Håkan Smith, November 3, 2015

1 NTV

Consider the momentum equation summed over species,

$$\boldsymbol{J}\times\boldsymbol{B}-\nabla\cdot\mathbf{P}=\nabla\cdot(\rho\boldsymbol{V}\boldsymbol{V})+\frac{\partial(\rho\boldsymbol{V})}{\partial t}. \tag{1}$$

We write

$$\begin{aligned}\boldsymbol{B} &= \boldsymbol{B}_0+\boldsymbol{B}_1, &(2)\\ \boldsymbol{J} &= \boldsymbol{J}_0+\boldsymbol{J}_1, &(3)\\ \mathbf{P} &= p_0(\psi)\mathbf{I}+p_1(\psi)\mathbf{I}+\mathbf{\Pi}, &(4)\end{aligned}$$

where

$$\boldsymbol{J}_0\times\boldsymbol{B}_0=p_0'(\psi)\nabla\psi. \tag{5}$$

Subtracting Eq. (5) from Eq. (1), we obtain

$$\frac{\partial\rho\boldsymbol{V}}{\partial t}=\boldsymbol{J}_1\times\boldsymbol{B}_0+\boldsymbol{J}_0\times\boldsymbol{B}_1-\nabla p_1-\nabla\cdot(\rho\boldsymbol{V}\boldsymbol{V}+\mathbf{M}+\mathbf{\Pi}). \tag{6}$$

where we have used the vector identity $\nabla B_1^2=2\boldsymbol{B}_1\cdot\nabla\boldsymbol{B}_1+2\boldsymbol{B}_1\times\nabla\times\boldsymbol{B}_1$ to write $\boldsymbol{J}_1\times\boldsymbol{B}_1=-\nabla\cdot\mathbf{M}$, with

$$\mathbf{M}=\frac{1}{\mu_0}\left(\frac{1}{2}B_1^2\mathbf{I}-\boldsymbol{B}_1\boldsymbol{B}_1\right). \tag{7}$$

The two components of Eq. (6) in the directions along $\boldsymbol{B}_0$ and $\boldsymbol{J}_0$ are particularly interesting as we shall see, because several terms disappear upon taking the flux surface average. Observe that

$$\begin{aligned}\langle\boldsymbol{B}_0\cdot\boldsymbol{J}_0\times\boldsymbol{B}_1\rangle &= \langle\boldsymbol{B}_1\cdot\nabla p_0\rangle=-\langle(\nabla\times\boldsymbol{A}_1)\cdot\nabla p_0\rangle=\langle\nabla\cdot(\nabla p_0\times\boldsymbol{A}_1)\rangle=0, &(8)\\ \langle\boldsymbol{J}_0\cdot\boldsymbol{J}_1\times\boldsymbol{B}_0\rangle &= -\langle\boldsymbol{J}_1\cdot\nabla p_0\rangle=\frac{1}{\mu_0}\langle(\nabla\times\boldsymbol{B}_1)\cdot\nabla p_0\rangle=-\langle\nabla\cdot(\nabla p_0\times\boldsymbol{B}_1)\rangle=0, &(9)\\ \langle\boldsymbol{B}_0\cdot\nabla p_1\rangle &= \langle\nabla\cdot(\boldsymbol{B}_0p_1)\rangle=0, &(10)\\ \langle\boldsymbol{J}_0\cdot\nabla p_1\rangle &= \langle\nabla\cdot(\boldsymbol{J}_0p_1)\rangle=0 &(11)\end{aligned}$$

if we neglect the displacement current. The flux surface average of the scalar product of Eq. (6) with $\boldsymbol{B}_0$ and $\boldsymbol{J}_0$ yields

$$\begin{aligned}\frac{\partial\langle\rho\boldsymbol{V}\cdot\boldsymbol{B}_0\rangle}{\partial t} &= -\left\langle\boldsymbol{B}_0\cdot\nabla\cdot(\rho\boldsymbol{V}\boldsymbol{V}+\mathbf{M}+\mathbf{\Pi})\right\rangle, &(12)\\ \frac{\partial\langle\rho\boldsymbol{V}\cdot\boldsymbol{J}_0\rangle}{\partial t} &= -\left\langle\boldsymbol{J}_0\cdot\nabla\cdot(\rho\boldsymbol{V}\boldsymbol{V}+\mathbf{M}+\mathbf{\Pi})\right\rangle. &(13)\end{aligned}$$

Consequently, the same type of expression also holds for any linear combination of $\boldsymbol{B}_0$ and $\boldsymbol{J}_0$, which we can denote in general as $\boldsymbol{L}=\alpha\boldsymbol{B}_0+\beta\boldsymbol{J}_0$, where α and β are flux functions, and

$$\frac{\partial\langle\rho\boldsymbol{V}\cdot\boldsymbol{L}\rangle}{\partial t}=-\left\langle\boldsymbol{L}\cdot\nabla\cdot(\rho\boldsymbol{V}\boldsymbol{V}+\mathbf{M}+\mathbf{\Pi})\right\rangle. \tag{14}$$

$\boldsymbol{V}$ is also a linear combination $\boldsymbol{V}=\alpha_V\boldsymbol{B}_0+\beta_V\boldsymbol{J}_0$, so one sees that $\boldsymbol{J}_0\times\boldsymbol{B}_0=\nabla p$ leads to

$$\begin{aligned}\langle\boldsymbol{L}\cdot\nabla\cdot\rho\boldsymbol{V}\boldsymbol{V}\rangle &= \rho\langle\boldsymbol{L}\cdot[\alpha_V^2\nabla(B_0^2/2+\mu_0p)+\beta_V^2(\nabla J_0^2/2-\boldsymbol{J}_0\times\nabla\times\boldsymbol{J}_0)]\rangle=\\ &= \rho\langle\alpha\beta_V^2\nabla\cdot(\boldsymbol{J}_0\times\nabla p)\rangle=0 &(15)\end{aligned}$$

1.1 Axisymmetry

In axisymmetry we can express $\boldsymbol{B}$ and $\boldsymbol{J}$ as

$$\begin{aligned}\boldsymbol{B}_0 &= F(\psi)\nabla\phi + \iota\nabla\phi\times\nabla\psi, &(16)\\ \iota\boldsymbol{J}_0 &= -p_0'(\psi)\left(\frac{\partial\boldsymbol{r}}{\partial\phi} + K(\psi)\boldsymbol{B}_0\right), &(17)\end{aligned}$$

where ϕ is the geometrical toroidal angle and ψ the toroidal flux. The flux function $K(\psi)$ can be obtained from the flux surface average of the parallel component of Eq. (17), yielding $K(\psi) = (\langle\iota\boldsymbol{J}_0\cdot\boldsymbol{B}_0\rangle/p_0' + F)/\langle B^2\rangle$. We are interessed in the toroidal torque on the plasma, so in Eq. (14) we choose

$$\boldsymbol{L} = R\hat{\phi} = \frac{\partial\boldsymbol{r}}{\partial\phi} = -\frac{\iota}{p_0'}\boldsymbol{J}_0 - K\boldsymbol{B}_0 \tag{18}$$

1.2 Non-axisymmetry

For an arbitrary 3D magnetic configuration, we would like to define a generalised vector $\boldsymbol{L}$, which in the limit of axisymmetry becomes $R\hat{\phi}$. One requirement for $\boldsymbol{L}$ to be parallel to $\hat{\phi}$ in this limit is that the streamlines of $\boldsymbol{L}$ close on themselves toroidally. This also makes sense when we are not exactly at axisymmentry because we are not interested in any net poloidal torque component.

