi_{OL} \leq \Delta + 1 - \lfloor \frac{1}{4} \lg(\Delta)) \rfloor$.

1 Introduction

In [10], Kostochka and Yancey applied the Kernel Lemma to a coloring problem in a novel manner. The following lemma generalizes and strengthens their idea. The basic idea is that given an independent set that is incident to many edges, we can find a reducible configuration. In this way, we can reduce coloring problems to the mere existence of a large independent set.

Before stating the Main Lemma we need to define f-choosable. Let G be a graph. A list assignment on G is a function L from $V(G)$ to the subsets of $\mathbb{N}$. A graph G is *L-colorable* if there is $\pi\colon V(G) \to \mathbb{N}$ such that $\pi(v) \in L(v)$ for each $v \in V(G)$ and $\pi(x) \neq \pi(y)$ for each $xy \in E(G)$. For $f\colon V(G) \to \mathbb{N}$, a list assignment L is an *f-assignment* if $|L(v)| = f(v)$ for each $v \in V(G)$. We say that G is *f-choosable* if G is L-colorable for every f-assignment L.

Main Lemma. *Let G be a nonempty graph and $f\colon V(G) \to \mathbb{N}$ with $f(v) \leq d_G(v) + 1$ for all $v \in V(G)$. If there is independent $A \subseteq V(G)$ such that*

$$\|A, G - A\| \geq \sum_{v \in V(G)} d_G(v) + 1 - f(v),$$

then G has a nonempty induced subgraph H that is (online) f_H-choosable where $f_H(v) := f(v) + d_H(v) - d_G(v)$ for $v \in V(H)$.

*School of Mathematical and Statistical Sciences, Arizona State University

The Main Lemma holds for online choosability as well which was independently introduced by Zhu [20] and Schauz [18] (Schauz called it *paintability*). Let G be a graph and $f \colon V(G) \to \mathbb{N}$. We say that G is *online f-choosable* if $f(v) \geq 1$ for all $v \in V(G)$ and for every $S \subseteq V(G)$ there is an independent set $I \subseteq S$ such that $G - I$ is online f'-choosable where $f'(v) := f(v)$ for $v \in V(G) - S$ and $f'(v) := f(v) - 1$ for $v \in S - I$.

As a simple first application, we show that Brooks' Theorem reduces to the bound it gives on the independence number. In particular, the proof shows that Brooks' Theorem for list coloring and online list coloring follow form Brooks' theorem for ordinary coloring.

Brooks' Theorem for Independence Number. *If G is a graph with $\Delta(G) \geq 3$ and $K_{\Delta(G)+1} \not\subseteq G$, then $\alpha(G) \geq \frac{|G|}{\Delta(G)}$.*

To prove Brooks' Theorem, consider a minimal counterexample G. Then G is regular and hence for any maximum independent set A in G, Brooks' Theorem for Independence Number gives $\|A, G - A\| \geq |A|\Delta(G) \geq |G|$. Applying the Main Lemma with $f(v) := d_G(v)$ gives a nonempty induced subgraph H of G that is (online) d_H-choosable. So, after $\Delta(G)$-coloring $G - H$ by minimality of G, we can finish the coloring on H, a contradiction.

A bound like Brooks' theorem in terms of the Ore-degree was given by Kierstead and Kostochka [8] and subsequently the required lower bound on Δ was improved in [15, 12, 17]. For example, we have the following.

Definition 1. The *Ore-degree* of an edge xy in a graph G is $\theta(xy) := d(x) + d(y)$. The *Ore-degree* of a graph G is $\theta(G) := \max_{xy \in E(G)} \theta(xy)$.

Theorem 1.1. *Every graph with $\theta \geq 10$ and $\omega \leq \frac{\theta}{2}$ is $\left\lfloor \frac{\theta}{2} \right\rfloor$-colorable.*

Another method for achieving the tightest of these results on Ore-degree was given by Kostochka and Yancey [10]. Their proof combined their new lower bound on the number of edges in a color critical graph together with their list coloring lemma derived via the kernel lemma. Main Lemma improves this latter lemma and, in a similar way, we use it in combination with our lower bound on the number of edges in online list-critical graphs [9] to prove an Ore-degree version of Brooks' Theorem for online list coloring.

The Main Lemma can be used to give another lower bound on the number of edges in a critical graph.

Definition 2. Let $\mathrm{mic}(G)$ be the maximum of $\sum_{v \in I} d_G(v)$ over all independent sets I of G.

