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Further improvements

Improving Brooks' theorem

Landon Rabern

Arizona State University

October 28, 2011

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- 2 Some background
- 3 The Ore-degree
- 4 Rephrasing the problem
- 5 Solving the rephrased problem

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A prison problem

A prison problem

You are a warden in a prison with five large cells. You need to put all the inmates into the cells, but to prevent fighting you cannot put a pair of inmates that have fought before into the same cell. Each inmate in the prison has fought with at most six other inmates and none of the inmates who have fought with six others have fought with each other. Under what conditions can you complete your task?

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- plainly, if there is a group of six inmates who have all fought one another, then you cannot complete your task
- is this simple necessary condition sufficient?

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Greedy coloring

• $C := \{c_1, c_2, c_3, \ldots\}$ an infinite set of colors

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- go through the vertices in order, coloring v_i with the first color not used on a neighbor of v_i

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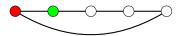
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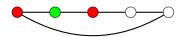
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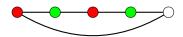
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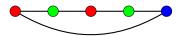
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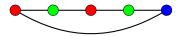
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Further improvements

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- G has vertices ordered v_1, v_2, \ldots, v_n
- go through the vertices in order, coloring v_i with the first color not used on a neighbor of v_i

For example, say $C := \{ red, green, blue, cyan, \ldots \}$ and G is the 5-cycle:



• if G has maximum degree k, then v_i has at most k colored neighbors, so greedy coloring uses at most k+1 colors

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Brooks' theorem

 χ(G) := the minimum number of colors needed to color the vertices of G so that adjacent vertices receive different colors

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Brooks' theorem

- χ(G) := the minimum number of colors needed to color the vertices of G so that adjacent vertices receive different colors
- $\omega(G)$:= the number of vertices in a largest complete subgraph of G

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- ω(G) := the number of vertices in a largest complete subgraph of G
- $\Delta(G) :=$ the maximum degree of G

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Brooks' theorem

- χ(G) := the minimum number of colors needed to color the vertices of G so that adjacent vertices receive different colors
- ω(G) := the number of vertices in a largest complete subgraph of G
- $\Delta(G)$:= the maximum degree of G

Theorem (Brooks 1941)

Every graph with $\Delta \geq 3$ satisfies $\chi \leq \max\{\omega, \Delta\}$.

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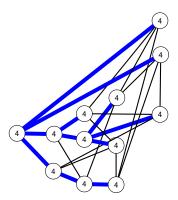
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Proof sketch

Any incomplete 2-connected graph with $\Delta \geq 3$ has a spanning tree where the root has two nonadjacent leaves as neighbors.



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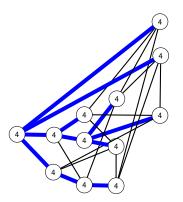
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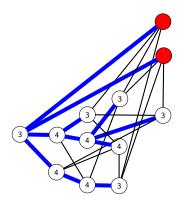
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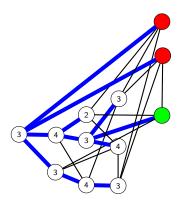
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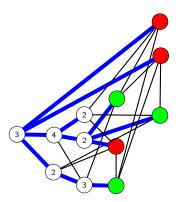
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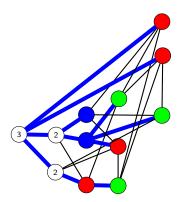
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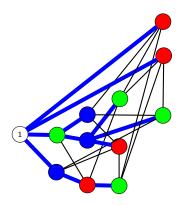
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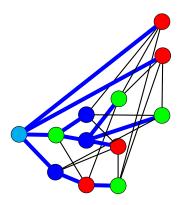
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The Ore-degree

Definition

The Ore-degree of an edge xy in a graph G is

$$\theta(xy) := d(x) + d(y).$$

The *Ore-degree* of a graph *G* is

$$\theta(G) := \max_{xy \in E(G)} \theta(xy).$$

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• every graph satisfies $\left|\frac{\theta}{2}\right| \leq \Delta$

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- every graph satisfies $\left|\frac{\theta}{2}\right| \leq \Delta$
- greedy coloring (in any order) shows that every graph satisfies $\chi \leq \left|\frac{\theta}{2}\right| + 1$

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Kierstead and Kostochka's generalization

Theorem (Kierstead and Kostochka 2009)

Every graph with $\theta \geq 12$ satisfies $\chi \leq \max \left\{ \omega, \left\lfloor \frac{\theta}{2} \right\rfloor \right\}$.

