PLANE AND SPHERICAL

TRIGONOMETRY,

AND

SURVEYING

BY

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Teachers' Edition.

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PREFACE.

THIS edition is intended for teachers, *and for them only.* The publishers will under no circumstances sell the book except to teachers of Wentworth's Trigonometry; and every teacher must consider himself in honor bound not to leave his copy where pupils can have access to it, and not to sell his copy except to the publishers, Messrs. Ginn & Company.

It is hoped that young teachers will derive great advantage from studying the systematic arrangement of the work, and that all teachers who are pressed for time will find great relief by not being obliged to work out every problem in the Trigonometry and Surveying.

G. A. WENTWORTH.

TRIGONOMETRY.

Exercise I. Page 5.

1. What are the functions of the other acute angle B of the triangle ABC (Fig. 2)?

$\sin B = \frac{b}{c}$, $\cos B = \frac{a}{c}$,

$\tan B = \frac{b}{a}$, $\cot B = \frac{a}{b}$,

$\sec B = \frac{c}{a}$, $\csc B = \frac{c}{b}$.

2. Prove that if two angles, A and B, are complements of each other (*i.e.*, if $A + B = 90°$), then

$\sin A = \cos B$, $\cos A = \sin B$,
$\tan A = \cot B$, $\cot A = \tan B$,
$\sec A = \csc B$, $\csc A = \sec B$.

$\sin A = \frac{a}{c}$, $\cos B = \frac{a}{c}$,

$\cos A = \frac{b}{c}$, $\sin B = \frac{b}{c}$,

$\tan A = \frac{a}{b}$, $\cot B = \frac{a}{b}$,

$\cot A = \frac{b}{a}$, $\tan B = \frac{b}{a}$,

$\sec A = \frac{c}{b}$, $\csc B = \frac{c}{b}$,

$\csc A = \frac{c}{a}$. $\sec B = \frac{c}{a}$.

3. Find the values of the functions of A, if a, b, c respectively have the following values:

(i.) 3, 4, 5. (iv.) 9, 40, 41.
(ii.) 5, 12, 13. (v.) 3.9, 8, 8.9.
(iii.) 8, 15, 17. (vi.) 1.19, 1.20, 1.69.

(i.) $\sin A = \frac{3}{5}$, $\cos A = \frac{4}{5}$, $\tan A = \frac{3}{4}$, $\cot A = \frac{4}{3}$, $\sec A = \frac{5}{4}$, $\csc A = \frac{5}{3}$.

(ii.) $\sin A = \frac{5}{13}$, $\cos A = \frac{12}{13}$, $\tan A = \frac{5}{12}$, $\cot A = \frac{12}{5}$, $\sec A = \frac{13}{12}$, $\csc A = \frac{13}{5}$.

(iii.) $\sin A = \frac{8}{17}$, $\cos A = \frac{15}{17}$, $\tan A = \frac{8}{15}$, $\cot A = \frac{15}{8}$, $\sec A = \frac{17}{15}$, $\csc A = \frac{17}{8}$.

(iv.) $\sin A = \frac{9}{41}$, $\cos A = \frac{40}{41}$, $\tan A = \frac{9}{40}$, $\cot A = \frac{40}{9}$, $\sec A = \frac{41}{40}$, $\csc A = \frac{41}{9}$.

(v.) $\sin A = \frac{39}{89}$, $\cos A = \frac{80}{89}$,

$\tan A = \frac{39}{80}$, $\cot A = \frac{80}{39}$,

$\sec A = \frac{89}{80}$, $\csc A = \frac{89}{39}$.

(vi.) $\sin A = \frac{119}{169}$, $\cos A = \frac{120}{169}$,

$\tan A = \frac{119}{120}$, $\cot A = \frac{120}{119}$,

$\sec A = \frac{169}{120}$, $\csc A = \frac{169}{119}$.

4. What condition must be fulfilled by the lengths of the three lines a, b, c (Fig. 2) in order to make them the sides of a right triangle? Is this condition fulfilled in Example 3?

$$a^2 + b^2 = c^2.$$

5. Find the values of the functions of A, if a, b, c respectively have the following values:

(i.) $2mn$, $m^2 - n^2$, $m^2 + n^2$.

(ii.) $\frac{2xy}{x-y}$, $x+y$, $\frac{x^2+y^2}{x-y}$.

(iii.) pqr, qrs, rsp.

(iv.) $\frac{mn}{pq}$, $\frac{mv}{sq}$, $\frac{nr}{ps}$.

