For a k-edge-coloring π of a graph G, let $\pi(x) = \{\pi(xy) \mid xy \in E(G)\}$ for $x \in V(G)$ and for $i \in [k]$ put $\pi_i = \{x \in N(v) \mid i \notin \pi(x)\}$.

Lemma 1. *If G is a simple graph and there exists $k \in \mathbb{N}$ and $v \in V(G)$ such that each of the following hold:*

1. $\chi'(G - v) \leqslant k$;
2. $d(v) \leqslant k$;
3. $d(x) \leqslant k$ *for all* $x \in N(v)$;
4. $d(x) = k$ *for at most one* $x \in N(v)$.

Then $\chi'(G) \leqslant k$.

Proof. Assume not and choose a counterexample G, vertex $v \in V(G)$ and $k \in \mathbb{N}$ minimizing k. Then v satisfies each of (1), (2), (3) and (4). By adding dummy pendant edges to v and its neighbors if necessary, we may assume that $d(v) = k$, $d(x) = k$ for exactly one $x \in N(v)$ and $d(y) = k - 1$ for $y \in N(v) - \{x\}$.

Choose a k-edge-coloring π of $G - v$ minimizing $\sum_{i \in [k]} |\pi_i|^2$. First, assume $|\pi_i| \neq 1$ for all $i \in [k]$. Then, we have $\sum_{i \in [k]} |\pi_i| = |\{(i, x) \in [k] \times N(v) \mid i \notin \pi(x)\}| = \sum_{x \in N(v)} (k - d_{G-v}(x)) = 2d(v) - 1 < 2k$. Hence there exists $a \in [k]$ such that $|\pi_a| = 0$. Also, since $2d(v) - 1$ is odd, there must be $b \in [k]$ such that $|\pi_b|$ is odd and hence at least 3. Pick $z \in \pi_b$ and consider a maximum length path zPw with edges alternating between color a and color b starting at z. Exchange colors a and b on P to get a new k-edge-coloring π' of $G - v$. Note that for any internal vertex x of P we have $\pi'(x) = \pi(x)$. Since every vertex in $N(v)$ is incident with color a, if $w \in N(v)$, then by maximality of P, the last edge of P must be colored a. Hence, in any case, $|\pi'_a|^2 + |\pi'_b|^2 < |\pi_a|^2 + |\pi_b|^2$ contradicting our minimality assumption on π.

Hence, we may assume $\pi_i = \{z\}$ for some $z \in N(v)$ and $i \in [k]$. Make a graph H by removing vz as well as all $e \in E(G)$ with $\pi(e) = i$ from G. Then $H - v$ is $(k-1)$-edge-colored and we have removed exactly one neighbor from v and each of its neighbors. Hence, by minimality of k, we must have $\chi'(H) \leqslant k - 1$. But then adding back in the edges we removed all colored with the same new color gives a k-edge-coloring of G. This final contradiction completes the proof. □

Vizing's Simple Theorem. *Every simple graph satisfies* $\Delta \leqslant \chi' \leqslant \Delta + 1$.

Proof. Let G be a simple graph. Plainly, $\chi'(G) \geqslant \Delta(G)$. Applying Lemma 1 inductively with $k = \Delta(G) + 1$ proves that $\chi'(G) \leqslant \Delta(G) + 1$. □

Lemma 2. *If G is a multigraph and there exists $k \in \mathbb{N}$ and $v \in V(G)$ such that each of the following hold:*

1. $\chi'(G - v) \leqslant k$;
2. $d(v) \leqslant k$;

3. $d(x) + \mu(vx) \leqslant k + 1$ for all $x \in N(v)$;

4. $d(x) + \mu(vx) = k + 1$ for at most one $x \in N(v)$.

Then $\chi'(G) \leqslant k$.

Proof. Assume not and choose a counterexample G, vertex $v \in V(G)$ and $k \in \mathbb{N}$ minimizing k. Then v satisfies each of (1), (2), (3) and (4). By adding dummy pendant edges to v and its neighbors if necessary, we may assume that $d(v) = k$, $d(x) + \mu(vx) = k + 1$ for exactly one $x \in N(v)$ and $d(y) + \mu(vy) = k$ for $y \in N(v) - \{x\}$.

Choose a k-edge-coloring π of $G - v$ minimizing $\sum_{i \in [k]} |\pi_i|^2$. First, assume $|\pi_i| \neq 1$ for all $i \in [k]$. Then, we have $\sum_{i \in [k]} |\pi_i| = |\{(i, x) \in [k] \times N(v) \mid i \notin \pi(x)\}| = \sum_{x \in N(v)} (k - d_{G-v}(x)) = -1 + \sum_{x \in N(v)} 2\mu(vx) = 2d(v) - 1 < 2k$. Hence there exists $a \in [k]$ such that $|\pi_a| = 0$. Also, since $2d(v) - 1$ is odd, there must be $b \in [k]$ such that $|\pi_b|$ is odd and hence at least 3. Pick $z \in \pi_b$ and consider a maximum length path zPw with edges alternating between color a and color b starting at z. Exchange colors a and b on P to get a new k-edge-coloring π' of $G - v$. Note that for any internal vertex x of P we have $\pi'(x) = \pi(x)$. Since every vertex in $N(v)$ is incident with color a, if $w \in N(v)$, then by maximality of P, the last edge of P must be colored a. Hence, in any case, $|\pi'_a|^2 + |\pi'_b|^2 < |\pi_a|^2 + |\pi_b|^2$ contradicting our minimality assumption on π.

Hence, we may assume $\pi_i = \{z\}$ for some $z \in N(v)$ and $i \in [k]$. Make a multigraph H by removing one edge between v and z as well as all $e \in E(G)$ with $\pi(e) = i$ from G. Then $H - v$ is $(k - 1)$-edge-colored and we have removed exactly one neighbor from v and each of its neighbors. Hence, by minimality of k, we must have $\chi'(H) \leqslant k - 1$. But then adding back in the edges we removed all colored with the same new color gives a k-edge-coloring of G. This final contradiction completes the proof. $\square$

Vizing's Theorem. *Every multigraph satisfies $\Delta \leqslant \chi' \leqslant \Delta + \mu$.*

Proof. Let G be a multigraph. Plainly, $\chi'(G) \geqslant \Delta(G)$. Applying Lemma 2 inductively with $k = \Delta(G) + \mu(G)$ proves that $\chi'(G) \leqslant \Delta(G) + \mu(G)$. $\square$