November 15, 2016 Matt Landreman

Form of sonic flows in a tokamak or stellarator

Introduction

In these notes we consider what spatial pattern the plasma flow can take in a toroidal plasma (tokamak or stellarator) for 'sonic' flows, meaning speeds large enough to be comparable to the thermal speed. We will find several important results:

- If the flow is sonic, it must be in a direction along which the magnetic field strength B is constant.
- Therefore in a tokamak, a sonic flow must be toroidal, with no poloidal component. Each flux surface rotates rigidly.
- Sonic flows are not permitted in a general stellarator.
- Sonic flows are however permitted in the symmetry direction of a perfectly quasisymmetric stellarator, at least at this level of analysis.

References

Variations of the calculation here have been published previously in several papers:

- Hinton & Wong, Phys Fluids 28, 3082 (1985). See section II and IV.
- Helander, Phys Plasmas 14, 104501 (2007).
- Abel et al, Rep. Prog. Phys. 76, 116201 (2013).

Orderings

We will order everything using the basic expansion parameter

$$\rho_* \ll 1 , \tag{1}$$

where

$$\rho_* = \frac{\upsilon_i}{\Omega L} , \tag{2}$$

$\upsilon_i = \sqrt{2T/m}$ is the ion thermal speed, $\Omega = qB/m$ is the gyrofrequency, and L is any macroscopic scale length. The distribution function, fields, and potentials are expanded as

$$f = f_0 + f_1 + ... \tag{3}$$

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1 + ... \tag{4}$$

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1 + ... \tag{5}$$

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1 + ... \tag{6}$$

$$\Phi = \Phi_0 + \Phi_1 + ... \tag{7}$$

(Note the difference in notation from Hinton-Wong, who begin some expansions with -1.) In this calculation we will consider the 'MHD' ordering:

$$\left|\mathbf{v}_E\right| \sim \upsilon_i . \tag{8}$$

For leading order quantities, we order time derivatives as

$$\frac{\partial\left(f_0,B_0,E_0,\ldots\right)}{\partial t}\ll\frac{\upsilon_i}{L}\left(f_0,B_0,E_0,\ldots\right)\ . \tag{9}$$

We order collisions in the usual way in ρ_*, as

$$C\sim\frac{\upsilon_i}{L}. \tag{10}$$

If we write the electric field in terms of the potentials,

$$\mathbf{E}=-\nabla\phi-\frac{\partial\mathbf{A}}{\partial t}. \tag{11}$$

Using $\mathbf{A}\sim BL$, the contribution of $\partial\mathbf{A}/\partial t$ to the ExB drift is

$$\frac{1}{B^2}\left(-\frac{\partial\mathbf{A}}{\partial t}\right)\times\mathbf{B}\ll\frac{1}{B^2}\left(\frac{\upsilon_i}{L}LB\right)B\sim\upsilon_i. \tag{12}$$

Therefore, for consistency with (8), the electric field must be electrostatic to leading order:

$$\mathbf{E}_0=-\nabla\Phi_0. \tag{13}$$

Velocity shift

The original Fokker-Planck equation is

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial\mathbf{r}}+\frac{q}{m}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial f}{\partial\mathbf{v}}=C\left\{f\right\}. \tag{14}$$

Since the ExB drift is comparable to the thermal speed, the gyro-orbits do not close in the lab frame. In order to make the gyro-orbits close to leading order, we use a shifted velocity coordinate

$$\mathbf{w}=\mathbf{v}-\mathbf{u}\left(\mathbf{r},t\right) \tag{15}$$

(same notation as in Abel) where

$$\mathbf{u}=\frac{1}{B^2}\mathbf{E}_0\times\mathbf{B}_0+u_{||}\mathbf{b} \tag{16}$$

where $B=\left|\mathbf{B}_0\right|$ and $\mathbf{b}=\mathbf{B}_0/B$, and $u_{||}$ is anything we like. We don't need to include $u_{||}$, but it is no extra work to include, and it will turn out to be convenient later. We order

$$\mathbf{u}\sim\upsilon_i. \tag{17}$$

Let us change variables to the shifted velocity $\mathbf{w}$:

$$\dot{t}\left(\frac{\partial f}{\partial t}\right)_{\mathbf{r},\mathbf{w}}+\dot{\mathbf{r}}\cdot\left(\frac{\partial f}{\partial\mathbf{r}}\right)_{t,\mathbf{w}}+\dot{\mathbf{w}}\cdot\left(\frac{\partial f}{\partial\mathbf{w}}\right)_{t,\mathbf{r}}=C\left\{f\right\}, \tag{18}$$

where

$$\dot{t}=\frac{\partial t}{\partial t}+\mathbf{v}\cdot\frac{\partial t}{\partial\mathbf{r}}+\frac{q}{m}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial t}{\partial\mathbf{v}}=1, \tag{19}$$

$$\dot{\mathbf{r}}=\frac{\partial\mathbf{r}}{\partial t}+\mathbf{v}\cdot\frac{\partial\mathbf{r}}{\partial\mathbf{r}}+\frac{q}{m}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial\mathbf{r}}{\partial\mathbf{v}}=\mathbf{v}, \tag{20}$$

$$\begin{aligned}\dot{\mathbf{w}} &= \frac{\partial(\mathbf{v}-\mathbf{u})}{\partial t} + \mathbf{v}\cdot\frac{\partial(\mathbf{v}-\mathbf{u})}{\partial \mathbf{r}} + \frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial(\mathbf{v}-\mathbf{u})}{\partial \mathbf{v}} \\ &= -\frac{\partial \mathbf{u}}{\partial t} - \mathbf{v}\cdot\frac{\partial \mathbf{u}}{\partial \mathbf{r}} + \frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B}).\end{aligned} \tag{21}$$

Thus,

$$\begin{aligned}&\left(\frac{\partial f}{\partial t}\right)_{\mathbf{r},\mathbf{w}} + (\mathbf{w}+\mathbf{u})\cdot\left(\frac{\partial f}{\partial \mathbf{r}}\right)_{t,\mathbf{w}} \\ &+\left[-\frac{\partial \mathbf{u}}{\partial t} - (\mathbf{w}+\mathbf{u})\cdot\frac{\partial \mathbf{u}}{\partial \mathbf{r}} + \frac{q}{m}\left(\mathbf{E}+[\mathbf{w}+\mathbf{u}]\times\mathbf{B}\right)\right]\cdot\left(\frac{\partial f}{\partial \mathbf{w}}\right)_{t,\mathbf{r}} = C\{f\}.\end{aligned} \tag{22}$$

This equation is equivalent to (3) in Hinton & Wong.