Definition 3. A graph G is *OC-reducible* to H if H is a nonempty induced subgraph of G which is online f_H-choosable where $f_H(v) := \delta(G) + d_H(v) - d_G(v)$ for all $v \in V(H)$. If G is not OC-reducible to any nonempty induced subgraph, then it is *OC-irreducible.*

Theorem 2.4. *If G is an OC-irreducible graph, then* $2\,\|G\| \geq (\delta(G) - 1)\,|G| + \mathrm{mic}(G) + 1$.

This quickly gives the aforementioned Ore degree version of Brooks' Theorem for list coloring.

Theorem 4.11. *Every graph with $\theta \geq 18$ and $\omega \leq \frac{\theta}{2}$ is $\left\lfloor \frac{\theta}{2} \right\rfloor$-choosable.*

Note that using Kostochka and Stiebitz's lower bound on the number of edges in a list critical graph [13] gives Theorem 4.11 with $\theta \geq 54$. Similarly, we get the online version.

Theorem 4.10. *Every graph with $\theta \geq 18$ and $\omega \leq \frac{\theta}{2}$ is online $\left\lfloor \frac{\theta}{2} \right\rfloor$-choosable.*

We expect that Theorems 4.10 and 4.11 actually hold for $\theta \geq 10$. In the regular coloring case, it was shown in [12] that the only exception when $\theta \geq 8$ is the graph O_5; again, the expectation is that the same result will hold for Theorems 4.10 and 4.11.

Finally, a simple probabilistic argument gives a reasonable bound on $\mathrm{mic}(G)$ for triangle-free graphs and we get the following.

Corollary 6.3. *Triangle-free graphs are online $\left(\Delta + 1 - \left\lfloor \frac{1}{4} \lg(\Delta) \right\rfloor\right)$-choosable.*

2 Proving the Main Lemma

A *kernel* in a digraph D is an independent set $I \subseteq V(D)$ such that each vertex in $V(D)-I$ has an edge into I. A digraph in which every induced subdigraph has a kernel is *kernel-perfect*.

Kernel Lemma. *Let G be a graph and $f\colon V(G) \to \mathbb{N}$. If G has a kernel-perfect orientation such that $f(v) \geq d^+(v) + 1$ for each $v \in V(G)$, then G is online f-choosable.*

Lemma 2.1 (Kostochka and Yancey [10])**.** *Let A be an independent set in a graph G and let $B := V(G - A)$. Any digraph D created from G by replacing each edge in $G[B]$ by a pair of opposite arcs and orienting the edges between A and B arbitrarily is kernel-perfect.*

Proof. Let G be a minimum counterexample, and let D be a digraph created from G that is not kernel-perfect. To get a contradiction it suffices to construct a kernel in D, since each subdigraph has a kernel by minimality. Either A is a kernel or there is some $v \in B$ which has no outneighbors in A. In the latter case, each neighbor of v in G has an inedge to v, so a kernel in $D - v - N(v)$ together with v is a kernel in D. □

The following lemma is folklore and can be derived from Hall's theorem by vertex splitting. It also follows by taking an arbitrary orientation and repeatedly reversing paths if doing so gets a gain.

Lemma 2.2. *Let G be a graph and $g\colon V(G) \to \mathbb{N}$. Then G has an orientation such that $d^-(v) \geq g(v)$ for all $v \in V(G)$ iff for every induced subgraph H of G, we have*

$$\|H\| + \|H, G - H\| \geq \sum_{v \in V(H)} g(v).$$

For independent $A \subseteq V(G)$, we write G_A for the bipartite subgraph $G - E(G - A)$ of G, so just the edges between A and $G - A$ remain.

Lemma 2.3. *Let G be a graph and $f\colon V(G) \to \mathbb{N}$ with $f(v) \leq d_G(v) + 1$ for all $v \in V(G)$. If there is independent $A \subseteq V(G)$ such that for each induced subgraph Q of G_A, we have*

$$\|Q\| + \|Q, G_A - Q\| \geq \sum_{v \in V(Q)} d_G(v) + 1 - f(v).$$

then G is online f-choosable.

