AMERICAN JOURNAL of PHYSICS

Determining the eccentricity of the Moon's orbit without a telescope

Kevin Krisciunas

Citation: Am. J. Phys. **78**, 834 (2010); doi: 10.1119/1.3369968
View online: http://dx.doi.org/10.1119/1.3369968
View Table of Contents: http://ajp.aapt.org/resource/1/AJPIAS/v78/i8
Published by the American Association of Physics Teachers

Determining the eccentricity of the Moon's orbit without a telescope

Kevin Krisciunas[a)]
Department of Physics and Astronomy, George P. and Cynthia Woods Mitchell Institute for Fundamental Physics and Astronomy, Texas A&M University, 4242 TAMU, College Station, Texas 77843-4242

(Received 21 December 2009; accepted 2 March 2010)

Prior to the invention of the telescope many astronomers worked out models of the motion of the Moon to predict the position of the Moon in the sky. These geometrical models implied a certain range of distances of the Moon from Earth. Ptolemy's most quoted model predicted that the Moon was nearly twice as far away at apogee than at perigee. Measurements of the angular size of the Moon were within the capabilities of pretelescopic astronomers. Such measurements could have helped refine the models of the motion of the Moon, but hardly anyone seems to have made any measurements that have come down to us. We use a piece of cardboard with a small hole in it which slides up and down a yardstick to show that it is possible to determine the eccentricity $\epsilon \approx 0.039 \pm 0.006$ of the Moon's orbit. A typical measurement uncertainty of the Moon's angular size is ± 0.8 arc min. Because the Moon's angular size ranges from 29.4 to 33.5 arc min, carefully taken naked eye data are accurate enough to demonstrate periodic variations of the Moon's angular size.

[DOI: 10.1119/1.3369968]

I. INTRODUCTION

The small angle formula is one of the simple geometrical relations discussed in elementary astronomy classes. An object of diameter d viewed at distance D subtends an angle $\theta = d/D$ in radians if $D \gg d$. Two objects that satisfy this criterion are the Sun and Moon.

The history and significance of estimating the angular diameter of the Moon are a curious one. Aristarchus of Samos (approximately 310–230 BCE) wrote a treatise called *On the Sizes and Distances of the Sun and Moon*, which has survived.[1] In it Aristarchus wrote that the Sun and Moon must have about the same angular size, because during a total solar eclipse, the Moon just barely covers the Sun. He also wrote that the angular size of the Moon is 2 deg, though this value was not a result of an actual measurement. It is just a small angle he adopted while illustrating his method of determining the Moon's distance using the geometry of lunar eclipses.

Archimedes (approximately 287–212 BCE) wrote that Aristarchus also used another value for the angular size of the Moon, 1/720th part of a circle, or 0.5 deg.[2,3] This value is close to the accepted mean value of 0.5182 deg $=31' 5.''5$.[4] (Dating back to the Babylonians, 1 deg $=60$ arc min, or $60'$, and 1 arc min $=60$ arc sec, or $60''$.)

Hipparchus (approximately 190–120 BCE) wrote a treatise on the sizes and distances of the Sun and Moon, which has not survived, but its contents are known from two sources.[5] Hipparchus obtained a mean lunar distance of 67.33 Earth radii, with a range of 62–72.67. In the *Almagest* (IV,9) Ptolemy (approximately 100–170 CE) quotes Hipparchus's value for the Moon's mean angular size of 1/650th of a circle, or $33'14''$.[6]

Ptolemy's values for the minimum and maximum angular sizes of the Moon were $31'20''$ and $35'20''$, respectively.[7] Ptolemy's model for the position of the Moon in the sky implies that its minimum distance is a mere 33.55 Earth radii, while its maximum distance is 64.17 Earth radii.[8] Given the importance of the *Almagest* from ancient times until the Renaissance, the implication that the Moon's distance (and hence, angular size) ranges by nearly a factor of two was known to all those who worked on models of the Moon's motion, although a serious observer of the Moon would have known that its range of angular size is considerably smaller.

