Coloring claw-free graphs with $\Delta - 1$ colors

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Abstract

We prove that every claw-free graph G that doesn't contain a clique on $\Delta(G) \geq 9$ vertices can be $\Delta(G) - 1$ colored.

1 Introduction

The first non-trivial result about coloring graphs with around Δ colors is Brooks' theorem from 1941.

Theorem 1.1 (Brooks [4]). *Every graph with $\Delta \geq 3$ satisfies $\chi \leq \max\{\omega, \Delta\}$.*

In 1977, Borodin and Kostochka conjectured that a similar result holds for $\Delta - 1$ colorings.

Conjecture 1.2 (Borodin and Kostochka [3]). *Every graph with $\Delta \geq 9$ satisfies $\chi \leq \max\{\omega, \Delta - 1\}$.*

Counterexamples exist (see Figure 1) showing that the $\Delta \geq 9$ condition is necessary. Using probabilistic methods, Reed [16] proved the conjecture for $\Delta \geq 10^{14}$.

In [8], Dhurandhar proved the Borodin-Kostochka Conjecture for a superset of line graphs of *simple* graphs defined by excluding the claw, $K_5 - e$ and another graph D as induced subgraphs. Kierstead and Schmerl [11] improved this by removing the need to exclude D. The aim of this paper is to remove the need to exclude $K_5 - e$; that is, to prove the Borodin-Kostochka Conjecture for claw-free graphs.

Theorem 4.5. *Every claw-free graph satisfying $\chi \geq \Delta \geq 9$ contains a K_Δ.*

This also generalizes the result of Beutelspacher and Hering [1] that the Borodin-Kostochka conjecture holds for graphs with independence number at most two. The value of 9 in Theorem 4.5 is best possible since the counterexample for $\Delta = 8$ in Figure 1 is claw-free. Theorem 4.5 is also optimal in the following sense. We can reformulate the statement as: every claw-free graph with $\Delta \geq 9$ satisfies $\chi \leq \max\{\omega, \Delta - 1\}$. Consider a similar statement with $\Delta - 1$

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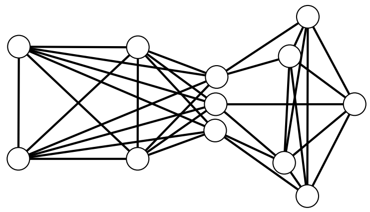
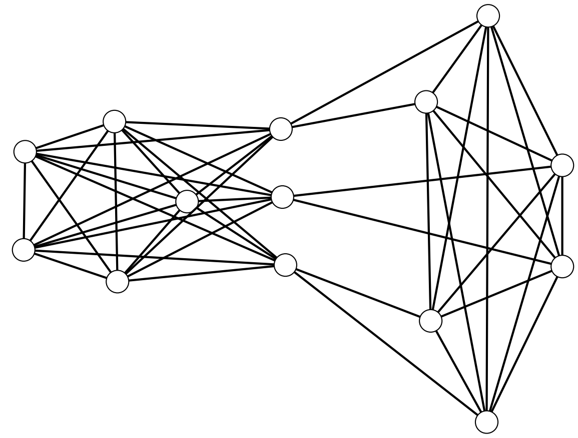
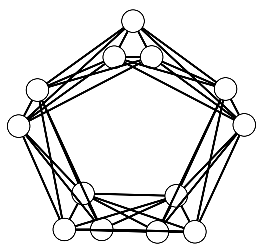


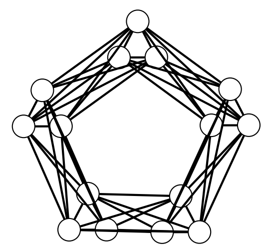
(a) $\Delta = 6$



(b) $\Delta = 7$



(c) $\Delta = 7$



(d) $\Delta = 8$

Figure 1: Counterexamples to the Borodin-Kostochka Conjecture for small Δ.

replaced by $f(\Delta)$ for some $f\colon \mathbb{N} \to \mathbb{N}$ and 9 replaced by Δ_0. We show that $f(x) \geq x - 1$ for $x \geq \Delta_0$. Consider $G_t := K_t * C_5$. We have $\chi(G_t) = t + 3$, $\omega(G_t) = t + 2$ and $\Delta(G_t) = t + 4$ and G_t is claw-free. Hence for $t \geq \Delta_0 - 4$ we have $t + 3 \leq \max\{t + 2, f(t + 4)\} \leq f(t + 4)$ giving $f(x) \geq x - 1$ for $x \geq \Delta_0$.

As shown in [15], the situation is very different for line graphs of multigraphs which satisfy $\chi \leq \max\{\omega, \frac{7\Delta+10}{8}\}$. There it was conjectured that $f(x) := \frac{5\Delta+8}{6}$ works for line graphs of multigraphs; this would be best possible. The example of $K_t * C_5$ is claw-free, but it isn't quasi-line.

Question. What is the situation for quasi-line graphs? That is, what is the optimal f such that every quasi-line graph with large enough maximum degree satisfies $\chi \leq \max\{\omega, f(\Delta)\}$.

Borodin and Kostochka conjectured (to themselves) [14] that their conjecture also holds for list coloring.

Conjecture 1.3 (Borodin and Kostochka [14])**.** *Every graph with $\Delta \geq 9$ satisfies $\chi_l \leq \max\{\omega, \Delta - 1\}$.*

We make some progress on this conjecture for claw-free graphs, proving it for circular interval graphs and severely restricting line graph counterexamples. These two classes are the base cases of the structure theorem for quasi-line graphs of Chudnovsky and Seymour [6] that we use. Finally, we prove the following.

Theorem 4.6. *If every quasi-line graph satisfying $\chi_l \geq \Delta \geq 9$ contains a K_Δ, then the same statement holds for every claw-free graph.*

In [10], Gravier and Maffray conjecture the following strengthening of the list coloring conjecture. Conjecture 1.3 for claw-free graphs would be an immediate consequence.

Conjecture 1.4 (Gravier and Maffray [10])**.** *Every claw-free graph satisfies $\chi_l = \chi$.*

Now we introduce some notation and terminology that will be used through the paper. If G is a vertex critical graph with $\chi = \Delta$, then every vertex in G has degree $\Delta - 1$ or Δ. We call the former vertices *low* and the latter vertices *high*. For vertices x, y in G, we write $x \leftrightarrow y$ if $xy \in E(G)$ and $x \not\leftrightarrow y$ if $xy \notin E(G)$. All the definitions for list coloring that we use are at the start of Section 6.

2 Circular interval graphs

A *representation* of a graph G in a graph H consists of:

- an injection $f \colon V(G) \hookrightarrow V(H)$;
- for each $xy \in E(G)$, a choice of path $p_{xy} \subseteq H$ from $f(x)$ to $f(y)$ such that $f^{-1}(V(p_{xy}))$ is a clique in G.

A graph is a *circular interval graph* if it has a representation in a cycle. We note that this class coincides with the class of proper circular arc graphs. A graph is a *linear interval graph* if it has a representation in a path.

Lemma 2.1. *Every circular interval graph satisfying $\chi_l \geq \Delta \geq 9$ contains a K_Δ.*

Proof. Suppose the contrary and choose a counterexample G minimizing $|G|$. Put $\Delta := \Delta(G)$. Then $\chi_l(G) = \Delta$, $\omega(G) \leq \Delta - 1$, $\delta(G) \geq \Delta - 1$ and $\chi_l(G - v) \leq \Delta - 1$ for all $v \in V(G)$. Since G is a circular interval graph, by definition G has a representation in a cycle $v_1 v_2 \ldots v_n$. Let K be a maximum clique in G. By symmetry we may assume that $V(K) = \{v_1, v_2, \ldots, v_t\}$ for some $t \leq \Delta - 1$; further, if possible we label the vertices so that $v_{t-3} \leftrightarrow v_{t+1}$ and the edge goes through v_{t-2}, v_{t-1}, v_t.

Claim 1. *$v_1 \not\leftrightarrow v_{t+1}$ and $v_2 \not\leftrightarrow v_{t+2}$ and $v_1 \not\leftrightarrow v_{t+2}$.* Assume the contrary. Clearly we can't have $v_1 \leftrightarrow v_{t+1}$ and have the edge go through $v_2, v_3, \ldots, v_t$ (since then we get a clique of size $t+1$). Similarly, we can't have $v_2 \leftrightarrow v_{t+2}$ and have the edge go through $v_3, v_4, \ldots, v_{t+1}$. So assume the edge $v_1 v_{t+2}$ exists and goes around the other way. If $v_1 \leftrightarrow v_{t+1}$, then let $G' = G \setminus \{v_1\}$ and if $v_1 \not\leftrightarrow v_{t+1}$, then let $G' = G \setminus \{v_1, v_{t+1}\}$. Now let $V_1 = \{v_2, v_3, \ldots, v_t\}$ and $V_2 = V(G') \setminus V_1$. Let $K' = G[V_1]$ and $L' = G[V_2]$; note that K' and L' are each cliques of size at most $\Delta - 2$. Now for each $S \subseteq V_2$, we have $|N_{\overline{G}}(S) \cap V_1| \geq |S|$ (otherwise we get a clique of size t in G' and a clique of size $t+1$ in G). Now by Hall's Theorem, we have a matching in $\overline{G}$ between V_1 and V_2 that saturates V_2. This implies that $G' \subseteq E_2^{\Delta-2}$, which in turn gives $G \subseteq E_2^{\Delta-1}$. By Lemma 6.13, G is $(\Delta - 1)$-choosable, which is a contradiction.