Proof. Applying Lemma 2.2 on G_A with $g(v) := d_G(v) + 1 - f(v)$ for all $v \in V(G_A)$ gives an orientation of G_A where $d^-(v) \geq d_G(v) + 1 - f(v)$ for each $v \in V(G_A)$. Make an orientation D of G by using this orientation of G_A for the edges between A and $V(G - A)$ and replacing each edge in $G - A$ by a pair of opposite arcs. For $v \in V(D)$ we have $d^+(v) \leq d_{G-A}(v) + d_{G_A}(v) - (d_G(v) + 1 - f(v)) = f(v) - 1$ and hence $f(v) \geq d^+(v) + 1$. By Lemma 2.1, D is kernel-perfect, so the Kernel Lemma shows that G is online f-choosable. □

Proof of Main Lemma. Let $A \subseteq V(G)$ be an independent set with

$$\|A, G - A\| \geq \sum_{v \in V(G)} (d_G(v) + 1 - f(v)).$$

Choose a nonempty induced subgraph H of G with $\|H_A\| \geq \sum_{v \in V(H)} (d_H(v) + 1 - f_H(v))$ minimizing $|H|$ (we can make this choice since G is a such a subgraph). Suppose H is not online f_H-choosable. Then, by Lemma 2.3, we have an induced subgraph Q of H_A with $\|Q\| + \|Q, H_A - Q\| < \sum_{v \in V(Q)} (d_H(v) + 1 - f_H(v))$. Now $Q \neq H$ by our assumption on $\|H_A\|$, hence $Z := H - Q$ is a nonempty induced subgraph of G with

$$\begin{aligned}
\|Z_A\| &= \|H_A\| - \|Q\| - \|Q, H_A - Q\| \\
&> \sum_{v \in V(H)} (d_H(v) + 1 - f_H(v)) - \sum_{v \in V(Q)} (d_H(v) + 1 - f_H(v)) \\
&= \sum_{v \in V(Z)} (d_Z(v) + 1 - f_Z(v)),
\end{aligned}$$

contradicting the minimality of $|H|$. □

As a special case we get the following lower bound on the number of edges. Recall that $\text{mic}(G)$ is the maximum of $\sum_{v \in I} d_G(v)$ over all independent sets I of G.

Theorem 2.4. *If G is an OC-irreducible graph, then* $2\|G\| \geq (\delta(G) - 1)|G| + \text{mic}(G) + 1$.

3 Classification of graphs with minimum mic

For a graph G, we define $d_0 : V(G) \to \mathbb{N}$ by $d_0(v) := d_G(v)$. The d_0-choosable graphs were first characterized by Borodin [3] and independently by Erdős, Rubin and Taylor [5]. The connected graphs which are not d_0-choosable are precisely the Gallai trees (connected graphs in which every block is complete or an odd cycle). Hladkỳ, Král and Schauz [7] generalized this classification to online d_0-choosable graphs. In fact, they proved a classification in terms of Alon-Tarsi orientations as follows. A subgraph H of a directed multigraph D is called *Eulerian* if $d_H^-(v) = d_H^+(v)$ for every $v \in V(H)$. We call H *even* if $\|H\|$ is even and *odd* otherwise. Let $EE(D)$ be the number of even, spanning, Eulerian subgraphs of D and $EO(D)$ the number of odd, spanning, Eulerian subgraphs of D. Note that the edgeless subgraph of D is even and hence we always have $EE(D) > 0$.

Let G be a graph and $f : V(G) \to \mathbb{N}$. We say that G is *f-Alon-Tarsi* (for brevity, *f-AT*) if G has an orientation D where $f(v) \geq d_D^+(v) + 1$ for all $v \in V(D)$ and $EE(D) \neq EO(D)$.

One simple way to achieve $EE(D) \neq EO(D)$ is to have D be acyclic since then we have $EE(D) = 1$ and $EO(D) = 0$. In this case, ordering the vertices so that all edges point the same direction and coloring greedily shows that G is f-choosable. If we require f to be constant, we get the familiar *coloring number* $\text{col}(G)$; that is, $\text{col}(G)$ is the smallest k for which G has an acyclic orientation D with $k \geq d_D^+(v) + 1$ for all $v \in V(D)$. Alon and Tarsi [2] generalized from the acyclic case to arbitrary f-AT orientations.

Lemma 3.1. *If a graph G is f-AT for $f\colon V(G) \to \mathbb{N}$, then G is f-choosable.*

Schauz [19] extended this result to online f-choosability.

Lemma 3.2. *If a graph G is f-AT for $f\colon V(G) \to \mathbb{N}$, then G is online f-choosable.*

Hladkỳ, Král and Schauz [7] proved the following.

Theorem 3.3. *A connected graph is d_0-AT if and only if it is not a Gallai tree.*

Acyclic orientations are also a special case of kernel-perfect orientations. A graph H is f-KP if H has a kernel-perfect oriented supergraph H' where $f(v) > d_{H'}^+(v)$ for all $v \in V(H')$. This supergraph for f-KP gives us more power, for example a $K_4 - e$ has no kernel-perfect orientation showing it is degree-choosable, but if we double the edge in two triangles, there is such an orientation. We could allow a supergraph for f-AT as well, but this doesn't give us any more power. We will now prove the same classification for connected d_0-KP graphs. The following theorem about how small $\text{mic}(G)$ can be implies this classification. First, a useful fact about $\text{mic}(G)$.