In his commentary on Aristotle's *De Caelo*, Simplicius (6th century CE) wrote, "…if we observe the moon by means of an instrument… it is at one time a disk of eleven finger-breadths, and again at another time a disk of twelve finger-breadths." Here a "finger-breadth" (digit or *daktylos*) is not the angular size of a finger at some distance. It is a Babylonian unit equal to 1/12 of a degree, but in this context it might be 1/12 of the maximum angular size of the Moon. Taken at face value, the statement by Simplicius implies the existence of observations that give just about the right range of the Moon's angular size. But the data and the identity of the observer are not given.[9]

In the 14th century two astronomers showed an interest in actually making some measurements. Levi ben Gerson (1288–1344), also known as Gersonides, was a versatile and accomplished scholar.[10] He invented the staff of Jacob, which consists of a calibrated ruler that slides perpendicularly along another calibrated staff. With it the angular separation of two stars in the sky or the height of building could be determined. By using a calibrated staff and a pinhole camera, he determined that the Sun ranges in angular size from $27'50''$ to $30'0''$.[11] He commented on Ptolemy's factor-of-two range of the Moon's angular size and used the staff and pinhole camera to find a lunar diameter at quadrature (first/third quarter) only slightly larger than at opposition (full Moon).[12] He did not carry out observations at all lunar phases.

Another pretelescopic astronomer to comment specifically on the range of lunar angular size as a consequence of a model to explain its varying position was Ibn al-Shatir (1304–1375/6 CE).[13] The implied range was $29'2''$ to $37'58''$.

Regiomontanus (1436–1476) doubted the very large range of lunar angular size implied by Ptolemy's model, basing his criticism on al-Battani's apparent diameters at syzygy (new/full Moon).[14] In *De Revolutionibus* Copernicus (1473–1543) gave a range of $30'$ to $35''38''$ for the Moon's angular size.[15]

Tycho Brahe (1546–1601), the greatest pretelescopic in-

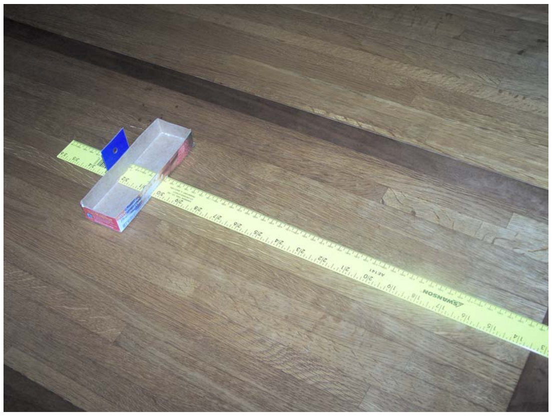

Fig. 1. A simple Moon measuring device made from the bottom of a box of pancake mix.

strument designer and observer, devoted considerable effort to the orbit of the Moon, but he was concerned only with the Moon's varying position. Still, his model implied the maximum lunar distance at syzygy is 5.8% greater than the minimum lunar distance at syzygy.[16] So the range of Moon's distance is $\pm 2.9\%$ with respect to its mean distance.

We now can model the orbit of the Moon very accurately. But we should not simply say that the eccentricity of the Moon is $\epsilon=0.05490$,[4] implying that the Moon's maximum distance is 5.5% greater than its mean distance, and that its minimum distance is 5.5% less than its mean distance. Contrary to most introductory astronomy textbooks, the Earth does not orbit the Sun on a simple elliptical orbit, nor does the Moon orbit the Earth on a simple elliptical orbit. The Earth-Moon barycenter orbits the Sun on a nearly Keplerian ellipse. The Earth and Moon do something else. Let $\delta=0.011$ be the "amplitude of Ptolemy's evection" (see Ref. 17). To first order the maximum deviation from uniform angular motion of the Moon varies from $\pm 2(\epsilon-\delta)$ when the Moon is new or full and $\pm 2(\epsilon+\delta)$ for the first and third quarters. The distance of the Moon from the Earth (to first order) varies by $(\epsilon+\delta)=\pm 6.6\%$.[17] Three modern ephemerides use 669, 921, and 915 terms, respectively, to calculate the distance to the Moon.[18]