Claim 2. *$v_{t-3} \leftrightarrow v_{t+1}$ and the edge passes through v_{t-2}, v_{t-1}, v_t.* Assume the contrary. If $t \geq 7$, then since $t \leq \Delta - 1$, v_4 has some neighbor outside of K; by (reflectional) symmetry we could have labeled the vertices so that $v_{t-3} \leftrightarrow v_{t+1}$. So we must have $t \leq 6$. Each vertex v that is high has either at least $\lceil \Delta/2 \rceil$ clockwise neighbors or at least $\lceil \Delta/2 \rceil$ counterclockwise neighbors. This gives a clique of size $1 + \lceil \Delta/2 \rceil \geq 6$. If v_3 is high, then either v_3 has at least 4 clockwise neighbors, so $v_3 \leftrightarrow v_7$, or else v_3 has at least 6 counterclockwise neighbors, so $|K| \geq 7$. Thus, we may assume that v_3 is low; by symmetry (and our choice of labeling prior to Claim 1) v_4 is also low. Now since v_4 has only 3 counterclockwise neighbors, we get $v_4 \leftrightarrow v_7$ (in fact, we get $v_4 \leftrightarrow v_9$). Thus, $\{v_3, v_4, v_5, v_6, v_7\}$ induces $K_3 * E_2$ with a low degree vertex in both the K_3 and the E_2, which contradicts Lemma 6.11.

Claim 3. $v_{t-2} \not\leftrightarrow v_{t+2}$. Assume the contrary. By Claim 1 the edge goes through v_{t-1}, v_t, v_{t+1}. If $v_{t-3} \leftrightarrow v_{t+2}$, then $\{v_1, v_2, v_{t-3}, v_{t-2}, v_{t-1}, v_t, v_{t+1}, v_{t+2}\}$ induces $K_4 * B$, where B is not almost complete; this contradicts Lemma 6.5. If $v_{t-3} \not\leftrightarrow v_{t+2}$, then we get a $K_3 * P_4$ induced by $\{v_1, v_{t-3}, v_{t-2}, v_{t-1}, v_t, v_{t+1}, v_{t+2}\}$, which contradicts Lemma 6.9.

Claim 4. $v_{t-1} \not\leftrightarrow v_{t+2}$. Suppose the contrary. Now $\{v_1, v_{t-3}, v_{t-2}, v_{t-1}, v_t, v_{t+1}, v_{t+2}\}$ induces $K_2 *$ antichair (with v_{t-1}, v_t in the K_2), which contradicts Lemma 6.8.

Claim 5. *G is $(\Delta - 1)$-choosable.* Let $S = \{v_{t-3}, v_{t-2}, v_{t-1}, v_t\}$. If any vertex of S is low, then $S \cup \{v_1, v_{t+1}\}$ induces $K_4 * E_2$ with a low vertex in the K_4, which contradicts Lemma 6.10. So all of S is high. If $v_t \not\leftrightarrow v_{t+2}$, then $\{v_t, v_{t-1}, \ldots, v_{t-\Delta+1}\}$ (subscripts are modulo n) induces K_Δ. So $v_t \leftrightarrow v_{t+2}$. Since $v_{t-1} \not\leftrightarrow v_{t+2}$ and all of S is high, we get $v_n \in (\cap_{v \in (S \setminus \{v_t\})} N(v)) \setminus N(v_t)$. Now we must have $v_n \not\leftrightarrow v_{t+1}$ (for otherwise G is $(\Delta - 1)$-choosable, as in Claim 1). So we get $K_3 * P_4$ induced by $\{v_{t+1}, v_t, v_{t-1}, v_{t-2}, v_{t-3}, v_1, v_n\}$, which contradicts Lemma 6.6. $\square$

3 Quasi-line graphs

A graph is *quasi-line* if every vertex is bisimplicial (its neighborhood can be covered by two cliques). We apply a version of Chudnovsky and Seymour's structure theorem for quasi-line graphs from King's thesis [12]. The undefined terms will be defined after the statement.

Lemma 3.1. *Every connected skeletal quasi-line graph is a circular interval graph or a composition of linear interval strips.*

A *homogeneous pair of cliques* (A_1, A_2) in a graph G is a pair of disjoint nonempty cliques such that for each $i \in [2]$, every vertex in $G - (A_1 \cup A_2)$ is either joined to A_i or misses all of A_i and $|A_1| + |A_2| \geq 3$. A homogeneous pair of cliques (A_1, A_2) is *skeletal* if for any $e \in E(A, B)$ we have $\omega(G[A \cup B] - e) < \omega(G[A \cup B])$. A graph is *skeletal* if it contains no nonskeletal homogeneous pair of cliques.

Generalizaing a lemma of Chudnovsky and Fradkin [5], King proved a lemma allowing us to handle nonskeletal homogeneous pairs of cliques.

Lemma 3.2 (King [12])**.** *If G is a nonskeletal graph, then there is a proper subgraph G' of G such that:*

1. *G' is skeletal;*
2. *$\chi(G') = \chi(G)$;*
3. *If G is claw-free, then so is G';*

4. If G is quasi-line, then so is G'.

It remains to define the generalization of line graphs introduced by Chudnovsky and Seymour [6]; this is the notion of *compositions of strips* (for a more detailed introduction, see Chapter 5 of [12]). We use the modified definition from King and Reed [13]. A *strip* (H, A_1, A_2) is a claw-free graph H containing two cliques A_1 and A_2 such that for each $i \in [2]$ and $v \in A_i$, $N_H(v) - A_i$ is a clique. If H is a linear interval graph, then (H, A_1, A_2) is a *linear interval strip*. Now let H be a directed multigraph (possibly with loops) and suppose for each edge e of H we have a strip (H_e, X_e, Y_e). For each $v \in V(H)$ define

$$C_v := \left(\bigcup \{X_e \mid e \text{ is directed out of } v\}\right) \cup \left(\bigcup \{Y_e \mid e \text{ is directed into } v\}\right)$$

The graph formed by taking the disjoint union of $\{H_e \mid e \in E(H)\}$ and making C_v a clique for each $v \in V(H)$ is the composition of the strips (H_e, X_e, Y_e). Any graph formed in such a manner is called a *composition of strips*. It is easy to see that if for each strip (H_e, X_e, Y_e) in the composition we have $V(H_e) = X_e = Y_e$, then the constructed graph is just the line graph of the multigraph formed by replacing each $e \in E(H)$ with $|H_e|$ copies of e.

It will be convenient to have notation and terminology for a strip together with how it attaches to the graph. An *interval 2-join* in a graph G is an induced subgraph H such that:

1. H is a (nonempty) linear interval graph,
2. The ends of H are (not necessarily disjoint) cliques A_1, A_2,
3. $G - H$ contains cliques B_1, B_2 (not necessarily disjoint) such that A_1 is joined to B_1 and A_2 is joined to B_2,
4. there are no other edges between H and $G - H$.

Note that A_1, A_2, B_1, B_2 are uniquely determined by H, so we are justified in calling both H and the quintuple (H, A_1, A_2, B_1, B_2) the interval 2-join. An interval 2-join (H, A_1, A_2, B_1, B_2) is *trivial* if $V(H) = A_1 = A_2$ and *canonical* if $A_1 \cap A_2 = \emptyset$. A canonical interval 2-join (H, A_1, A_2, B_1, B_2) with leftmost vertex v_1 and rightmost vertex v_t is *reducible* if H is incomplete and $N_H(A_1) \setminus A_1 = N_H(v_1) \setminus A_1$ or $N_H(A_2) \setminus A_2 = N_H(v_t) \setminus A_2$. We call such a canonical interval 2-join reducible because we can *reduce* it as follows. Suppose H is incomplete and $N_H(A_1) \setminus A_1 = N_H(v_1) \setminus A_1$. Put $C := N_H(v_1) \setminus A_1$ and then $A'_1 := C \setminus A_2$ and $A'_2 := A_2 \setminus C$. Since H is not complete $v_t \in A'_2$ and hence $H' := G[A'_1 \cup A'_2]$ is a nonempty linear interval graph that gives the reduced canonical interval 2-join $(H', A'_1, A'_2, A_1 \cup (C \cap A_2), B_2 \cup (C \cap A_2)$.

