Håkan Smith, December 5, 2017

1 Cosine-sine series (used in the matlab class `mnmat`)

The following holds when the spatial discretisation includes the point $\zeta = \theta = 0$.

1.1 M even, N even

$$
\begin{aligned}
f(\theta_\kappa, \zeta_\lambda) = F^c_{00} + \sum_{n=1}^{N/2} F^c_{0,n} \cos(n\mathcal{N}\zeta_\lambda) + \sum_{n=1}^{N/2-1} F^s_{0,n} \sin(n\mathcal{N}\zeta_\lambda) + \\
+ \sum_{m=1,n=-N/2+1}^{M/2-1,N/2} \{F^c_{mn} \cos(m\theta_\kappa - n\mathcal{N}\zeta_\lambda) + F^s_{mn} \sin(m\theta_\kappa - n\mathcal{N}\zeta_\lambda)\} + \\
+ \sum_{n=0}^{N/2} F^c_{\frac{M}{2}n} \cos(\frac{M}{2}\theta_\kappa - n\mathcal{N}\zeta_\lambda) + \sum_{n=1}^{N/2-1} F^s_{\frac{M}{2}n} \sin(\frac{M}{2}\theta_\kappa - n\mathcal{N}\zeta_\lambda)
\end{aligned} \tag{1}
$$

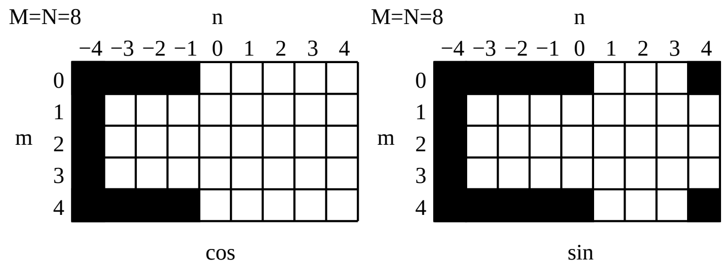

Figure 1: Unique cosinus (left) and sinus components (right) for the case $M = N = 8$.

1.2 M even, N odd

$$
\begin{aligned}
f(\theta_\kappa, \zeta_\lambda) = F^c_{00} + \sum_{n=1}^{(N-1)/2} F^c_{0,n} \cos(n\mathcal{N}\zeta_\lambda) + \sum_{n=1}^{(N-1)/2} F^s_{0,n} \sin(n\mathcal{N}\zeta_\lambda) + \\
+ \sum_{m=1,n=-(N-1)/2}^{M/2-1,(N-1)/2} \{F^c_{mn} \cos(m\theta_\kappa - n\mathcal{N}\zeta_\lambda) + F^s_{mn} \sin(m\theta_\kappa - n\mathcal{N}\zeta_\lambda)\} + \\
+ \sum_{n=0}^{(N-1)/2} F^c_{\frac{M}{2}n} \cos(\frac{M}{2}\theta_\kappa - n\mathcal{N}\zeta_\lambda) + \sum_{n=1}^{(N-1)/2} F^s_{\frac{M}{2}n} \sin(\frac{M}{2}\theta_\kappa - n\mathcal{N}\zeta_\lambda)
\end{aligned} \tag{2}
$$

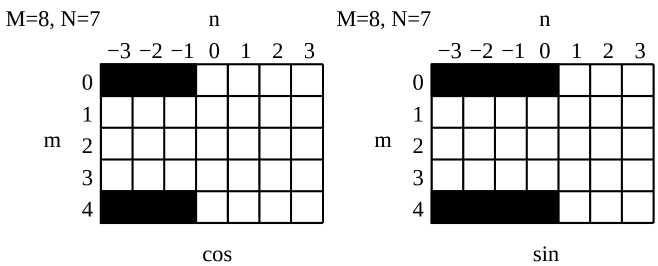

Figure 2: Unique cosinus (left) and sinus components (right) for the case $M = 8, N = 7$.

1.3 M odd, N even

$$f(\theta_\kappa, \zeta_\lambda) = F^c_{00} + \sum_{n=1}^{N/2} F^c_{0,n} \cos(n\mathcal{N}\zeta_\lambda) + \sum_{n=1}^{N/2-1} F^s_{0,n} \sin(n\mathcal{N}\zeta_\lambda) +$$
$$+ \sum_{m=1,n=-N/2+1}^{(M-1)/2,N/2} \left\{F^c_{mn} \cos\left(m\theta_\kappa - n\mathcal{N}\zeta_\lambda\right) + F^s_{mn} \sin\left(m\theta_\kappa - n\mathcal{N}\zeta_\lambda\right)\right\} \tag{3}$$

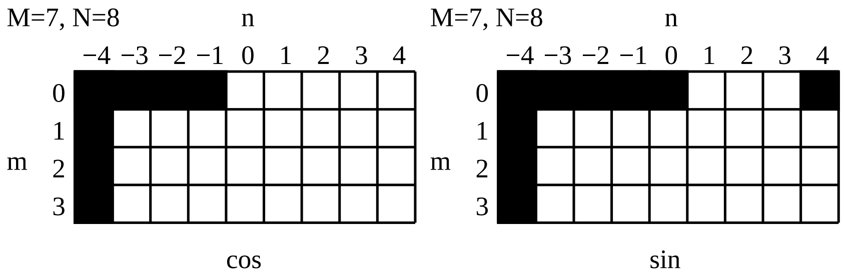

Figure 3: Unique cosinus (left) and sinus components (right) for the case $M = 7, N = 8$.