Leading order terms

Let us see how big all the terms are:

$$\left(\frac{\partial f}{\partial t}\right)_{\mathbf{r},\mathbf{w}} \ll \frac{\upsilon_i}{L} f_0, \tag{23}$$

$$(\mathbf{w}+\mathbf{u})\cdot\left(\frac{\partial f}{\partial \mathbf{r}}\right)_{t,\mathbf{w}} \sim \frac{\upsilon_i}{L} f_0, \tag{24}$$

$$\left[-\frac{\partial \mathbf{u}}{\partial t}\right]\cdot\left(\frac{\partial f}{\partial \mathbf{w}}\right)_{t,\mathbf{r}} \ll \frac{\upsilon_i}{L} f_0, \tag{25}$$

$$\left[-(\mathbf{w}+\mathbf{u})\cdot\frac{\partial \mathbf{u}}{\partial \mathbf{r}}\right]\cdot\left(\frac{\partial f}{\partial \mathbf{w}}\right)_{t,\mathbf{r}} \sim \frac{\upsilon_i}{L} f_0, \tag{26}$$

$$\frac{q}{m}\mathbf{E}\cdot\left(\frac{\partial f}{\partial \mathbf{w}}\right)_{t,\mathbf{r}} \sim \frac{E}{B}\frac{qB}{m}\frac{f_0}{\upsilon_i} \sim \Omega f_0 \sim \frac{1}{\rho_*}\frac{\upsilon_i}{L} f_0, \tag{27}$$

$$\frac{q}{m}[\mathbf{w}+\mathbf{u}]\times\mathbf{B}\cdot\left(\frac{\partial f}{\partial \mathbf{w}}\right)_{t,\mathbf{r}} \sim \frac{qB}{m} f_0 \sim \Omega f_0, \tag{28}$$

$$C\{f\} \sim \frac{\upsilon_i}{L} f_0. \tag{29}$$

So the largest terms are

$$\frac{q}{m}\left(\mathbf{E}_0 + [\mathbf{w}+\mathbf{u}]\times\mathbf{B}_0\right)\cdot\left(\frac{\partial f_0}{\partial \mathbf{w}}\right)_{t,\mathbf{r}} = 0. \tag{30}$$

Since

$$\mathbf{u}\times\mathbf{B}_0 = -\mathbf{E}_{0\perp}, \tag{31}$$

where $\mathbf{E}_0 = E_{0||}\mathbf{b} + \mathbf{E}_{0\perp}$, a cancellation occurs to leave

$$\frac{q}{m}\left(E_{0\|}\mathbf{b}+\mathbf{w}\times\mathbf{B}_0\right)\cdot\left(\frac{\partial f_0}{\partial \mathbf{w}}\right)_{t,\mathbf{r}}=0. \tag{32}$$

Implications of the leading-order equation

Equation (32) says that f_0 must be constant along the characteristic curves in velocity space $\mathbf{w}(\tau)$, which satisfy

$$\frac{d\mathbf{w}}{d\tau}=\frac{q}{m}\left[E_{0\|}\mathbf{b}+\mathbf{w}\times\mathbf{B}_0\right], \tag{33}$$

and τ parameterizes the curves. Let's compute these characteristic curves. Using coordinates (x,y,z) with $\mathbf{B}_0=B\mathbf{e}_z$,

$$\frac{dw_x}{d\tau}=\Omega w_y, \tag{34}$$

$$\frac{dw_y}{d\tau}=-\Omega w_x, \tag{35}$$

$$\frac{dw_z}{d\tau}=\frac{q}{m}E_{0z}, \tag{36}$$

where $\Omega=qB/m$. From (36),

$$w_z=w_{z0}+\frac{q}{m}E_{0z}\tau. \tag{37}$$

Applying $\partial/\partial\tau$ to (34) and substituting in (35) gives

$$\frac{d^2w_x}{d\tau^2}=\Omega\frac{dw_y}{d\tau}=-\Omega^2 w_x, \tag{38}$$

The solution of (38) is

$$w_x=A\cos(\Omega\tau-\gamma), \tag{39}$$

where A and γ are integration constants. Then from (34),

$$w_y=-A\sin(\Omega\tau-\gamma). \tag{40}$$

In other words,

$$\mathbf{w}(\tau)=\begin{pmatrix} A\cos(\Omega\tau-\gamma) \\ -A\sin(\Omega\tau-\gamma) \\ w_{z0}+\frac{q}{m}E_{-1z}\tau \end{pmatrix}. \tag{41}$$

Thus, if $E_{0\|}$ is nonzero these characteristic curves are helices in velocity space, oriented along $\mathbf{B}_0$, whereas the curves are closed circles if $E_{0\|}$ is zero. If $E_{0\|}$ is nonzero, the helices extend to infinity, and since f_0 must be constant along these curves, then f_0 must vanish everywhere. To avoid this unphysical result, then, we must have

$$\mathbf{E}_0 \cdot \mathbf{B}_0 = 0 . \tag{42}$$

Therefore

$$\mathbf{B}_0 \cdot \nabla \Phi_0 = 0 , \tag{43}$$

and so Φ_0 is a flux function.

Equation (32) has simplified to

$$\mathbf{w} \times \mathbf{B}_0 \cdot \left(\frac{\partial f_0}{\partial \mathbf{w}} \right)_{t,\mathbf{r}} = 0 . \tag{44}$$

Cylindrical velocity variables

Let us introduce cylindrical velocity coordinates in the usual way:

$$\mathbf{w} = w_{||} \mathbf{b} + \mathbf{w}_\perp , \tag{45}$$

$$\mathbf{w}_\perp = w_\perp \left(\mathbf{e}_1 \cos\varphi + \mathbf{e}_2 \sin\varphi \right), \tag{46}$$

where φ is called the gyrophase or gyroangle. It will be useful to transform the velocity-space gradient $\partial / \partial \mathbf{w}$ into $\left(w_{||}, w_\perp, \varphi \right)$ coordinates. To do this we first write a general change of variables:

$$\frac{\partial Y}{\partial \mathbf{w}} = \left(\frac{\partial w_{||}}{\partial \mathbf{w}} \right) \left(\frac{\partial Y}{\partial w_{||}} \right)_{w_\perp, \varphi} + \left(\frac{\partial w_\perp}{\partial \mathbf{w}} \right) \left(\frac{\partial Y}{\partial w_\perp} \right)_{w_{||}, \varphi} + \left(\frac{\partial \varphi}{\partial \mathbf{w}} \right) \left(\frac{\partial Y}{\partial \varphi} \right)_{w_{||}, w_\perp} \tag{47}$$

for any quantity Y. We obtain the first right-hand-side term by applying $\partial / \partial \mathbf{w}$ to $w_{||} = \mathbf{b} \cdot \mathbf{w}$, yielding

$$\frac{\partial w_{||}}{\partial \mathbf{w}} = \mathbf{b} \cdot \frac{\partial \mathbf{w}}{\partial \mathbf{w}} = \mathbf{b} . \tag{48}$$