Lemma 3.3. *If (H, A_1, A_2, B_1, B_2) is an irreducible canonical interval 2-join in a vertex critical graph G with $\chi(G) = \Delta(G) \geq 9$, then $B_1 \cap B_2 = \emptyset$ and $|A_1|, |A_2| \leq 3$. Moreover, if G is skeletal, then H is complete.*

Proof. Let (H, A_1, A_2, B_1, B_2) be an irreducible canonical interval 2-join in a vertex critical graph G with $\chi(G) = \Delta(G) \geq 9$. Put $\Delta := \Delta(G)$.

Note that, since it is vertex critical, G contains no K_Δ and in particular G has no simplicial vertices. Label the vertices of H left-to-right as $v_1, \ldots, v_t$. Say $A_1 = \{v_1, \ldots, v_L\}$ and $A_2 = \{v_R, \ldots, v_t\}$. For $v \in V(H)$, define $r(v) := \max\{i \in [t] \mid v \leftrightarrow v_i\}$ and $l(v) := \min\{i \in [t] \mid v \leftrightarrow v_i\}$. These are well-defined since $|H| \geq 2$ and H is connected by the following claim.

Claim 1. *$A_1, A_2, B_1, B_2 \neq \emptyset$, $B_1 \not\subseteq B_2$, $B_2 \not\subseteq B_1$ and H is connected.* Otherwise G would contain a clique cutset.

Claim 2. *If H is complete, then $R - L = 1$.* Suppose $V(H) \neq A_1 \cup A_2$. Then any $v \in V(H) \setminus A_1 \cup A_2$ would be simplicial in G, which is impossible. Hence $R - L = 1$.

Claim 3. *If H is not complete, then $r(v_L) = r(v_1) + 1$ and $l(v_R) = l(v_t) - 1$. In particular, v_1, v_t are low and $|A_1|, |A_2| \geq 2$.* Suppose otherwise that H is not complete and $r(v_L) \neq r(v_1) + 1$. By definition, $N_H(v_1) \subseteq N_H(v_L)$ and v_1, v_L have the same neighbors in $G \setminus H$. Hence if $r(v_L) > r(v_1) + 1$, then $d(v_L) - d(v_1) \geq 2$, impossible. So we must have $r(v_L) = r(v_1)$ and hence $N_H(A_1) \setminus A_1 = N_H(v_1) \setminus A_1$. Thus the 2-join is reducible, a contradiction. Therefore $r(v_L) = r(v_1) + 1$. Similarly, $l(v_R) = l(v_t) - 1$.

Claim 4. *$|A_1|, |A_2| \leq 3$.* Suppose otherwise that $|A_1| \geq 4$. First, suppose H is complete. By Claim 2, $V(H) = A_1 \cup A_2$. If v_1 is low, then for any $w_1 \in B_1 \setminus B_2$ the vertex set $\{v_1, \ldots, v_4, v_t, w_1\}$ induces a $K_4 * E_2$ violating Lemma 6.10. Hence v_1 is high. If $|A_2| \geq 2$ and $|B_1 \setminus B_2| \geq 2$, then for any $w_1, w_2 \in B_1 \setminus B_2$, the vertex set $\{v_1, \ldots, v_4, v_{t-1}, v_t, w_1, w_2\}$ induces a $K_4 * 2K_2$, which is impossible by Lemma 6.5. Hence either $|A_2| = 1$ or $|B_1 \setminus B_2| = 1$. Suppose $|A_2| = 1$. Then, since $A_1 \cup B_1$ induces a clique and $|A_1 \cup B_1| = d(v_1)$, v_1 must be low, impossible. Hence we must have $|B_1 \setminus B_2| = 1$. Thus $|B_1 \cap B_2| = |B_1| - 1$. Hence $V(H) \cup B_1 \cap B_2$ induces a clique with $|A_1| + |A_2| + |B_1| - 1 = d(v_1) = \Delta$ vertices, impossible.

Therefore H must be incomplete. By Claim 3, v_1 is low. But then as above for any $w_1 \in B_1 \setminus B_2$ the vertex set $\{v_1, \ldots, v_4, v_{L+1}, w_1\}$ induces a $K_4 * E_2$ violating Lemma 6.10. Hence we must have $|A_1| \leq 3$. Similarly, $|A_2| \leq 3$.

Claim 5. *$R - L = 1$.* Suppose otherwise that $R - L \geq 2$. Then by Claim 2, H is incomplete. Hence by Claim 3, $r(v_L) = r(v_1) + 1$, $l(v_R) = l(v_t) - 1$, v_1, v_t are low and $|A_1|, |A_2| \geq 2$.

Subclaim 5a. *$L + \Delta - 2 \leq r(v_{L+1}) \leq L + \Delta - 1$.* Since v_{L+1} has exactly L neighbors to the left, we have $r(v_{L+1}) \leq L + 1 + \Delta - L = \Delta + 1 \leq L + \Delta - 1$. If v_{L+1} is high, the previous computation is exact and $r(v_{L+1}) = \Delta + 1 \geq L + \Delta - 2$. Suppose v_{L+1} is low. If $L = 3$, then for some $w_1 \in B_1$ the vertex set $\{v_1, v_2, v_3, v_4, w_1\}$ induces a $K_3 * E_2$ violating Lemma 6.11. Hence $L = 2$ and $r(v_{L+1}) = L + 1 + \Delta - 1 - L = \Delta \geq L + \Delta - 2$.

Subclaim 5b. *$L + \Delta - 2 \leq r(v_{L+2}) \leq L + \Delta$.* By Subclaim 5a, $r(v_{L+2}) \geq L + \Delta - 2$. Since H contains no Δ-clique, v_{L+2} has at least 2 neighbors to the left if it is high and at least 1 neighbor to the left if it is low. Thus $r(v_{L+2}) \leq L + 2 + \Delta - 2 = L + \Delta$.

Subclaim 5c. *If v_{L+4} is high, then $l(v_{L+4}) \leq L$.* Suppose otherwise. Recall that $v_{L+1} \leftrightarrow v_{L+4}$. Then v_{L+4} has exactly 3 neighbors to the left, so $r(v_{L+4}) = L + \Delta + 1$. Consider the subgraph induced on $\{v_{L+1}, v_{L+2}, v_{L+4}, v_{L+5}, v_{L+6}, v_{L+7}, v_{L+9}, v_{L+10}\}$. By Subclaim 5a and Subclaim 5b, this induces a subgraph violating Lemma 6.5.

Subclaim 5d. *$l(v_{L+3}) \leq L$.* Suppose otherwise. Since $v_{L+1} \leftrightarrow v_{L+3}$, vertex v_{L+3} has exactly 2 neighbors to the left, so $r(v_{L+3}) \geq L + \Delta$. By Subclaim 5c, v_{L+4} is low. By Subclaim 5a, $L + \Delta - 2 \leq r(v_{L+1}) \leq L + \Delta - 1$. Therefore $\{v_{L+1}, v_{L+3}, v_{L+4}, v_{L+5}, v_{L+6}, v_{L+9}\}$ induces a $K_4 * E_2$ violating Lemma 6.10.

Subclaim 5e. $l(v_{L+2}) \le L-1$. By Subclaim 5d $r(v_L) \ge L+3$ and hence by Claim 3, $r(v_1) \ge L+2$. Hence $l(v_{L+2}) \le L-1$.

Subclaim 5f. *Claim 5 is true.* Let π be a $(\Delta-1)$-coloring of $G \setminus H$ and define a list assignment J on H by $J(v) := [\Delta-1] - \pi(N_{G \setminus H}(v))$. Then $|J(v)| \ge d_H(v) - 1$ for all $v \in V(H)$ and since v_1 is low, $|J(v_1)| \ge d_H(v_1)$. Pick $w \in B_1$. Note that $\pi(w) \notin J(v_i)$ for $i \in [L]$. Since $J(v_{L+1}) = [\Delta-1]$, we may color v_{L+1} with $\pi(w)$ to get a new list assignment J' on $H' := H - v_{L+1}$. Then, since $\pi(w) \notin J(v_i)$ for $i \in [L]$, we have $|J'(v_i)| \ge d_{H'}(v_i)$ for $i \in [L]$ and $|J'(v_1)| \ge d_{H'}(v_1) + 1$. Now color the vertices of H' greedily from their lists in the order $v_t, v_{t-1}, \ldots, v_1$. Since G has no Δ-clique, we must have $N(v_t) \not\subseteq A_2 \cup B_2$, so $l(v_t) \le R-1$. Since $l(v_R) = l(v_t) - 1$, each of $v_R, \ldots, v_t$ have at least two neighbors to the left in H'. For $L+4 \le i < R$, since G doesn't contain K_Δ, every high v_i has at least two neighbors to the left in H' and every low v_i at least one neighbor. By Subclaim 5d, the same holds for v_{L+3} and by Subclaim 5e, it holds for v_{L+2}. Hence each vertex will have a color free to use when we encounter it, so we can complete the $(\Delta-1)$-coloring to all of G, a contradiction.