In Hamada coordinates (V,ϑ,φ), the streamlines of both $\boldsymbol{B}_0$ and $\boldsymbol{J}_0$ are straight, i.e. $\boldsymbol{B}_0$ and $\boldsymbol{J}_0$ are linear combinations of $\partial\boldsymbol{r}/\partial\vartheta$ and $\partial\boldsymbol{r}/\partial\varphi$ and vice versa. The sought linear combination of $\boldsymbol{B}_0$ and $\boldsymbol{J}_0$ whose streamlines close on themselves toroidally is thus $\boldsymbol{e}_\varphi = \partial\boldsymbol{r}/\partial\varphi$. Because of the above mentioned reasons, $\boldsymbol{e}_\varphi$ becomes parallel to $\hat{\phi}$ in the limit of axisymmetry (note that $\nabla\varphi$ does not become parallel to $\hat{\phi}$). Therefore, since $\nabla\cdot\boldsymbol{e}_\varphi = 0$ and $\nabla\cdot R\hat{\phi} = 0$, we conclude that $\boldsymbol{e}_\varphi \to cR\hat{\phi}$, where the constant $c = 1$ because φ and ϕ are both 2π periodic.

We now want to determine the constants α and β in $\boldsymbol{L} = \boldsymbol{e}_\varphi = \alpha\boldsymbol{B}_0 + \beta\boldsymbol{J}_0$. In Hamada coordinates, we can express $\boldsymbol{B}_0$ and $\boldsymbol{J}_0$ as

$$\begin{aligned}\boldsymbol{B}_0 &= \nabla\psi\times\nabla\vartheta + \iota(\psi)\nabla\varphi\times\nabla\psi = I(\psi)\nabla\vartheta + G(\psi)\nabla\varphi + \nabla H(\psi,\vartheta,\varphi), &(19)\\ \mu_0\boldsymbol{J}_0 &= I'(\psi)\nabla\psi\times\nabla\vartheta - G'(\psi)\nabla\varphi\times\nabla\psi. &(20)\end{aligned}$$

Moreover, the Jacobian is a flux function, so

$$\begin{aligned}V'(\psi) &= \int_0^{2\pi} d\vartheta \int_0^{2\pi} d\varphi \frac{1}{\nabla\psi\cdot\nabla\vartheta\times\nabla\varphi} = \frac{4\pi^2}{\nabla\psi\cdot\nabla\vartheta\times\nabla\varphi} &(21)\\ \boldsymbol{e}_\varphi &= \frac{\nabla\psi\times\nabla\vartheta}{\nabla\psi\cdot\nabla\vartheta\times\nabla\varphi} = \frac{V'}{4\pi^2}\nabla\psi\times\nabla\vartheta, &(22)\end{aligned}$$

and equilibrium implies

$$\begin{aligned}\boldsymbol{J}_0\times\boldsymbol{B}_0 &= -\frac{1}{\mu_0}(G' + \iota I')(\nabla\psi\cdot\nabla\vartheta\times\nabla\varphi)\nabla\psi = p_0'\nabla\psi &(23)\\ G' + \iota I' &= -\frac{V'}{4\pi^2}\mu_0 p_0' &(24)\end{aligned}$$

The equation $\boldsymbol{e}_\varphi = \alpha\boldsymbol{B}_0 + \beta\boldsymbol{J}_0$ becomes

$$\frac{V'}{4\pi^2}\nabla\psi\times\nabla\vartheta = \alpha\left(\nabla\psi\times\nabla\vartheta + \iota\nabla\varphi\times\nabla\psi\right) + \frac{\beta}{\mu_0}\left(I'\nabla\psi\times\nabla\vartheta - G'\nabla\varphi\times\nabla\psi\right) \tag{25}$$

yielding

$$\alpha = \frac{V'}{4\pi^2}\frac{G'}{G'+\iota I'} = -\frac{G'}{\mu_0 p_0'}, \tag{26}$$

$$\beta = \frac{V'}{4\pi^2}\frac{\iota}{G'+\iota I'} = -\frac{\iota}{p_0'}, \tag{27}$$

i.e.,

$$\boldsymbol{e}_\varphi = -\frac{\iota}{p_0'}\boldsymbol{J}_0 - \frac{G'}{\mu_0 p_0'}\boldsymbol{B}_0. \tag{28}$$

Note that in the expressions for α and β, the flux functions $G(\psi)$ and $I(\psi)$ are the same as in Boozer coordinates (ψ, θ, ζ), where

$$\boldsymbol{B}_0 = \nabla\psi\times\nabla\theta + \iota\nabla\zeta\times\nabla\psi = I(\psi)\nabla\theta + G(\psi)\nabla\zeta + \kappa(\psi,\theta,\zeta)\nabla\psi. \tag{29}$$

1.2.1 The parallel current

Let us divide the parallel current in its Pfirsch-Schlüter and Ohmic parts and thereby also define a quantity u in the following way

$$J_\parallel = \frac{p'}{\iota}Bu + \frac{\langle J_\parallel B\rangle}{\langle B^2\rangle}B. \tag{30}$$

In axisymmetry one would have $u = F\left[\left\langle B^2\right\rangle^{-1} - B^{-2}\right]$ (and $F = G$). In general, we see that u must fulfill the restriction $\langle uB^2\rangle = 0$. The total current

$$\boldsymbol{J} = \boldsymbol{J}_\perp + J_\parallel\boldsymbol{B} = \frac{p'}{B^2}\boldsymbol{B}\times\nabla\psi + \left(\frac{p'}{\iota}u + \frac{\langle J_\parallel B\rangle}{\langle B^2\rangle}\right)\boldsymbol{B} \tag{31}$$

must be divergence-free, which implies that we can determine u from the equation

$$\boldsymbol{B}\cdot\nabla u = 2\iota B^{-3}\boldsymbol{B}\times\nabla\psi\cdot\nabla B. \tag{32}$$

Henceforth, we employ Boozer coordinates, in which

$$\boldsymbol{B}\cdot\nabla = \frac{1}{\sqrt{g}}\left(\iota\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\zeta}\right) \tag{33}$$

$$\boldsymbol{B}\times\nabla\psi\cdot\nabla B = \frac{1}{\sqrt{g}}\left(G\frac{\partial B}{\partial\theta} - I\frac{\partial B}{\partial\zeta}\right) \tag{34}$$

so that equation (32) corresponds to

$$\left(\iota\frac{\partial u}{\partial\theta} + \frac{\partial u}{\partial\zeta}\right) = 2\frac{\iota}{B^3}\left(G\frac{\partial B}{\partial\theta} - I\frac{\partial B}{\partial\zeta}\right), \tag{35}$$

and u can thus easily be solved for using Fourier decompositions of u and B^{-2}. Note that the $m = n = 0$ component of u in such a decomposition must be zero because of $\langle uB^2\rangle = 0$.