(i.)

$$\sin A = \frac{a}{c} = \frac{2mn}{m^2+n^2},$$

$$\cos A = \frac{b}{c} = \frac{m^2-n^2}{m^2+n^2},$$

$$\tan A = \frac{a}{b} = \frac{2mn}{m^2-n^2},$$

$$\cot A = \frac{b}{a} = \frac{m^2-n^2}{2mn},$$

$$\sec A = \frac{c}{b} = \frac{m^2+n^2}{m^2-n^2},$$

$$\csc A = \frac{c}{a} = \frac{m^2+n^2}{2mn}.$$

(ii.)

$$\sin A = \frac{2xy}{x-y} \times \frac{x-y}{x^2+y^2} = \frac{2xy}{x^2+y^2},$$

$$\cos A = (x+y) \times \frac{x-y}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2},$$

$$\tan A = \frac{2xy}{x-y} \times \frac{1}{x+y} = \frac{2xy}{x^2-y^2},$$

$$\cot A = \frac{x-y}{2xy} \times (x+y) = \frac{x^2-y^2}{2xy},$$

$$\sec A = \frac{1}{x+y} \times \frac{x^2+y^2}{x-y} = \frac{x^2+y^2}{x^2-y^2},$$

$$\csc A = \frac{x-y}{2xy} \times \frac{x^2+y^2}{x-y} = \frac{x^2+y^2}{2xy}.$$

(iii.)

$\sin A = \frac{pqr}{rsp} = \frac{q}{s}$, $\cos A = \frac{qrs}{rsp} = \frac{q}{p}$,

$\tan A = \frac{pqr}{qrs} = \frac{p}{s}$, $\cot A = \frac{qrs}{pqr} = \frac{s}{p}$,

$\sec A = \frac{rsp}{qrs} = \frac{p}{q}$, $\csc A = \frac{rsp}{pqr} = \frac{s}{q}$.

(iv.)

$$\sin A = \frac{mn}{pq} \times \frac{ps}{nr} = \frac{ms}{qr},$$

$$\cos A = \frac{mv}{sq} \times \frac{ps}{nr} = \frac{mpv}{nqr},$$

$$\tan A = \frac{n}{pq} \times \frac{sq}{mv} = \frac{ns}{pv},$$

$$\cot A = \frac{pq}{mn} \times \frac{mv}{sq} = \frac{pv}{ns},$$

$$\sec A = \frac{sq}{mv} \times \frac{nr}{ps} = \frac{nqr}{mpv},$$

$$\csc A = \frac{pq}{mn} \times \frac{nr}{ps} = \frac{qr}{ms}.$$

6. Prove that the values of a, b, c, in (i.) and (ii.), Example 5, satisfy the condition necessary to make them the sides of a right triangle.

(i.)

$$a^2 + b^2 = c^2,$$
$$(2mn)^2 + (m^2 - n^2)^2 = (m^2 + n^2)^2,$$
$$4m^2n^2 + m^4 - 2m^2n^2 + n^4 = m^4 + 2m^2n^2 + n^4,$$
$$m^4 + 2m^2n^2 + n^4 = m^4 + 2m^2n^2 + n^4.$$

(ii.)

$$\left(\frac{2xy}{x-y}\right)^2 + (x+y)^2 = \left(\frac{x^2+y^2}{x-y}\right)^2,$$
$$\frac{4x^2y^2}{x^2 - 2xy + y^2} + x^2 + 2xy + y^2 = \frac{x^4 + 2x^2y^2 + y^4}{x^2 - 2xy + y^2},$$
$$4x^2y^2 + x^4 - 2x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4,$$
$$x^4 + 2x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4.$$

7. What equations of condition must be satisfied by the values of a, b, c, in (iii.) and (iv.), Example 5, in order that the values may represent the sides of a right triangle.

(iii.)

$$p^2q^2r^2 + q^2r^2s^2 = r^2s^2p^2,$$
or $$p^2q^2 + q^2s^2 = p^2s^2.$$

(iv.)

$$\frac{m^2n^2}{p^2q^2} + \frac{m^2v^2}{s^2q^2} = \frac{n^2r^2}{p^2s^2},$$
or $$m^2n^2s^2 + m^2p^2v^2 = n^2q^2r^2.$$

8. Compute the functions of A and B when $a = 24$, $b = 143$.

$$c = \sqrt{(24)^2 + (143)^2} = \sqrt{21025} = 145.$$

$$\sin A = \frac{24}{145} = \cos B,$$
$$\cos A = \frac{143}{145} = \sin B,$$
$$\tan A = \frac{24}{143} = \cot B,$$
$$\cot A = \frac{143}{24} = \tan B,$$
$$\sec A = \frac{145}{143} = \csc B,$$
$$\csc A = \frac{145}{24} = \sec B.$$

9. Compute the functions of A and B when $a = 0.264$, $c = 0.265$.

$$b^2 = c^2 - a^2 = 0.070225 - 0.069696 = 0.000529.$$
$$\therefore b = 0.023.$$

$$\sin A = \frac{a}{c} = \frac{264}{265} = \cos B,$$
$$\cos A = \frac{b}{c} = \frac{23}{265} = \sin B,$$
$$\tan A = \frac{a}{b} = \frac{264}{23} = \cot B,$$
$$\cot A = \frac{b}{a} = \frac{23}{264} = \tan B,$$
$$\sec A = \frac{c}{b} = \frac{265}{23} = \csc B,$$
$$\csc A = \frac{c}{a} = \frac{265}{264} = \sec B.$$