Claim 6. $B_1 \cap B_2 = \emptyset$. Suppose otherwise that we have $w \in B_1 \cap B_2$.

Subclaim 6a. *Each $v \in V(H)$ is low, $|B_1| = |B_2|$, $|B_1 \setminus B_2| = |B_2 \setminus B_1| = 1$, $d(v) = |A_1| + |A_2| + |B_1| - 1$ for each $v \in V(H)$ and H is complete.* By Claim 5, we have $d(v) \le |A_1| + |A_2| + |B_1| - 1$ for $v \in A_1$ and $d(v) \le |A_1| + |A_2| + |B_2| - 1$ for $v \in A_2$. Also, as $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$, we have $d(w) \ge \max\{|B_1|, |B_2|\} + |A_1| + |A_2|$. So $d(w) \ge d(v) + 1$ for any $v \in V(H)$. This implies that each $v \in V(H)$ is low, $|B_1| = |B_2|$, $|B_1 \setminus B_2| = |B_2 \setminus B_1| = 1$, $d(v) = |A_1| + |A_2| + |B_1| - 1$ for each $v \in V(H)$ and hence H is complete.

Subclaim 6b. $|B_1 \cap B_2| \le 3$. Suppose otherwise that $|B_1 \cap B_2| \ge 4$. Pick $w_1 \in B_1 \setminus B_2$, $w_2 \in B_2 \setminus B_1$ and $z_1, z_2, z_3, z_4 \in B_1 \cap B_2$. Then the set $\{z_1, z_2, z_3, z_4, w_1, w_2, v_1, v_t\}$ induces a subgraph violating Lemma 6.5. Hence $|B_1 \cap B_2| \le 3$.

Subclaim 6c. *Claim 6 is true.* By Subclaim 6a and Subclaim 6b we have $3 \ge |B_1 \cap B_2| = |B_1| - 1$ and hence $|B_1| = |B_2| \le 4$. Suppose $|A_1|, |A_2| \le 2$. Then $\Delta - 1 = d(v_1) \le 3 + |B_1| \le 7$, a contradiction. Hence by symmetry we may assume that $|A_1| \ge 3$. But then for $w_1 \in B_1 \setminus B_2$, the set $\{v_1, v_2, v_3, v_t, w_1\}$ induces a $K_3 * E_2$ violating Lemma 6.11.

Claim 7. *If G is skeletal, then H is complete.* Suppose G is skeletal and H is incomplete. By Claim 5, $R - L = 1$. Then, by Claim 3 $r(v_L) = r(v_1) + 1$ and $l(v_R) = l(v_t) - 1$. Since v_1 is not simplicial, $r(v_1) \ge L + 1 = R$. Hence $l(v_R) = 1$ and thus $l(v_t) = 2$. Similarly, $r(v_1) = t - 1$. So, H is K_t less an edge. But (A_1, A_2) is a homogeneous pair of cliques with $|A_1|, |A_2| \ge 2$ and hence there is an edge between A_1 and A_2 that we can remove without decreasing $\omega(G[A_1 \cup A_2])$. This contradicts the fact that G is skeletal. □

Lemma 3.4. *An interval 2-join in a vertex critical graph satisfying $\chi = \Delta \ge 9$ is either trivial or canonical.*

Proof. Let (H, A_1, A_2, B_1, B_2) be an interval 2-join in a vertex critical graph satisfying $\chi = \Delta \ge 9$. Suppose H is nontrivial; that is, $A_1 \ne A_2$. Put $C := A_1 \cap A_2$. Then $(H \setminus C, A_1 \setminus C, A_2 \setminus C, C \cup B_1, C \cup B_2)$ is a canonical interval 2-join. Reduce this 2-join until we get an irreducible canonical interval 2-join $(H', A_1', A_2', B_1', B_2')$ with $H' \trianglelefteq H \setminus C$. Since C is joined to $H - C$, it is also joined to H'. Hence $C \subseteq B_1' \cap B_2' = \emptyset$ by Lemma 3.3. Hence $A_1 \cap A_2 = C = \emptyset$ showing that H is canonical. □

Theorem 3.5. *Every quasi-line graph satisfying $\chi \geq \Delta \geq 9$ contains a K_Δ.*

Proof. We will prove the theorem by reducing to the case of line graphs, i.e., for every strip (H, A_1, A_2) we have $A_1 = A_2$. Suppose not and choose a counterexample G minimizing $|G|$. Plainly, G is vertex critical. By Lemma 3.2, we may assume that G is skeletal. By Lemma 2.1, G is not a circular interval graph. Therefore, by Lemma 3.1, G is a composition of linear interval strips. Choose such a composition representation of G using the maximum number of strips.

Let (H, A_1, A_2) be a strip in the composition. Suppose $A_1 \neq A_2$. Put $B_1 := N_{G\setminus H}(A_1)$ and $B_2 := N_{G\setminus H}(A_2)$. Then (H, A_1, A_2, B_1, B_2) is an interval 2-join. Since $A_1 \neq A_2$, H is canonical by Lemma 3.4. Suppose H is reducible. By symmetry, we may assume that $N_H(A_1) \setminus A_1 = N_H(v_1) \setminus A_1$. But then replacing the strip (H, A_1, A_2) with the two strips $(G[A_1], A_1, A_1)$ and $(H \setminus A_1, N_H(A_1) \setminus A_1, A_2)$ gives a composition representation of G using more strips, a contradiction. Hence H is irreducible. By Lemma 3.3, H is complete and thus replacing the strip (H, A_1, A_2) with the two strips $(G[A_1], A_1, A_1)$ and $(G[A_2], A_2, A_2)$ gives another contradiction.

Therefore, for every strip (H, A_1, A_2) in the composition we must have $V(H) = A_1 = A_2$. Hence G is a line graph of a multigraph. But this is impossible by Lemma 5.1. □

4 Claw-free graphs

In this section we reduce the Borodin-Kostochka conjecture for claw-free graphs to the case of quasi-line graphs. We first show that a certain graph cannot appear in the neighborhood of any vertex in our counterexample.

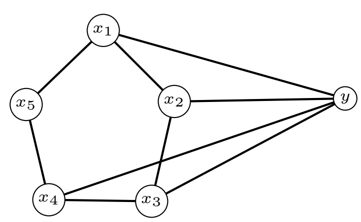


Figure 2: The graph N_6.

Lemma 4.1. *The graph $K_1 * N_6$ where N_6 is the graph in Figure 2 is d_1-choosable.*

Proof. Suppose not and let L be a minimal bad d_1-assignment on $K_1 * N_6$. Then, by the Small Pot Lemma, $|Pot(L)| \leq 6$. Let v be the vertex in the K_1. Note that $|L(v)| = 5$, $|L(y)| = 4$, $|L(x_5)| = 2$, and $|L(x_i)| = 3$ for all $i \in [4]$. Since $\sum_{i=1}^{5} |L(x_i)| = 14 > |Pot(L)|\omega(C_5)$, we see that two nonadjacent x_i's have a common color. Hence, by Lemma 6.3, we have $|Pot(L)| \leq 5$. Thus we have $c \in L(y) \cap L(x_5)$. Also, $L(x_1) \cap L(x_4) \neq \emptyset$, $L(x_1) \cap L(x_3) \neq \emptyset$ and $L(x_2) \cap L(x_4) \neq \emptyset$. By Lemma 6.12, the common color in all of these sets must be c. Hence c is in all the lists.

Now consider the list assignment L' where $L'(z) = L(z) - c$ for all $z \in N_6$. Then $|Pot(L')| = 4$ and since $\sum_{i=1}^{5} |L'(x_i)| = 9 > |Pot(L')|\omega(C_5)$, we see that that nonadjacent x_i's have a common color different than c. Now appling Lemma 6.12 gives a final contradiction. $\square$

By a *thickening* of a graph G, we just mean a graph formed by replacing each $x \in V(G)$ by a complete graph T_x such that $|T_x| \geq 1$ and for $x, y \in V(G)$, T_x is joined to T_y iff $x \leftrightarrow y$.

Lemma 4.2. *Any graph H with $\alpha(H) \leq 2$ such that every induced subgraph of $K_1 * H$ is not d_1-choosable can either be covered by two cliques or is a thickening of C_5.*

Proof. Suppose not and let H be a counterexample.

Claim 1. *H contains an induced C_4 or an induced C_5.* Suppose not. Then H must be chordal since $\alpha(H) \leq 2$. In particular, H contains a simplicial vertex x. But then $\{x\} \cup N_H(x)$ and $V(H) - N_H(x) - \{x\}$ are two cliques covering H, a contradiction.

