Calculation of physical constants using Johnson-Nyquist Noise

Lab Report #2

Landon Osborne (7159452) - *Physics BS*
Partner : Brian Ly

November 5th, 2022

Abstract

In this experiment our goal was to use the properties of Johnson-Nyquist noise in resistors to measure the values of physical constants. The constants we measured were the temperate of absolute zero in Celsius and the value of the Boltzmann constant, k_B. Our experiment consisted of measuring the noise produced by a set of resistors when exposed to two different temperatures, the temperature of the room we conducted our experiment in and the temperature of liquid Nitrogen. From our data we calculated absolute zero to be -226.74 ± 10.54 °C and $k_B = 1.153 \times 10^{-23} \pm 4.913 \times 10^{-25} \frac{J}{K}$ when our experiment was conducted at room temperature. When at the temperature of liquid Nitrogen we calculated absolute zero to be -258.98 ± 2.65 °C and $k_B = 1.129 \times 10^{-23} \pm 4.784 \times 10^{-25}$. Comparing these to the actual values for absolute zero, -273.15 °C, and the Boltzmann constant, $k_B = 1.381 \times 10^{-23}$ it can be seen that while our calculations were on the same order of magnitude as the correct values, there is still a substantial difference between them, and the range of uncertainty in our measurements does not overlap with any of the expected values for the physical constants. Ultimately while close to the expected results our experiment did not accurately determine the values of the physical constants.

Introduction

Johnson-Nyquist Noise is a thermodynamic property of resistors in all circuits. It is the electronic noise generated by thermal vibrations of the electrons inside of the conductors in a circuit. The Johnson noise increases with temperature, meaning that the warmer the conductor, the higher the voltage of the noise produced. This phenomenon was first discovered and measured by John B. Johnson in 1926 and quantified mathematically by Harry Nyquist in a paper in 1928 [1]. Nyquist's Theorem for the RMS voltage V_j across any resistor R at temperature T over a change in frequency df is

$$dV_j^2 = 4Rk_BTdf \tag{1}$$

By measuring the voltage created by the noise for different values for resistance and temperature, this equation can be used to draw conclusions about the values of physical constants involved, such as the temperature of absolute zero and the Boltzmann constant, k_B.

Measuring the noise voltage creates a frequency dependent resistance R_f caused by the system's capacitance creating a low pass filter. This can be modeled by

$$R_f = \frac{R}{1 + (2\pi fCR)^2}$$

In most cases the voltage produced from noise is too small to measure directly, so in order to properly measure the noise through one resistor we needed to amplify this noise voltage using an amplifier. This creates an amplified voltage dependent on the gain function of the amplifier, $g(f)$, and the noise voltage, shown in the equation

$$dV^2 = [g(f)]^2 dV_j^2$$

Substituting this equation into equation 1 and integrating gives the expected voltage as seen by the measuring device

$$V^2 = 4Rk_BTG \tag{2}$$

where the Band Gain, G, of the system is

$$G = \int_0^\infty \frac{|g(f)|^2}{1 + (2\pi fCR)^2} df \tag{3}$$

and

$$V^2 = V_{meas}^2 - V_{system}^2 \tag{4}$$

By combining the measurements of noise voltage, resistance, capacitance, and Gain we can use their relation in equation 2 to extrapolate the other values, like the temperature, T, or k_B.

Experimental Methods

The procedure for our experiment can be broken down into several different processes that are as follows. Setting up the equipment, determining the characteristics of the system, measuring the noise at room temperature, and measuring the noise at the temperature of liquid Nitrogen.

Setup

This experiment uses a pre-amplifier, filter, white noise generator, and a spectrum analyzer, and they all need to be set up and connected under the proper settings to avoid any damage to the equipment or opportunity for injury. Most important is calibrating the spectrum analyzer and avoiding overloading the voltage on the pre-amplifier, which will damage it. To calibrate the spectral analyzer, first turn it on with nothing attached to the input. Then in the input range of the spectral analyzer increase the attenuation of the analyzer until the curve is flat and there are no peaks (there will always be a peak around 0 kHz because of the pink noise, but this is inconsequential for our experiment so ignore it). As for the pre-amplifier it will overload if the voltage through it is too high, which can be seen by a bright red **OVLD** light flashing. When this happens either reduce the amplifier gain or the input voltage until the light stops flashing.