One might think that Hipparchus or some other pretelescopic astronomer had actually measured the variation of the apparent angular size of the Moon over a number of lunations (that is, cycle of its phases, 29.53 days on average). Even if such measurements had been done, no data set or analysis of them has come down to us. With an instrument as simple as a quarter-inch diameter hole viewed at some distance down a yardstick, is it possible to show that the Moon's angular size varies from 29 to 33 arc min in a quasi-sinusoidal way?

II. OBSERVATIONS

Hipparchus measured the angular size of the Moon using a *dioptra*. Such a device uses a round object of small angular size to *occult* (cover up) the Moon. As did Levi ben Gerson,[19] we have found that it is better to use a sighting hole.

We fashioned a cross piece of cardboard that can slide up and down a yardstick, such as that pictured in Fig. 1. The yardstick is calibrated in centimeters and millimeters along one edge. The purpose of the cross piece is to hold a thin piece of cardboard that has a small hole punched through it. We estimate that the diameter of the sighting hole is 6.2 mm.

In Table I we present a series of observations carried out during 2009. All of the data were taken at College Station, Texas, which is about 100 m above sea level. All observations were made with my left eye, which is my better eye. In Table I the age of the Moon is the number of days since the previous new Moon.[20] The "true angular size" is the geocentric angular diameter interpolated from values given in the *Astronomical Almanac*.[21] Observations from 11:15 to 13:15 UT and from 22:37 to 23:43 UT were made during twilight or daytime. Starting on 4 August 2009 our data values in column 5 of Table I are typically the average of two measurements, the first obtained while moving the cross piece out from a position much too close to the eye and the second obtained by moving the cross piece in from the far end of the yardstick. A pair of such observations made on the same occasion typically exhibits a difference of 8 mm. Thus, a typical uncertainty of one of the averages in column 5 is ± 4 mm.

Let D be the distance down the yardstick that the 6.2 mm sighting hole is viewed. We use the small angle approximation to obtain the observed angular size of the Moon as

$$\theta_{\rm obs}(\text{arc min}) = \left(\frac{6.2}{D}\right)\left(\frac{180}{\pi}\right)60. \tag{1}$$

Taken at face value, our result for the mean angular size of the Moon is 25.9 arc min. Thus, our preliminary data exhibit a systematic error of 5.2 arc min. However, the pupil has a nonzero size and is comparable to the size of the sighting hole.

We therefore devised a simple calibration. A disk of diameter 90.44 mm would subtend an angle of 31.09 arc min if situated at a distance of 10 m. We cut out a disk that was 91 mm in diameter. At 10 m the angular size was 31.28 arc min. If it were not for the finite size of the pupil, a 6.2 mm hole viewed at 681.3 mm would subtend an equal angle. If we place the sighting hole at this distance, we can see beyond the left side of the disk from the right side of the pupil, and we can see beyond the right side of the disk from the left side of the pupil. Hence, we need to place the sighting hole at a greater distance from the eye to make the best apparent fit of the disk as seen through the hole. I found that I had to place the sighting hole 821 mm from my left eye to produce the best match to the angular size of the 91 mm disk. (This calibration was done on 5 August 2009 in an illuminated hallway.) We thus obtain a correction factor of $821/681.3=1.205$, which should be used to scale our preliminary angular sizes. That is, from April through November 2009 when these lunar observations were made, the correction for my eye is given by

$$\theta_{\rm corr} = 1.205\,\theta_{\rm obs}. \tag{2}$$

Further justification of the correction factor is shown in Fig. 2, in which we plot the ratios of true (geocentric) angular sizes to the corresponding observed values. One outlier is eliminated from the analysis. A simple regression line gives a nonzero slope at the 2.3σ level of significance. Thus, there is marginal (but statistically insignificant) evidence that I was measuring the Moon to be larger over time. The average value of the ratios is 1.199 ± 0.005, which is close to the correction factor derived from a measurement of the 91 mm

Table I. Measurements of the Moon's angular size and associated data.