Claim 2. *H does not contain an induced C_5 together with a vertex joined to at least 4 vertices in the C_5.* Suppose the contrary. If the vertex is joined to all of the C_5, then we have a induced $K_2 * C_5$, which is d_1-choosable by Lemma 6.7. If the vertex is joined to only four vertices in the C_5, we have an induced $K_1 * N_6$, impossible by Lemma 4.1.

Claim 3. *H contains no induced C_4.* Suppose otherwise that H contains an induced C_4, say $x_1x_2x_3x_4x_1$. Put $R := V(H) - \{x_1, x_2, x_3, x_4\}$. Let $y \in R$. As $\alpha(H) \leq 2$, y has a neighbor in $\{x_1, x_3\}$ and a neighbor in $\{x_2, x_4\}$. If y is adjacent to all of $x_1, \ldots, x_4$, then $K_1 * H$ contains $K_2 * C_4$ which is d_1-choosable, impossible. If y is adjacent to three of $x_1, \ldots, x_4$, then $K_1 * H$ contains $E_2 * \text{paw}$ which is d_1-choosable, impossible.

Thus every $y \in R$ is adjacent to all and only the vertices on one side of the C_4. We show that any two vertices in R must be adjacent to the same or opposite side and this gives the desired covering by two cliques. If this doesn't happen, then by symmetry we may suppose we have $y_1, y_2 \in R$ such that $y_1 \leftrightarrow x_1, x_2$ and $y_2 \leftrightarrow x_2, x_3$. We must have $y_1 \leftrightarrow y_2$ for otherwise $\{y_1, y_2, x_4\}$ is an independent set. But now $x_1y_1y_2x_3x_4x_1$ is an induced C_5 in which x_2 has 4 neighbors, impossible by Claim 2.

Claim 4. *H does not exist.* By Claim 1 and Claim 3, H contains an induced C_5. That H is a thickening of this C_5 is now is immediate from $\alpha(H) \leq 2$ and Claim 2. This final contradiction completes the proof. $\square$

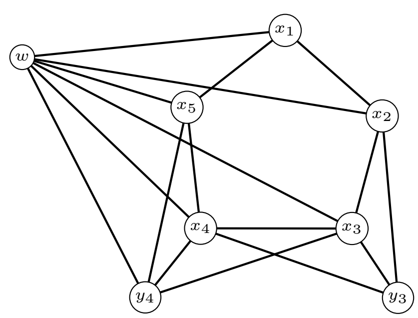


Figure 3: The graph D_8.

Lemma 4.3. *The graph D_8 is d_1-choosable.*

Proof. Suppose not and let L be a minimal bad d_1-assignment on $G := D_8$.

Claim 1. $|Pot(L)| \le 6$. By the Small Pot Lemma, we know that $|Pot(L)| \le 7$. Suppose $|Pot(L)| = 7$. Say $Pot(L) - L(w) = \{a, b\}$.

We must have $L(y_3) = \{a, b\}$. Otherwise we could color y_3 from $L(y_3) - \{a, b\}$ and note that $G - y_3 - w$ is d_0-choosable and hence has a coloring from its lists. Then we can easily modify this coloring to use both a and b at least once. But now we can color w.

If there exist distinct vertices $u, v \in V(G) - y_3$ such that $a \in L(u)$, $b \in L(v)$ and $\{u, v\} \not\subseteq \{x_2, x_3, x_4\}$, then we can color G as follows. Color y_3 arbitrarily to leave a available on u and b available on v. Again, $G - y_3 - w$ has a coloring. We can modify it to use a and b, then color w. Thus, a and b each appear only on some subset of $\{y_3, x_2, x_3, x_4\}$.

If $a \in L(x_2) \cap L(x_4)$, then we use a on x_2 and x_4 and color greedily y_3, x_3, y_4, x_1, x_5, w (actually any order will work if y_3 is first and w is last). If a appears only on y_3 and exactly one neighbor x_i, then we violate Lemma 6.2 since $|Pot_{y_3,x_i}(L)| < 7$. So now a appears precisely on either y_3, x_2, x_3 or y_3, x_3, x_4. Similarly b appears precisely on either y_3, x_2, x_3 or y_3, x_3, x_4.

If $\{a, b\} \cap L(x_2) = \emptyset$, then we use a on y_3 and b on x_3, then greedily color y_4, x_4, x_5, x_1, w, x_2. By symmetry, we may assume that $a \in L(x_2)$. But then since $\{a, b\} \subseteq L(x_3)$ we have $|Pot_{y_3,x_2,x_3}(L)| < 7$ violating Lemma 6.2. Hence $|Pot(L)| \le 6$.

Claim 2. $|Pot(L)| \le 5$. Suppose $|Pot(L)| = 6$. Choose $a \in Pot(L) - L(w)$ and $b \in L(w) \cap L(y_3)$. Put $H := G - y_3 - w$.

First we show that $b \in L(x_2) \cap L(x_3) \cap L(x_4)$. If not, we use b on y_3 and w, then greedily color x_1, x_5, y_4. Now we can finish by coloring last the x_i such that $b \notin L(x_i)$.

We must have $a \in L(y_3)$ or else we color x_2, x_4 with b and something else in H with a (since G_a contains an edge by Lemma 6.2) and finish. Now $a \notin L(x_1), L(x_5), L(y_4)$, for otherwise we color x_2, x_4 with b, y_3 with a and then color x_1, x_5, y_4, x_3 in order using a when we can, then color w. Now a is on y_3 and at least two of x_2, x_3, x_4 or else we violate Lemma 6.2. Now $a \notin L(x_2) \cap L(x_4)$ since otherwise we color x_2, x_4 with a, then y_3 with b, then greedily color x_1, x_5, y_4, x_3, w. Also $a \notin L(x_2) \cap L(x_3)$ since then $\{a, b\} \subseteq L(y_3) \cap L(x_2) \cap L(x_3)$ and hence $|Pot_{y_3,x_2,x_3}(L)| < 6$ violating Lemma 6.2. Therefore $V(G_a) = \{y_3, x_3, x_4\}$.

Now $|Pot_{y_3,x_3,x_4}|(L) \le 6$ and hence $L(x_3) \cap L(x_4) = \{a, b\}$ for otherwise we violate Lemma 6.2. Say $L(x_3) = \{a, b, c, d\}$ and $L(x_4) = \{a, b, e, f\}$. Then by symmetry $L(x_1)$ contains either c or e. If $c \in L(x_1)$, color x_1, x_3 with c, x_4 with a and y_3 with b. Now we can greedily finish. If $e \in L(x_1)$, color x_1, x_4 with e, x_3 with a and y_3 with b, again we can greedily finish. Hence $|Pot(L)| \le 5$.

Claim 3. *L does not exist.* Since $|Pot(L)| \le 5$ we see that x_3, x_5 have two colors in common and x_2, x_4 have two colors in common as well. In fact, these sets of common colors must be the same and equal $L(y_3) := \{a, b\}$ or we can finish the coloring. Similarly, we may assume that $a \in L(y_4)$ (if $\{a, b\} \cap L(y_4) = \emptyset$, then we have $L(x_2) \cap L(y_3) \cap (Pot(L) \setminus \{a, b\}) \neq \emptyset$ and color a on x_3, x_5, so we can color y_3 with b and then finish by Lemma 6.12). Similarly, $L(x_1)$ contains a or b. But it can't contain a for then we could color y_3, y_4, x_1 with a, and x_2, x_4 with b, and then finish greedily. Say $L(x_4) = \{a, b, c, d\}$. Then as no nonadjacent pair has a color in common that is in $Pot(L) - \{a, b\}$ we have $L(x_2) = \{a, b, e\}$, then by symmetry of c and d we have $L(x_5) = \{a, b, c\}$. Then $L(x_3) = \{a, b, d, e\}$ and hence $L(x_1) = \{a, b\}$, which contradicts that $a \notin L(x_1)$. We conclude that L cannot exist. □

Lemma 4.4. *Let H be a thickening of C_5 such that $|H| \geq 6$. Then $K_1 * H$ is f-choosable where $f(v) \geq d(v)$ for the v in the K_1 and $f(x) \geq d(x) - 1$ for $x \in V(H)$.*

Proof. Suppose not and let L be a minimal bad f-assignment on $K_1 * H$. By the Small Pot Lemma, $|Pot(L)| \leq |H|$. Note that H is d_0-choosable since it contains an induced diamond. Let $x_1, \ldots, x_5$ be the vertices of an induced C_5 in H. Then $\sum_i |L(x_i)| = \sum_i d_H(x_i) = 3\,|H| - 5 > 2\,|H| \geq \omega(H[x_1, \ldots, x_5])\,|Pot(L)|$ and hence some nonadjacent pair in $\{x_1, \ldots, x_5\}$ have a color in common. Now applying Lemma 6.4 gives a contradiction. □

We are now in a position to finish the proof of Borodin-Kostochka for claw-free graphs.

