Serebv Beull

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MATH 753.01

HW #07

$\boxed{5.1.6}$ Use 3-pt. centered difference for second derivative to approximate $f''(0)$ where $f(x) = \cos(x)$ for:

- General form: $f''(x) = \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} - \frac{h^2}{12} f^{(4)}(c)$

a) $h = 0.1$

$$f''(0) = \frac{f(-0.1) - 2f(0) + \cancel{f(0.1)}}{(0.1)^2} - \frac{h^2}{12} f^{(4)}(c) \Rightarrow \frac{2f(0.1) - 2f(0)}{(0.01)} - \frac{(0.01)}{12} f^{(4)}(c)$$

" $\Rightarrow \boxed{f''(0) \approx -0.9992}$ ← see "Three Point Centered Difference 2nd Deriv.m"

- Actual error $= (-1.000) - (-0.9992) = \boxed{0.0008}$
- $\mathcal{E}_r(x) = 8.3333 \times 10^{-4}$ $\quad \mathcal{E}_r(x+h) = 8.2917 \times 10^{-4}$

b) $h = 0.01$ (use >>> format long)

$\boxed{f''(0) \approx -0.9999\ldots}$ $\boxed{\mathcal{E}_{act}(x) = +8.3333 \times 10^{-6}}$

c) $h = 0.001$

$\boxed{f''(0) \approx -0.9999\ldots}$ $\boxed{\mathcal{E}_{act}(x) = 8.3349 \times 10^{-8}}$

$\boxed{5.1.8}$ Prove second order formula for 1st derivative:

$$f'(x) = \frac{-f(x+2h) + 4f(x+h) - 3f(x)}{2h} + O[h^2]$$

- Expand terms (Assuming 3x differentiable)

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2f''(x) + O[h^3]$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + O[h^3]$$

$$-f(x+2h) + 4f(x+h) = +3f(x) - 2hf'(x) + O[h^3]$$

- Solve for $f'(x)$

$$2hf'(x) = -f(x+2h) + 4f(x+h) - 3f(x) + O[h^3]$$

$$\Downarrow$$

$$f'(x) = \frac{1}{2h}\left[-f(x+2h) + 4f(x+h) - 3f(x)\right] + O[h^2] \checkmark$$

$\boxed{5.1.9}$ Develop 2nd-order formula for first derivative, $f'(x)$ in terms of $f(x)$ & $f(x-h)$ & $f(x-2h)$

- If $f(x)$ is 3x's differentiable

$$\begin{cases} f(x-2h) = f(x) - 2h f'(x) + 2h^2 f''(x) + O[h^3] \\ f(x-h) = f(x) - h f'(x) + \frac{h^2}{2} f''(x) + O[h^3] \end{cases}$$

- Solve for $f'(x)$
- Subtract $f(x-2h) - 4f(x-h)$:

$$f(x-2h) - 4f(x-h) = -3f(x) - 2hf'(x) + O[h^3]$$

$$\rightarrow -2hf'(x) = f(x-2h) - 4f(x-h) + 3f(x) + O[h^3]$$

order drop 1 b/c $\frac{1}{h}$

$$\rightarrow \boxed{f'(x) = \frac{1}{2h}\left[f(x-2h) - 4f(x-h) + 3f(x)\right] + O[h^2]}$$

$\boxed{5.1.15}$ Develop a 1st-order method for approximating $f''(x)$ that uses $f(x-h)$, $f(x)$ & $f(x+3h)$ only

- If $f(x)$ is 4x's continuously differentiable

$$f(x+3h) = f(x) + 3hf'(x) + \frac{3^2h^2}{2}f''(x) + \frac{3^3h^3}{6}f'''(x) + \frac{3^4h^4}{4!}f''''(c_1)$$

$$f(x-h) = f(x) - hf'(x) + \frac{1}{2}h^2f''(x) - \frac{1}{6}h^3f'''(x) + \frac{h^4}{4!}f''''(c_2)$$

- With: $x-h < c_1 < x < c_2 < x+3h$ ($f'(x)$ dies!)

$$\left\{ \frac{1}{3}f(x+3h) + f(x-h) = \frac{4}{3}f(x) + 2h^2f''(x) + \frac{4}{3}h^3f'''(x) + \underbrace{\frac{7}{12}h^4f''''(c)}_{\to \text{ Using IVT}} \right.$$

- Solve for $f''(x)$

$$f''(x) = \frac{1}{h^2}\left[\frac{1}{6}f(x+3h) + \frac{1}{2}f(x-h) - \frac{2}{3}f(x)\right] - \frac{2}{3}hf'''(x) - \frac{7}{24}h^4f''''(c)$$

- Please note: The "IVT" term in the brackets gave me some trouble, so I found a similar problem from Pearson's Website and adapted the solution accordingly.

