February 26, 2015 Matt Landreman

Equations implemented in the minimal solver for stellarator monoenergetic transport coefficients

This code solves the following normalized kinetic equation:

$$\xi B\left[\iota\frac{\partial f}{\partial\theta}+\frac{\partial f}{\partial\zeta}\right]-\left[1-\xi^2\right]\frac{1}{2}\left[\iota\frac{\partial B}{\partial\theta}+\frac{\partial B}{\partial\zeta}\right]\frac{\partial f}{\partial\xi}-\frac{\nu}{2}\frac{\partial}{\partial\xi}\left[1-\xi^2\right]\frac{\partial f}{\partial\xi}=\left[1+\xi^2\right]\frac{1}{B}\left[G\frac{\partial B}{\partial\theta}-I\frac{\partial B}{\partial\zeta}\right]. \quad (1)$$

This is a 3D linear inhomogeneous equation for $f(\theta,\zeta,\xi)$. The independent variables θ and ζ are periodic spatial coordinates (with period 2π), and the independent variable ξ is the cosine of the pitch angle in velocity space. The domain of ξ is $[-1,\ 1]$. In (1), ν is a constant scaling the diffusive term. The constants G, I, and ι are related to the geometry of the magnetic field. The magnetic field strength $B(\theta,\zeta)$ is a specified function. The model used in the code is

$$B(\theta,\zeta)=1+\varepsilon_t\cos\theta+\varepsilon_h\cos(h\theta-N\zeta) \quad (2)$$

where h is called `helicity_l` in the code and N is called `Nperiods`. Since the system has a discrete symmetry in ζ, the code only simulates the domain $\zeta\in[0,\ 2\pi/N]$ rather than the full 2π.

For a discretization of the ξ coordinate that is both sparse and spectrally accurate, the unknown f is expanded in Legendre polynomials $P_\ell(\xi)$:

$$f(\theta,\zeta,\xi)=\sum_\ell f_\ell(\theta,\zeta)\ P_\ell(\xi). \quad (3)$$

We discretize (1) by applying the operation

$$\frac{2L+1}{2}\int_{-1}^{1}d\xi\ P_L(\xi)(\quad\cdot\quad). \quad (4)$$

Thus, (1) becomes

$$\sum_L M_{L,\ell}f_\ell=R_L \quad (5)$$

where the main linear operator has become

$$\begin{aligned}M_{L,\ell}=&\left[\frac{L+1}{2L+3}\delta_{L+1,\ell}+\frac{L}{2L-1}\delta_{L-1,\ell}\right]B\left[\iota\frac{\partial}{\partial\theta}+\frac{\partial}{\partial\zeta}\right]\\&-\frac{1}{2}\left[\iota\frac{\partial B}{\partial\theta}+\frac{\partial B}{\partial\zeta}\right]\left[\frac{(L+1)(L+2)}{2L+3}\delta_{L+1,\ell}-\frac{(L-1)L}{2L-1}\delta_{L-1,\ell}\right]\\&+\frac{\nu}{2}L[L+1]\delta_{L,\ell}\end{aligned} \quad (6)$$

and the right-hand side is

$$R_L=\frac{1}{B}\left[G\frac{\partial B}{\partial\theta}-I\frac{\partial B}{\partial\zeta}\right]\left[\frac{4}{3}\delta_{L,0}+\frac{2}{3}\delta_{L,2}\right]. \quad (7)$$

The PDE (1) has a single null solution, $f=$ constant. We eliminate this null solution by imposing one extra equation in an additional row of the matrix. For physical reasons, we choose this condition to be

$$\int d\theta \int d\zeta \int d\xi \, \frac{f}{B^2} = 0 \,. \tag{8}$$

In the Legendre modal expansion, (8) becomes

$$\int d\theta \int d\zeta \, \frac{f_0}{B^2} = 0 \,. \tag{9}$$

To keep the linear system square, we also add one extra unknown, a number λ, which is added to the left-hand side of (1). (λ is independent of all 3 independent variables.) Equivalently, to the left-hand side of (5) we add $\lambda\delta_{L,0}$. We can think of λ as a source in the kinetic equation (1), and it plays a role like a Lagrange multiplier. By applying the operation

$$\int d\theta \int d\zeta \int d\xi \, \frac{1}{B^2}(\ldots) \tag{10}$$

to (1) and integrating by parts in several places, it can be seen that all terms vanish or cancel except λ, forcing $\lambda = 0$. Thus, we are not really changing our original equation (1) by introducing λ, since it will always end up vanishing.

Combining (5) and (9), our final linear system has the 2×2 block structure

$$\begin{pmatrix} M_{L,\ell} & \delta_{L,0} \\ \int d\theta \int d\zeta \left(1/B^2\right) & 0 \end{pmatrix} \begin{pmatrix} f_\ell \\ \lambda \end{pmatrix} = \begin{pmatrix} R_L \\ 0 \end{pmatrix}. \tag{11}$$