Theorem 4.5. *Every claw-free graph satisfying $\chi \geq \Delta \geq 9$ contains a K_Δ.*

Proof. Suppose not and choose a counterexample G minimizing $|G|$. Then G is vertex critical and not quasi-line by Lemma 3.5. Hence G contains a vertex v that is not bisimplicial. By Lemma 4.2, $G_v := G[N(v)]$ is a thickening of a C_5. Also, by Lemma 4.4, v is high. Pick a C_5 in G_v and label its vertices $x_1, \ldots, x_5$ in clockwise order. For $i \in [5]$, let T_i be the thickening clique containing x_i. Also, let S be those vertices in $V(G) - N(v) - \{v\}$ that have a neighbor in $\{x_1, \ldots, x_5\}$. First we establish a few properties of vertices in S.

Claim 1. *For $z \in S$ we have $N(z) \cap \{x_1, \ldots, x_5\} \in \{\{x_i, x_{i+1}\}, \{x_i, x_{i+1}, x_{i+2}\}\}$ for some $i \in [5]$.* Let $z \in S$ and put $N := N(z) \cap \{x_1, \ldots, x_5\}$. If $|N| \geq 4$, then some subset of $\{v, z\} \cup N$ induces the d_1-choosable graph $E_2 * P_4$. Hence $|N| \leq 3$. Since G is claw-free, the vertices in N must be contiguous.

Claim 2. *If $z \in S$ is adjacent to x_i, x_{i+1}, x_{i+2}, then $|T_i| = |T_{i+1}| = |T_{i+2}| = 1$.* Suppose not. First, lets deal with the case when $|T_{i+1}| \geq 2$. Pick $y \in T_{i+1} - x_{i+1}$. If $y \not\leftrightarrow z$, then $\{x_i, y, z, x_{i-1}\}$ induces a claw, impossible. Thus $y \leftrightarrow z$ and $\{v, z, x_i, x_{i+1}, x_{i+2}, y\}$ induces the d_1-choosable graph $E_2 * \text{diamond}$.

Hence, by symmetry, we may assume that $|T_i| \geq 2$. If $y \not\leftrightarrow z$, then $\{v, x_1, \ldots, x_5, y, z\}$ induces a D_8 contradicting Lemma 4.3. Hence $y \leftrightarrow z$ and $\{v, z, x_i, x_{i+1}, x_{i+2}, y\}$ induces the d_1-choosable graph $E_2 * \text{paw}$, a contradiction.

Claim 3. *For $i \in [5]$, let B_i be the $z \in S$ with $N(z) \cap \{x_1, \ldots, x_5\} = \{x_i, x_{i+1}\}$. Then $B_i \cup B_{i+1}$ and $B_i \cup T_i \cup T_{i+1}$ both induce cliques for any $i \in [5]$.* Otherwise there would be a claw.

Claim 4. *$|T_i| \leq 2$ for all $i \in [5]$.* Suppose otherwise that we have i such that $|T_i| \geq 3$. Put $A_i := N(x_i) \cap S$. By Claim 2, $A_i \subseteq B_{i-1} \cup B_i$ and A_i is joined to T_i. Thus T_i is joined to $F_i := \{v\} \cup A_i \cup T_{i-1} \cup T_{i+1}$. If $A_i \neq \emptyset$, then F_i induces a graph that is connected and not almost complete, so this is impossible by Lemma 6.6. If $A_i = \emptyset$, then x_i must have at least $\Delta - 2$ neighbors in $T_{i-1} \cup T_i \cup T_{i+1}$. But that leaves at most one vertex for $T_{i-2} \cup T_{i+2}$, impossible.

Claim 5. *G does not exist.* Since $d(v) = \Delta \geq 9$, by symmetry we may assume that $|T_i| = 2$ for all $i \in [4]$. As in the proof of Claim 4, we get that T_2 is joined to F_2. Since $|T_i| \leq 2$ for all i, we must have $A_i \neq \emptyset$ (for all i, but in particular for A_2). Since $A_i \subseteq B_{i-1} \cup B_i$, by symmetry, we may assume that $A_2 \cap B_2 \neq \emptyset$. Pick $z \in A_2 \cap B_2$ and $y_i \in T_i - x_i$ for $i \in [3]$. Then F_2 has the graph in Figure 4 as an induced subgraph, but this is impossible by Lemma 6.7. □

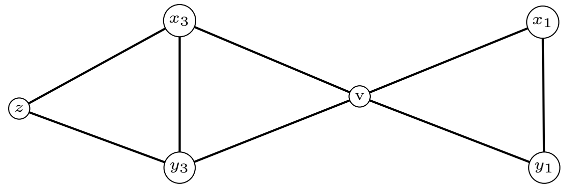


Figure 4: K_2 joined to this graph is d_1-choosable

We note that this reduction to the quasi-line case also works for the Borodin-Kostochka conjecture for list coloring; that is, we have the following result.

Theorem 4.6. *If every quasi-line graph satisfying $\chi_l \geq \Delta \geq 9$ contains a K_Δ, then the same statement holds for every claw-free graph.*

5 Line graphs

In [15], the second author proved the Borodin-Kostochka conjecture for line graphs of multigraphs.

Theorem 5.1 (Rabern [15])**.** *Every line graph of a multigraph satisfying $\chi \geq \Delta \geq 9$ contains a K_Δ.*

Some of the techniques used in the proof of this theorem carry over to the Borodin-Kostochka conjecture for list coloring; unfortunately, a key part of the proof used the fan equation and we don't have that for list coloring. Our aim in this section is to lay out what we can prove about the list version of the Borodin-Kostochka conjecture for line graphs (of multigraphs).

Lemma 5.2. *Fix $t \geq 2$ and $j \in \{0, 1\}$. Let B be the complement of a bipartite graph with $\omega(B) < |B| - j$. Let L be a list assignment on $G := K_t * B$ with $|L(v)| \geq d(v) - j$ for each $v \in V(K_t)$ and $|L(v)| \geq d(v) - 1$ for each $v \in V(B)$. If G is not L-colorable, then:*

- *$t = 3$ and B is the disjoint union of two complete subgraphs; or,*
- *$t = 2$ and B is the disjoint union of two complete subgraphs; or,*
- *$t = 2$ and B is formed by adding an edge between two disjoint complete subgraphs; or,*
- *$t = 2$, B has a dominating vertex v and $B - v$ is the disjoint union of two complete subgraphs.*

Proof. If $t \geq 4$, then by Lemma 6.5, B is almost complete and hence $j = 0$. But then Lemma 6.10 gives a contradiction. Hence $t \leq 3$.

Suppose $t = 3$. By Lemma 6.6, B is either almost complete or $K_r + K_{|B|-r}$. Suppose B is almost complete. Then $j = 0$. Let $z \in V(B)$ be the vertex outside of the $|B| - 1$ clique and $x \in V(B)$ some nonneighbor of x. Then $|L(x)| + |L(z)| \geq d(x) + d(z) - 2 = d_B(z) + |B| + 2$. By the Small Pot Lemma (see Section 6), $|Pot(L)| \leq |B| + 2$. Hence if $d_B(z) > 0$, we could color x and z the same and then greedily complete the coloring to the rest of G, impossible. So, B is $K_1 + K_{|B|-1}$.

Now suppose $t = 2$. If B has no dominating vertex, then by Lemma 6.7, B is the disjoint union of two complete subgraphs or B is formed by adding an edge between two disjoint complete subgraphs. Otherwise B has a dominating vertex v and hence $B = K_3 * B - v$. Similarly to the $t = 3$ case, we conclude that $B - v$ is the disjoint union of two complete subgraphs. □

Lemma 5.3. *Let H be a multigraph and let G be the line graph of H such that $\omega(G) < \chi_l(G) = \Delta(G)$. Suppose we have a bad $(\Delta(G)-1)$-assignment L on G, and that G is vertex critical with respect to L. Then $\mu(H) \leq 3$ and no multiplicity 3 edge is in a triangle. Let $xy \in E(G)$ have $\mu(xy) = 2$. Then xy is contained in at most one triangle. Moreover, this triangle is either 4-sided or 5-sided. If the triangle is 5-sided, then one of x or y has all its neighbors in the triangle and in particular has degree at most 4 in H.*

Proof. Put $\Delta := \Delta(G)$. Let $xy \in E(H)$ be an edge in H. Let A be the set of all edges incident with both x and y. Let B be the set of edges incident with either x or y but not both. Then, in G, A is a clique joined to B and B is the complement of a bipartite graph. Put $F := G[A \cup B]$. Since xy is L-critical, we can color $G - F$ from L. Doing so leaves a list assignment J on F with $|J(v)| = \Delta - 1 - (d_G(v) - d_F(v)) = d_F(v) - 1 + \Delta - d_G(v)$ for each $v \in V(F)$. Put $j := d_G(xy)+1-\Delta$. Since $d_G(xy)+1 = |A|+|B|$ and $\Delta > \omega(G) \geq \omega(A)+\omega(B) = |A|+\omega(B)$, we have $\omega(B) < |B| - j$.