Next, we apply $\partial / \partial \mathbf{w}$ to $\mathbf{w} \cdot \mathbf{w} = w_{||}^2 + w_\perp^2$ to obtain

$$\mathbf{w} = w_{||} \frac{\partial w_{||}}{\partial \mathbf{w}} + w_\perp \frac{\partial w_\perp}{\partial \mathbf{w}} . \tag{49}$$

Using (48) then gives

$$\frac{\partial w_\perp}{\partial \mathbf{w}} = \frac{\mathbf{w}_\perp}{w_\perp} . \tag{50}$$

Next, we apply $\partial / \partial \mathbf{w}$ to

$$\mathbf{e}_1 \cdot \mathbf{w} = w_\perp \cos\varphi \tag{51}$$

and

$$\mathbf{e}_2 \cdot \mathbf{w} = w_\perp \sin\varphi \tag{52}$$

yielding

$$\mathbf{e}_1 = \frac{\partial w_\perp}{\partial \mathbf{w}} \cos\varphi - w_\perp \sin\varphi \frac{\partial \varphi}{\partial \mathbf{w}} = \frac{\mathbf{w}_\perp}{w_\perp} \cos\varphi - w_\perp \sin\varphi \frac{\partial \varphi}{\partial \mathbf{w}} \tag{53}$$

and

$$\mathbf{e}_2 = \frac{\partial w_\perp}{\partial \mathbf{w}} \sin\varphi + w_\perp \cos\varphi \frac{\partial \varphi}{\partial \mathbf{w}} = \frac{\mathbf{w}_\perp}{w_\perp} \sin\varphi + w_\perp \cos\varphi \frac{\partial \varphi}{\partial \mathbf{w}} . \tag{54}$$

Forming $-\sin\varphi$ (53)+ $\cos\varphi$ (54) then gives

$$-\mathbf{e}_1 \sin\varphi + \mathbf{e}_2 \cos\varphi = w_\perp \frac{\partial \varphi}{\partial \mathbf{w}} . \tag{55}$$

Noting that

$$\mathbf{b} \times \mathbf{w} = w_\perp \left(\mathbf{e}_2 \cos\varphi - \mathbf{e}_1 \sin\varphi \right), \tag{56}$$

then (55) gives

$$\frac{\partial \varphi}{\partial \mathbf{w}} = \frac{\mathbf{b} \times \mathbf{w}}{w_\perp^2} . \tag{57}$$

Combining (47), (48), (50), and (57), we obtain a formula that is often useful in drift- and gyro-kinetics:

$$\frac{\partial Y}{\partial \mathbf{w}} = \mathbf{b} \left(\frac{\partial Y}{\partial w_{||}} \right)_{w_\perp, \varphi} + \frac{\mathbf{w}_\perp}{w_\perp} \left(\frac{\partial Y}{\partial w_\perp} \right)_{w_{||}, \varphi} + \frac{\mathbf{b} \times \mathbf{w}}{w_\perp^2} \left(\frac{\partial Y}{\partial \varphi} \right)_{w_{||}, w_\perp} . \tag{58}$$

Applying this result to (44), we obtain

$$\left(\frac{\partial f_0}{\partial \varphi} \right)_{w_{||}, w_\perp} = 0 , \tag{59}$$

so f_0 is independent of gyrophase in the shifted velocity frame.

Next order kinetic equation

Next, let us look for the terms in (22) of size $\left(\upsilon_i / L \right) f_0$:

$$\left(\mathbf{w} + \mathbf{u} \right) \cdot \left(\frac{\partial f_0}{\partial \mathbf{r}} \right)_{\mathbf{w}} - \left[\left(\mathbf{w} + \mathbf{u} \right) \cdot \nabla \mathbf{u} \right] \cdot \frac{\partial f_0}{\partial \mathbf{w}} + \frac{q}{m} \left(\mathbf{E}_1 + \left[\mathbf{w} + \mathbf{u} \right] \times \mathbf{B}_1 \right) \cdot \frac{\partial f_0}{\partial \mathbf{w}} + \frac{q}{m} \mathbf{w} \times \mathbf{B}_0 \cdot \frac{\partial f_1}{\partial \mathbf{w}} = C\left\{ f_0 \right\} . \tag{60}$$

We can anticipate gyro-averaging to annihilate the unknown f_1 term. In preparation for this averaging, let us change from Cartesian to cylindrical velocity variables. For the first term in (60), we have

$$\begin{aligned} \left(\frac{\partial f_0}{\partial \mathbf{r}} \right)_{\mathbf{w}} &= \left(\frac{\partial f_0}{\partial \mathbf{r}} \right)_{w_{||}, w_\perp, \varphi} + \left(\frac{\partial w_{||}}{\partial \mathbf{r}} \right)_{\mathbf{w}} \left(\frac{\partial f_0}{\partial w_{||}} \right)_{\mathbf{r}, w_\perp, \varphi} \\ &\quad + \left(\frac{\partial w_\perp}{\partial \mathbf{r}} \right)_{\mathbf{w}} \left(\frac{\partial f_0}{\partial w_\perp} \right)_{\mathbf{r}, w_{||}, \varphi} + \left(\frac{\partial \varphi}{\partial \mathbf{r}} \right)_{\mathbf{w}} \underbrace{\left(\frac{\partial f_0}{\partial \varphi} \right)_{\mathbf{r}, w_{||}, w_\perp}}_{=0} . \end{aligned} \tag{61}$$

Since $w_{||} = \mathbf{b} \cdot \mathbf{w}$,

$$\left(\frac{\partial w_{\parallel}}{\partial \mathbf{r}}\right)_{\mathbf{w}} = \left(\frac{\partial\left(\mathbf{b}\cdot\mathbf{w}\right)}{\partial \mathbf{r}}\right)_{\mathbf{w}} = \left(\nabla\mathbf{b}\right)\cdot\mathbf{w}\,. \tag{62}$$

Then applying $\left(\partial/\partial\mathbf{r}\right)_{\mathbf{w}}$ to

$$w_{\perp}^2 = w^2 - w_{\parallel}^2 \tag{63}$$

we get

$$w_{\perp}\left(\frac{\partial w_{\perp}}{\partial \mathbf{r}}\right)_{\mathbf{w}} = -w_{\parallel}\left(\frac{\partial w_{\parallel}}{\partial \mathbf{r}}\right)_{\mathbf{w}} = -w_{\parallel}\left(\nabla\mathbf{b}\right)\cdot\mathbf{w} \tag{64}$$