Lemma 3.4. *For a graph G and any non-isolated $v \in V(G)$, we have* $\text{mic}(G) \geq \text{mic}(G - v) + 1$.

Proof. Let A be an independent set in $G - v$ with $||A, G - v - A|| = \text{mic}(G - v)$. Put $A' := A \cup \{v\}$ if $||v, A|| = 0$ and $A' := A$ otherwise. Then A' is independent in G and $\text{mic}(G) \geq ||A', G - A'|| \geq \text{mic}(G - v) + 1$. □

The following lemma has many different proofs [5, 4, 7]. Although it is known as Rubin's Block Lemma [5], the lemma was implicit in the much earlier work of Gallai [6] and Dirac.

Rubin's Block Lemma. *If G is a 2-connected graph that is not complete and not an odd cycle, then G contains an even cycle with at most one chord.*

Theorem 3.5. *For a connected graph G, we have* $\text{mic}(G) \geq |G| - 1$ *with equality just in case G is a Gallai tree.*

Proof. Applying Lemma 3.4 with $v \in V(G)$ a noncutvertex shows that $\text{mic}(G) \geq \text{mic}(G - v) + 1$. Doing induction on $|G|$, we conclude that $\text{mic}(G - v) \geq |G| - 2$ and hence $\text{mic}(G) \geq |G| - 1$.

Suppose G has an induced subgraph H that is a cycle with at most one chord. Since $\alpha(H) = \frac{|H|}{2}$ and $\delta(H) = 2$, we have $\text{mic}(H) \geq |H|$. Hence, by Lemma 3.4, we have $\text{mic}(G) \geq |G|$. So, if $\text{mic}(G) = |G| - 1$, applying Rubin's Block Lemma shows that G is a Gallai tree. It remains to show that if G is a Gallai tree, then $\text{mic}(G) = |G| - 1$. We could prove this directly, but it is easier to note that if G is a Gallai tree, then it is not d_0-choosable and hence $\text{mic}(G) \leq |G| - 1$ by the Main Lemma. □

Lemma 3.6. *A connected graph G is d_0-KP if and only if some nonempty induced subgraph of G is d_0-KP.*

Proof. The 'only if' direction is trivial. For the other direction, suppose G has a nonempty induced subgraph that is d_0-KP and choose H to be a maximal such subgraph. If $H = G$, we are done, so suppose not. Let $S = N_G(V(H)) \setminus V(H)$. Then $S \neq \emptyset$ since G is connected. We show that $G[V(H) \cup S]$ is d_0-KP, violating maximality of H. Start with a kernel-perfect oriented supergraph H' of H where $d_H(v) > d^+_{H'}(v)$ for all $v \in V(H')$. Now create an oriented supergraph Q' of $Q := G[V(H) \cup S]$ by directing all edges from $V(H)$ to S into S and replacing each edge in $Q[S]$ with arcs going both ways. Clearly, this oriented graph is kernel-perfect. Each vertex in $Q'[S]$ has at least one in-edge coming from $V(H)$, so we have $d_Q(v) > d^+_{Q'}(v)$ for all $v \in V(Q')$ as desired. □

Corollary 3.7. *A connected graph is d_0-KP if and only if it is not a Gallai tree.*

Proof. If G is a Gallai tree, then it is not d_0-choosable and hence not d_0-KP by the Kernel Lemma. For the other direction, let G be a connected graph that is not a Gallai tree. By Theorem 3.5, we have $\text{mic}(G) \geq |G|$. Applying Main Lemma with $f(v) := d_G(v)$ for all $v \in V(G)$ gives a nonempty induced subgraph H of G that is f_H-KP where $f_H(v) := f(v) + d_H(v) - d_G(v) = d_H(v)$ for $v \in V(H)$. Now Lemma 3.6 shows that G is d_0-KP. □

4 Ore Brooks for online list coloring

For a graph G, let $\mathcal{H}(G)$ be the subgraph of G induced on the vertices of degree greater than $\delta(G)$ and $\mathcal{L}(G)$ the subgraph of G induced on the vertices of degree $\delta(G)$.