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Kierstead and Kostochka's generalization

Theorem (Kierstead and Kostochka 2009)

Every graph with $\theta \ge 12$ satisfies $\chi \le \max \{\omega, \lfloor \frac{\theta}{2} \rfloor \}$.

Kierstead and Kostochka conjectured that the 12 could be reduced to 10. That this would be best possible can be seen from the following example which has $\theta=9$, $\omega=4$ and $\chi=5$.

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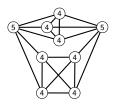


Figure: O_5 , a counterexample with $\theta = 9$.

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Definition

A graph G is called *vertex critical* if $\chi(G - v) < \chi(G)$ for each $v \in V(G)$.

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Definition

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Definition

Let G be a vertex critical graph. The *low vertex subgraph* $\mathcal{L}(G)$ is the graph induced on the vertices of degree $\chi(G)-1$. The *high vertex subgraph* $\mathcal{H}(G)$ is the graph induced on the vertices of degree at least $\chi(G)$.

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Problem

Prove that $K_{\Delta(G)+1}$ is the only vertex critical graph G with $\chi(G) \geq \Delta(G) \geq 6$ such that $\mathcal{H}(G)$ is edgeless.

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Kierstead and Kostochka's proof

 the proof is high-tech and clean, it uses both of the following

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- using these it is basically just a counting argument
- unfortunately, it only works for $\Delta \geq 7$

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To get down to $\Delta=6$, go low-tech and get dirty.

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To get down to $\Delta = 6$, go low-tech and get dirty.

Theorem (Rabern 2010)

 $K_{\Delta(G)+1}$ is the only vertex critical graph G with

$$\chi(G) \ge \Delta(G) \ge 6$$
 and $\omega(\mathcal{H}(G)) \le \left\lfloor \frac{\Delta(G)}{2} \right\rfloor - 2$.

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• setting $\omega(\mathcal{H}(G))=1$ proves Kierstead and Kostochka's conjecture

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- setting $\omega(\mathcal{H}(G))=1$ proves Kierstead and Kostochka's conjecture
- equivalently, as long as there is no group of six inmates who have all fought one another, you (the warden) can complete your inmate-cell-assignment task

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- start with a minimal counterexample G
- for any induced subgraph H, $\Delta-1$ coloring G-H leaves a list assignment L on H where $|L(v)| \geq \deg(v)-1$

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Goal

Construct a subgraph H for which such a list assignment can always be completed.

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Goal

Construct a subgraph H for which such a list assignment can always be completed.

 we need H to have large degrees to get large lists, so H will be "dense"

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- first, use minimality of G to exclude some troublesome H's

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Construct a subgraph H for which such a list assignment can always be completed.

- we need H to have large degrees to get large lists, so H will be "dense"
- first, use minimality of G to exclude some troublesome H's
- run the following recoloring algorithm to construct H

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Partitioned colorings

Definition

Let G be a vertex critical graph. Let $a \ge 1$ and r_1, \ldots, r_a be such that $1 + \sum_i r_i = \chi(G)$. By a (r_1, \ldots, r_a) -partitioned coloring of G we mean a proper coloring of G of the form

$$\{\{x\}, L_{11}, L_{12}, \dots, L_{1r_1}, L_{21}, L_{22}, \dots, L_{2r_2}, \dots, L_{a1}, L_{a2}, \dots, L_{ar_a}\}.$$

Here $\{x\}$ is a singleton color class and each L_{ij} is a color class.