Thus,

$$\left(\frac{\partial f_0}{\partial \mathbf{r}}\right)_{\mathbf{w}} = \left(\frac{\partial f_0}{\partial \mathbf{r}}\right)_{w_{\parallel},w_{\perp},\varphi} + \left(\nabla\mathbf{b}\right)\cdot\mathbf{w}\left[\left(\frac{\partial f_0}{\partial w_{\parallel}}\right)_{\mathbf{r},w_{\perp},\varphi} - \frac{w_{\parallel}}{w_{\perp}}\left(\frac{\partial f_0}{\partial w_{\perp}}\right)_{\mathbf{r},w_{\parallel},\varphi}\right]. \tag{65}$$

Then (58) can be written

$$\begin{aligned}
&\left(\mathbf{w}+\mathbf{u}\right)\cdot\frac{\partial f_0}{\partial \mathbf{r}} + \left(\mathbf{w}+\mathbf{u}\right)\cdot\left(\nabla\mathbf{b}\right)\cdot\mathbf{w}\left[\left(\frac{\partial f_0}{\partial w_{\parallel}}\right) - \frac{w_{\parallel}}{w_{\perp}}\left(\frac{\partial f_0}{\partial w_{\perp}}\right)\right] \\
&-\left[\left(\mathbf{w}+\mathbf{u}\right)\cdot\nabla\mathbf{u}\right]\cdot\left[\mathbf{b}\frac{\partial f_0}{\partial w_{\parallel}} + \frac{\mathbf{w}_{\perp}}{w_{\perp}}\frac{\partial f_0}{\partial w_{\perp}}\right] \\
&+\frac{q}{m}\left(\mathbf{E}_1 + \left[\mathbf{w}+\mathbf{u}\right]\times\mathbf{B}_1\right)\cdot\left[\mathbf{b}\frac{\partial f_0}{\partial w_{\parallel}} + \frac{\mathbf{w}_{\perp}}{w_{\perp}}\frac{\partial f_0}{\partial w_{\perp}}\right] \\
&-\Omega\frac{\partial f_1}{\partial \phi} = C\left\{f_0\right\}
\end{aligned} \tag{66}$$

where we have used (59). We get a constraint by annihilating the f_1 term, which is accomplished by gyroaveraging. The gyroaverage of any quantity Y is defined as

$$\left\langle Y\right\rangle_{\varphi} = \frac{1}{2\pi}\int_0^{2\pi} Y\; d\varphi \tag{67}$$

where the integral is performed at fixed $\left(t,\mathbf{r},w_{\parallel},w_{\perp}\right)$. We will use the identity

$$\begin{aligned}
\left\langle \mathbf{w}\mathbf{w}\right\rangle_{\varphi} &= w_{\parallel}^2\mathbf{b}\mathbf{b} + \frac{w_{\perp}^2}{2}\left(\mathbf{I}-\mathbf{b}\mathbf{b}\right) \\
&= \frac{w_{\perp}^2}{2}\mathbf{I} + \left(w_{\parallel}^2 - \frac{w_{\perp}^2}{2}\right)\mathbf{b}\mathbf{b}.
\end{aligned} \tag{68}$$

The gyroaverages of various terms in (66) are

$$\left\langle \left(\mathbf{w}+\mathbf{u}\right)\cdot\frac{\partial f_0}{\partial \mathbf{r}}\right\rangle_{\varphi} = \left(w_{\parallel}\mathbf{b}+\mathbf{u}\right)\cdot\frac{\partial f_0}{\partial \mathbf{r}}, \tag{69}$$

$$\left\langle \mathbf{u}\cdot\left(\nabla\mathbf{b}\right)\cdot\mathbf{w}\left[\left(\frac{\partial f_0}{\partial w_{||}}\right)-\frac{w_{||}}{w_\perp}\left(\frac{\partial f_0}{\partial w_\perp}\right)\right]\right\rangle_\varphi = \mathbf{u}\cdot\underbrace{\left(\nabla\mathbf{b}\right)\cdot\mathbf{b}}_{=\nabla\left(b^2/2\right)=0} w_{||}\left[\left(\frac{\partial f_0}{\partial w_{||}}\right)-\frac{w_{||}}{w_\perp}\left(\frac{\partial f_0}{\partial w_\perp}\right)\right]=0 \qquad (70)$$

,

$$\begin{aligned}
\left\langle \mathbf{w}\cdot\left(\nabla\mathbf{b}\right)\cdot\mathbf{w}\left[\left(\frac{\partial f_0}{\partial w_{||}}\right)-\frac{w_{||}}{w_\perp}\left(\frac{\partial f_0}{\partial w_\perp}\right)\right]\right\rangle_\varphi &= \left\langle \mathbf{ww}\right\rangle_\varphi : \nabla\mathbf{b}\left[\left(\frac{\partial f_0}{\partial w_{||}}\right)-\frac{w_{||}}{w_\perp}\left(\frac{\partial f_0}{\partial w_\perp}\right)\right] \\
&= \left[\frac{w_\perp^2}{2}\mathbf{I}+\left(w_{||}^2-\frac{w_\perp^2}{2}\right)\mathbf{bb}\right]:\nabla\mathbf{b}\left[\left(\frac{\partial f_0}{\partial w_{||}}\right)-\frac{w_{||}}{w_\perp}\left(\frac{\partial f_0}{\partial w_\perp}\right)\right] \qquad (71) \\
&= \frac{w_\perp^2}{2}\left(\nabla\cdot\mathbf{b}\right)\left[\left(\frac{\partial f_0}{\partial w_{||}}\right)-\frac{w_{||}}{w_\perp}\left(\frac{\partial f_0}{\partial w_\perp}\right)\right],
\end{aligned}$$

$$\left\langle -\left[\left(\mathbf{w}+\mathbf{u}\right)\cdot\nabla\mathbf{u}\right]\cdot\left[\mathbf{b}\frac{\partial f_0}{\partial w_{||}}\right]\right\rangle_\varphi = -\left(w_{||}\mathbf{b}+\mathbf{u}\right)\cdot\left(\nabla\mathbf{u}\right)\cdot\mathbf{b}\frac{\partial f_0}{\partial w_{||}} \qquad (72)$$