Therefore we may apply Lemma 5.2. We conclude $\mu(xy) \leq 3$. Also, if B is a disjoint union of two cliques in G, then xy is in no triangle. Now suppose $\mu(xy) = 2$. If B has no dominating vertex in G, then xy is in exactly one triangle and it is 4-sided. Otherwise, by symmetry we may assume that B has a dominating vertex xz. Then y must have all its edges to x and z and y must have at least one edge to z for otherwise we would have a Δ clique in G. Since $B - xz$ is the disjoint union of two cliques, we must have $\mu(xz) = 1$. Also $\mu(yz) \leq 2$ and hence $d_H(y) \leq 4$. □

Lemma 5.4. *Let H be a multigraph and let G be the line graph of H such that $\omega(G) < \chi_l(G) = \Delta(G)$. Suppose we have a bad $(\Delta(G)-1)$-assignment J on G, and that G is vertex critical with respect to J. Then H cannot have triple edges uv and vw, such that $d(u) \geq 6$, $d(w) \geq 6$, and $d(v) \geq 7$ (or $d(v) \geq 6$ and every edge incident to v in H is low in G).*

Proof. Assume the contrary and let H be a counterexample. Recall from Lemma 2.2 above that the maximum edge multiplicity of H is at most 3.

Let a_1, a_2, a_3 be three edges incident to u but not v; let b_1, b_2, b_3 be the edges incident to u and v; let c be incident to v but not u or w; let d_1, d_2, d_3 be incident to v and w; let e_1, e_2, e_3 be incident to w (but not u or v). We use these names for both the edges of H and the vertices of G, interchangeably.

By criticality of G, we can color $V(G) \setminus \{a_1, a_2, a_3, b_1, b_2, b_3, c, d_1, d_2, d_3, e_1, e_2, e_3\}$ from J. Let L denote the list of remaining colors on the uncolored vertices. Note that $|L(a_i)| \geq 4$, $|L(e_i)| \geq 4$, $|L(c)| \geq 5$, $|L(b_i)| \geq 8$, and $|L(d_i)| \geq 8$. We may assume that equality holds in each case.

Claim 1. *If there exist $\alpha \in L(a_1) \cap L(c)$, then we can color G from its lists.* Suppose such an α exists. We use α on a_1 and c. This saves a color on each of b_1, b_2, b_3. Now

$|L(d_1) \setminus \{\alpha\}| + |L(a_2) \setminus \{\alpha\}| \geq 7 + 3 > 8 = |L(b_1)|$, so we can color d_1 and a_2 to save an additional color on b_1. Now we greedily color e_1, e_2, e_3, d_2, d_3, c, a_2, a_3, b_3, b_2, b_1.

Claim 2. *If there exists $\alpha \in L(a_1) \cap L(d_1)$, then we can color G from its lists.* Suppose such an α exists. If $\alpha \in L(c)$, then we proceed as above. Otherwise we use α on a_1 and d_1. This saves a color on b_1, b_2, and b_3. We may assume that $\alpha \in L(b_1)$, since otherwise we can color greedily toward b_1. Now we get $|L(a_2) \setminus \{\alpha\}| + |L(c)| \geq 3 + 5 > 7 = |L(b_1) \setminus \{\alpha\}|$. Thus, we can color a_2 and c to save a color on b_1. Afterwards we color greedily toward b_1.

Claim 3. *We may assume that $L(b_1) = L(b_2) = L(b_3) = L(d_1) = L(d_2) = L(d_3)$; otherwise we can color G from its lists.* Suppose to the contrary that (by symmetry) there exists $\alpha \in L(d_1) \setminus L(b_1)$. If we also have $\alpha \notin L(b_2)$, then we may use α on d_1 (color a_1 arbitrarily) and proceed as in Claim 2. So now we have $\alpha \in L(b_2)$. By Claim 2 and symmetry, we have $\alpha \notin \cup L(e_i)$. Thus we use α on d_1 (without reducing the $L(e_i)$). Since we have $|L(c) \setminus \{\alpha\}| + |L(a_1)| > |L(b_2) \setminus \{\alpha\}|$, we can color c and a_1 to save a color on b_2. Now we color d_2 and a_2 to save a second color on b_1. Finally, we color greedily toward b_1.

Claim 4. *We can color G.* By Claim 2, we know that $L(a_1) \cap L(d_1) = \emptyset$. By Claim 3, we know that $L(b_1) = L(d_1)$; thus, $L(a_1) \cap L(b_1) = \emptyset$. By symmetry, we get $L(a_i) \cap L(b_j) = \emptyset$ for all $i, j \in \{1, 2, 3\}$. Now we can color the a_i arbitrarily, which saves 3 colors on each of the b_i. Finally, we color greedily towards b_1. This proves the lemma. □

As an application of the lemma above, we show that if H has minimum degree at least 7 and G is the line graph of H, then the list version of the Borodin-Kostochka Conjecture holds for G. We need the following theorem, due to Borodin, Kostochka, and Woodall.

Theorem 5.5 (Borodin, Kostochka, Woodall [2])**.** *Let G be a bipartite multigraph with partite sets X, Y, so that $V = X \cup Y$. G is edge-f-choosable, where $f(e) := \max\{d(x), d(y)\}$ for each edge $e = xy$.*

Theorem 5.6. *Let H be a multigraph with $\delta(H) \geq 7$ and let G be the line graph of H. Then $\chi_l(G) \leq \max\{\omega(G), \Delta(G) - 1\}$.*

Proof. Suppose the contrary, and let G_0 (and H_0) be a counterexample with list assignment L. Let G (and H) be a vertex critical subgraph with respect to L. It suffices to color G from L. Note that $\Delta(G) = \Delta(G_0)$, since otherwise we can color G from L by the list version of Brooks' Theorem. Since G is L-critical, we have $\delta(G) \geq \Delta(G) - 1$. Thus, we have $d_H(u) \geq d_{H_0}(u) - 1$ for all $u \in V(H)$ so $\delta(H) \geq 6$. In particular, if uv is high in G, then $d_H(u) = d_{H_0}(u)$ and $d_H(v) = d_{H_0}(v)$. Note that if $\mu_H(xy) = 3$, then in H_0 each of x and y is incident to only one triple edge (by Lemma 5.4, since G is critical with respect to L).

Claim. *If xy is an edge of H with multiplicity 3 and $d(x) = 7$, then vertex xy is low in G.* Suppose the contrary. Since G is χ_l-critical, for each edge $xy \in H$, we have $d_H(x) + d_H(y) = \Delta(G) + \mu(xy) + 1$ if xy is high and $d_H(x) + d_H(y) = \Delta(G) + \mu(xy)$ if xy is low. Suppose that $\mu(xy) = 3$, $d(x) = 7$, and xy is high. Now we get $d_{H_0}(y) = \Delta(G) - 3$. By the last sentence of the previous paragraph, we know that every edge incident to y other than xy has multiplicity at most 2. Let z be a neighbor of y other than x. By the degree condition above, we get $d_{H_0}(y) + d_{H_0}(z) \leq \Delta(G) + \mu_{H_0}(xy) + 1 \leq \Delta(G) + 3$. This implies that $d_{H_0}(z) \leq 6$, which is a contradiction. This proves the claim. More simply, for any vertex x with $d_H(x) = 6$, we see that every edge xy incident to x must be low in G.

Now for each triple edge of H that is high in G, we delete one of the edges; call the resulting graph G' (and H'). Clearly, we have $\delta(H') \geq 6$. By the previous Lemma and the claim, $d_{H'}(x) \geq 7$ for every vertex x incident to a triple edge or an edge xy that corresponds to a high vertex in G. For if xy is a triple edge and $d(x) = 7$, then edge xy is low in G. Similarly, if $d(x) = 6$, then every edge xy is low in G. Otherwise, each vertex of H that is incident to a triple edge has degree at least 8 and is incident to exactly one triple edge.

Let B be a maximum bipartite subgraph of H'. For each vertex $x \in H$, we have $d_B(x) \geq d_{H'}(x)/2$ (otherwise B has more edges if we move x to the other part); thus $\delta(B) \geq 3$ and $d_B(x) \geq 4$ for each vertex incident to a triple edge or an edge xy that is high in G. Let G_B denote the line graph of B. Since G is critical with respect to L, we can color $G - V(G_B)$ from L. So to show that G is $(\Delta-1)$-choosable, it suffices to show that we can color G_B when each vertex v that is high in G gets a list of size $d_{G_B}(v) - 1$ and each vertex v that is low in G gets a list of size $d_{G_B}(v)$. Consider a high vertex xy in G; recall that $d_{H'}(x) \geq 7$ and $d_{H'}(y) \geq 7$, so $d_B(x) \geq 4$ and $d_B(y) \geq 4$. The degree of xy in G_B is $d_B(x) + d_B(y) - \mu_B(xy) - 1$. Since $\mu_B(xy) \leq 2$, we see that xy has a list of size at least $d_B(x) + d_B(y) - 4 \geq \max\{d_B(x), d_B(y)\}$. Each low vertex xy has a list of size at least $d_B(x) + d_B(y) - \mu(xy) - 1 \geq \max\{d_B(x), d_B(y)\}$. So by Theorem 5.5, we can color G_B from its remaining lists. □

Theorem 2 is best possible in the sense that if we replace "$\delta(H) \geq 7$" by "$\delta(H) \geq 6$", then the theorem is false. One counterexample is when H is a 5-cycle in which each edge has multiplicity 3, shown in Figure 1 (d).