To setup the experiment we connect the source of noise into the input of the pre-amplifier, which boosts the noise voltage. This is then fed into the input of the filter, which for our experiment was set to a bandpass filter from 3 kHz to 10 kHz. This then outputs into the spectrum analyzer that measures the voltage at different frequencies. A diagram for the setup of the experiment can be seen below in figure 1.

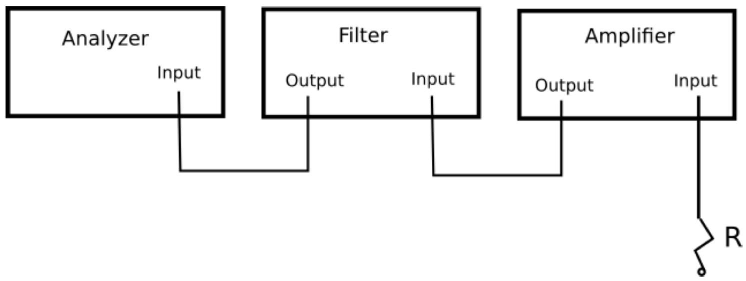

Figure 1: Diagram of the setup for our experiment taken from the lab manual [2]. The noise from the source R is amplified and feeds into the filter and the spectral analyzer. To characterize the gain function of the amplifier/filter the source of the noise is the white noise generator, but for the experiments we used different resistors to create the noise. The temperature of the resistor directly affects the noise it produces. In the second part of our experiment the resistor was placed in a dewer filled with liquid Nitrogen, lowering its temperature to 77 K.

System Characteristics

Before being able to quantify the way the noise changes under different conditions first we need to measure and quantify the characteristics of the system. These characteristics include the gain function of the amplifier/filter, the total capacitance of the system, and the resistance of the various resistors used in the experiment.

First, the gain function of the amplifier/filter. This is found using a white noise generator to create an equally distribution of noise voltage across all frequencies, and comparing the noise to when it is put through the system as follows. The procedure is to first calibrate the spectral analyzer to remove the background level of noise as described in the Setup section above. Then plug the white noise generator directly into the spectrum analyzer and increase the generators amplitude until it is higher than the background of the spectral analyzer. Then move the white noise generator into the input of the pre-amplifier and adjust the gain settings to be at the maximum possible without overloading. Record the settings of the white noise generator and pre-amplifier. Next take several linear averaged data sets of the spectrum analyzer when the white noise generator is plugged directly into the spectrum analyzer and again when the white noise generator is plugged into the amplifier and the system is set up as shown in figure 1.

Using these two data sets we can create a gain function for the system by using the equation

$$g(f) = \frac{V_o(f)}{V_i(f)}$$

where $V_o(f)$ is the amplified white noise data and $V_i(f)$ is the white noise data.

This data for the system gain is then fitted to a Gaussian function so that gain function has a mathematical representation of

$$g(f) = Ae^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{5}$$

For the values of resistance measure each of the resistors using a DMM or other probe and for the total capacitance measure the resistors, the BNC connector cables, and the electronics themselves, and then add them in series with the formula

$$\frac{1}{C_{tot}} = \frac{1}{C_1} + \frac{1}{C_2} + ...$$

to find the total capacitance of the system C_{tot}.

Noise Room Temperature

For this experiment we take different resistors and plug them into the input of the pre-amplifier, which displays the noise voltage on the spectrum analyzer. By setting the spectral analyzer to band analysis mode we can find the V_{rms} of the 3 kHz to 10kHz frequency band. We took 5 points of data for each resistor and then averaged them to one value of V_{rms} for each value of resistor. Later we analyze this data to extrapolate the value of absolute zero and the Boltzmann constant.

Noise Liquid Nitrogen

The procedure for this experiment is the same as the previous at room temperature, except that the resistors are placed in a dewer filled with liquid Nitrogen until their temperature is lowered to 77 K. We took 5 points of data for each resistor and then averaged them to one value of V_{rms} for each value of resistor. Later we analyze this data to extrapolate the value of absolute zero and the Boltzmann constant.

Raw Data

System Characteristics

The data collected for our system characteristics include the data for the gain function, seen in figure 2, the values of the resistors, seen in table 1, and the system's capacitance, seen in table 2. Our setup used a white noise generator voltage of amplitude 15 mV and a gain setting of 2×10^3.