,

$$\begin{aligned}
\left\langle -\left[\left(\mathbf{w}+\mathbf{u}\right)\cdot\nabla\mathbf{u}\right]\cdot\left[\frac{\mathbf{w}_\perp}{w_\perp}\frac{\partial f_0}{\partial w_\perp}\right]\right\rangle_\varphi &= -\left\langle \left[\mathbf{w}\cdot\nabla\mathbf{u}\right]\cdot\left[\frac{\mathbf{w}_\perp}{w_\perp}\frac{\partial f_0}{\partial w_\perp}\right]\right\rangle_\varphi \\
&= -\left\langle \mathbf{w}\mathbf{w}_\perp\right\rangle_\varphi : \left(\nabla\mathbf{u}\right)\frac{1}{w_\perp}\frac{\partial f_0}{\partial w_\perp} \\
&= -\left[\frac{w_\perp^2}{2}\mathbf{I}-\frac{w_\perp^2}{2}\mathbf{bb}\right]:\left(\nabla\mathbf{u}\right)\frac{1}{w_\perp}\frac{\partial f_0}{\partial w_\perp} \qquad (73) \\
&= -\left[\frac{w_\perp^2}{2}\nabla\cdot\mathbf{u}-\frac{w_\perp^2}{2}\mathbf{b}\cdot\left(\nabla\mathbf{u}\right)\cdot\mathbf{b}\right]\frac{1}{w_\perp}\frac{\partial f_0}{\partial w_\perp},
\end{aligned}$$

$$\begin{aligned}
\left\langle \frac{q}{m}\left(\mathbf{E}_1+\left[\mathbf{w}+\mathbf{u}\right]\times\mathbf{B}_1\right)\cdot\left[\mathbf{b}\frac{\partial f_0}{\partial w_{||}}\right]\right\rangle_\varphi &= \frac{q}{m}\left(\mathbf{E}_1+\left[w_{||}\mathbf{b}+\mathbf{u}\right]\times\mathbf{B}_1\right)\cdot\mathbf{b}\frac{\partial f_0}{\partial w_{||}} \\
&= \frac{q}{m}\left(\mathbf{E}_1+\mathbf{u}\times\mathbf{B}_1\right)\cdot\mathbf{b}\frac{\partial f_0}{\partial w_{||}} \qquad (74) \\
&= \frac{q}{m}\tilde{E}_{||}\frac{\partial f_0}{\partial w_{||}},
\end{aligned}$$

where we have defined

$$\tilde{E}_{||} = \left(\mathbf{E}_1+\mathbf{u}\times\mathbf{B}_1\right)\cdot\mathbf{b}\,, \qquad (75)$$

and

$$\left\langle \frac{q}{m}\left(\mathbf{E}_1+\left[\mathbf{w}+\mathbf{u}\right]\times\mathbf{B}_1\right)\cdot\left[\frac{\mathbf{w}_\perp}{w_\perp}\frac{\partial f_0}{\partial w_\perp}\right]\right\rangle_\varphi =\left\langle \frac{q}{m}\left(\mathbf{w}\times\mathbf{B}_1\right)\cdot\left[\frac{\mathbf{w}_\perp}{w_\perp}\frac{\partial f_0}{\partial w_\perp}\right]\right\rangle_\varphi$$
$$=\left\langle \frac{q}{m}\left(w_{\|}\mathbf{b}\times\mathbf{B}_1\right)\cdot\left[\frac{\mathbf{w}_\perp}{w_\perp}\frac{\partial f_0}{\partial w_\perp}\right]\right\rangle_\varphi =0. \tag{76}$$

Combining the above results, we get

$$\left(w_{\|}\mathbf{b}+\mathbf{u}\right)\cdot\frac{\partial f_0}{\partial \mathbf{r}}+\frac{w_\perp^2}{2}\left(\nabla\cdot\mathbf{b}\right)\left[\left(\frac{\partial f_0}{\partial w_{\|}}\right)-\frac{w_{\|}}{w_\perp}\left(\frac{\partial f_0}{\partial w_\perp}\right)\right]-\left(w_{\|}\mathbf{b}+\mathbf{u}\right)\cdot\left(\nabla\mathbf{u}\right)\cdot\mathbf{b}\frac{\partial f_0}{\partial w_{\|}}$$
$$-\frac{w_\perp}{2}\left[\nabla\cdot\mathbf{u}-\mathbf{b}\cdot\left(\nabla\mathbf{u}\right)\cdot\mathbf{b}\right]\frac{\partial f_0}{\partial w_\perp}+\frac{q}{m}\tilde{E}_{\|}\frac{\partial f_0}{\partial w_{\|}}=\left\langle C\left\{f_0\right\}\right\rangle_\varphi \tag{77}$$

which matches eq (44) in Hinton & Wong. (The color is for future reference.)

Entropy equation

We next multiply (77) by $\ln f_0$. Note that this factor can be moved inside the gyroaverage of the last (collision) term. We then apply the operation

$$2\pi\int_{-\infty}^{\infty}dw_{\|}\int_0^{\infty}dw_\perp w_\perp\left(\ldots\right) \tag{78}$$

which amount to integrating over velocity space, $\int d^3\upsilon$. Notice that this operation annihilates the $\tilde{E}_{\|}$ term:

$$2\pi\int_{-\infty}^{\infty}dw_{\|}\int_0^{\infty}dw_\perp w_\perp\left(\ln f_0\right)\left(\frac{q}{m}\tilde{E}_{\|}\frac{\partial f_0}{\partial w_{\|}}\right)=\frac{q}{m}\tilde{E}_{\|}2\pi\int_0^{\infty}dw_\perp w_\perp\int_{-\infty}^{\infty}dw_{\|}\frac{\partial}{\partial w_{\|}}\left(f_0\ln f_0-f_0\right)$$
$$=0. \tag{79}$$

For the same reason, the red terms in (77) also give no contribution. We are left with

$$\int d^3\upsilon\ln f_0 C\left\{f_0\right\}=Q \tag{80}$$

where

$$Q=2\pi\int_{-\infty}^{\infty}dw_{\|}\int_0^{\infty}dw_\perp w_\perp\left(\ln f_0\right)\left\{\begin{aligned}&\left(w_{\|}\mathbf{b}+\mathbf{u}\right)\cdot\frac{\partial f_0}{\partial\mathbf{r}}-\frac{w_\perp w_{\|}}{2}\left(\nabla\cdot\mathbf{b}\right)\left(\frac{\partial f_0}{\partial w_\perp}\right)\\&-w_{\|}\mathbf{b}\cdot\left(\nabla\mathbf{u}\right)\cdot\mathbf{b}\frac{\partial f_0}{\partial w_{\|}}-\frac{w_\perp}{2}\left[\nabla\cdot\mathbf{u}+\mathbf{b}\cdot\left(\nabla\mathbf{u}\right)\cdot\mathbf{b}\right]\frac{\partial f_0}{\partial w_\perp}\end{aligned}\right\}. \tag{81}$$