1.4 M odd, N odd

$$f(\theta_\kappa, \zeta_\lambda) = F^c_{00} + \sum_{n=1}^{(N-1)/2} F^c_{0,n} \cos(n\mathcal{N}\zeta_\lambda) + \sum_{n=1}^{(N-1)/2} F^s_{0,n} \sin(n\mathcal{N}\zeta_\lambda) +$$
$$+ \sum_{m=1,n=-(N-1)/2}^{(M-1)/2,(N-1)/2} \left\{F^c_{mn} \cos\left(m\theta_\kappa - n\mathcal{N}\zeta_\lambda\right) + F^s_{mn} \sin\left(m\theta_\kappa - n\mathcal{N}\zeta_\lambda\right)\right\} \tag{4}$$

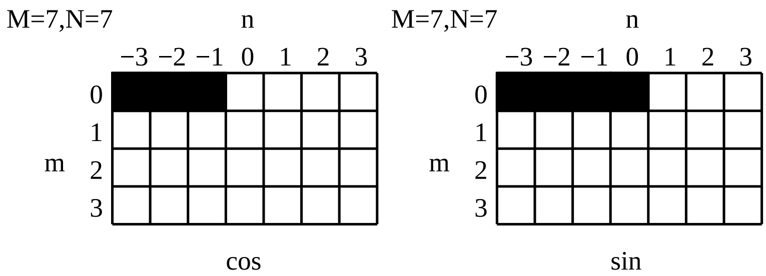

Figure 4: Unique cosinus (left) and sinus components (right) for the case $M = N = 7$.

2 FFT

The command `G=fft2(g)` in matlab corresponds to

$$G_{k,l} = \sum_{\kappa,\lambda=1}^{M,N} g(\theta_\kappa,\zeta_\lambda) e^{-i\Delta_\theta(k-1)(\kappa-1)-i\Delta_{\mathcal{N}\zeta}(l-1)(\lambda-1)} \tag{5}$$

Now, let

$$\begin{aligned} n &= \mod(1-l+\lfloor N/2\rfloor, N) - \lfloor N/2\rfloor \\ m &= k-1 \end{aligned} \tag{6}$$

The command `F=fliplr(fftshift(fft2(f),2))` in matlab corresponds to

$$\begin{aligned} F_{mn} &= \sum_{\kappa,\lambda=1}^{M,N} f(\theta_\kappa,\zeta_\lambda) e^{-i\Delta_\theta m(\kappa-1)+i\Delta_{\mathcal{N}\zeta} n(\lambda-1)} = \\ &= \sum_{\kappa,\lambda=1}^{M,N} f(\theta_\kappa,\zeta_\lambda)\left\{\cos\left[m\theta_\kappa - n\mathcal{N}\zeta_\lambda\right] - i\sin\left[m\theta_\kappa - n\mathcal{N}\zeta_\lambda\right]\right\} = \\ &\approx \frac{1}{\Delta_\theta\Delta_{\mathcal{N}\zeta}}\int_0^{2\pi/\mathcal{N}}\int_0^{2\pi} d\theta d\zeta f(\theta,\zeta)\left\{\cos\left[m\theta - n\mathcal{N}\zeta\right] - i\sin\left[m\theta - n\mathcal{N}\zeta\right]\right\}. \end{aligned} \tag{7}$$

If we let

$$n_{\min} = -\lfloor N/2\rfloor + 1 \tag{8}$$

$$n_{\max} = \lfloor N/2\rfloor \tag{9}$$

if L is even and

$$n_{\min} = -\lfloor N/2\rfloor \tag{10}$$

$$n_{\max} = \lfloor N/2\rfloor \tag{11}$$

if L is odd, we can write the inverse transform

$$\begin{aligned} f(\theta_\kappa,\zeta_\lambda) = f_{\kappa\lambda} &= \frac{1}{MN}\sum_{m=0,n=n_{\min}}^{M-1,n_{\max}} F_{mn} e^{i\Delta_\theta(\kappa-1)m - i\Delta_{\mathcal{N}\zeta}(\lambda-1)n} = \\ &= \frac{1}{MN}\sum_{m=0,n=n_{\min}}^{M-1,n_{\max}} F_{mn}\left\{\cos\left[m\theta_\kappa - n\mathcal{N}\zeta_\lambda\right] + i\sin\left[m\theta_\kappa - n\mathcal{N}\zeta_\lambda\right]\right\} \end{aligned} \tag{12}$$

We see that for $1 \le m \le \lfloor (M-1)/2\rfloor$ we can extract F^c_{mn} and F^s_{mn} as defined in section 1.

$$F^c_{mn} = \frac{2}{MN}\Re(F_{mn}) \tag{13}$$

$$F^s_{mn} = -\frac{2}{MN}\Im(F_{mn}). \tag{14}$$

For $m = 0$ and $m = M/2$ we get

$$F^c_{m0} = \frac{1}{MN}\Re(F_{mn}) \tag{15}$$
$$F^s_{m0} = \text{not defined} \tag{16}$$
$$F^c_{m\frac{N}{2}} = \frac{1}{MN}\Re(F_{m\frac{N}{2}}) \tag{17}$$
$$F^s_{m\frac{N}{2}} = \text{not defined} \tag{18}$$
$$F^c_{mn} = \frac{1}{MN}\Re(F_{mn} + F_{m,-n}) = \frac{2}{MN}\Re(F_{mn}), \quad 1 \leq n < \frac{N}{2} \tag{19}$$
$$F^s_{mn} = -\frac{1}{MN}\Im(F_{mn} + F_{m,-n}) = \frac{2}{MN}\Im(F_{mn}), \quad 1 \leq n < \frac{N}{2} \tag{20}$$