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Lemma (Mozhan 1983)

Let G be a vertex critical graph. Let $a \ge 1$ and r_1, \ldots, r_a be such that $1 + \sum_i r_i = \chi(G)$. Of all (r_1, \ldots, r_a) -partitioned colorings of G pick one minimizing

$$\sum_{i=1}^{a} \left\| G \left[\bigcup_{j=1}^{r_i} L_{ij} \right] \right\|.$$

Remember that $\{x\}$ is a singleton color class in the coloring. Put $U_i := \bigcup_{j=1}^{r_i} L_{ij}$ and let $Z_i(x)$ be the component of x in $G[\{x\} \cup U_i]$. If $d_{Z_i(x)}(x) = r_i$, then $Z_i(x)$ is complete if $r_i \ge 3$ and $Z_i(x)$ is an odd cycle if $r_i = 2$.

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The recoloring algorithm

• take a $(\lfloor \frac{\Delta-1}{2} \rfloor, \lceil \frac{\Delta-1}{2} \rceil)$ -partitioned coloring minimizing the above function

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- take a $\left(\left\lfloor\frac{\Delta-1}{2}\right\rfloor, \left\lceil\frac{\Delta-1}{2}\right\rceil\right)$ -partitioned coloring minimizing the above function
- prove that we may assume that x is a low vertex

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Further improvement

- take a $\left(\left\lfloor\frac{\Delta-1}{2}\right\rfloor, \left\lceil\frac{\Delta-1}{2}\right\rceil\right)$ -partitioned coloring minimizing the above function
- prove that we may assume that x is a low vertex
- by Mozhan's lemma, the neighborhood of x in each part induces a clique or an odd cycle

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A prison problem

Some backgroun

Ore-degre

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Kierstead an Kostochka's proof

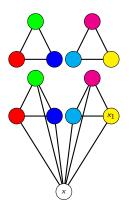
Problem solved

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swap x with a low vertex x₁ in the right part

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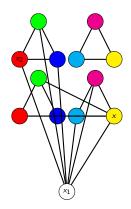
Kierstead an Kostochka's proof

Problem solved

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- swap x with a low vertex x_1 in the right part
- swap x_1 with a low vertex x_2 in the left part

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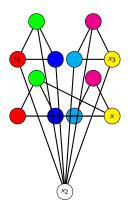
Kierstead and Kostochka's proof

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- swap x with a low vertex x_1 in the right part
- swap x_1 with a low vertex x_2 in the left part
- continue swapping back and forth until you wrap around

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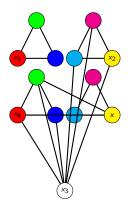
Kierstead an Kostochka's proof

Problem solved

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Further improvement



- swap x with a low vertex x_1 in the right part
- swap x_1 with a low vertex x_2 in the left part
- continue swapping back and forth until you wrap around

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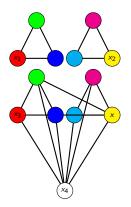
problem Kierstead and Kostochka's proof

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- swap x with a low vertex x_1 in the right part
- swap x_1 with a low vertex x_2 in the left part
- continue swapping back and forth until you wrap around

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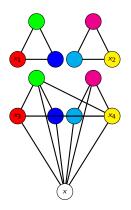
Kierstead and Kostochka's proof Problem solve

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- swap x with a low vertex x_1 in the right part
- swap x_1 with a low vertex x_2 in the left part
- continue swapping back and forth until you wrap around

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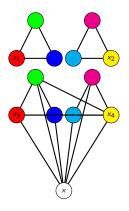
Problem solved

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 use the fact that you wrapped around to show that there are many edges between the two cliques

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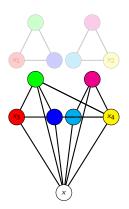
Kierstead and Kostochka's proof Problem solv

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- use the fact that you wrapped around to show that there are many edges between the two cliques
- we have now constructed the desired large "dense" subgraph

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Further

Generalizing maximum degree

Definition

For $0 \le \epsilon \le 1$, define $\Delta_{\epsilon}(G)$ as

$$\left[\max_{xy\in E(G)}(1-\epsilon)\min\{d(x),d(y)\}+\epsilon\max\{d(x),d(y)\}\right].$$

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Further improvements

Generalizing maximum degree

Definition

For $0 \leq \epsilon \leq 1$, define $\Delta_{\epsilon}(G)$ as

$$\left[\max_{xy\in E(G)}(1-\epsilon)\min\{d(x),d(y)\}+\epsilon\max\{d(x),d(y)\}\right].$$

Note that $\Delta_1 = \Delta$, $\Delta_{\frac{1}{2}} = \lfloor \frac{\theta}{2} \rfloor$.