We now return to the quantity $\langle J_{\|} B \rangle$ in Eq. (30). To derive it we first calculate $\boldsymbol{J}$ from the Boozer representation of $\boldsymbol{B}$ (29),

$$
\begin{aligned}
\mu_0 \boldsymbol{J} &= \left(I' - \frac{\partial \kappa}{\partial \theta}\right) \nabla\psi \times \nabla\theta - \left(G' + \frac{\partial \kappa}{\partial \zeta}\right) \nabla\zeta \times \nabla\psi && (36)\\
\mu_0 \boldsymbol{B} \cdot \boldsymbol{J} &= \frac{1}{\sqrt{g}} \left[\left(I' - \frac{\partial \kappa}{\partial \theta}\right) G - \left(G' + \frac{\partial \kappa}{\partial \zeta}\right) I\right] && (37)\\
\mu_0 \langle J_{\|} B \rangle &= \frac{1}{V'} \int d\theta d\zeta \left[\left(I' - \frac{\partial \kappa}{\partial \theta}\right) G - \left(G' + \frac{\partial \kappa}{\partial \zeta}\right) I\right] = \frac{4\pi^2}{V'} \left(I'G - G'I\right) && (38)
\end{aligned}
$$

We can derive the useful relation

$$
\iota \mu_0 \frac{\langle J_{\|} B \rangle}{\langle B^2 \rangle} + G' \;=\; \frac{\iota I'G - \iota G'I}{G + \iota I} + \frac{G'G + \iota IG'}{G + \iota I} = \frac{G' + \iota I'}{G + \iota I} = -\mu_0 p' \frac{G}{\langle B^2 \rangle}, \qquad (39)
$$

where we have used that $\langle B^2 \rangle = (G + \iota I)\, 4\pi^2/V'$ and $\mu_0 p' = -(G' + \iota I')\, 4\pi^2/V'$. One can also write

$$
\begin{aligned}
\boldsymbol{e}_\varphi &= -\frac{\iota}{p'} \boldsymbol{J} + \left(\frac{G}{\langle B^2 \rangle} + \frac{\iota}{p'} \frac{\langle J_{\|} B \rangle}{\langle B^2 \rangle}\right) \boldsymbol{B}, && (40)\\
\boldsymbol{e}_\varphi \cdot \boldsymbol{B} &= G \frac{B^2}{\langle B^2 \rangle} - B^2 u && (41)
\end{aligned}
$$

1.3 The toroidal viscosity

We now turn our attention to the toroidal viscosity term in Eq. (14),

$$
\langle \boldsymbol{e}_\varphi \cdot \nabla \cdot \boldsymbol{\Pi} \rangle = \alpha \langle \boldsymbol{B} \cdot \nabla \cdot \boldsymbol{\Pi} \rangle + \beta \langle \boldsymbol{J} \cdot \nabla \cdot \boldsymbol{\Pi} \rangle. \qquad (42)
$$

where we have dropped the index 0 on $\boldsymbol{B}_0$, $\boldsymbol{J}_0$ and p_0. First, we note that $\boldsymbol{\Pi}$ consists of contributions from both pressure anisotropy and radial electric field, $\boldsymbol{\Pi} = \boldsymbol{\Pi}_p + \boldsymbol{\Pi}_E$. The latter is formally smaller, but might still be of importance if we are close to axi-symmetry, so it will be calculated as well. The two terms are (see the SFINCS paper)

$$
\begin{aligned}
\boldsymbol{\Pi}_p &= \tilde{p}(\mathbf{I}/3 - B^{-2} \boldsymbol{B}\boldsymbol{B}), && (43)\\
\boldsymbol{\Pi}_E &= mnV_{\|} B^{-3} (\boldsymbol{B}\boldsymbol{B} \times \nabla\psi + \boldsymbol{B} \times \nabla\psi \boldsymbol{B}) && (44)
\end{aligned}
$$

1.3.1 Torque from pressure anisotropy

We start with the contribution from pressure anisotropy. $\boldsymbol{\Pi}_p = \tilde{p}(\mathbf{I}/3 - B^{-2} \boldsymbol{B}\boldsymbol{B})$, where $\tilde{p} \equiv p_\perp - p_{\|}$, which implies that

$$
\begin{aligned}
\nabla \cdot \boldsymbol{\Pi}_p &= \frac{1}{3} \nabla\tilde{p} - \nabla \frac{\tilde{p}}{B^2} \cdot \boldsymbol{B}\boldsymbol{B} - \tilde{p} \left(B^{-1} \nabla B + \frac{\mu_0}{B^2} \nabla p\right) && (45)\\
\boldsymbol{B} \cdot \nabla \cdot \boldsymbol{\Pi}_p &= -\frac{2}{3} \boldsymbol{B} \cdot \nabla\tilde{p} + \frac{\tilde{p}}{B} \nabla B && (46)\\
\langle \boldsymbol{B} \cdot \nabla \cdot \boldsymbol{\Pi}_p \rangle &= \langle \tilde{p} \nabla_{\|} B \rangle && (47)\\
\langle \boldsymbol{J} \cdot \nabla \cdot \boldsymbol{\Pi}_p \rangle &= -\left\langle J_{\|} B \boldsymbol{B} \cdot \nabla \frac{\tilde{p}}{B^2} \right\rangle - \left\langle \frac{\tilde{p}}{B} \boldsymbol{J} \cdot \nabla B \right\rangle && (48)
\end{aligned}
$$

In the last expression, we can replace $\boldsymbol{J}$ with $\boldsymbol{J} = J_\| B^{-1}\boldsymbol{B} + p' B^{-2}(\boldsymbol{B}\times\nabla\psi)$ where $\nabla\cdot\boldsymbol{J}=0$ gives $\boldsymbol{B}\cdot\nabla(J_\|/B) = 2p'B^{-3}\boldsymbol{B}\times\nabla\psi\cdot\nabla B$. If we also use that $\langle a\boldsymbol{B}\cdot\nabla b\rangle = -\langle b\boldsymbol{B}\cdot\nabla a\rangle$, we can write

$$\begin{aligned}
\langle \boldsymbol{J}\cdot\nabla\cdot\boldsymbol{\Pi}_p\rangle &= \left\langle \frac{\tilde{p}}{B^2}\boldsymbol{B}\cdot\nabla(J_\| B)\right\rangle - \left\langle \frac{\tilde{p}}{B^2}J_\|\boldsymbol{B}\cdot\nabla B\right\rangle - p'\left\langle \frac{\tilde{p}}{B^3}\boldsymbol{B}\times\nabla\psi\cdot\nabla B\right\rangle = \\
&= \left\langle \frac{\tilde{p}}{B^2}J_\|\boldsymbol{B}\cdot\nabla B\right\rangle + p'\left\langle \frac{\tilde{p}}{B^3}\boldsymbol{B}\times\nabla\psi\cdot\nabla B\right\rangle = \\
&= \frac{1}{2}\left\langle \frac{\tilde{p}}{B^2}\boldsymbol{B}\cdot\nabla(J_\| B)\right\rangle. \qquad (49)
\end{aligned}$$

We obtain the torque

$$\tau_p \equiv -\langle \boldsymbol{e}_\varphi\cdot\nabla\cdot\boldsymbol{\Pi}_p\rangle = \frac{G'}{\mu_0 p'}\left\langle \tilde{p}\nabla_\| B\right\rangle + \frac{\iota}{2p'}\left\langle \frac{\tilde{p}}{B}\nabla_\|(J_\| B)\right\rangle \qquad (50)$$

Note that the two terms cancel in axisymmetry, because equating Eqs. (18) and (28), we get $K = G'/(\mu_0 p')$, and the scalar product of Eq. (17) with $\boldsymbol{B}$ gives $-\iota J_\| B/p' = F + KB^2$. To calculate τ in non-axisymmetry, we first need to determine $J_\|$.