Equivalently,

$$Q=2\pi\int_{-\infty}^{\infty}dw_{\|}\int_0^{\infty}dw_\perp w_\perp\left\{\begin{aligned}&\left(w_{\|}\mathbf{b}+\mathbf{u}\right)\cdot\frac{\partial F}{\partial\mathbf{r}}-\frac{w_\perp w_{\|}}{2}\left(\nabla\cdot\mathbf{b}\right)\left(\frac{\partial F}{\partial w_\perp}\right)\\&-w_{\|}\mathbf{b}\cdot\left(\nabla\mathbf{u}\right)\cdot\mathbf{b}\frac{\partial F}{\partial w_{\|}}-\frac{w_\perp}{2}\left[\nabla\cdot\mathbf{u}+\mathbf{b}\cdot\left(\nabla\mathbf{u}\right)\cdot\mathbf{b}\right]\frac{\partial F}{\partial w_\perp}\end{aligned}\right\} \tag{82}$$

where

$$F = f_0 \ln f_0 - f_0 . \tag{83}$$

Let us integrate each of the terms in (82) by parts, to remove all the derivatives on F:

$$\begin{aligned} 2\pi\int_{-\infty}^{\infty} dw_{\|}\int_0^{\infty} dw_{\perp} w_{\perp}\left(w_{\|}\mathbf{b}+\mathbf{u}\right)\cdot\frac{\partial F}{\partial \mathbf{r}} &= \left[\frac{\partial}{\partial \mathbf{r}}\cdot\int d^3 w F\left(w_{\|}\mathbf{b}+\mathbf{u}\right)\right] \\ &\quad -2\pi\int_{-\infty}^{\infty} dw_{\|}\int_0^{\infty} dw_{\perp} F\frac{\partial}{\partial \mathbf{r}}\cdot\left[w_{\perp}\left(w_{\|}\mathbf{b}+\mathbf{u}\right)\right] \\ &= -\nabla\cdot\mathbf{G} - 2\pi\int_{-\infty}^{\infty} dw_{\|}\int_0^{\infty} dw_{\perp} F\left[w_{\perp}\left(w_{\|}\nabla\cdot\mathbf{b}+\nabla\cdot\mathbf{u}\right)\right] \end{aligned} \tag{84}$$

where

$$G = -\int d^3 w f_0 \ln f_0 \left(w_{\|}\mathbf{b}+\mathbf{u}\right) \tag{85}$$

as in Helander’s equation following (7).

$$\begin{aligned} 2\pi\int_{-\infty}^{\infty} dw_{\|}\int_0^{\infty} dw_{\perp} w_{\perp}\left\{-\frac{w_{\perp}w_{\|}}{2}\left(\nabla\cdot\mathbf{b}\right)\left(\frac{\partial F}{\partial w_{\perp}}\right)\right\} &= 2\pi\int_{-\infty}^{\infty} dw_{\|}\int_0^{\infty} dw_{\perp} F\frac{\partial\left(w_{\perp}^2\right)}{\partial w_{\perp}}\frac{w_{\|}}{2}\left(\nabla\cdot\mathbf{b}\right) \\ &= 2\pi\int_{-\infty}^{\infty} dw_{\|}\int_0^{\infty} dw_{\perp} F w_{\perp} w_{\|}\left(\nabla\cdot\mathbf{b}\right). \end{aligned} \tag{86}$$

$$\begin{aligned} 2\pi\int_{-\infty}^{\infty} dw_{\|}\int_0^{\infty} dw_{\perp} w_{\perp}\left\{-w_{\|}\mathbf{b}\cdot\left(\nabla\mathbf{u}\right)\cdot\mathbf{b}\frac{\partial F}{\partial w_{\|}}\right\} &= 2\pi\int_{-\infty}^{\infty} dw_{\|}\int_0^{\infty} dw_{\perp} w_{\perp} F\mathbf{b}\cdot\left(\nabla\mathbf{u}\right)\cdot\mathbf{b}\frac{\partial w_{\|}}{\partial w_{\|}} \\ &= 2\pi\int_{-\infty}^{\infty} dw_{\|}\int_0^{\infty} dw_{\perp} w_{\perp} F\mathbf{b}\cdot\left(\nabla\mathbf{u}\right)\cdot\mathbf{b}. \end{aligned} \tag{87}$$

$$\begin{aligned} 2\pi\int_{-\infty}^{\infty} dw_{\|}\int_0^{\infty} dw_{\perp} w_{\perp}\left\{-\frac{w_{\perp}}{2}\left[\nabla\cdot\mathbf{u}-\mathbf{b}\cdot\left(\nabla\mathbf{u}\right)\cdot\mathbf{b}\right]\frac{\partial F}{\partial w_{\perp}}\right\} &= 2\pi\int_{-\infty}^{\infty} dw_{\|}\int_0^{\infty} dw_{\perp}\left\{\frac{F}{2}\left[\nabla\cdot\mathbf{u}-\mathbf{b}\cdot\left(\nabla\mathbf{u}\right)\cdot\mathbf{b}\right]\frac{\partial w_{\perp}^2}{\partial w_{\perp}}\right\} \\ &= 2\pi\int_{-\infty}^{\infty} dw_{\|}\int_0^{\infty} dw_{\perp} w_{\perp} F\left[\nabla\cdot\mathbf{u}-\mathbf{b}\cdot\left(\nabla\mathbf{u}\right)\cdot\mathbf{b}\right]. \end{aligned} \tag{88}$$

Adding the terms, there are many cancellations, leaving just $Q = -\nabla\cdot\mathbf{G}$, or equivalently

$$\nabla\cdot\mathbf{G} = -\int d^3\upsilon \ln f_0 C\left\{f_0\right\}. \tag{89}$$

This result is equivalent to (48) in Hinton & Wong, and to (7) in Helander. The right-hand side represents the rate of production of entropy.