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Theorem (Rabern 2010)

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Theorem (Rabern 2010)

For every $0 < \epsilon \le 1$, there exists t_{ϵ} such that every graph with $\Delta_{\epsilon} \ge t_{\epsilon}$ satisfies $\chi \le \max\{\omega, \Delta_{\epsilon}\}$.

• the proof uses a recoloring algorithm similar to the above

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Theorem (Rabern 2010)

- the proof uses a recoloring algorithm similar to the above
- it would be interesting to determine, for each ϵ , the smallest t_{ϵ} that works

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Theorem (Rabern 2010)

- the proof uses a recoloring algorithm similar to the above
- it would be interesting to determine, for each ε, the smallest t_ε that works
- that $t_1 = 3$ is smallest is Brooks' theorem

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- the proof uses a recoloring algorithm similar to the above
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- that $t_1 = 3$ is smallest is Brooks' theorem
- the graph O_5 shows that $t_\epsilon=6$ is smallest for $rac{1}{2} \leq \epsilon < 1$

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Theorem (Rabern 2010)

- the proof uses a recoloring algorithm similar to the above
- it would be interesting to determine, for each ϵ , the smallest t_{ϵ} that works
- that $t_1 = 3$ is smallest is Brooks' theorem
- the graph O_5 shows that $t_\epsilon=6$ is smallest for $\frac{1}{2} \leq \epsilon < 1$
- best known general bounds, $\frac{2}{\epsilon} + 1 \le t_{\epsilon} \le \frac{4}{\epsilon} + 2$

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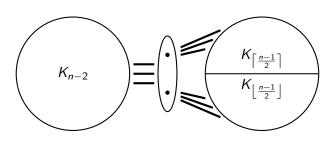


Figure: The graph O_n .

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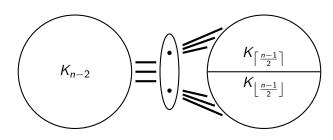


Figure: The graph O_n .

•
$$\chi(O_n)=n>\omega(O_n)$$
 and $\Delta(O_n)=\left\lceil \frac{n-1}{2} \right\rceil +n-2$

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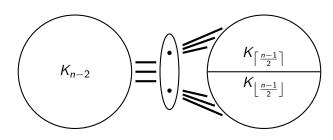


Figure: The graph O_n .

- $\chi(O_n) = n > \omega(O_n)$ and $\Delta(O_n) = \left\lceil \frac{n-1}{2} \right\rceil + n 2$
- $\mathcal{H}(O_n)$ is edgeless

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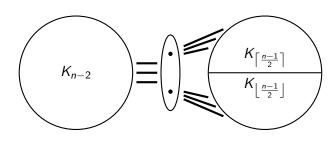


Figure: The graph O_n .

- $\chi(O_n) = n > \omega(O_n)$ and $\Delta(O_n) = \left\lceil \frac{n-1}{2} \right\rceil + n 2$
- $\mathcal{H}(O_n)$ is edgeless
- computing Δ_{ϵ} gives $t_{\epsilon} \geq rac{2}{\epsilon} + 1$

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What about Δ_0 ?

 \bullet the above proofs only work for $\epsilon>0$

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Further improvement

What about Δ_0 ?

- \bullet the above proofs only work for $\epsilon>0$
- what happens when $\epsilon=0$?

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What about Δ_0 ?

- ullet the above proofs only work for $\epsilon>0$
- what happens when $\epsilon = 0$?
- the parameter Δ_0 has already been investigated by Stacho under the name Δ_2

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What about Δ_0 ?