6 List coloring lemmas

Let G be a graph. A *list assignment* to the vertices of G is a function from $V(G)$ to the finite subsets of $\mathbb{N}$. A list assignment L to G is *good* if G has a coloring c where $c(v) \in L(v)$ for each $v \in V(G)$. It is *bad* otherwise. We call the collection of all colors that appear in L, the *pot* of L. That is $Pot(L) := \bigcup_{v \in V(G)} L(v)$. For a subset A of $V(G)$ we write $Pot_A(L) := \bigcup_{v \in A} L(v)$. Also, for a subgraph H of G, put $Pot_H(L) := Pot_{V(H)}(L)$. For $S \subseteq Pot(L)$, let G_S be the graph $G[\{v \in V(G) \mid L(v) \cap S \neq \emptyset\}]$. We also write G_c for $G_{\{c\}}$. For $f : V(G) \to \mathbb{N}$, an f-assignment on G is an assignment L of lists to the vertices of G such that $|L(v)| = f(v)$ for each $v \in V(G)$. We say that G is *f-choosable* if every f-assignment on G is good.

In [7], we proved that many graphs can be excluded as induced subgraphs of a vertex critical graph with $\chi = \Delta$. The idea is simple. Let F be an induced subgraph of a vertex critical graph G with $\chi(G) = \Delta(G)$. Then we can $(\Delta(G) - 1)$-color $G - F$. Doing so leaves a list assignment on F where each $v \in V(F)$ has a list of size at least $d_F(v) - 1$. We call a graph that is f-choosable where $f(v) := d(v) - 1$ a d_1-choosable graph. From the above, d_1-choosable graphs cannot be induced subgraphs of vertex critical graphs with $\chi = \Delta$. We restate some of the results on d_1-choosable graphs from [7] that we need here. Additionally, we prove further results on d_1-choosability.

We need the following lemmas from [7]. Given a graph G and $f : V(G) \to \mathbb{N}$, we have a partial order on the f-assignments to G given by $L < L'$ iff $|Pot(L)| < |Pot(L')|$. When we talk of *minimal* f-assignments, we mean minimal with respect to this partial order.

Small Pot Lemma. *Let G be a graph and $f\colon V(G) \to \mathbb{N}$ with $f(v) < |G|$ for all $v \in V(G)$. If G is not f-choosable, then G has a bad f-assignment L such that $|Pot(L)| < |G|$.*

The core of the Small Pot Lemma is the following. We will prove a lemma that gets more when $|S| = 1$.

Lemma 6.1. *Let G be a graph and $f\colon V(G) \to \mathbb{N}$. Suppose G is not f-choosable and let L be a minimal bad f-assignment. Assume $L(v) \neq Pot(L)$ for each $v \in V(G)$. Then, for each nonempty $S \subseteq Pot(L)$, any coloring of G_S from L uses some color not in S.*

When $|S| = 1$, we can say more. We will use the following lemma in the proof that the graph D_8 in Figure 3 is d_1-choosable. It should be useful elsewhere as well.

Lemma 6.2. *Let G be a graph and $f\colon V(G) \to \mathbb{N}$. Suppose G is not f-choosable and let L be a minimal bad f-assignment. Then for any $c \in Pot(L)$, there is a component H of G_c such that $Pot_H(L) = Pot(L)$. In particular, $Pot_{G_c}(L) = Pot(L)$.*

Proof. Suppose otherwise that we have $c \in Pot(L)$ such that $Pot_H(L) \subsetneq Pot(L)$ for all components H of G_c. Say the components of G_c are $H_1, \dots, H_t$. For $i \in [t]$, choose $\alpha_i \in Pot(L) - Pot_{H_i}(L)$. Now define a list assignment L' on G by setting $L'(v) := L(v)$ for all $v \in V(G) - V(G_c)$ and for each $i \in [t]$ setting $L'(v) := (L(v) - c) \cup \{\alpha_i\}$ for each $v \in V(H_i)$. Then $|Pot(L')| < |Pot(L)|$ and hence by minimality L we have an L'-coloring π of G. Plainly $Q := \{v \in V(G_c) \mid \pi(v) = \alpha_i \text{ for some } i \in [t]\}$ is an independent set. Since c doesn't appear outside G_c, we can recolor all vertices in Q with c to get an L-coloring of G. This contradicts the fact that L is bad. $\square$

The next two lemmas allow us to color pairs in H without worrying about completing the coloring to H.

Lemma 6.3. *Let H be a d_0-choosable graph such that $G := K_1 * H$ is not d_1-choosable and L a minimal bad d_1-assignment on G. If some nonadjacent pair in H have intersecting lists, then $|Pot(L)| \leq |H| - 1$.*

With the same proof, we have the following.

Lemma 6.4. *Let H be a d_0-choosable graph such that $G := K_1 * H$ is not f-choosable where $f(v) \geq d(v)$ for the v in the K_1 and $f(v) \geq d(x) - 1$ for $x \in V(H)$. If L is a minimal bad f-assignment on G, then all nonadjacent pairs in H have disjoint lists.*

Lemma 6.5. *Let A be a connected graph with $|A| \geq 4$ and B an arbitrary graph. If $A * B$ is not d_1-choosable, then B is $E_3 * K_{|B|-3}$ or almost complete.*

Lemma 6.6. *$K_3 * B$ is not d_1-choosable iff B is almost complete, $K_t + K_{|B|-t}$, $K_1 + K_t + K_{|B|-t-1}$, $E_3 + K_{|B|-3}$, or $|B| \leq 5$ and $B = E_3 * K_{|B|-3}$.*

Lemma 6.7. *If $K_2 * B$ is not d_1-choosable, then B consists of a disjoint union of complete subgraphs, together with at most one incomplete component H. If H has a dominating vertex v, then $K_2 * H = K_3 * (H - v)$, so by Lemma 6.6 we can completely describe H. Otherwise H is formed either by adding an edge between two disjoint cliques or by adding a single pendant edge incident to each of two distinct vertices of a clique. Furthermore, all graphs formed in this way are not d_1-choosable.*

Pulling out some particular cases makes for easier application. A *chair* is formed from $K_{1,3}$ by subdividing an edge. An *antichair* is the complement of a chair.

Lemma 6.8. *$K_2 * $ antichair is d_1-choosable.*

Lemma 6.9. *$K_3 * P_4$ is d_1-choosable.*

We often need to handle low vertices in our proofs, this corresponds to a vertex with $|L(v)| \geq d(v)$ when we try to complete the partial coloring.

Lemma 6.10. *Let A be a graph with $|A| \geq 4$. Let L be a list assignment on $G := E_2 * A$ such that $|L(v)| \geq d(v) - 1$ for all $v \in V(G)$ and each component D of A has a vertex v such that $|L(v)| \geq d(v)$. Then L is good on G.*

Lemma 6.11. *Let A be a graph with $|A| \geq 3$. Let L be a list assignment on $G := E_2 * A$ such that $|L(v)| \geq d(v) - 1$ for all $v \in V(G)$, $|L(v)| \geq d(v)$ for some v in the E_2 and each component D of A has a vertex v such that $|L(v)| \geq d(v)$. Then L is good on G.*

Lemma 6.12. *Let A and B be graphs such that $G := A * B$ is not d_1-choosable. If either $|A| \geq 2$ or B is d_0-choosable and L is a bad d_1-assignment on G, then*

1. *for any independent set $I \subseteq V(B)$ with $|I| = 3$, we have $\bigcap_{v \in I} L(v) = \emptyset$; and*
2. *for disjoint nonadjacent pairs $\{x_1, y_1\}$ and $\{x_2, y_2\}$ at least one of the following holds*
   - *(a) $L(x_1) \cap L(y_1) = \emptyset$;*
   - *(b) $L(x_2) \cap L(y_2) = \emptyset$;*
   - *(c) $|L(x_1) \cap L(y_1)| = 1$ and $L(x_1) \cap L(y_1) = L(x_2) \cap L(y_2)$.*

Let E_2^n denote the join of n copies of E_2, i.e., E_2^n is isomorphic to $K_{2n} - E(M)$, where M is a perfect matching. The following lemma first appeared in [9]. We also prove it in [7].

Lemma 6.13. *E_2^n is n-choosable.*

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