Apply a flux surface average $\left\langle\ \right\rangle_{\psi}$, using the property

$$\left\langle\nabla\cdot\mathbf{G}\right\rangle = \frac{1}{dV/d\psi}\frac{d}{d\psi}\left[\frac{dV}{d\psi}\mathbf{G}\cdot\nabla\psi\right] \tag{90}$$

where $V\left(\psi\right)$ is the flux surface volume, and ψ is any flux surface label. Noting $\mathbf{G}\cdot\nabla\psi = 0$, we obtain

$$\left\langle\int d^3\upsilon \ln f_0 C\left\{f_0\right\}\right\rangle_{\psi} = 0. \tag{91}$$

H theorem

It can be shown that

$$-\int d^3\upsilon \ln f_0 C\left\{f_0\right\} \geq 0, \tag{92}$$

and so (91) can only be true if (92) vanishes at every point on the surface:

$$\int d^3\upsilon \ln f_0 C\left\{f_0\right\} = 0. \tag{93}$$

It can also be shown that (93) implies f_0 is a Maxwellian (which may depend on position):

$$f_0 = n(\mathbf{r})\left[\frac{m}{2\pi T(\mathbf{r})}\right]\exp\left(-\frac{m}{2T(\mathbf{r})}\left[\mathbf{w}-\mathbf{W}(\mathbf{r})\right]^2\right) \tag{94}$$

where $\mathbf{W}(\mathbf{r})$ is the mean flow. We know $\partial f_0/\partial\varphi = 0$, so $\mathbf{W}$ must be parallel to $\mathbf{b}$. We are free to choose $u_{||}$ (from (16)) so the mean flow in the $\mathbf{w}$ frame vanishes, i.e. so $\mathbf{W}=0$. Hence,

$$f_0 = n(\mathbf{r})\left[\frac{m}{2\pi T(\mathbf{r})}\right]^{3/2}\exp\left(\frac{mw^2}{2T(\mathbf{r})}\right) = n(\mathbf{r})\left[\frac{m}{2\pi T(\mathbf{r})}\right]^{3/2}\exp\left(-\frac{m}{2T(\mathbf{r})}\left[w_{||}^2+w_{\perp}^2\right]\right). \tag{95}$$

Notice that the velocity shift $\mathbf{u}(\mathbf{r})$ has become precisely the mean velocity of the plasma in the lab frame.

Constraints on the density, temperature, and flow

We now plug (95) back into (77). The collision term vanishes since f_0 is Maxwellian. We will need the derivatives

$$\frac{\partial f_0}{\partial w_{||}} = -f_0\frac{mw_{||}}{T}, \tag{96}$$

$$\frac{\partial f_0}{\partial w_{\perp}} = -f_0\frac{mw_{\perp}}{T}, \tag{97}$$

$$\nabla f_0 = f_0\left[\frac{1}{n}\nabla n + \left(\frac{m\left[w_{||}^2+w_{\perp}^2\right]}{2T}-\frac{3}{2}\right)\frac{1}{T}\nabla T\right]. \tag{98}$$

We thus obtain, after a cancellation,

$$\begin{aligned}&\left(w_{||}\mathbf{b}+\mathbf{u}\right)\cdot\left[\frac{1}{n}\nabla n + \left(\frac{m\left[w_{||}^2+w_{\perp}^2\right]}{2T}-\frac{3}{2}\right)\frac{1}{T}\nabla T\right] + \left(w_{||}\mathbf{b}+\mathbf{u}\right)\cdot\left(\nabla\mathbf{u}\right)\cdot\mathbf{b}\frac{mw_{||}}{T}\\ &+\frac{w_{\perp}}{2}\left[\nabla\cdot\mathbf{u}-\mathbf{b}\cdot\left(\nabla\mathbf{u}\right)\cdot\mathbf{b}\right]\frac{mw_{\perp}}{T}-\frac{q}{m}\tilde{E}_{||}\frac{mw_{||}}{T}=0.\end{aligned} \tag{99}$$

This equation must hold for all values of $\left(w_{||},w_{\perp}\right)$, so the terms with each power of $w_{||}$ and $w_{\perp}$ must separately vanish:

$$O\left(w_{||}^0 w_{\perp}^0\right):\quad \mathbf{u}\cdot\left[\frac{1}{n}\nabla n - \frac{3}{2}\frac{1}{T}\nabla T\right]=0, \tag{100}$$

$$O\left(w_{\|}^{1}w_{\perp}^{0}\right):\quad \mathbf{b}\cdot\left[\frac{1}{n}\nabla n-\frac{3}{2}\frac{1}{T}\nabla T\right]+\mathbf{u}\cdot\left(\nabla\mathbf{u}\right)\cdot\mathbf{b}\frac{m}{T}-\tilde{E}_{\|}\frac{q}{T}=0, \tag{101}$$

$$O\left(w_{\|}^{2}w_{\perp}^{0}\right):\quad \mathbf{u}\cdot\nabla\ln T+\mathbf{b}\cdot\left(\nabla\mathbf{u}\right)\cdot\mathbf{b}=0, \tag{102}$$

$$O\left(w_{\|}^{0}w_{\perp}^{2}\right):\quad \mathbf{u}\cdot\nabla\ln T-\nabla\cdot\mathbf{u}+\mathbf{b}\cdot\left(\nabla\mathbf{u}\right)\cdot\mathbf{b}=0, \tag{103}$$

$$\left(w_{\|}^{1}w_{\perp}^{2}\right):\quad \mathbf{b}\cdot\nabla T=0. \tag{104}$$

This last equation tells us $T=T\left(\psi\right)$, so also $\mathbf{u}\cdot\nabla T=0$. Then (102) gives

$$\mathbf{b}\cdot\left(\nabla\mathbf{u}\right)\cdot\mathbf{b}=0, \tag{105}$$

and then (103) gives

$$\nabla\cdot\mathbf{u}=0. \tag{106}$$

(We do not need (100) or (101), which determine the density variation on a surface.)

Form of the flow

To exploit (105)-(106), we first note

$$\nabla\times\left(\mathbf{u}\times\mathbf{B}_0\right)=\nabla\times\left(\frac{1}{B^2}\left[\mathbf{B}_0\times\nabla\Phi_0\right]\times\mathbf{B}_0\right)=\nabla\times\left(\nabla\Phi_0-\frac{1}{B^2}\mathbf{B}_0\underbrace{\mathbf{B}_0\cdot\nabla\Phi_0}_{=0}\right)=0. \tag{107}$$

Dotting (107) with $\mathbf{B}_0\cdot\left(...\right)$,

$$\begin{aligned}0&=\mathbf{B}_0\cdot\nabla\times\left(\mathbf{u}\times\mathbf{B}_0\right)=\mathbf{B}_0\cdot\left[\mathbf{u}\underbrace{\left(\nabla\cdot\mathbf{B}_0\right)}_{=0}+\mathbf{B}_0\cdot\nabla\mathbf{u}-\mathbf{B}_0\underbrace{\left(\nabla\cdot\mathbf{u}\right)}_{=0}-\mathbf{u}\cdot\nabla\mathbf{B}_0\right]\\&=\mathbf{B}_0\cdot\nabla\mathbf{u}\cdot\mathbf{B}_0-\mathbf{u}\cdot\nabla\mathbf{B}_0\cdot\mathbf{B}_0.\end{aligned} \tag{108}$$

The penultimate term vanishes due to (105), so

$$0=\mathbf{u}\cdot\nabla\mathbf{B}_0\cdot\mathbf{B}_0=\mathbf{u}\cdot\nabla\left(B^2/2\right)=B\mathbf{u}\cdot\nabla B, \tag{109}$$

and hence

$$\boxed{\mathbf{u}\cdot\nabla B=0}. \tag{110}$$

This is one of our primary results: if a plasma carries a sonic flow, it must be in a direction along which the field strength B is constant.