- ullet the above proofs only work for $\epsilon>0$
- what happens when $\epsilon = 0$?
- the parameter Δ_0 has already been investigated by Stacho under the name Δ_2

Definition (Stacho 2001)

For a graph G define

$$\Delta_0(G) := \max_{xy \in E(G)} \min\{d(x), d(y)\}.$$

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Further improvements

Facts about Δ_0

• greedy coloring (in any order) shows that every graph satisfies $\chi \leq \Delta_0 + 1$

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What about Δ_0 ?

Further improvement

Facts about Δ_0

- greedy coloring (in any order) shows that every graph satisfies $\chi \leq \Delta_0 + 1$
- for any fixed $t\geq 3$, the problem of determining whether or not $\chi(G)\leq \Delta_0(G)$ for graphs with $\Delta_0(G)=t$ is NP-complete (Stacho 2001)

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A tempting thought

A tempting thought

There exists t such that every graph with $\Delta_0 \geq t$ satisfies $\chi \leq \max\{\omega, \Delta_0\}$.

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A tempting thought

A tempting thought

There exists t such that every graph with $\Delta_0 \geq t$ satisfies $\chi \leq \max\{\omega, \Delta_0\}$.

• since $t_{\epsilon} \geq \frac{2}{\epsilon} + 1$, we see that $t_{\epsilon} \to \infty$ as $\epsilon \to 0$

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Further improvement

A tempting thought

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- since $t_{\epsilon} \geq \frac{2}{\epsilon} + 1$, we see that $t_{\epsilon} \to \infty$ as $\epsilon \to 0$
- thus, t₀ does not exist and the tempting thought cannot hold

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A tempting thought

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- there is a cute algorithmic way to see this assuming $P \neq NP$

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A tempting thought

A tempting thought

There exists t such that every graph with $\Delta_0 \geq t$ satisfies $\chi \leq \max\{\omega, \Delta_0\}$.

- since $t_{\epsilon} \geq \frac{2}{\epsilon} + 1$, we see that $t_{\epsilon} \to \infty$ as $\epsilon \to 0$
- ullet thus, t_0 does not exist and the tempting thought cannot hold
- there is a cute algorithmic way to see this assuming $P \neq NP$
- we use Lovász's ϑ parameter which can be appoximated in polynomial time and has the property that $\omega(\mathcal{G}) \leq \vartheta(\mathcal{G}) \leq \chi(\mathcal{G})$

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A polynomial-time algorithm

ullet assume the tempting thought holds for some $t\geq 3$

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Further improvement

- assume the tempting thought holds for some $t \ge 3$
- ullet take any arbitrary graph with $\Delta_0 \geq t$

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Further improvement

- assume the tempting thought holds for some $t \ge 3$
- ullet take any arbitrary graph with $\Delta_0 \geq t$
- first, compute Δ_0 in polynomial time

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Further improvements

- assume the tempting thought holds for some $t \ge 3$
- ullet take any arbitrary graph with $\Delta_0 \geq t$
- first, compute Δ_0 in polynomial time
- second, compute x such that $x \frac{1}{2} < \vartheta < x + \frac{1}{2}$ in polynomial time

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on t_{ϵ} What about Δ_0 ?

Further improvements

- assume the tempting thought holds for some $t \ge 3$
- ullet take any arbitrary graph with $\Delta_0 \geq t$
- first, compute Δ_0 in polynomial time
- second, compute x such that $x \frac{1}{2} < \vartheta < x + \frac{1}{2}$ in polynomial time
- if $x \ge \Delta_0 + \frac{1}{2}$, then $\chi \ge \vartheta > \Delta_0$ and hence $\chi = \Delta_0 + 1$

Further

A polynomial-time algorithm

- assume the tempting thought holds for some $t \ge 3$
- ullet take any arbitrary graph with $\Delta_0 \geq t$
- first, compute Δ_0 in polynomial time
- second, compute x such that $x \frac{1}{2} < \vartheta < x + \frac{1}{2}$ in polynomial time
- if $x \ge \Delta_0 + \frac{1}{2}$, then $\chi \ge \vartheta > \Delta_0$ and hence $\chi = \Delta_0 + 1$
- if $x < \Delta_0 + \frac{1}{2}$, then $\omega \le \vartheta < \Delta_0 + 1$, and hence $\omega \le \Delta_0$