UT date (2009)	UT (hh:mm)	JD[a] (days)	Age[b] (days)	D[c] (mm)	θ_{corr} (arc min)	θ_{true} [d] (arc min)
21 Apr	11:23	4942.9743	25.80	874.0	29.39	30.69
06 May	03:55	4957.6632	11.02	840.0	30.58	31.03
29 May	01:33	4980.5646	4.56	810.0	31.71	32.52
31 May	01:33	4982.5646	6.56	794.0	32.35	31.82
14 Jul	11:15	5026.9688	21.65	828.0	31.02	30.65
16 Jul	12:02	5029.0014	23.69	812.0	31.63	31.65
29 Jul	01:33	5041.5646	6.96	816.0	31.47	30.62
04 Aug	02:08	5047.5889	12.98	859.0	29.90	29.42
07 Aug	12:01	5051.0007	16.39	846.5	30.34	29.72
10 Aug	11:58	5053.9986	19.39	846.5	30.34	30.40
13 Aug	12:04	5057.0028	22.40	796.0	32.27	31.48
14 Aug	11:57	5057.9979	23.39	817.0	31.44	31.90
15 Aug	12:08	5059.0056	24.40	816.0	31.47	32.31
17 Aug	10:00	5060.9167	26.31	805.0	31.90	32.96
27 Aug	02:48	5070.6167	6.70	810.0	31.71	30.26
30 Aug	03:00	5073.6250	9.71	870.5	29.50	29.70
02 Sep	01:12	5076.5500	12.63	899.0	28.57	29.56
07 Sep	02:48	5081.6167	17.70	811.0	31.67	30.83
09 Sep	11:45	5083.9896	20.07	795.0	32.31	31.23
16 Sep	11:46	5090.9903	27.07	804.0	31.94	32.80
27 Sep	23:30	5102.4792	9.20	848.0	30.29	29.54
29 Sep	23:42	5104.4875	11.21	865.0	29.69	29.68
03 Oct	03:51	5107.6604	14.38	859.5	29.88	30.41
08 Oct	03:40	5112.6528	19.37	841.0	30.54	31.73
08 Oct	13:00	5113.0417	19.76	802.0	32.02	31.81
15 Oct	11:54	5119.9958	26.72	761.5	33.73	32.23
23 Oct	01:27	5127.5604	4.83	765.0	33.57	29.95
23 Oct	23:29	5128.4785	5.75	840.0	30.58	29.74
24 Oct	23:42	5129.4875	6.76	843.0	30.47	29.60
25 Oct	22:37	5130.4424	7.71	868.0	29.59	29.56
27 Oct	23:43	5132.4882	9.76	819.0	31.36	29.75
31 Oct	23:28	5136.4778	13.75	817.0	31.44	30.98
03 Nov	12:51	5139.0354	16.30	792.0	32.43	31.85
05 Nov	12:42	5141.0292	18.30	773.0	33.23	32.26
06 Nov	12:55	5142.0382	19.31	789.5	32.53	32.35
10 Nov	13:15	5146.0521	23.32	800.0	32.10	32.11

[a]Julian date minus 2450000.
[b]Time in days since previous new Moon.
[c]Distance from the eye of the 6.2 mm diameter sighting hole. Typical internal error is ±4 mm.
[d]Geocentric angular diameter of the Moon, interpolated from data given in Ref. 21.

disk at 10 m. Ideally, we would do this calibration after each measurement of the Moon under lighting conditions as similar as possible. Maybe there is a set of three correction factors for each observer, one for nighttime, one for twilight, and one for daytime observations. But such an investigation is beyond the scope of this paper.