We are now in the position to calculate τ from Eqs. (30) and (50),

$$\begin{aligned}
\tau_p &= \frac{G'}{\mu_0 p'}\left\langle \tilde{p}\nabla_\| B\right\rangle + \frac{1}{2}\left\langle \frac{\tilde{p}}{B}\nabla_\|(B^2 u)\right\rangle + \frac{\iota}{p'}\frac{\langle J_\| B\rangle}{\langle B^2\rangle}\left\langle \tilde{p}\nabla_\| B\right\rangle = \{\text{use Eq. (39)}\} = \\
&= -\frac{G}{\langle B^2\rangle}\left\langle \tilde{p}\nabla_\| B\right\rangle + \frac{1}{2}\left\langle \frac{\tilde{p}}{B}\nabla_\|(B^2 u)\right\rangle = \\
&= -\left\langle \left(\frac{G}{\langle B^2\rangle} - u\right)\tilde{p}\nabla_\| B\right\rangle + \left\langle \tilde{p}\frac{1}{2}\boldsymbol{B}\cdot\nabla u\right\rangle = \\
&= \left\langle (\gamma+u)\,\tilde{p}\nabla_\| B\right\rangle + \left\langle \tilde{p}\,\iota B^{-3}\boldsymbol{B}\times\nabla\psi\cdot\nabla B\right\rangle, \qquad (51)
\end{aligned}$$

where we have denoted

$$\gamma \equiv -\frac{G}{\langle B^2\rangle}. \qquad (52)$$

Now, use Eqs. (33-34) to obtain a more programming-friendly expression for τ,

$$\begin{aligned}
\tau_p &= \left\langle (\gamma+u)\frac{\tilde{p}}{B\sqrt{g}}\left(\iota\frac{\partial B}{\partial\theta} + \frac{\partial B}{\partial\zeta}\right)\right\rangle + \left\langle \frac{\tilde{p}\,\iota}{B^3\sqrt{g}}\left(G\frac{\partial B}{\partial\theta} - I\frac{\partial B}{\partial\zeta}\right)\right\rangle = \\
&= \frac{1}{V'}\int d\theta d\zeta\frac{\tilde{p}}{B^2}\left[(\gamma+u)B\left(\iota\frac{\partial B}{\partial\theta} + \frac{\partial B}{\partial\zeta}\right) + \frac{\iota}{B}\left(G\frac{\partial B}{\partial\theta} - I\frac{\partial B}{\partial\zeta}\right)\right]. \qquad (53)
\end{aligned}$$

In a tokamak $\tau_p = 0$ because $\gamma + u = -G/B^2$ and $\partial B/\partial\zeta = \partial B/\partial\phi = 0$. For historical reason I defined NTV $\equiv -\tau$ in Sfincs.

1.3.2 Torque from radial electric field

We now continue with the viscosity tensor contribution from the radial electric field. Denote $\boldsymbol{\Pi}_E = \frac{d\Phi}{d\psi}mn\boldsymbol{K}$, where

$$\begin{aligned}
\boldsymbol{K} &= \frac{V_\|}{B^3}(\boldsymbol{B}\boldsymbol{B}\times\nabla\psi + \boldsymbol{B}\times\nabla\psi\boldsymbol{B}) \qquad (54) \\
\nabla\cdot\boldsymbol{K} &= \frac{V_\|}{B^3}\nabla\cdot(\boldsymbol{B}\boldsymbol{B}\times\nabla\psi + \boldsymbol{B}\times\nabla\psi\boldsymbol{B}) + \left(\boldsymbol{B}\cdot\nabla\frac{V_\|}{B^3}\right)\boldsymbol{B}\times\nabla\psi + \left(\boldsymbol{B}\times\nabla\psi\cdot\nabla\frac{V_\|}{B^3}\right)\boldsymbol{B}
\end{aligned}$$

We first note that

$$\begin{aligned}
\nabla\cdot(\boldsymbol{B}\boldsymbol{B}\times\nabla\psi+\boldsymbol{B}\times\nabla\psi\boldsymbol{B}) &= -\boldsymbol{B}\times\nabla\times(\boldsymbol{B}\times\nabla\psi)-\boldsymbol{B}\times\nabla\psi\times\mu_0\boldsymbol{J} && (55)\\
\boldsymbol{B}\cdot\nabla\cdot(\boldsymbol{B}\boldsymbol{B}\times\nabla\psi+\boldsymbol{B}\times\nabla\psi\boldsymbol{B}) &= 0 && (56)\\
\boldsymbol{J}\cdot\nabla\cdot(\boldsymbol{B}\boldsymbol{B}\times\nabla\psi+\boldsymbol{B}\times\nabla\psi\boldsymbol{B}) &= -\boldsymbol{J}\times\boldsymbol{B}\cdot\nabla\times(\boldsymbol{B}\times\nabla\psi)=-\nabla p\cdot\nabla\times(\boldsymbol{B}\times\nabla\psi)=\\
&= \nabla\cdot(\nabla p\times(\boldsymbol{B}\times\nabla\psi))=p'\boldsymbol{B}\cdot\nabla g^{\psi\psi} && (57)
\end{aligned}$$

so that

$$\begin{aligned}
\langle\boldsymbol{B}\cdot\nabla\cdot\boldsymbol{K}\rangle &= \left\langle B^2\boldsymbol{B}\times\nabla\psi\cdot\nabla\frac{V_\parallel}{B^3}\right\rangle && (58)\\
\langle\boldsymbol{J}\cdot\nabla\cdot\boldsymbol{K}\rangle &= \left\langle p'\frac{V_\parallel}{B^3}\boldsymbol{B}\cdot\nabla g^{\psi\psi}+p'g^{\psi\psi}\boldsymbol{B}\cdot\nabla\frac{V_\parallel}{B^3}\right\rangle+\left\langle J_\parallel B\left(\boldsymbol{B}\times\nabla\psi\cdot\nabla\frac{V_\parallel}{B^3}\right)\right\rangle=\\
&= \left\langle\left(\frac{p'}{\iota}u+\frac{\langle J_\parallel B\rangle}{\langle B^2\rangle}\right)B^2\left(\boldsymbol{B}\times\nabla\psi\cdot\nabla\frac{V_\parallel}{B^3}\right)\right\rangle && (59)
\end{aligned}$$