Lemma 4.1. *If G is an OC-irreducible graph such that $\mathcal{H}(G)$ is edgeless and $\Delta(G) = \delta(G) + 1$, then $2\|G\| < \left(\delta(G) + \frac{1}{\delta(G)}\right)|G|$.*

Proof. Put $\delta := \delta(G)$ and suppose for contradiction that $2\|G\| \geq \left(\delta + \frac{1}{\delta}\right)|G|$. Then $|\mathcal{H}(G)| + \delta|G| = 2\|G\| \geq \left(\delta + \frac{1}{\delta}\right)|G|$ and hence $|G| \leq \delta|\mathcal{H}(G)|$. Therefore $\|\mathcal{L}(G), \mathcal{H}(G)\| \geq (\delta + 1)|\mathcal{H}(G)| \geq |\mathcal{H}(G)| + |G|$. Plugging into Theorem 2.4 gives $2\|G\| \geq 2\|G\| + 1$, which is impossible. □

To break up our computations we reformulate Lemma 4.1 as an upper bound on σ where

$$\sigma(G) := \left(\delta(G) - 1 + \frac{2}{\delta(G)}\right)|\mathcal{L}(G)| - 2\|\mathcal{L}(G)\|.$$

Lemma 4.2. *If G is an OC-irreducible graph such that $\mathcal{H}(G)$ is edgeless and $\Delta(G) = \delta(G) + 1$, then $\sigma(G) < \left(4 - \frac{2}{\delta(G)}\right)|\mathcal{H}(G)|$.*

Proof. Put $\delta := \delta(G)$. By Lemma 4.1, we have $2\|G\| < \left(\delta + \frac{1}{\delta}\right)|G|$. We have $\delta|G| + |\mathcal{H}(G)| = 2\|G\| < \left(\delta + \frac{1}{\delta}\right)|G|$ giving $|\mathcal{H}(G)| < \frac{|G|}{\delta}$ and hence $|\mathcal{L}(G)| > (\delta - 1)|\mathcal{H}(G)|$. The lemma follows from the following computation:

$$\begin{aligned}
\sigma(G) &= 2\delta\left|\mathcal{L}(G)\right| - 2\left\|\mathcal{L}(G)\right\| - \left(\delta + 1 - \frac{2}{\delta}\right)\left|\mathcal{L}(G)\right| \\
&= 2\left\|G\right\| - \left(\delta + 1 - \frac{2}{\delta}\right)\left|\mathcal{L}(G)\right| \\
&< \left(\delta + \frac{1}{\delta}\right)\left|G\right| - \left(\delta + 1 - \frac{2}{\delta}\right)\left|\mathcal{L}(G)\right| \\
&= -\left(1 - \frac{3}{\delta}\right)\left|\mathcal{L}(G)\right| + \left(\delta + \frac{1}{\delta}\right)\left|\mathcal{H}(G)\right| \\
&< -\left(1 - \frac{3}{\delta}\right)(\delta - 1)\left|\mathcal{H}(G)\right| + \left(\delta + \frac{1}{\delta}\right)\left|\mathcal{H}(G)\right| \\
&= \left(4 - \frac{2}{\delta}\right)\left|\mathcal{H}(G)\right|.
\end{aligned}$$

□

We need the following bound from [9]. Put $\alpha_k := \frac{1}{2} - \frac{1}{(k-1)(k-2)}$ and let $c(G)$ be the number of components in G.

Corollary 4.3. *If G is an OC-irreducible graph with $\delta(G) \geq 6$ and $\omega(G) \leq \delta(G)$ such that $\mathcal{H}(G)$ is edgeless, then $\sigma(G) \geq (\delta(G) - 2)\alpha_{\delta(G)+1}\left|\mathcal{H}(G)\right| + 2(1 - \alpha_{\delta(G)+1})c(\mathcal{L}(G))$.*

By combining Lemma 4.2 with Corollary 4.3 we can prove the Ore version of Brooks' theorem for online list coloring for $\Delta \geq 11$. With a bit more work we will improve this to $\Delta \geq 10$. First, we can actually get a bit more out of Theorem 2.4 by considering independent sets of low vertices that have no high neighbors. Such sets can be added to $V(\mathcal{H}(G))$ to get a cut with more edges. To apply this idea we need the following counting lemma. For a graph G and $t \in \mathbb{N}$, let $\beta_t(G)$ be the size of a largest independent set of degree t vertices in G; that is, $\beta_t(G) := \alpha(G[x|d(x) = t])$.