Axisymmetry

If $\mathbf{B}_0$ is axisymmetric, then

$$\mathbf{B}_0=\nabla\zeta\times\nabla\psi_p+G\left(\psi_p\right)\nabla\zeta \tag{111}$$

where $2\pi\psi_p$ is the poloidal flux, and ζ is the usual toroidal angle. The identity

$$
\begin{aligned}
\mathbf{B}_0 \times \nabla \psi_p &= -\left|\nabla \psi_p\right|^2 \nabla \zeta + G \nabla \zeta \times \nabla \psi_p \\
&= -\left|\nabla \psi_p\right|^2 \nabla \zeta + \left(\mathbf{B}_0 - G \nabla \zeta\right) G \\
&= \mathbf{B}_0 G - \left(G^2 + \left|\nabla \psi_p\right|^2\right) \nabla \zeta \\
&= \mathbf{B}_0 G - R^2 B^2 \nabla \zeta
\end{aligned}
\tag{112}
$$

follows. Applying this identity to

$$
\mathbf{u} = \frac{1}{B^2} \frac{d\Phi_0}{d\psi_p} \mathbf{B}_0 \times \nabla \psi_p + \frac{u_{||}}{B} \mathbf{B}_0 \tag{113}
$$

gives

$$
\begin{aligned}
\mathbf{u} &= \frac{1}{B^2} \frac{d\Phi_0}{d\psi_p} \left[\mathbf{B}_0 G - R^2 B^2 \nabla \zeta\right] + \frac{u_{||}}{B} \mathbf{B}_0 \\
&= \left[\frac{G}{B^2} \frac{d\Phi_0}{d\psi_p} + \frac{u_{||}}{B}\right] \mathbf{B}_0 - R^2 \nabla \zeta \frac{d\Phi_0}{d\psi_p}.
\end{aligned}
\tag{114}
$$

Applying (110) then gives

$$
0 = \mathbf{u} \cdot \nabla B = \left[\frac{G}{B^2} \frac{d\Phi_0}{d\psi_p} + \frac{u_{||}}{B}\right] \mathbf{B}_0 \cdot \nabla B, \tag{115}
$$

so the quantity in square brackets must vanish. Then (114) implies

$$
\boxed{\mathbf{u} = -R^2 \nabla \zeta \frac{d\Phi_0}{d\psi_p} = -R \mathbf{e}_\zeta \frac{d\Phi_0}{d\psi_p}}, \tag{116}
$$

so the rotation is purely toroidal. Each flux surface rotates rigidly with an angular frequency $-d\Phi_0 / d\psi_p$.

Stellarators

Let us return to the case of general geometry. We know the flow $\mathbf{u}$ is perpendicular both to $\nabla \psi$ and to ∇B, so we must have

$$
\mathbf{u} = g \nabla \psi \times \nabla B \tag{117}
$$

for some $g(\mathbf{r})$. Then

$$
\frac{1}{B^2} \frac{d\Phi_0}{d\psi} \mathbf{B}_0 \times \nabla \psi + u_{||} \mathbf{b} = g \nabla \psi \times \nabla B. \tag{118}
$$

The $\mathbf{B}_0 \times \nabla \psi$ component of (118) gives

$$
\frac{1}{B^2} \frac{d\Phi_0}{d\psi} \left(\mathbf{B}_0 \times \nabla \psi\right) \cdot \left(\mathbf{B}_0 \times \nabla \psi\right) = g \left(\nabla \psi \times \nabla B\right) \cdot \left(\mathbf{B}_0 \times \nabla \psi\right). \tag{119}
$$

A vector identity then gives

$$g = \frac{1}{\mathbf{B}_0 \cdot \nabla B} \frac{d\Phi_0}{d\psi}. \tag{120}$$

Plugging this result into (117),

$$\mathbf{u} = \frac{\nabla\psi \times \nabla B}{\mathbf{B}_0 \cdot \nabla B} \frac{d\Phi_0}{d\psi}. \tag{121}$$

Then (106) implies

$$\begin{aligned} 0 &= \nabla \cdot \left(\frac{\nabla\psi \times \nabla B}{\mathbf{B}_0 \cdot \nabla B} \frac{d\Phi_0}{d\psi} \right) \\ &= \frac{d^2\Phi_0}{d\psi^2} \underbrace{\nabla\psi \cdot \left(\frac{\nabla\psi \times \nabla B}{\mathbf{B}_0 \cdot \nabla B} \right)}_{=0} + \frac{d\Phi_0}{d\psi} \nabla \cdot \left(\frac{\nabla\psi \times \nabla B}{\mathbf{B}_0 \cdot \nabla B} \right). \end{aligned} \tag{122}$$

Therefore

$$\begin{aligned} 0 = \nabla \cdot \left(\frac{\nabla\psi \times \nabla B}{\mathbf{B}_0 \cdot \nabla B} \right) &= \nabla\psi \times \nabla B \cdot \nabla \left(\frac{1}{\mathbf{B}_0 \cdot \nabla B} \right) + \frac{1}{\mathbf{B}_0 \cdot \nabla B} \underbrace{\nabla \cdot \left(\nabla\psi \times \nabla B \right)}_{=0} \\ &= -\left(\frac{1}{\mathbf{B}_0 \cdot \nabla B} \right)^2 \nabla\psi \times \nabla B \cdot \nabla \left(\mathbf{B}_0 \cdot \nabla B \right). \end{aligned} \tag{123}$$

Thus,

$$\nabla\psi \times \nabla B \cdot \nabla \left(\mathbf{B}_0 \cdot \nabla B \right) = 0, \tag{124}$$

which is one of the equivalent definitions of quasisymmetry (see e.g. Simakov & Helander PPCF (2011).)

Thus, in a perfectly quasisymmetric stellarator, it does appear possible for there to be a sonic flow, at the level of analysis we have carried out here. Sugama et al (Physics of Plasmas 18, 082505 (2011)) have carried the analysis further, and found that in fact a different constraint prevents sonic flows in quasisymmetric stellarators that are not truly axisymmetric.