By using Eq. (39) we now obtain

$$\begin{aligned}
\tau_E\equiv-\langle\boldsymbol{e}_\varphi\cdot\nabla\cdot\boldsymbol{\Pi}_E\rangle &= \frac{d\Phi}{d\psi}mn\left[\frac{\iota}{p'}\langle\boldsymbol{J}\cdot\nabla\cdot\boldsymbol{K}\rangle+\frac{G'}{\mu_0 p'}\langle\boldsymbol{B}\cdot\nabla\cdot\boldsymbol{K}\rangle\right]=\\
&= \frac{d\Phi}{d\psi}mn\left\langle(u+\gamma)B^2\left(\boldsymbol{B}\times\nabla\psi\cdot\nabla\frac{V_\parallel}{B^3}\right)\right\rangle, && (60)
\end{aligned}$$

which vanishes in axisymmetry, as it should. It is instructive to rewrite this in terms of $V_\parallel/B$. Note therefore that

$$\begin{aligned}
\boldsymbol{B}\times\nabla\psi\cdot\nabla\frac{V_\parallel}{B^3} &= B^{-2}\boldsymbol{B}\times\nabla\psi\cdot\nabla\frac{V_\parallel}{B}-\frac{V_\parallel}{B}2B^{-3}\boldsymbol{B}\times\nabla\psi\cdot\nabla B=\\
&= B^{-2}\boldsymbol{B}\times\nabla\psi\cdot\nabla\frac{V_\parallel}{B}-\frac{V_\parallel}{B}\iota^{-1}\boldsymbol{B}\cdot\nabla u, && (61)
\end{aligned}$$

so that

$$\tau_E=\frac{d\Phi}{d\psi}mn\left\langle-\frac{V_\parallel}{B}\boldsymbol{B}\times\nabla\psi\cdot\nabla u-V_\parallel B(u+\gamma)\iota^{-1}\boldsymbol{B}\cdot\nabla u\right\rangle, \tag{62}$$

Using Eqs. (41) and $\boldsymbol{V}=\varpi(\psi)\boldsymbol{e}_\varphi+\kappa(\psi)\boldsymbol{B}$ we can write $V_\parallel=B\left[\kappa-\varpi(u+\gamma\rangle)\right]$, which means that the first term in (63) is zero.

$$\tau_E=-\frac{d\Phi}{d\psi}mn\left\langle B^2\left[\kappa-\varpi(u+\gamma\rangle)\right](u+\gamma)\iota^{-1}\boldsymbol{B}\cdot\nabla u\right\rangle, \tag{63}$$

In axisymmetry, $B^2(u+\gamma\rangle)=G$, so we still get zero in that case. For calculations it is somtimes more useful to start from

$$\tau_E=-\frac{d\Phi}{d\psi}mn\left\langle B^2\left[\kappa-\varpi(u+\gamma\rangle)\right](u+\gamma)2B^{-3}\boldsymbol{B}\times\nabla\psi\cdot\nabla B\right\rangle. \tag{64}$$

1.4 Determination of the radial electric field

The radial electric field $\boldsymbol{E}_r=-d\Phi/d\psi\,\nabla\psi$ is required for the calculation, so we need to obtain it from measurements. It is not measured in itself, but it is related to the measured toroidal plasma rotation. From the radial component of the momentum balance we see that

$$\boldsymbol{V}_a\times\boldsymbol{B}=-\boldsymbol{E}_r+\frac{\nabla p_a}{eZ_a n_a}+\frac{1}{eZ_a n_a}\left(\nabla\cdot\pi_a-\boldsymbol{R}_a+m_a n_a d\boldsymbol{V}_a/dt\right)\cdot\frac{\nabla\psi}{|\nabla\psi|}. \tag{65}$$

The terms in the bracket are small unless the plasma rotation velocity is of the order of the thermal speed, in which case the term $d\boldsymbol{V}_a/dt = (\boldsymbol{V}_a \cdot \nabla)\boldsymbol{V}_a$ would have had to be kept. For slow rotation, we have

$$\boldsymbol{V}_a \times \boldsymbol{B} = -\boldsymbol{E}_r + \frac{\nabla p_a}{eZ_a n_a}. \tag{66}$$

For $\nabla \cdot \boldsymbol{V}_a = 0$ to hold we must be able to write $\boldsymbol{V}_a$ as a linear combination of $\boldsymbol{e}_\varphi$ (or $\boldsymbol{J}$) and $\boldsymbol{B}$,

$$\boldsymbol{V}_a = \varpi_a(\psi)\boldsymbol{e}_\varphi + \kappa_a(\psi)\boldsymbol{B}, \tag{67}$$

from which we obtain $\boldsymbol{V}_a \times \boldsymbol{B} = -\iota\varpi_a \nabla\psi$ and

$$-\iota\varpi_a(\psi) = \frac{d\Phi}{d\psi} + \frac{1}{eZ_a n_a}\frac{dp_a}{d\psi} \tag{68}$$

Eq. (67) yields (if we use that $\langle \boldsymbol{e}_\varphi \cdot \boldsymbol{B}\rangle = G$),

$$\langle \boldsymbol{V}_a \cdot \boldsymbol{B}\rangle = \varpi_a G + \kappa_a \langle B^2\rangle = -\frac{G}{\iota}\frac{d\Phi}{d\psi} - \frac{G}{eZ_a n_a \iota}\frac{dp_a}{d\psi} + \kappa_a\langle B^2\rangle. \tag{69}$$

We can now show that κ_a is independent of the radial electric field in axi-symmetry. The argument is that if we write $\langle \boldsymbol{V}_a \cdot \boldsymbol{B}\rangle$ in terms of the transport matrix and thermodynamical forces we get

$$\langle \boldsymbol{V}_a \cdot \boldsymbol{B}\rangle = \frac{G}{\iota}\frac{d\Phi}{d\psi}L_{31} + \text{ terms proportional to } \frac{dn}{d\psi}, \frac{dT}{d\psi} \text{ and } E_\parallel. \tag{70}$$

The symmetric one-species transport matrix is defined by

$$\begin{pmatrix} \frac{\iota q(G+\iota I)}{nTG}\left\langle \int d^3v\, f \boldsymbol{v}_\mathrm{d}\cdot\nabla\psi\right\rangle \\ \frac{\iota q(G+\iota I)}{nTG}\left\langle \int d^3v\, f \frac{mv^2}{2T}\boldsymbol{v}_\mathrm{d}\cdot\nabla\psi\right\rangle \\ \frac{\langle \boldsymbol{V}\cdot\boldsymbol{B}\rangle}{v_T B_0} \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} \frac{TG}{\iota q B_0 v_T}\tilde{A}_1 \\ \frac{TG}{\iota q B_0 v_T}\tilde{A}_2 \\ \frac{q(G+\iota I)}{T}\frac{\langle \boldsymbol{E}\cdot\boldsymbol{B}\rangle}{\langle B^2\rangle} \end{pmatrix} \tag{71}$$

where

$$\tilde{A}_1 \equiv \frac{d\ln n}{d\psi} + \frac{q}{T}\frac{d\Phi}{d\psi} - \frac{3}{2}\frac{d\ln T}{d\psi} \tag{72}$$

$$\tilde{A}_2 \equiv \frac{d\ln T}{d\psi}. \tag{73}$$

In axi-symmetry, one can show that $L_{31} = -1$ in all collisionality regimes, and in the relevant close-to-axi-symmetric ASDEX equilibria, we have $L_{31} + 1 \ll 10^{-3}$. With this value of L_{31}, Eq. (69) gives

$$\kappa_a\langle B^2\rangle = \frac{G}{eZ_a n_a \iota}\frac{dp_a}{d\psi} + \text{ terms proportional to } \frac{dn}{d\psi}, \frac{dT}{d\psi} \text{ and } E_\parallel, \tag{74}$$

which is independent of E_r. We can thus argue that close to axi-symmetry we can choose any radial electric field to calculate κ_a. For simplicity, we take $E_r = 0$, and determine the resulting parallel flux $\langle \boldsymbol{V}_a \cdot \boldsymbol{B}\rangle_{E_r=0}$ with SFINCS. According to Eq. (69), κ_a is then given by

$$\kappa_a\langle B^2\rangle = \langle \boldsymbol{V}_a \cdot \boldsymbol{B}\rangle_{E_r=0} + \frac{G}{\iota e Z_a n_a}\frac{dp_a}{d\psi}. \tag{75}$$