The correction factor eliminates any systematic error in the adopted size of the sighting hole. If the true diameter of the sighting hole were really 6.4 mm, then the correction factor would be correspondingly smaller. For the calibration we have given and observations with my left eye, I can use the following relation for the Moon's angular size:

$$\theta_{\mathrm{corr}} = 31.28\left(\frac{821}{D}\right). \tag{3}$$

Thanks to the correction factor, my mean value for the angular size is 31.18 arc min, close to the accepted mean value of 31.09 arc min.

III. ANALYSIS AND DISCUSSION

Even before the ancient Greeks, the Babylonians knew the length of the *anomalistic month*, which is the time between occurrences of maximum daily motion against the background of stars (that is, the time from perigee to perigee), 27.55455 days.[4] We have taken data over 7 lunations=7.5 anomalistic months. Because a lunation takes about 29.53 days, the apogee/perigee occurs 2 days earlier each subsequent lunation. Ideally, we should have a data set that extends over 14 lunations=15 anomalistic months. In that case every phase would occur at some time over the range of observations at perigee and at apogee.

Because all observations presented in Table I were made by the same observer with the same eye and the same sighting hole, we can derive a value of the eccentricity of the Moon's orbit from an analysis of the distances down the yardstick given in column 5, or we can use the uncorrected

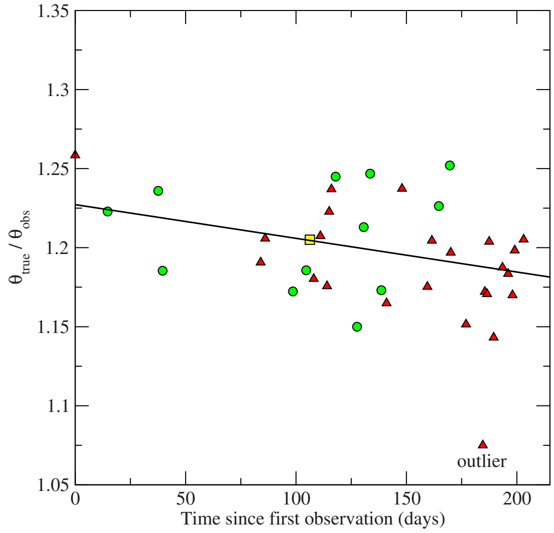

Fig. 2. Ratios of the true (geocentric) angular sizes of the Moon to the corresponding measured (uncorrected) values. Triangles represent observations made during twilight or daytime. Dots represent observations made at nighttime when the observer's pupil was presumably smaller. The square represents a calibration done with a 91 mm disk viewed at 10 m, which matches the true mean angular size of the Moon. The slope of the regression line is nonzero at the 2.3σ level, which is to say it is not statistically significantly different than zero.

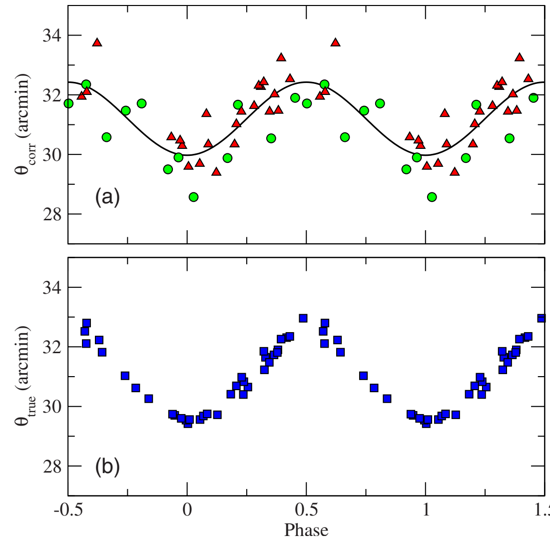

Fig. 3. Angular diameter of the Moon versus its phase in its cycle from aphelion to aphelion. (a) Measured values. Data represented by triangles were made during twilight or daytime. Data represented by dots were made at nighttime. The solid line is a single sine wave whose amplitude and mean value were given by the program PERDET. (b) True geocentric values of the Moon's angular size, as interpolated from Ref. 21.

