Nuclear-norm Regularized Model Let $D \in \mathbb{R}^{n\times t}$ be the reshaped video with n pixels and t frames. Assume that $\Phi_s \in \mathbb{R}^{n\times n}$ and $\Phi_t \in \mathbb{R}^{t\times t}$ are graph Laplacians in the spatial and temporal domains, respectively. Now we consider a foreground-background separation model of the form

$$\min_{L,S\in\mathbb{R}^{n\times t}} \frac{1}{2}\|D-L-S\|_F^2 + \lambda_1\|L\|_* + \lambda_2\|S\|_1 + \frac{\gamma_1}{2}\operatorname{tr}(L^T\Phi_s L) + \frac{\gamma_2}{2}\operatorname{tr}(L\Phi_t L^T).$$

By introducing an auxiliary variable U, we rewrite the above problem

$$\min_{L,S,U} \frac{1}{2}\|D-L-S\|_F^2 + \lambda_1\|U\|_* + \lambda_2\|S\|_1 + \frac{\gamma_1}{2}\operatorname{tr}(L^T\Phi_s L) + \frac{\gamma_2}{2}\operatorname{tr}(L\Phi_t L^T), \qquad \text{s.t.} \qquad U = L.$$

Define the augmented Lagrangian

$$\mathcal{L}(L,S,U) = \frac{1}{2}\|D-L-S\|_F^2 + \lambda_1\|U\|_* + \lambda_2\|S\|_1 + \frac{\gamma_1}{2}\operatorname{tr}(L^T\Phi_s L) + \frac{\gamma_2}{2}\operatorname{tr}(L\Phi_t L^T) + \frac{\rho}{2}\left\|U-L+\widetilde{L}\right\|_F^2.$$

Based on the alternating direction method of multipliers (ADMM), we derive the algorithm

$$\begin{cases} L \leftarrow \operatorname*{argmin}_L \frac{1}{2}\|D-L-S\|_F^2 + \frac{\gamma_1}{2}\operatorname{tr}(L^T\Phi_s L) + \frac{\gamma_2}{2}\operatorname{tr}(L\Phi_t L^T) + \frac{\rho}{2}\left\|U-L+\tilde{L}\right\|_F^2 \\ S \leftarrow \operatorname*{argmin}_S \lambda_2\|S\|_1 + \frac{1}{2}\|D-L-S\|_F^2 \\ U \leftarrow \operatorname*{argmin}_U \lambda_1\|U\|_* + \frac{\rho}{2}\left\|U-L+\widetilde{L}\right\|_F^2 = \operatorname*{argmin}_U \frac{\lambda_1}{\rho}\|U\|_* + \frac{1}{2}\left\|U-L+\widetilde{L}\right\|_F^2 \\ \widetilde{L} \leftarrow \widetilde{L} + (U-L) \end{cases}$$

The first L-subproblem is a least-square problem with the normal equation

$$L + S - D + \gamma_1\Phi_s L + \gamma_2 L\Phi_t + \rho(L-U-\widetilde{L}) = 0$$

Note that $\frac{d}{dX}\operatorname{tr}(X^TAX) = (A+A^T)X$, $\frac{d}{dX}\operatorname{tr}(XAX^T) = X(A+A^T)$ and both Φ_t and Φ_s are symmetric. The above equation can be rewritten in the Sylvester equation form

$$(I + \gamma_1\Phi_s + \gamma_2 I + \rho I)L + L(\gamma_2\Phi_t) = D - S + \rho(U+\widetilde{L}).$$

This can give us a unique solution under certain condition via `scipy.linalg.solve_sylvester(A,B,Q)` for solving the standard Sylvester equation of the form $AX + XB = Q$

$$L = \texttt{sylvester\_solver}(A,B,Q),$$

where $A = (1+\gamma_2+\rho)I + \gamma_1\Phi_s$, $B = \gamma_2\Phi_t$, and $Q = D - S + \rho(U+\widetilde{L}))$. The S-subproblem has the closed-form solution

$$S \leftarrow \operatorname{shrink}(D-L, \lambda_2).$$

Here the shrinkage operator is defined as $(\operatorname{shrink}(A,\mu))_{ij} = \operatorname{sign}(a_{ij})\cdot\max\{|a_{ij}|-\mu, 0\}$ where a_{ij} is the (i,j)-th entry of A. The U-subproblem has the closed-form solution via the singular value thresholding operator (SVT)

$$U \leftarrow A\widetilde{D}B, \quad \widetilde{D} = \operatorname{shrink}(D, \lambda_1/\rho)$$

where ADB is the SVD of the matrix $(L-\widetilde{L})$.