We are now ready to determine the radial electric field with the aid of the measured toroidal rotation frequency, which we take to be defined by $\omega_a = \langle \boldsymbol{V}_a \cdot \nabla\varphi \rangle$. From Eq. (67) we obtain

$$\omega_a = \varpi_a + \kappa_a \langle \nabla\psi \times \nabla\vartheta \cdot \nabla\varphi \rangle = \varpi_a + \kappa_a \frac{\langle B^2 \rangle}{G + \iota I} \tag{76}$$

and Eq. (68) finally gives the radial electric field

$$\begin{aligned} -\frac{d\Phi}{d\psi} &= \iota\varpi_a + \frac{1}{eZ_a n_a}\frac{dp_a}{d\psi} = \\ &= \iota\left(\omega_a - \kappa_a \frac{\langle B^2 \rangle}{G + \iota I}\right) + \frac{1}{eZ_a n_a}\frac{dp_a}{d\psi} = \\ &= \iota\left[\omega_a + \frac{1}{G + \iota I}\left(\frac{I}{eZ_a n_a}\frac{dp_a}{d\psi} - \langle \boldsymbol{V}_a \cdot \boldsymbol{B} \rangle_{E_r=0}\right)\right] \end{aligned} \tag{77}$$

In SFINCS version 3 normalisations (see SFINCS documentation) this expression becomes

$$-\frac{d\hat{\Phi}}{d\psi_N} = \iota\left[\frac{2\hat{\psi}_a}{\Delta\alpha}\frac{\omega_a \bar{R}}{\bar{v}} + \frac{1}{\left(\hat{G} + \iota\hat{I}\right)\hat{n}_a\alpha}\left(\frac{\hat{I}}{Z_a}\frac{d(\hat{n}_a\hat{T}_a)}{d\psi_N} - \frac{2\hat{\psi}_a}{\Delta}\text{FSABflow}\right)\right] \tag{78}$$

1.5 Damping and intrinsic rotation frequency

1.5.1 . . . from transport matrix elements

For the case of one particle species, the flux-force relation gives the torque

$$\tau = -qn\iota\langle \boldsymbol{V} \cdot \nabla\psi \rangle - qnG\frac{\langle E_\parallel B \rangle}{\langle B^2 \rangle}, \tag{79}$$

where $q = eZ$ and we have dropped the species subscript a. In the first thermodynamical force, we take the radial electric field from the calculation of κ above,

$$\frac{d\Phi}{d\psi} = -\iota\left(\omega - \kappa\frac{\langle B^2 \rangle}{G + \iota I}\right) - \frac{T}{q}\left(\frac{d\ln T}{d\psi} + \frac{d\ln n}{d\psi}\right). \tag{80}$$

The first thermodynamical force becomes

$$\tilde{A}_1 = -\frac{\iota q}{T}\left(\omega - \kappa\frac{\langle B^2 \rangle}{G + \iota I}\right) - \frac{5}{2}\frac{d\ln T}{d\psi}. \tag{81}$$

We can write κ in terms of transport matrix elements as

$$\begin{aligned} \kappa\langle B^2 \rangle &= \langle \boldsymbol{V} \cdot \boldsymbol{B} \rangle_{E_r=0} + \frac{G}{\iota qn}\frac{dp}{d\psi} = \\ &= \frac{GT}{\iota q}\left(\frac{d\ln n}{d\psi} - \frac{3}{2}\frac{d\ln T}{d\psi}\right)L_{31,E_r=0} + \frac{GT}{\iota q}\frac{d\ln T}{d\psi}L_{32,E_r=0} + \\ &\quad + \frac{qv_T B_0}{T}(G + \iota I)\frac{\langle E_\parallel B \rangle}{\langle B^2 \rangle}L_{33,E_r=0} + \frac{G}{\iota qn}\frac{dp}{d\psi} = \\ &= \frac{GT}{\iota q}\left(\frac{d\ln n}{d\psi} - \frac{3}{2}\frac{d\ln T}{d\psi}\right)[L_{31,E_r=0} + 1] + \frac{GT}{\iota q}\frac{d\ln T}{d\psi}\left[\frac{5}{2} + L_{32,E_r=0}\right] + \\ &\quad + \frac{qv_T B_0}{T}(G + \iota I)\frac{\langle E_\parallel B \rangle}{\langle B^2 \rangle}L_{33,E_r=0}. \end{aligned} \tag{82}$$

In axi-symmetry, $L_{31,E_r=0} = -1$ (see notes by Matt). Thus, close to axi-symmetry, we expect the first term to be negligible, and the result therefore independent of the density gradient. In fact, we already neglected a term proportional to $(L_{31}+1)$ in Eq. (75), so we must neglect the one appearing here too. We obtain

$$\begin{aligned}\tilde{A}_1 &= -\frac{q\iota\omega}{T} + \frac{G}{\iota(G+\iota I)}\frac{d\ln T}{d\psi}\left(\frac{5}{2} + L_{32,E_r=0}\right) - \frac{5}{2}\frac{d\ln T}{d\psi} + \iota\frac{q^2 v_T B_0}{T^2}\frac{\langle E_\parallel B\rangle}{\langle B^2\rangle}L_{33,E_r=0} = \\ &= -\frac{q\iota\omega}{T} + \frac{1}{G+\iota I}\frac{d\ln T}{d\psi}\left(GL_{32,E_r=0} - \frac{5}{2}\iota I\right) + \iota\frac{q^2 v_T B_0}{T^2}\frac{\langle E_\parallel B\rangle}{\langle B^2\rangle}L_{33,E_r=0}. \end{aligned} \tag{83}$$

The torque becomes

$$\begin{aligned}\tau &= -qn\iota\langle \boldsymbol{V}\cdot\nabla\psi\rangle - qnG\frac{\langle E_\parallel B\rangle}{\langle B^2\rangle} = \\ &= -\frac{nT^2G^2}{\iota q v_T B_0(G+\iota I)}\left[L_{11}\tilde{A}_1 + L_{12}\frac{d\ln T}{d\psi}\right] - qnG\frac{\langle E_\parallel B\rangle}{\langle B^2\rangle}(L_{13}+1) = \\ &= -\frac{nT^2G^2}{\iota q v_T B_0(G+\iota I)}\left\{-\frac{q}{T}L_{11}\iota\omega + \frac{d\ln T}{d\psi}\left[L_{12} + \frac{L_{11}}{G+\iota I}\left(GL_{32,E_r=0} - \frac{5}{2}\iota I\right)\right]\right\} + \\ &\quad + \frac{qnG^2}{G+\iota I}\frac{\langle E_\parallel B\rangle}{\langle B^2\rangle}L_{33,E_r=0}\,L_{11} \end{aligned} \tag{84}$$