Lemma 4.4. *Fix $k \geq 6$. Let G be a Gallai forest with maximum degree at most $k - 1$ not containing K_k. We have the following inequality:*

$$(k-1)\beta_{k-1}(G) + \sum_{v \in V(G)} k - 1 - d(v) \geq \frac{2(k-3)}{k-2}\left|G\right| - \frac{(k-1)(k-4)}{k-2}c(G).$$

Proof. It will suffice to prove that for any Gallai tree T with maximum degree at most $k-1$ we have:

$$(k-1)\beta_{k-1}(T) + \sum_{v \in V(T)} k - 1 - d(v) \geq \frac{2(k-3)}{k-2}\left|T\right| - \frac{(k-1)(k-4)}{k-2}.$$

Suppose not and choose a counterexample T minimizing $|T|$. First, if T has only one block it is easy to see that the inequality is satisfied. Let B be an endblock of T and say x is the cutvertex in B. Suppose $\chi(B) \leq k - 3$. Put $T' := T - (B - x)$. By minimality of $|T|$, T' satisfies the inequality. When we add $B - x$ back in, the left side increases by $(k-\chi(B))(|B|-1)-(|B|-1) \geq 2(|B|-1)$. But the right side increases by only $\frac{2(k-3)}{k-2}(|B|-1)$ and hence T is not a counterexample, a contradiction.

Hence B is either K_{k-2} or K_{k-1}. Consider $T' := T - B$. Suppose $d_T(x) = k-1$. Note that none of x's neighbors in T' have degree $k-1$ in T' and thus are in no maximum independent set of degree $k-1$ vertices in T'. Therefore, we can add x to any such independent set, giving $\beta_{k-1}(T) > \beta_{k-1}(T')$. Hence, after applying minimality to T', we see that adding back B increases the left side by $k-1+(k-2)-1$ if B is K_{k-1} and by $k-1+2(k-3)-2$ if B is K_{k-2}. Since the right side increases by only $\frac{2(k-3)}{k-2}|B|$ in both cases, T satisfies the inequality, a contradiction.

Therefore, it must be that B is K_{k-2} and $d_T(x) = k-2$. Now when we add B back, the left side increases by $2(k-3)+1-1$ and the right side increases by only $2(k-3)$ and again T satisfies the inequality, a contradiction. $\square$

Lemma 4.5. *If G is an OC-irreducible graph such that $\mathcal{H}(G)$ is edgeless, $\Delta(G) = \delta(G)+1 \geq 7$ and $K_{\Delta(G)} \not\subseteq G$, then $|\mathcal{H}(G)| < \frac{\delta(G)(\delta(G)-3)}{(\delta(G)-1)(\delta(G)-5)}c(\mathcal{L}(G))$.*

Proof. Put $\delta := \delta(G)$. By Theorem 2.4 we have $\text{mic}(G) < |\mathcal{L}(G)| + 2|\mathcal{H}(G)| < |\mathcal{L}(G)| + \frac{2}{\delta-1}|\mathcal{L}(G)| = \frac{\delta+1}{\delta-1}|\mathcal{L}(G)|$. But applying Lemma 4.4 to $\mathcal{L}(G)$ gives $\text{mic}(G) \geq 2\frac{\delta-2}{\delta-1}|\mathcal{L}(G)| - \frac{\delta(\delta-3)}{\delta-1}c(\mathcal{L}(G))$. Combining these inequalities and $|\mathcal{L}(G)| > (\delta-1)|\mathcal{H}(G)|$ proves the lemma. $\square$

Lemma 4.6. *Every OC-irreducible graph G with $\delta(G)+1 = \Delta(G) \geq 10$ such that $\mathcal{H}(G)$ is edgeless contains $K_{\Delta(G)}$.*

Proof. Suppose not and let G be a counterexample. Put $\delta := \delta(G)$. By Corollary 4.3 we have $\sigma(G) \geq (\delta-2)\alpha_{\delta+1}|\mathcal{H}(G)| + 2(1-\alpha_{\delta+1})c(\mathcal{L}(G))$. By Lemma 4.5 we have $c(\mathcal{L}(G)) > \frac{(\delta-1)(\delta-5)}{\delta(\delta-3)}|\mathcal{H}(G)|$. Also, by Lemma 4.2, we have $\sigma(G) < \left(4 - \frac{2}{\delta}\right)|\mathcal{H}(G)|$. Putting these together, we get $4 - \frac{2}{\delta} > \alpha_{\delta+1}(\delta-2) + 2(1-\alpha_{\delta+1})\frac{(\delta-1)(\delta-5)}{\delta(\delta-3)}$. But then $\delta \leq 8$, a contradiction. $\square$

We need the following lemma from [18] allowing us to patch together online list colorability of parts into online list colorability of the whole.