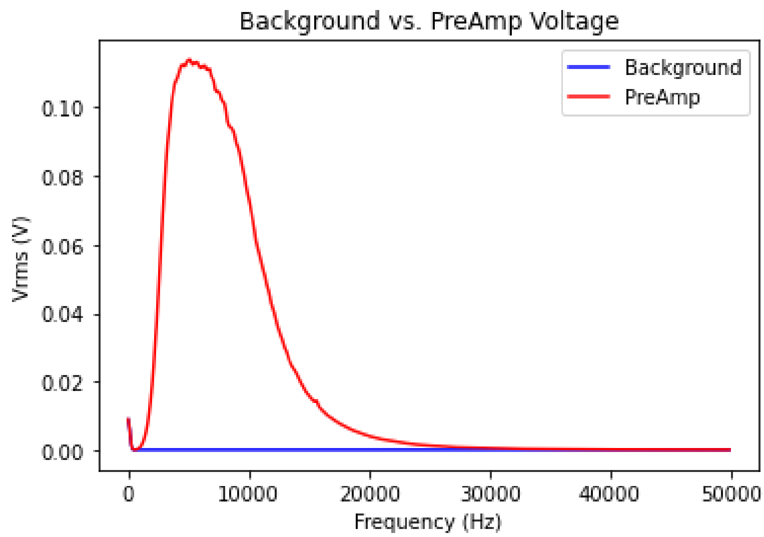

Figure 2: This graph shows the difference between the unamplified voltage of the white noise generator (background), and the amplified voltage from the white noise generator. The peak in the amplified voltage around 0 kHz is a consequence of the circuits pink noise, or $\frac{1}{f}$ interference, which is present in all electronics. From the data the bandpass filter can be seen as the amplified voltage falls off below 3 kHz and above 10kHz.

R (Ω)	δR (Ω)
996	12
9,990	130
20,100	180
35,220	260
48,650	320
100,500	1,300

Table 1: Measured resistance and associated uncertainty values. Measurements had an accuracy of $\pm(0.5\% + 8$ digits)

Source	Capacitance (pF)	δC (pF)
Resistors	75	0.1
BNC # 1	76	0.1
BNC # 2	76	0.1
Pre-Amplifier	25	0.1
Band Filter	50	0.1
Spectrum Analyzer	15	0.1

Table 2: List of the measured capacitance of different parts of the experiments circuit. For the resistors and BNC cables we used a conversion of $80\frac{\text{pF}}{\text{m}}$. For the electronics we used the rated values on the frame or in the manual.

From this data we found our system had a measured total capacitance of $C_{tot} = 6.0 \pm 0.05$ pF. δC_{tot} was found with the equation

$$\delta C_{tot} = |C_{tot}|\sqrt{\left(\frac{\delta C_1}{C_1}\right)^2 + \left(\frac{\delta C_2}{C_2}\right)^2 + ...}$$

Noise Room Temperature

Our data is the average of the band voltage for each of the resistors at room temperature. We measured the room temperature to be 22 °C for the entirety of the experiment.

R (Ω)	V_{rms} (V)	δV_{rms} (V)
0	3.914×10^{-4}	2.747×10^{-5}
996	5.974×10^{-4}	4.194×10^{-5}
9,990	1.456×10^{-3}	1.022×10^{-4}
20,100	2.017×10^{-3}	1.416×10^{-4}
35,220	2.630×10^{-3}	1.846×10^{-4}
48,650	3.820×10^{-3}	2.682×10^{-4}
100,500	4.241×10^{-3}	2.977×10^{-4}

Table 3: Measured values of V_{rms} of resistors at room temperature.

where the accuracy of the band V_{rms} is given by the manual for the SR760 spectrum analyzer [3] as

$$\frac{\delta V_{rms}}{|V_{rms}|} = \pm 7.02\%$$

Noise Liquid Nitrogen

Our data is the average of the band voltage for each of the resistors when submerged in liquid Nitrogen.

R (Ω)	V_{rms} (V)	δV_{rms} (V)
0	3.922×10^{-4}	2.753×10^{-5}
996	4.576×10^{-4}	3.212×10^{-5}
9,990	8.383×10^{-4}	5.885×10^{-5}
20,100	1.110×10^{-3}	7.791×10^{-5}
35,220	1.417×10^{-3}	9.947×10^{-5}
48,650	1.612×10^{-03}	1.132×10^{-4}
100,500	2.104×10^{-3}	1.477×10^{-4}

Table 4: Measured values of V_{rms} of resistors at temperature of liquid Nitrogen.

Results

Using our raw data in the previous section we calculated several quantities that we used in our final analysis. These included the band gain and the values of V^2 for both room temperature and in liquid Nitrogen.