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Further improvement

- assume the tempting thought holds for some $t \ge 3$
- ullet take any arbitrary graph with $\Delta_0 \geq t$
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- if $x \ge \Delta_0 + \frac{1}{2}$, then $\chi \ge \vartheta > \Delta_0$ and hence $\chi = \Delta_0 + 1$
- if $x < \Delta_0 + \frac{1}{2}$, then $\omega \le \vartheta < \Delta_0 + 1$, and hence $\omega \le \Delta_0$
- now, $\chi \leq \max\{\omega, \Delta_0\} \leq \Delta_0$

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Further improvements

- assume the tempting thought holds for some $t \ge 3$
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- now, $\chi \leq \max\{\omega, \Delta_0\} \leq \Delta_0$
- we just gave a polynomial time algorithm to determine whether or not $\chi \leq \Delta_0$ for graphs with $\Delta_0 \geq t$

Further improvement

A polynomial-time algorithm

- assume the tempting thought holds for some $t \ge 3$
- ullet take any arbitrary graph with $\Delta_0 \geq t$
- ullet first, compute Δ_0 in polynomial time
- second, compute x such that $x \frac{1}{2} < \vartheta < x + \frac{1}{2}$ in polynomial time
- if $x \ge \Delta_0 + \frac{1}{2}$, then $\chi \ge \vartheta > \Delta_0$ and hence $\chi = \Delta_0 + 1$
- if $x < \Delta_0 + \frac{1}{2}$, then $\omega \le \vartheta < \Delta_0 + 1$, and hence $\omega \le \Delta_0$
- now, $\chi \leq \max\{\omega, \Delta_0\} \leq \Delta_0$
- we just gave a polynomial time algorithm to determine whether or not $\chi \leq \Delta_0$ for graphs with $\Delta_0 \geq t$
- this is impossible unless *P*=*NP*

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Further improvements

What we can prove about Δ_0

Theorem (Rabern 2010)

Every graph with $\Delta \geq 3$ satisfies

$$\chi \leq \max \left\{ \omega, \Delta_0, rac{5}{6}(\Delta+1)
ight\}.$$

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Further improvements

What we can prove about Δ_0

Theorem (Rabern 2010)

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• the proof uses a recoloring algorithm similar to the above

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 Δ_0 ?

What we can prove about Δ_0

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Every graph with $\Delta \geq 3$ satisfies

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ight\}.$$

- the proof uses a recoloring algorithm similar to the above
- actually, all the above results about Δ_{ϵ} follow from this result

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In joint work with Kostochka and Stiebitz similar techniques were used to improve the bounds further. Highlights:

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Further improvements

In joint work with Kostochka and Stiebitz similar techniques were used to improve the bounds further. Highlights:

Theorem (Kostochka, Rabern and Stiebitz 2010)

Every graph with $\theta \geq 8$, except O_5 , satisfies $\chi \leq \max\left\{\omega, \left\lfloor \frac{\theta}{2} \right\rfloor\right\}$.

Further improvements

In joint work with Kostochka and Stiebitz similar techniques were used to improve the bounds further. Highlights:

Theorem (Kostochka, Rabern and Stiebitz 2010)

Every graph with $\theta \geq 8$, except O_5 , satisfies $\chi \leq \max\left\{\omega, \left\lfloor \frac{\theta}{2} \right\rfloor\right\}$.

Theorem (Kostochka, Rabern and Stiebitz 2010)

Every graph satisfies

$$\chi \leq \max \left\{ \omega, \Delta_0, rac{3}{4}(\Delta+2)
ight\}.$$

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Conjecture

Every graph satisfies

$$\chi \leq \max\left\{\omega, \Delta_0, \frac{2\Delta+5}{3}
ight\}.$$

The examples O_n above show that this would be tight.

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Further improvements



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