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MATH 753 01
HW 05

$\boxed{3.1.1}$ Use Lagrange Interp. to find a polynomial that passes through the points.

b) $(-1,0)$, $(2,1)$, $(3,1)$, $(5,2)$

$$P_3(x) = \cancel{y_1 L_1(x)}^{0} + y_2 L_2(x) + y_3 L_3(x) + y_4 L_4(x)$$

$$L_k(x) = \frac{(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

$$L_2(x) = \frac{(x+1)(x-3)(x-5)}{(2+1)(2-3)(2-5)} \qquad L_3 = \frac{(x+1)(x-2)(x-5)}{(3+1)(3-2)(3-5)}$$

$+3 \quad -1 \quad -3 = +9 \qquad\qquad 4 \quad +1 \quad -2 = -8$

$$L_4(x) = \frac{(x+1)(x-2)(x-3)}{(5+1)(5-2)(5-3)}$$

$6 \quad 3 \quad 2 = 36$

$$P(x) = (2)\frac{(x+1)(x-3)(x-5)}{9} + (3)\frac{(x+1)(x-2)(x-5)}{-8} + (5)\frac{(x+1)(x-2)(x-3)}{36}$$

c) $(0,-2),(2,1),(4,4)$

$$P_2(x) = y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x)$$

$$L_1(x) = \frac{(x-2)(x-4)}{(0-2)(0-4)} \qquad L_2(x) = \frac{(x-0)(x-4)}{(2-0)(2-4)}$$

$(-2)\ (-4) = 8 \qquad\qquad 2 \quad -2 \quad = -4$

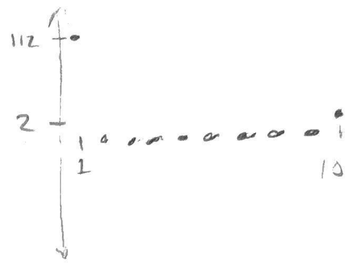
$$L_3(x) = \frac{(x-0)(x-2)}{(4-0)(4-2)}$$

$4 \quad 2 = 8$

$$\boxed{P_2(x) = (-2)\frac{(x-2)(x-4)}{8} + (2)\frac{x(x-4)}{-4} + (4)\frac{x(x-2)}{8}}$$

3.1.8 Let $P(x)$ be a deg-9 polynomal that has

$f(1) = 112$, $f(10) = 2$, $f(2) \rightarrow f(9) = 0$



$(1, 112)$
$(2, 0)$
$(3, 0)$
$\vdots$
$(9, 0)$
$(10, 2)$

$$P_9(x) = \sum_{i=1}^{9} y_i L_i(x)$$

$$L_K(x) = \frac{\prod_{i \neq K}(x - x_i)}{\prod_{i \neq K}(x_K - x_i)}$$

$$L_1 = \frac{(x-2)(x-3)\cdots(x-9)(x-10)}{(1-2)(1-3)\cdots(1-9)(1-10)}$$

$$L_{10} = \frac{(x-1)(x-2)\cdots(x-8)(x-9)}{(10-1)(10-2)\cdots(10-8)(10-9)}$$

$y_n L_n(x) = 0$
$\forall\ n \neq 1, 10$

$P(0) = (112)L_1(x=0) + (2)L_{10}(x=0)$ ← see Q318.m

$$\boxed{P(0) = 1218}$$

$\boxed{3.1.12}$ Can a degree-3 polynomial interesect a degree 4 polynomial in exactly 5 places? Explain.

- There is a uniquness in polynomial interpolation:
  - ↳ Only <u>one</u> polynomial of degree $(n-1)$ can be use to interpolate n points
- Thus it is impossible for a degree $n-2$ polynomial to pass through the same points
- So, a degree-3 polymomial & degree-4 polynomial can <u>never</u> interesect in 5 pts.

$\boxed{3.1.17}$ Find degree 3 polynomial of data & use it to estimate CO_2 in a) 1950 & b) 2050

$\{(1800, 280), (1850, 283), (1900, 291), (2000, 370)\}$

- Newton's Divided Difference

$$
\begin{array}{ll|llll}
1800 & \boxed{280} & & & & \\
 & & \frac{(283-280)}{1850-1800} = \frac{3}{50} = \boxed{0.06} & & & \\
1850 & 283 & & \frac{(0.16-0.06)}{100} = \boxed{0.001} & & \\
 & & \frac{(291-283)}{(1900-1850)} = \frac{8}{50} = 0.16 & & \boxed{1.6 \times 10^{-5}} & \\
1900 & 291 & & \frac{(0.79-0.16)}{150} = 0.0042 & & \\
 & & \frac{(370-291)}{(2000-1900)} = \frac{79}{100} = 0.79 & & & \\
2000 & 370 & & & &
\end{array}
$$

- Construct polynomial

$$P(x) = (280) + (0.06)(x-1800) + (0.001)(x-1800)(x-1850) +$$
$$\text{''} \ (1.6 \times 10^{-5})(x-1800)(x-1850)(x-1900)$$

a) Estimate 1950 (See Q3117.m)

$$\boxed{P(1950) = 316 \text{ ppm of } CO_2}$$

b) Estimate 2050 (See Q3117.m)

$$\boxed{P(2050) = 465 \text{ ppm of } CO_2}$$

↑ thats not good!