Band Gain

From our data visualized in figure 2 we calculated the gain function for our system and then fitted the data to the gaussian curve given by equation 5. This data for the gain and fit function is visualized in figure 3. Our calculated fit parameters were $A = 2382.21 \pm 29.474$, $\mu = 6761.75 \pm 48.836$, and $\sigma = 3411.83 \pm 50.754$

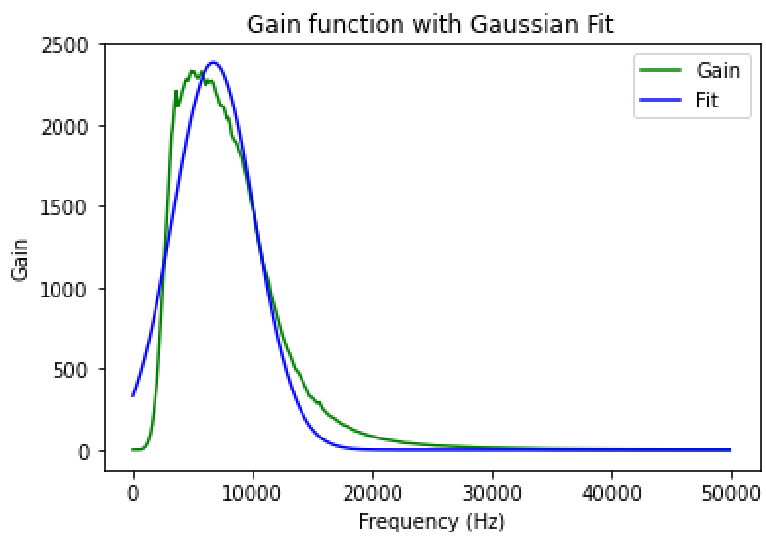

Figure 3: This graph shows our calculated gain function, $g(f)$, found by dividing the amplified voltage by the background, and the Gaussian fit for this data. From the data bandpass filter can be seen as the gain falls off below 3 kHz and above 10 kHz.

Using equation 3 we calculated the band gain, G, for our system's bandwidth with the formula

$$G = \int_{3\text{kHz}}^{10\text{kHz}} \frac{|g(f)|^2}{1 + (2\pi fCR)^2} df$$

for each value of resistor. This gave us the resulting band gain values of shown below in table 5.

R (Ω)	G	δG
996	2.919 $\times 10^10$	7.585 $\times 10^8$
9,990	2.919 $\times 10^10$	7.585 $\times 10^8$
20,100	2.919 $\times 10^10$	7.586 $\times 10^8$
35,220	2.919 $\times 10^10$	7.586 $\times 10^8$
48,650	2.919 $\times 10^10$	7.589 $\times 10^8$
100,500	2.918 $\times 10^10$	7.597 $\times 10^8$

Table 5: Calculated values of the band gain for each resistor and the associate uncertainty of G.

The value for δG was found by propagating error with the equation

$$\delta G = \sqrt{\left(\frac{\partial G}{\partial A}\delta A\right)^2 + \left(\frac{\partial G}{\partial \mu}\delta \mu\right)^2 + \left(\frac{\partial G}{\partial \sigma}\delta \sigma\right)^2 + \left(\frac{\partial G}{\partial C}\delta C\right)^2 + \left(\frac{\partial G}{\partial R}\delta R\right)^2}$$

Room Temperature

From our raw data for the band V_{rms} we then calculated the value for V_{rms}^2 by squaring the measured value and subtracting the value of the shorted resistor in quadrature, as shown in equation 4. These values are shown below in table 6.

R (Ω)	V_{rms}^2 (V)	δV_{rms}^2(V)
996	2.037×10^{-7}	2.022×10^{-8}
9,990	1.967×10^{-6}	1.953×10^{-7}
20,100	3.915×10^{-6}	3.886×10^{-7}
35,220	6.764×10^{-6}	6.715×10^{-7}
48,650	1.443×10^{-5}	1.433×10^{-6}
100,500	1.783×10^{-5}	1.770×10^{-6}

Table 6: Values for $V_{meas}^2 - V_{system}^2$ at room temperature with associate uncertainty values. Here the system refers to the value of the short resistor.

