Turán Graph. Let $r \leq n$ be positive integers. We write $T_{n,r}$ for the complete r-partite graph $K_{n_1,\ldots,n_r}$ where $\sum_i n_i = n$ and $|n_i - n_j| \leq 1$ for all i, j.

Turán's Theorem. *Let $r \leq n$ be positive integers. If G is a K_{r+1}-free graph with n vertices and the maximum number of edges, then $G = T_{n,r}$.*

Proof. Let G be a K_{r+1}-free graph with n vertices and the maximum number of edges.

First, assume G is a complete multipartite graph $K_{n_1,\ldots,n_s}$ with $n_i \geq n_j$ for $i \leq j$. Then $s \leq r$ since G is K_{r+1}-free. If $s < r$, then $n_1 \geq 2$ and $K_{1,n_1-1,n_2,\ldots,n_s}$ is K_{r+1}-free and has more edges. Thus $s = r$. If $n_1 - n_s \geq 2$, then $K_{n_1-1,n_2,\ldots,n_{s-1},n_s+1}$ is K_{r+1}-free and has more edges. Thus $G = T_{n,r}$ and we are done.

Therefore, we may assume that $\overline{G}$ is not a disjoint union of cliques. Hence G contains an induced $\overline{P_3}$, say with vertices x, y, z where $yz \in E(G)$ and $xy, xz \notin E(G)$.

First, assume $d(x) \geq d(y)$ and $d(x) \geq d(z)$. Create a new graph H by adding two copies of x to G and removing y and z. Plainly, H is K_{r+1}-free and $|E(H)| = |E(G)| + 2d(x) - (d(y) + d(z) - 1) > |E(G)|$. This is a contradiction.

Hence, without loss of generality, we may assume that $d(x) < d(y)$. Now create a new graph F by adding a copy of y to G and removing x. Plainly, F is K_{r+1}-free and $|E(F)| = |E(G)| + d(y) - d(x) > |E(G)|$. This final contradiction completes the proof. $\square$