Lemma 4.7. *Let G be a graph and $f\colon V(G) \to \mathbb{N}$. If H is an induced subgraph of G such that $G - H$ is online $f\big|_{V(G-H)}$-choosable and H is online f_H-choosable where $f_H(v) := f(v) + d_H(v) - d_G(v)$, then G is online f-choosable.*

Theorem 4.8. *If G is a graph with $\Delta(G) \geq 10$ not containing $K_{\Delta(G)}$ such that $\mathcal{H}(G)$ is edgeless, then G is online $(\Delta(G)-1)$-choosable.*

Proof. Suppose not and choose a counterexample G minimizing $|G|$. Then G is online f-critical where $f(v) := \Delta(G) - 1$ for all $v \in V(G)$. Hence $\delta(G) \geq \Delta(G) - 1$ and we may apply Lemma 4.6 to get a nonempty induced subgraph H of G that is online f_H-choosable where $f_H(v) := \Delta(G) - 1 + d_H(v) - d_G(v)$ for all $v \in V(H)$. But then applying Lemma 4.7 shows that G is $(\Delta(G)-1)$-choosable, a contradiction. $\square$

Combining Lemma 4.8 with the following version of Brooks' theorem for online list coloring (first proved in [7]) we get Theorem 4.10.

Lemma 4.9. *Every graph with $\Delta \geq 3$ not containing $K_{\Delta+1}$ is online Δ-choosable.*

Theorem 4.10. *Every graph with $\theta \geq 18$ and $\omega \leq \frac{\theta}{2}$ is online $\left\lfloor \frac{\theta}{2} \right\rfloor$-choosable.*

Proof. Suppose not and choose a counterexample G minimizing $|G|$. Put $k := \left\lfloor \frac{\theta(G)}{2} \right\rfloor$. Then G is online f-critical where $f(v) := k$ for all $v \in V(G)$. Hence $\delta(G) \geq k$ and thus $\Delta(G) \leq k+1$. If $\Delta(G) = k$, then the theorem follows from Lemma 4.9. Hence we must have $\Delta(G) = k+1$. Therefore $\mathcal{H}(G)$ is edgeless, $\Delta(G) \geq 10$ and $\omega(G) \leq \Delta(G) - 1$. Applying Theorem 4.8 shows that G is online $(\Delta(G) - 1)$-choosable, a contradiction. □

The same result for list coloring is an immediate consequence.

Theorem 4.11. *Every graph with $\theta \geq 18$ and $\omega \leq \frac{\theta}{2}$ is $\left\lfloor \frac{\theta}{2} \right\rfloor$-choosable.*

5 Ore Brooks for maximum degree four

Kostochka and Yancey's bound [10] shows that if G is 4-critical, then $\|G\| \geq \left\lceil \frac{5|G|-2}{3} \right\rceil$. If we try to analyze 4-critical graphs with edgeless high vertex subgraphs by putting this lower bound on the number of edges together with the results on orientations and list coloring obtained in [10], the bounds miss each other. Using the improved bound from Lemma 4.1 we get an exact bound on the number of edges in such a graph.

Lemma 5.1. *For a critical graph G with $\Delta(G) \leq \chi(G) = 4$ such that $\mathcal{H}(G)$ is edgeless we have $\|G\| = \left\lceil \frac{5|G|-2}{3} \right\rceil$ and $|G|$ is not a multiple of* 3.

Proof. Since G is 4-critical, applying Lemma 4.1 gives $2\|G\| < \left(3 + \frac{1}{3}\right)|G| = \frac{10}{3}|G|$. By Kostochka and Yancey's bound we have $\left\lceil \frac{5|G|-2}{3} \right\rceil \leq \|G\| < \frac{5}{3}|G|$. Hence $\|G\| = \left\lceil \frac{5|G|-2}{3} \right\rceil$ and $|G|$ is not a multiple of 3. □

It is easy to see that contracting a diamond in a critical graph G with $\Delta(G) \leq \chi(G) = 4$ such that $\mathcal{H}(G)$ is edgeless gives another such graph. So, the following characterization of these graphs is natural. Recently, Postle [14] proved this using an extension of the potential method of Kostochka and Yancey.

Theorem 5.2 (Postle [14])**.** *Every critical graph G with $\Delta(G) \leq \chi(G) = 4$ such that $\mathcal{H}(G)$ is edgeless, except K_4, has an induced diamond. In particular, any such G can be reduced to K_4 by a sequence of diamond contractions.*

6 Online choosability of triangle-free graphs

We write $\lg(x)$ for the base 2 logarithm of x. We can get a reasonably good lower bound on $\mathrm{mic}(G)$ for triangle-free graphs using a simple probabilistic technique of Shearer and its modification by Alon (see [1]).