The values for δV_{rms}^2 were propogated with the error propogation formula

$$\frac{\delta V_{rms}^2}{|V_{rms}^2|} = \sqrt{2\left(\frac{\delta V_{rms}}{V_{rms}}\right)^2}$$

From here the data was linearly fitted to the formula of

$$\frac{V_{rms}^2}{2k_B G} = R$$

where slope is the temperature T. This data and the fit line are visualized below in figure 4. For this equation we used the relation of

$$V_{peak}^2 = 2V_{rms}^2$$

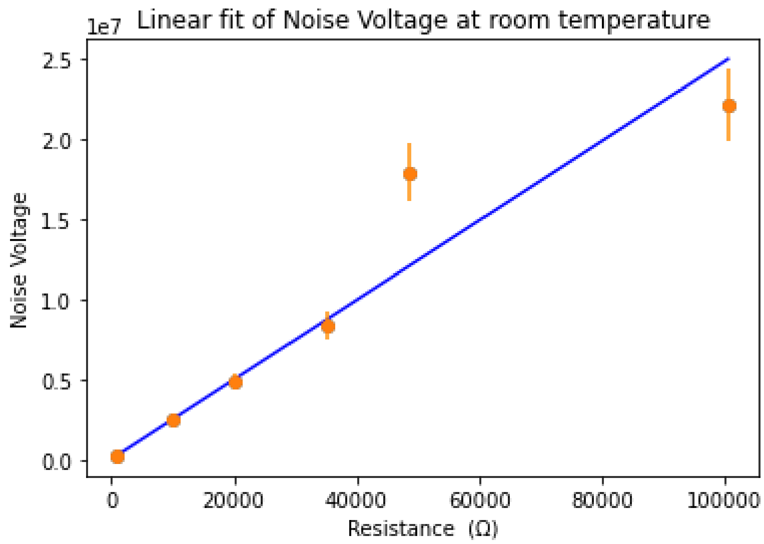

Figure 4: This graph shows the linear fit of the data we measured for resistance and noise voltage with our resistors at room temperature. From this fit the slope of the line is the calculated temperature of the resistor. The data points and associate error bars are shown in orange and the fitted line is blue. In this fit we found a slope of $T = 248.74 \pm 10.54$ °K.

Taking the value of T from this fitted data as an absolute scale in centigrade, and comparing to our measured value for room temperature of 22 °C we calculate that absolute zero is -226.74 ± 10.54 °C. The uncertainty for this calculation came from taking

$$Y = \frac{V_{rms}^2}{2k_B G}$$

then

$$\delta Y = |Y|\sqrt{\left(\frac{\delta V_{rms}^2}{V_{rms}^2}\right)^2 + \left(\frac{\delta G}{G}\right)^2}$$

and then using this value of δY as the value of uncertainty in our fit calculations.

Next we used our data to find the Boltzmann constant by plotting our data with the formula

$$\frac{V_{rms}^2}{2RG} = T$$

and fitting it to a line where the slope is the Boltzmann constant k_B. The uncertainty for this calculation came from taking

$$Y = \frac{V_{rms}^2}{2RG}$$

then

$$\delta Y = |Y|\sqrt{\left(\frac{\delta V_{rms}^2}{V_{rms}^2}\right)^2 + \left(\frac{\delta G}{G}\right)^2 + \left(\frac{\delta R}{R}\right)^2}$$

and then using this value of δY as the value of uncertainty in our fit calculations.

The results of this linear fit gave us a value of $k_B = 1.153 \times 10^{-23} \pm 4.913 \times 10^{-25} \frac{J}{K}$

Liquid Nitrogen

For our experiment with the resistors chilled by liquid Nitrogen we performed the same calculations to find V_{rms}^2 and fit our data the same way as described above. The calculations and data are visualized below in table 7 and figure 5.

R (Ω)	V_{rms}^2 (V)	δV_{rms}^2 (V)
996	5.558×10^{-8}	5.518×10^{-9}
9,990	5.489×10^{-7}	5.449×10^{-8}
20,100	1.077×10^{-6}	1.070×10^{-7}
35,220	1.854×10^{-6}	1.840×10^{-7}
48,650	2.446×10^{-6}	2.428×10^{-7}
100,500	4.274×10^{-6}	4.243×10^{-7}

Table 7: Values for $V_{meas}^2 - V_{system}^2$ in liquid Nitrogen with associated uncertainty values. Here the system refers to the value of the short resistor.