(observed) angular sizes of the Moon. The eccentricity of the orbit is just the amplitude of the sinusoid that best fits the data, divided by the mean value of the data. We used the corrected angular sizes given in column 6 of Table I because the correction factor eliminates a significant source of systematic error.

Note that although a Keplerian elliptical orbit is necessarily eccentric, without more accurate data than we present here, we cannot show that the Moon's orbit is elliptical, circular, ovoid, or some other shape. We have only demonstrated that the Earth does not reside at the center of the Moon's orbit, as was known by Ptolemy, Copernicus, and others.

Suppose we knew nothing of the value of the anomalistic month. What does our data set tell us is the time between perigees? We eliminated one data point from the analysis, the first datum from 23 October 2009, which is a 4.5σ outlier. Using a program for the analysis of variable star light curves,[22] we obtain a best fit period of 27.24 ± 0.29 days. The uncertainty of the period is to be considered a lower limit.[23] The amplitude of the variation is 1.23 ± 0.17 arc min. The root-mean-square scatter of the data with respect to that sinusoid is ± 0.74 arc min, which is the internal random error of the measurements. The implied eccentricity of the Moon's orbit is 0.039 ± 0.006. In Fig. 3(a) we show our data folded with the best period derivable from only our data set.

Figure 3(b) shows the geocentric angular size of the Moon (from Ref. 21) at the dates and times we took data, folded with the anomalistic period of the Moon. Note that the actual "true values" do not exhibit a simple sinusoid.

A comparison of our corrected angular sizes to the geocentric values from Ref. 21 indicates that a typical one of our individual data points is accurate to ± 0.80 arc min. This accuracy is only marginally larger than the root-mean-square scatter of the data with respect to the best fit sine wave, implying that the corrected angular sizes contain no serious systematic offset.

The data obtained when the age of the Moon is between 7.7 and 22.4 days (first quarter to third quarter) are a bit more accurate than observations of the crescent Moon. Twilight and daytime observations are more accurate than nighttime observations because the Moon's glare makes the observations more difficult.

It is remarkable that none of the great pretelescopic astronomers carried out simple observations like those presented here. Of course, before the 19th century there was no Fourier analysis or least-squares theory to derive some of the numbers presented here, but the Babylonians did know the duration of the anomalistic month to a high accuracy, and hence measurements of the angular diameter of the Moon could have been folded with that period using nothing more than simple arithmetic.

It could be asserted that because no known pretelescopic data set has come down to us, the analysis presented here is completely unhistorical and therefore of no interest to historians. Perhaps this paper will motivate scholars who can read Arabic or Hebrew to identify and scrutinize some previously unstudied manuscripts that contain data similar to those presented here.

The geometrical models of the Moon's motion were concerned with its ecliptic latitude and longitude, not with its distance. But observations of the Moon's angular size, which is to say measures related to its physical distance, could have nudged astronomers down the path of more physically realistic models of the solar system.

ACKNOWLEDGMENTS

The author thanks Dudley Herschbach for the suggestion to write this article. The author is particularly grateful to Noel Swerdlow for numerous fruitful discussions. Alexei Be-

lyanin, Gerald Handler, and an anonymous referee provided useful references. The author also thanks James Evans and Stephen O'Meara for useful comments.

[a]Electronic mail: krisciunas@physics.tamu.edu

[1]Thomas Heath, *Aristarchus of Samos: The Ancient Copernicus* (Clarendon, Oxford, 1913), pp. 351–414.

[2]Reference 1, pp. 353, 383.

[3]Albert Van Helden, *Measuring the Universe: Cosmic Dimensions from Aristarchus to Halley* (University of Chicago Press, Chicago, 1985), pp. 8–11.