Lemma 6.1. *If G is a triangle-free graph, then $\mathrm{mic}(G) \geq \frac{1}{4}\sum_{v \in V(G)} \lg(d(v))$.*

Proof. Let W be a random independent set in G chosen uniformly from all independent sets in G. For each $v \in V(G)$ put $X_v := d(v)\,|\{v\} \cap W| + |N(v) \cap W|$. We claim that $E(X_v) \geq \frac{1}{2}\lg(d(v))$. This implies the lemma since by linearity of expectation $2\,\mathrm{mic}(G) \geq E\left(\sum_{v \in V(G)} X_v\right) \geq \frac{1}{2}\sum_{v \in V(G)} \lg(d(v))$.

To prove the claim, let H be the subgraph of G induced on $V(G) - (N(v) \cup \{v\})$, fix an independent set S in H and let X be the set of all nonneighbors of S in $N(v)$. Put $x := |X|$. It will suffice to bound the conditional expectation for each possible S as follows:

$$E\left(X_v \mid W \cap V(H) = S\right) \geq \frac{\lg(d(v))}{2}.$$

For each S, there are exactly $2^x + 1$ possibilities for W and we see that the conditional expectation is exactly $\frac{d(v)+x2^{x-1}}{2^x+1}$. Suppose this is less than $\frac{\lg(d(v))}{2}$ for some x. Then $2^x\left(\frac{\lg(d(v))}{2} - \frac{x}{2}\right) > d(v) - \frac{\lg(d(v))}{2}$. Put $t := \lg(d(v)) - x$. We have $\frac{td(v)}{2^{t+1}} = \frac{d(v)}{2^t}\left(\frac{\lg(d(v))}{2} - \frac{\lg(d(v))-t}{2}\right) > d(v) - \frac{\lg(d(v))}{2} > \frac{d(v)}{2}$ and hence $\frac{t}{2^t} > 1$, a contradiction. □

Theorem 6.2. *If G is a triangle-free graph and $f\colon V(G) \to \mathbb{N}$ by $f(v) := d_G(v) + 1 - \left\lfloor \frac{1}{4}\lg(d_G(v))\right\rfloor$, then G has a nonempty induced subgraph H that is online f_H-choosable where $f_H(v) := f(v) + d_H(v) - d_G(v)$ for $v \in V(H)$.*

Proof. Immediate upon applying Main Lemma to G since

$$\sum_{v \in V(G)} d_G(v) + 1 - f(v) = \sum_{v \in V(G)} \left\lfloor \frac{1}{4}\lg(d_G(v))\right\rfloor \leq \mathrm{mic}(G).$$

□

Corollary 6.3. *If G is a triangle-free graph with $\Delta(G) \leq t$ for some $t \in \mathbb{N}$, then G is online $\left(t + 1 - \left\lfloor \frac{1}{4}\lg(t)\right\rfloor\right)$-choosable.*

Proof. Suppose not and choose a counterexample G and $t \in \mathbb{N}$ so as to minimize $|G|$. Put $f(v) := d_G(v) + 1 - \left\lfloor \frac{1}{4}\lg(d_G(v))\right\rfloor$. By Theorem 6.2, G has a nonempty induced subgraph H that is online f_H-choosable where $f_H(v) := f(v) + d_H(v) - d_G(v)$ for $v \in V(H)$. Since $t+1-\left\lfloor \frac{1}{4}\lg(t)\right\rfloor \geq d_G(v)+1-\left\lfloor \frac{1}{4}\lg(d_G(v))\right\rfloor$ for all $v \in V(G)$, we have that H is $g(v)$-choosable where $g(v) := t+1-\left\lfloor \frac{1}{4}\lg(t)\right\rfloor + d_H(v) - d_G(v)$. Now applying minimality of $|G|$ and Lemma 4.7 gives a contradiction. □

The best, known bounds for the chromatic number of triangle-free graphs are Kostochka's upper bound of $\frac{2}{3}\Delta + 2$ in [11] (see [16] for a proof in English) for small Δ and Johansson's upper bound of $\frac{9\Delta}{\ln(\Delta)}$ for large Δ. Johansson's proof also works for list coloring, but not for online list coloring. To the best of our knowledge Corollary 6.3 is the best, known-upper bound for online list colorings of triangle-free graphs. Additionally, Corollary 6.3 improves on Johansson's bound for list coloring for $\Delta \leq 8000$. The bound can surely be improved by a more complicated computation of $\mathrm{mic}(G)$, but not beyond around $\Delta + 1 - \lfloor 2\ln(\Delta) \rfloor$ via this method as can be seen by examples of triangle-free graphs with independence number near $\frac{2\ln(\Delta)}{\Delta}n$.

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