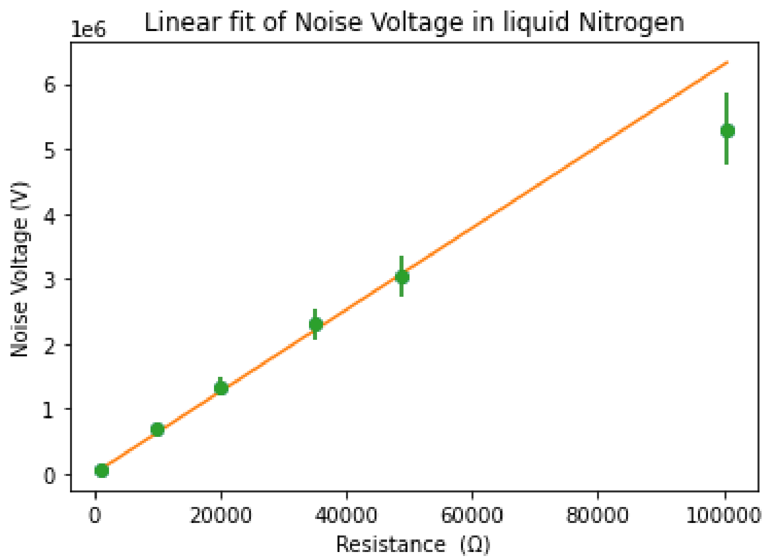

Figure 5: This graph shows the linear fit of the data we measured for resistance and noise voltage with our resistors in liquid Nitrogen. From this fit the slope of the line is the calculated temperature of the resistor. The data points and associate error bars are shown in green and the fitted line is orange. In this fit we found a slope of $T = 62.98 \pm 2.654$ K.

Taking the value of T from this fitted data as an absolute scale in centigrade, and comparing to our known value for the temperature of liquid Nitrogen of -196°C we calculate that absolute zero is -258.98 ± 2.65 °C.

Next we used our data to find the Boltzmann constant by plotting our data with the formula

$$\frac{V_{rms}^2}{2RG} = T$$

and fitting it to a line where the slope is the Boltzmann constant k_B. The results of this linear fit gave us a value of $k_B = 1.129 \times 10^{-23} \pm 4.784 \times 10^{-25} \frac{J}{K}$

Discussion

Comparing the results from our experiment to the expected values for absolute zero and the Boltzmann constant it can be seen that the range of uncertainty in our measurements does not overlap with the actual values. However it does seem that, at least for the calculation of absolute zero, our results were much closer when using the low temperature of liquid Nitrogen. This could be due to the reduced amount of background noise and interference making our measurements more accurate.

	Absolute Zero (°C)	Boltzmann Constant ($\frac{J}{K}$)
Room Temperature	-226.74 ± 10.54	$1.153 \times 10^{-23} \pm 4.913 \times 10^{-25}$
Liquid Nitrogen	-258.98 ± 2.65	$1.129 \times 10^{-23} \pm 4.784 \times 10^{-25}$
Actual Value	-273.15	1.381×10^{-23}

Table 8: Calculated values for absolute zero and the Boltzmann constant compared to the actual values

This gap between our results and the actual results could be a result of unexpected sources of noise in our equipment or a larger than calculated value of capacitance in our system. Both of these possibilities could have had a large effect on our measurements and calculations.

Another area that had a large uncertainty was in our fit of the gain to a Gaussian function. The fit that we used for our calculation had large values of uncertainty on the parameters and did not model the data for the gain function as well as it could have. Repeating this part of the experiment and taking the average of more data sets could have resulted in a more accurate fit. The gain function is important because it is used multiple times in our calculations as the band gain, and the band gain uncertainty, δG, is used in the formulas of error propagation for both fits of temperature and Boltzmann constant. So any uncertainty or error in our Gaussian model could drastically effect all of our calculations through the band gain.

If we were to redo this experiment, a greater focus would be placed on finding an accurate Gaussian model of the gain function. Also attempting to use even lower temperatures than liquid Nitrogen could prove useful, and possibly allow for more accurate measurements for the value of absolute zero and the Boltzmann constant.

References

[1] H. Nyquist, "Thermal agitation of electric charge in conductors," *Phys. Rev.*, vol. 32, pp. 110–113, Jul 1928. [Online]. Available: https://link.aps.org/doi/10.1103/PhysRev.32.110

[2] A. Kerr, Aug 2017. [Online]. Available: https://web.physics.ucsb.edu/~phys128/experiments/noise/johnson-noise%20v1.1.pdf

[3] "Chap 00 intro - thinksrs.com," Mar 2018. [Online]. Available: https://www.thinksrs.com/downloads/pdfs/manuals/SR760m.pdf