Arithmetic with Complex Numbers

Complex numbers use the special value i, which has the property that $i \times i = -1$. All of complex arithmetic follows from this property.

A complex number has two parts: real and imaginary. For example, in the number $2 + 3i$, the real part is 2 and the imaginary part is 3.

Ordinary real numbers are just complex numbers with a zero imaginary part:

$3 = 3 + 0i$

Conversely, imaginary numbers are complex numbers with a zero real part:

$3i = 0 + 3i$

Addition and Subtraction

You can add or subtract two complex numbers by simply adding or subtracting their real parts, and adding or subtracting their imaginary parts:

$(2 + 3i) + (4 - 5i)$

$(2 + 3i) + (4 - 5i) = (2+4) + (3-5)i$

$(2 + 3i) + (4 - 5i) = 6 - 2i$

The general formulas for addition and subtraction are:

$(a + bi) + (c + di) = (a+c) + (b+d)i$

$(a + bi) - (c + di) = (a-c) + (b-d)i$

Multiplication

Multiplication is also straightforward, as long as you remember that $i \times i = -1$. For example,

$(2 + 3i) \times (4 - 2i)$

$(2 + 3i) \times (4 - 2i) = 2\times4 + 2\times(-2i) + (3i)\times4 + (3i)\times(-2i)$

$(2 + 3i) \times (4 - 2i) = 2\times4 - 2\times2i + 3\times4i - 3\times2(i\times i)$

(2 + 3i) × (4 - 2i) = 2×4 + -2×2i + 3×4i + 3×2

(2 + 3i) × (4 - 2i) = (2×4 + 3×2) + (-2×2 + 3×4)i

(2 + 3i) × (4 - 2i) = 14 + 8i

The general formula for multiplication is:

(a + bi) × (c + di) = (ac - bd) + (ad + bc)i

Conjugate

To do division, we need to define the conjugate of a complex number: it's just the number, with the imaginary part negated. For example:

conjugate(*3 + 4i*) = *3 - 4i*

Division

How do you divide *(2 + 4i) / (1 + i)*?

First, multiply top and bottom by the conjugate of the denominator. That's *(1 - i)*.

(2 + 4i) / (1 + i) = [(2 + 4i)×(1 - i)] / [(1 + i)×(1 - i)]

Then

(2 + 4i) / (1 + i) = (6 + 2i) / (2 + 0i)

Notice that the denominator is a purely real number! You just use its real part:

(2 + 4i) / (1 + i) = (6 + 2i) / 2

Which gives us our final result:

(2 + 4i) / (1 + i) = 3 + i

Notice that multiplying the *(1 + i)* by *(1 - i)* yields an ordinary real number, with no imaginary part. Multiplying a number by its conjugate always yields a real number.

A bit of algebra will show that the general formula for division needs these steps:

numerator = (*a + bi*) × conjugate(*c + di*)

denominator = (*c + di*) × (*c - di*)

(a + bi) / (c + di) = numerator / denominator

In that last line, the numerator will be a complex number, but the denominator will be an ordinary real number, so the right-hand side simply involves dividing both parts of the numerator by the real part of the denominator.

Magnitude

The magnitude of a complex number uses the pythagorean formula:

magnitude($a + bi$) = $\sqrt{}(a^2 + b^2)$

We can use conjugates to get the magnitude of a complex number. Just multiply the number by its conjugate, which gives us a real number. Take the real part, and get its square root:

magnitude($a + bi$) = $[(a + bi)\times(a - bi)]^{0.5}$