$\boxed{5.1.17}$ Develop a 2nd-order method for approximating $f'(x)$ that uses $f(x-2h)$, $f(x)$ & $f(x+3h)$ only

- We are 3x's continuously differentiable:

$$f(x+3h) = f(x) + 3hf'(x) + \frac{9}{2}h^2 f''(x) + \frac{27}{6}h^3 f'''(c_1)$$

$$f(x-2h) = f(x) - 2hf'(x) + 2h^2 f''(x) - \frac{4}{3}h^3 f'''(c_2)$$

- With $(x-2h < c_1 < x < c_2 < x+3h)$
- Eliminate $f''(x)$ term

$$f(x+3h) - \frac{9}{4}f(x-2h) = -\frac{5}{4}f(x) + \frac{15}{2}hf'(x) + \underbrace{\frac{9}{2}h^3 f'''(c_1) + \overset{=\frac{6}{2}}{3}h^3 f'''(c_2)}_{\text{Combine w/ IVT again}} {}^{=\frac{15}{2}}$$

$$'' \quad = -\frac{5}{4}f(x) + \frac{15}{2}hf'(x) + \frac{15}{2}h^3 f'''(c)$$

- Now $(x-2h < c < x+3h)$

$$f'(x) = \frac{h}{30}\left[f(x+3h) - 9f(x-2h) + 5f(x) \right] - \cancel{\frac{15}{2}}\, h^2 f'''(c)$$

$\boxed{5.1.20}$ Prove 2nd-order formula for 3rd derivative

$$f'''(x) = \frac{f(x-3h) - 6f(x-2h) + 12f(x-h) - 10f(x) + 3f(x+h)}{2h^3} + O[h^2]$$

- From class: (Go up to T_5 to ge $O[h^2]$)

$$f(x \pm h) = f(x) \pm hf'(x) + \frac{h^2}{2}f''(x) \pm \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) \pm \frac{h^5}{5!}f^{(5)}(x) + O[h^6]$$

$$f(x+2h) = f(x) + 2hf'(x) + \frac{4h^2}{2}f''(x) + \frac{8h^3}{3!}f'''(x) + \frac{16h^4}{4!}f^{(4)}(x) + \frac{32h^5}{5!}f^{(5)}(x) + O[h^6]$$

- We exploit some symmetry (antisymmetry around orign)
- Compute

$$f(x+2h) - f(x-2h) = 4hf'(x) - \frac{16}{3!}h^3 f'''(x) - \frac{64}{5!}h^5 f^{(5)}(x) + O[h^6]$$

$$-2f(x+h) + 2f(x-h) = -4hf'(x) + \frac{-4}{3!}h^3 f'''(x) + \frac{-4}{5!}h^5 f^{(5)}(x) + O[h^6]$$

- Aronge

$$\frac{1}{2h^3}\left[2h^3 f'''(x) + \frac{h^5}{2}f^{(5)}(x) + O[h^6]\right] \equiv \boxed{f'''(x) + \underbrace{\frac{h^4}{4}f^{(5)}(x) + O[h^3]}_{\text{Error}}}$$

$\boxed{5.1.2}$ Computer problem

Make a table & plot error of 3pt. centered difference of $f'(1)$, where $f(x) = (1+x)^{-1}$

$\rightarrow f(x) = \frac{1}{1+x} \quad f'(x) = \frac{(1+x)(0) - (1)(1)}{(1+x)^2} = -\frac{1}{(1+x)^2} \quad f'''(x) = -\frac{6}{(1+x)^4}$

$f'(1) = -\frac{1}{4}$ ← known Solution

$\rightarrow$ 3Pt - CD : $f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{6} f'''(c) \quad (x+h < c < x-h)$

See "Q512.m"