[4]*Allen's Astrophysical Quantities*, edited by Arthur N. Cox (Springer-Verlag, New York, 2000), pp. 308–309. The "mean Moon diameter" (2×1738.2 km) divided by the mean distance (384401 km) times $180/\pi=0.5182$ deg$=31.09$ arc min. Given the extreme range of the Moon's distance (356400–406700 km), its geocentric angular size ranges from 29.39 to 33.53 arc min. The mean distance of the Moon divided by the equatorial radius of the Earth, 6378 km, gives a mean geocentric distance of 60.27 Earth radii. Note that the maximum distance is 5.8% greater than the mean distance, and the minimum distance is 7.3% less than the mean distance. For more information on the variation of the time between successive occurrences of new/full Moon with respect to the mean value of 29.53 days, see R. L. Reese, G. Y. Chang, and D. L. Dupuy, "The oscillation of the synodic period of the Moon: A 'beating' phenomenon," Am. J. Phys. **57**, 802–807 (1989).

[5]G. J. Toomer, "Hipparchus on the distances of the Sun and Moon," Arch. Hist. Exact Sci. **14**, 126–142 (1974).

[6]G. J. Toomer, *Ptolemy's Almagest* (Springer-Verlag, Berlin, 1984), p. 205.

[7]Reference 6, pp. 254, 284.

[8]Reference 6, pp. 251, 259. In Sec. 5.13 Ptolemy gives the mean distance of the Moon at syzygy (new/full Moon) of 59 Earth radii. The mean distance at quadrature (first or third quarter) is $38\frac{43}{60}$ Earth radii, and the radius of the epicycle is $5\frac{10}{60}$ Earth radii. It follows that the greatest distance occurs at syzygy and equals $59+5\frac{10}{60}=64.17$ Earth radii. The minimum distance occurs at quadrature and is $38\frac{43}{60}$ minus $5\frac{10}{60}=33.55$ Earth radii. See also Ref. 17.

[9]Thomas L. Heath, *Greek Astronomy* (Dover, New York, 1991), pp. xvii, 69. See also the anonymous article, "Babylonian measures and the *daktylos*," Observatory **42**, 46–51 (1919).

[10]Bernard R. Goldstein, *The Astronomy of Levi ben Gerson (1288–1344)* (Springer-Verlag, New York, 1985).

[11]Reference 10, p. 113.

[12]Reference 10, pp. 105, 186.

[13]V. Roberts, "The solar and lunar theory of Ibn ash-Shatir: A pre-Copernican model," Isis **48**, 428–432 (1957).

[14]Johannes Regiomontanus, *Epytoma in Almagesti Ptolemei* (Venice, 1496), Sec. 5.22. There is no published translation of this work. al-Battani lived from about 858 to 929 CE.

[15]Edward Rosen, *Nicholas Copernicus: On the Revolutions* (Johns Hopkins U. P., Baltimore, 1992), p. 209.

[16]N. Swerdlow, "The lunar theories of Tycho Brahe and Christian Longomontanus in the Progymnasmata and Astronomia Danica," Ann. Sci. **66**, 5–58 (2009).

[17]M. C. Gutzwiller, "Moon-Earth-Sun: The oldest three-body problem," Rev. Mod. Phys. **70**, 589–639 (1998). Compared to uniform motion against the background of stars, the new/full Moon can be 5 deg ahead or behind. At first/third quarter the Moon can be 7.5 deg ahead or behind the mean motion. To quote Gutzwiller: "This new feature is known as the *evection*. Ptolemy found a mechanical analog for this peculiar complication, called the crank model. It describes the angular coupling between the Sun and Moon correctly, but it has the absurd consequence of causing the distance of the Moon from the Earth to vary by almost a factor of 2."

[18]Reference 17, p. 628.

[19]Reference 10, p. 156.

[20]See: ⟨aa.usno.navy.mil/data/docs/MoonPhase.phpy2009⟩.

[21]*The Astronomical Almanac for the Year 2009* (Nautical Almanac Office, Washington, DC, 2009).

[22]M. Breger, "PERDET: Multiple PERiod DETermination user manual," Communications in Astroseismology, No. 6 (1989).

[23]M. H. Montgomery and D. O'Donoghue, "A derivation of the errors for least squares fitting to time series data," Delta Scuti Star Newsletter No. 13, pp. 28–32 (1999).