We want to express the torque as

$$\tau = -\nu m n\langle g_{\varphi\varphi}\rangle(\omega - \omega_{\text{in}}), \tag{85}$$

where ν is the rotation relaxation rate and ω_{in} is the intrinsic rotation frequency. For $E_\parallel = 0$, we obtain

$$\nu = -L_{11}\frac{TG^2}{m v_T B_0\langle g_{\varphi\varphi}\rangle(G+\iota I)} \tag{86}$$

$$\omega_{\text{in}} = \frac{T}{q\iota}\frac{d\ln T}{d\psi}\left[\frac{L_{12}}{L_{11}} + \frac{1}{G+\iota I}\left(GL_{32,E_r=0} - \frac{5}{2}\iota I\right)\right] \tag{87}$$

Two important things to note:

- The intrinsic frequency ω_{in} depends on E_r through the first term, and it is thus a function $\omega_{\text{in}} = \omega_{\text{in}}(\omega)$. Equation (85) does therefore NOT imply that the plasma rotation frequency ω will exponentially approach $\omega_{\text{in}}(\omega_{\text{initial}})$. Instead, the end state (assuming that the equilibrium does not evolve) will be the solution to the equation $\omega = \omega_{\text{in}}(\omega)$. Solving this equation is equivalent to searching for the radial electric field which gives zero radial flux.

- Such a procedure will not give the physical end state for the rotation, because this whole standard neoclassical calculation is based on that the radial transport of momentum can be neglected, being of higher order in ρ_{L}/L. However, the closeness to axi-symmetry can make the NTV torque small enough to be comparable to radial transport of momentum. The steady-state equation is then not $\tau = 0$, but instead

$$\frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle \Pi_\varphi^\psi\right\rangle_{\text{axi-sym}} = \tau. \tag{88}$$

  In practice, it is not only the NTV torque that influences the rotation. Collisions with neutrals and NBI injection are two additional effects that provide torque, and these could compete with the NTV torque.

1.5.2 ...from three SFINCS runs

1. The first run is performed to determine κ and thereby E_r at the measured ion rotation frequency ω_i as in Eq. (77).

$$-\left.\frac{d\Phi}{d\psi}\right|_\mathrm{LHS} = \iota\left[\omega_\mathrm{i} + \frac{1}{G+\iota I}\left(\frac{I}{q_\mathrm{i} n_\mathrm{i}}\frac{dp_\mathrm{i}}{d\psi} - \langle \boldsymbol{V}_\mathrm{i}\cdot\boldsymbol{B}\rangle_{E_r=0}\right)\right] \tag{89}$$

2. The next step is made to calculate the relaxation rate ν_t. To this end we should run with $d\Phi/d\psi|_\mathrm{LHS}$ on the left hand side but with $-d\Phi/d\psi = \iota\omega_\mathrm{i}$ and $dn_a/d\psi = dT_a/d\psi = E_\| = 0$ in the thermodynamical forces and determine the torque. To account for the different values of the radial electric field on the two sides of the equation we must alter the input density gradients as follows

$$\left.\frac{d\ln n_\mathrm{i}}{d\psi}\right|_\mathrm{input} = -\frac{q_\mathrm{i}}{T_\mathrm{i}}\left(\left.\frac{d\Phi}{d\psi}\right|_\mathrm{LHS} + \iota\omega_\mathrm{i}\right) = \frac{q_\mathrm{i}}{T_\mathrm{i}}\frac{\iota}{G+\iota I}\left(\frac{I}{q_\mathrm{i} n_\mathrm{i}}\frac{dp_\mathrm{i}}{d\psi} - \langle \boldsymbol{V}_\mathrm{i}\cdot\boldsymbol{B}\rangle_{E_r=0}\right) \tag{90}$$

$$\left.\frac{d\ln n_\mathrm{e}}{d\psi}\right|_\mathrm{input} = -\frac{q_\mathrm{e}}{T_\mathrm{e}}\left(\left.\frac{d\Phi}{d\psi}\right|_\mathrm{LHS} + \iota\omega_\mathrm{i}\right) = \frac{q_\mathrm{e}}{T_\mathrm{e}}\frac{\iota}{G+\iota I}\left(\frac{I}{q_\mathrm{i} n_\mathrm{i}}\frac{dp_\mathrm{i}}{d\psi} - \langle \boldsymbol{V}_\mathrm{i}\cdot\boldsymbol{B}\rangle_{E_r=0}\right). \tag{91}$$

The corresponding normalised expressions in version 3 SFINCS are

$$\left.\frac{d\hat{n}_\mathrm{i}}{d\psi_N}\right|_\mathrm{input} = -\frac{Z_\mathrm{i}\hat{n}_\mathrm{i}}{\hat{T}_\mathrm{i}}\left(\alpha\left.\frac{d\hat{\Phi}}{d\psi_N}\right|_\mathrm{LHS} + \iota\frac{2}{\Delta}\frac{\omega_\mathrm{i}\bar{R}\hat{\psi}_a}{\bar{v}}\right) \tag{92}$$

$$\left.\frac{d\hat{n}_\mathrm{e}}{d\psi_N}\right|_\mathrm{input} = -\frac{(-1)\hat{n}_\mathrm{e}}{\hat{T}_\mathrm{e}}\left(\alpha\left.\frac{d\hat{\Phi}}{d\psi_N}\right|_\mathrm{LHS} + \iota\frac{2}{\Delta}\frac{\omega_\mathrm{i}\bar{R}\hat{\psi}_a}{\bar{v}}\right) \tag{93}$$

The resulting torque, denoted τ_1, is related to ν_t through

$$\tau_1 = -\nu_\mathrm{t} m_\mathrm{i} n_\mathrm{i}\langle g_{\varphi\varphi}\rangle\,\omega_\mathrm{i}. \tag{94}$$

3. The last step is to run with $d\Phi/d\psi|_\mathrm{LHS}$ on the left hand side but with $d\Phi/d\psi = d\Phi/d\psi|_\mathrm{LHS} + \iota\omega_\mathrm{i}$ in the thermodynamical forces, keeping gradients and $E_\|$ at their real values. To do this we need to use the altered density gradients

$$\left.\frac{dn_\mathrm{i}}{d\psi}\right|_\mathrm{input} = \left.\frac{dn_\mathrm{i}}{d\psi}\right|_\mathrm{real} + \frac{q_\mathrm{i}}{T_\mathrm{i}}\iota\omega_\mathrm{i} \tag{95}$$

$$\left.\frac{dn_\mathrm{e}}{d\psi}\right|_\mathrm{input} = \left.\frac{dn_\mathrm{e}}{d\psi}\right|_\mathrm{real} + \frac{q_\mathrm{e}}{T_\mathrm{e}}\iota\omega_\mathrm{i} \tag{96}$$

The corresponding normalised expressions in version 3 SFINCS are

$$\left.\frac{d\hat{n}_\mathrm{i}}{d\psi_N}\right|_\mathrm{input} = \left.\frac{d\hat{n}_\mathrm{i}}{d\psi_N}\right|_\mathrm{real} + \frac{Z_\mathrm{i}\hat{n}_\mathrm{i}}{\hat{T}_\mathrm{i}}\iota\frac{2}{\Delta}\frac{\omega_\mathrm{i}\bar{R}\hat{\psi}_a}{\bar{v}} \tag{97}$$

$$\left.\frac{d\hat{n}_\mathrm{e}}{d\psi_N}\right|_\mathrm{input} = \left.\frac{d\hat{n}_\mathrm{e}}{d\psi_N}\right|_\mathrm{real} + \frac{(-1)\hat{n}_\mathrm{e}}{\hat{T}_\mathrm{e}}\iota\frac{2}{\Delta}\frac{\omega_\mathrm{i}\bar{R}\hat{\psi}_a}{\bar{v}} \tag{98}$$

The resulting torque, τ_in is used to determine the intrinsic rotation frequency through

$$\omega_\mathrm{in} = -\omega_\mathrm{i}\ \tau_\mathrm{in}/\tau_1, \tag{99}$$

so that

$$\tau = \tau_1 + \tau_\mathrm{in} = -\nu_\mathrm{t} m_\mathrm{i} n_\mathrm{i}\langle g_{\varphi\varphi}\rangle(\omega_\mathrm{i} - \omega_\mathrm{in}) \tag{100}$$

1.6 Implementation

This part deals with the implementation of Eq. (53) in SFINCS, and uses the internal normalisations in the code, which are described in the code documentation. The following is therefore probably only interesting if you work with the code development.