:(

$\boxed{3.2.4}$ Conside $f(x) = \frac{1}{(x+5)}$ w/ interpolation nodes @ $x = 0, 2, 4, 6, 8, 10$. Find upper bound on interp-error at values:

a) $x = 1$

$$|f(x) - P(x)| = \frac{(x-0)(x-2)(x-4)(x-6)(x-8)(x-10)}{6!} f^{(6)}(c)$$

$$f^{(6)}(c) \Rightarrow \frac{-1}{(x+5)^2} \rightarrow \frac{+2}{(x+5)^3} \rightarrow \frac{-6}{(x+5)^4} \rightarrow \frac{+24}{(x+5)^5} \rightarrow \frac{-120}{(x+5)^6}$$

$$f^{(6)}(c) \Rightarrow \frac{+720}{(x+5)^7} \quad \text{w/ } (0 < c < 10)$$

$$f^{(6)}(1) = \frac{720}{(6)^7} \Rightarrow \frac{\cancel{720}}{(279{,}936)\ \cancel{6!}}$$

$$\boxed{|f(1) - P(1)| = \left|\frac{x(x-2)(x-4)(x-6)(x-8)(x-10)}{(279{,}936)}\right|}$$

b) $x = 5$

$$f^{(6)}(5) = \frac{720}{(10)^7} \Rightarrow \frac{1}{10^7}\ \frac{\cancel{720}}{1}\ \frac{1}{\cancel{6!}}$$

$$\boxed{|f(5) - P(5)| = \left|\frac{x(x-2)(x-4)(x-6)(x-8)(x-10)}{10^7}\right|}$$

$\boxed{3.2.6}$

Polynomial $P_5(x)$ interpolates a function $f(x)$ w/ six samples $(x_i, f(x_i))$

$x_1 = 0 \quad x_2 = 0.2 \quad x_3 = 0.4 \quad x_4 = 0.6, \ x_5 = 0.8, \ x_6 = 1$

- Interp-error @ $x = 0.3$ is $|f(0.3) - P_5(0.3)| = 0.01$

Estimate new interp-error of $|f(0.3) - P_7(0.3)|$ w/ new:

$x_6 = 0.1$ & $x_7 = 0.5$

$$|f(x) - P_5(x)| = \left| \frac{(x)(x-0.2)(x-0.4)(x-0.6)(x-0.8)(x-1.0)}{6!} \right| f^{(6)}(c) = 0.01$$

$$|f(x) - P_7(x)| = |f(x) - P_7(x)| \frac{(x-0.1)(x-0.5)}{1} \quad @ \ x = 0.3 \quad = 0.01$$

$$|f(0.3) - P_7(0.03)| = (0.01) \frac{(0.3-0.1)(0.3-0.5)}{(7 \cdot 8)} = \boxed{+9.9 \times 10^{-9}} \ ???$$

$\boxed{3.3.4}$ Use Chebyshev Polynomial, $Q_5(x)$ on interval $x \in [0.6, 1.0]$ for $f(x) = e^x$. Use interp to find worst-case error $|e^x - Q_5(x)|$ 6 pts.

- Error:

$$\varepsilon = |f(x) - P_5(x)| = \frac{(x-x_1)(x-x_2)\cdots(x-x_6)}{6!} f^{(6)}(c)$$

$x_1 = \cos\left(\frac{\pi}{12}\right)$ $x_2 = \cos\left(\frac{3\pi}{12}\right)$ $x_3 = \cos\left(\frac{5\pi}{12}\right)$

$x_4 = \cos\left(\frac{7\pi}{12}\right)$ $x_5 = \cos\left(\frac{9\pi}{12}\right)$ $x_6 = \cos\left(\frac{11\pi}{12}\right)$

- By Thm 3.6: $|(x-x_1)(x-x_2)\cdots(x-x_6)| \le \frac{1}{2^5}$ & $f^{(6)} \le e^1$

- Error: $|e^x - P_5(x)| \le \frac{e}{(2^5)(6!)} \approx \boxed{1.1798 \times 10^{-4}}$

$\boxed{3.2.2}$ Computer Problems

Plot the interpolation error of the sin1 Key in program 3.3 on $x \in [-2\pi, +2\pi]$

$\varepsilon_p = |f(x) - P(x)|$ → See "Sin Error.png"

$\boxed{3.3.2}$ Computer problems

Build MATLAB to eval $\cos(x)$ to 10 decimals w/ Chebyshev - interp.

- Interp on domain $x \in [0, \pi/2]$ & extend to $\pm 10^4$

Chebyshev base pts are $x_i = \cos\left(\frac{(2i-1)\pi}{2N}\right)$, $N = 10$

on intv. $[a,b]$ $x_i = \frac{b+a}{2} + \frac{b-a}{2}\cos\left(\frac{(2i-1)\pi}{2N}\right)$

$\underbrace{\frac{\pi}{2} + \frac{\pi}{2}\cos\left(\frac{(2i-1)\pi}{2N}\right)}_{[0,\pi/2]} \rightarrow \underbrace{((2 \times 10^4) + 0}_{[-10^4, +10^4]}$;

see "cos2.m"