We first need to calculate u by solving Eq. (35). In SFINCS normalisation, we have $G = \hat{G}\bar{R}\bar{B}$, $B = \hat{B}\bar{B}$, $u = \hat{u}\bar{R}/\bar{B}$, so the normalised equation which is solved in the code is

$$\left(\iota\frac{\partial \hat{u}}{\partial \theta} + \frac{\partial \hat{u}}{\partial \zeta}\right) = 2\frac{\iota}{\hat{B}^3}\left(\hat{G}\frac{\partial \hat{B}}{\partial \theta} - \hat{I}\frac{\partial \hat{B}}{\partial \zeta}\right). \tag{101}$$

Furthermore, we need to calculate the pressure anisotropy $\tilde{p}$ from the distribution function obtained by the code. In the normalisations used in the SFINCS single species documentation,

$$\begin{aligned}\tilde{p} = p_\perp - p_\parallel = m\int d^3v\ f\left(\frac{v_\perp^2}{2} - v_\parallel^2\right) = \frac{2\Delta\hat{T}^{3/2}n}{\sqrt{\pi}\hat{\psi}_a}\int_{-1}^{1} d\xi\int_0^\infty dx\ x^2\hat{f}m\left(\frac{v_\perp^2}{2} - v_\parallel^2\right) = \\ = \left\{m\left(\frac{v_\perp^2}{2} - v_\parallel^2\right) = \hat{T}\bar{T}x^2(1-3\xi^2) = -2\hat{T}\bar{T}x^2P_2(\xi)\right\} = \\ = -\bar{T}n\frac{4\Delta\hat{T}^{5/2}}{\sqrt{\pi}\hat{\psi}_a}\int_{-1}^{1} d\xi\int_0^\infty dx\ x^4P_2(\xi)\hat{f}.\end{aligned} \tag{102}$$

Note the following about the Legendre polynomial P_2,

$$\int_{-1}^{1} d\xi P_2^2(\xi) = \frac{2}{5}, \tag{103}$$

so that with

$$\hat{f} = \sum_{l=0}^{\infty} f_l P_l(\xi) \tag{104}$$

we obtain

$$\int_{-1}^{1} d\xi P_2(\xi)\hat{f} = \frac{2}{5}f_2. \tag{105}$$

If we define $\gamma = \hat{\gamma}\bar{R}/\bar{B}$ we can write

$$\tau_p \ = \ \bar{T}n\frac{4\Delta\hat{T}^{5/2}\bar{R}}{V'\sqrt{\pi}\hat{\psi}_a\bar{B}}\int d\theta d\zeta\frac{1}{\hat{B}^2}\left[(\hat{\gamma}+\hat{u})\hat{B}\left(\iota\frac{\partial \hat{B}}{\partial \theta} + \frac{\partial \hat{B}}{\partial \zeta}\right) + \frac{\iota}{\hat{B}}\left(\hat{G}\frac{\partial \hat{B}}{\partial \theta} - \hat{I}\frac{\partial \hat{B}}{\partial \zeta}\right)\right]\frac{2}{5}\int_0^\infty dx\ x^4 f_2 \tag{106}$$

Define $\hat{V}' = \int d\theta d\zeta\ \hat{B}^{-2}$ as in early SFINCS versions (version three it is different) and note that

$$V' = \int d\theta d\zeta\ \frac{G+\iota I}{B^2} = \frac{\bar{R}}{\bar{B}}\left(\hat{G}+\iota\hat{I}\right)\hat{V}', \tag{107}$$

so that

$$\hat{\gamma} = -\frac{\hat{V}'\hat{G}}{4\pi^2} \tag{108}$$

We define the normalised torque in the single species code

$$\begin{aligned}-\hat{\tau}_p^{\text{single}} = \frac{1}{n\bar{T}}\langle \boldsymbol{e}_\varphi \cdot \nabla \cdot \boldsymbol{\Pi}_p\rangle\frac{\hat{\psi}_a(\hat{G}+\iota\hat{I})\hat{V}'}{2\Delta\iota} = \\ = \frac{2\hat{T}^{5/2}}{\iota\sqrt{\pi}}\int d\theta d\zeta\frac{1}{\hat{B}^2}\left[(\hat{\gamma}+\hat{u})\hat{B}\left(\iota\frac{\partial \hat{B}}{\partial \theta} + \frac{\partial \hat{B}}{\partial \zeta}\right) + \frac{\iota}{\hat{B}}\left(\hat{G}\frac{\partial \hat{B}}{\partial \theta} - \hat{I}\frac{\partial \hat{B}}{\partial \zeta}\right)\right]\frac{2}{5}\int_0^\infty dx\ x^4 f_2.\end{aligned} \tag{109}$$

With this normalisation, the so-called flux-friction relations imply that $-\hat{\tau}^{\text{single}}$ equals the radial particle flux in the single species Sfincs definition (when momentum is conserved and $E_\parallel = 0$).

In the multi-species code some normalisations are different, in particular, the definition of $\hat{f}$ differs in the following way,

$$\hat{f}^{\,\text{multi}} = \frac{\bar{v}^3}{\bar{n}} f = \frac{\hat{m}^{3/2}\hat{n}}{\pi^{3/2}\hat{\psi}_a}\hat{f}. \tag{110}$$

A suitable normalised torque in the multiple species code is

$$\begin{aligned}
-\hat{\tau}_p^{\,\text{multi}} &= \frac{1}{\bar{n}\bar{T}}\langle \boldsymbol{e}_\varphi \cdot \nabla \cdot \boldsymbol{\Pi}_p \rangle = \\
&= \frac{4\pi \hat{T}^{5/2}}{\hat{m}^{3/2}(\hat{G}+\iota\hat{I})\hat{V}'}\int d\theta d\zeta \frac{1}{\hat{B}^2}\left[(\hat{\gamma}+\hat{u})\hat{B}\left(\iota\frac{\partial\hat{B}}{\partial\theta}+\frac{\partial\hat{B}}{\partial\zeta}\right)+\frac{\iota}{\hat{B}}\left(\hat{G}\frac{\partial\hat{B}}{\partial\theta}-\hat{I}\frac{\partial\hat{B}}{\partial\zeta}\right)\right]\frac{2}{5}\int_0^\infty dx\; x^4 f_2^{\text{multi}}
\end{aligned} \tag{111}$$