10. Compute the functions of A and B when $b = 9.5$, $c = 19.3$.

$$a^2 = c^2 - b^2 = 372.49 - 90.25 = 282.24.$$
$$\therefore a = 16.8.$$

$$\sin A = \frac{a}{c} = \frac{168}{193} = \cos B,$$

$$\cos A = \frac{b}{c} = \frac{95}{193} = \sin B,$$

$$\tan A = \frac{a}{b} = \frac{168}{95} = \cot B,$$

$$\cot A = \frac{b}{a} = \frac{95}{168} = \tan B,$$

$$\sec A = \frac{c}{b} = \frac{193}{95} = \csc B,$$

$$\csc A = \frac{c}{a} = \frac{193}{168} = \sec B.$$

11. Compute the functions of A and B when

$$a = \sqrt{p^2 + q^2},\ b = \sqrt{2pq}.$$

$$a^2 + b^2 = c^2,$$

$$p^2 + 2pq + q^2 = c^2.$$

$$\therefore\ p + q = c.$$

$$\sin A = \frac{a}{c} = \frac{\sqrt{p^2 + q^2}}{p+q} = \cos B,$$

$$\cos A = \frac{b}{c} = \frac{\sqrt{2pq}}{p+q} = \sin B,$$

$$\tan A = \frac{a}{b} = \frac{\sqrt{p^2 + q^2}}{\sqrt{2pq}} = \cot B,$$

$$\cot A = \frac{b}{a} = \frac{\sqrt{2pq}}{\sqrt{p^2 + q^2}} = \tan B,$$

$$\sec A = \frac{c}{b} = \frac{p+q}{\sqrt{2pq}} = \csc B,$$

$$\csc A = \frac{c}{a} = \frac{p+q}{\sqrt{p^2 + q^2}} = \sec B.$$

12. Compute the functions of A and B when

$$a = \sqrt{p^2 + pq},\ c = p + q.$$

$$b^2 = c^2 - a^2$$

$$= q^2 + pq.$$

$$\therefore\ b = \sqrt{q^2 + pq}.$$

$$\sin A = \frac{a}{c} = \frac{\sqrt{p^2 + pq}}{p+q} = \cos B,$$

$$\cos A = \frac{b}{c} = \frac{\sqrt{q^2 + pq}}{p+q} = \sin B,$$

$$\tan A = \frac{a}{b} = \frac{\sqrt{p^2 + pq}}{\sqrt{q^2 + pq}} = \cot B,$$

$$\cot A = \frac{b}{a} = \frac{\sqrt{q^2 + pq}}{\sqrt{p^2 + pq}} = \tan B,$$

$$\sec A = \frac{c}{b} = \frac{p+q}{\sqrt{q^2 + pq}} = \csc B,$$

$$\csc A = \frac{c}{a} = \frac{p+q}{\sqrt{p^2 + pq}} = \sec B.$$

13. Compute the functions of A and B when

$$b = 2\sqrt{pq},\ c = p + q.$$

$$a^2 + b^2 = c^2,$$

$$a^2 + 4pq = p^2 + 2pq + q^2,$$

$$a^2 = p^2 - 2pq + q^2,$$

$$a = p - q.$$

$$\sin A = \frac{a}{c} = \frac{p-q}{p+q} = \cos B,$$

$$\cos A = \frac{b}{c} = \frac{2\sqrt{pq}}{p+q} = \sin B,$$

$$\tan A = \frac{a}{b} = \frac{p-q}{2\sqrt{pq}} = \cot B,$$

$$\cot A = \frac{b}{a} = \frac{2\sqrt{pq}}{p-q} = \tan B,$$

$$\sec A = \frac{c}{b} = \frac{p+q}{2\sqrt{pq}} = \csc B,$$

$$\csc A = \frac{c}{a} = \frac{p+q}{p-q} = \sec B.$$

14. Compute the functions of A when $a = 2b$.

$$a = 2b,$$

$$a^2 + b^2 = c^2,$$

$$4b^2 + b^2 = c^2,$$

$$5b^2 = c^2,$$

$$c = b\sqrt{5}.$$

$$\sin A = \frac{a}{c} = \frac{2b}{b\sqrt{5}} = \tfrac{2}{5}\sqrt{5} = 0.89443,$$

$$\cos A = \frac{b}{c} = \frac{b}{b\sqrt{5}} = \tfrac{1}{5}\sqrt{5},$$

$$\tan A = \frac{a}{b} = \frac{2b}{b} = 2,$$

$$\cot A = \frac{b}{a} = \frac{1}{2},$$

$$\sec A = \frac{c}{b} = \frac{b\sqrt{5}}{b} = \sqrt{5},$$

$$\csc A = \frac{c}{a} = \frac{b\sqrt{5}}{2b} = \tfrac{1}{2}\sqrt{5}.$$

15. Compute the functions of A when $a = \frac{2}{3}c$.

$$\begin{aligned} a &= \tfrac{2}{3}c, \\ c &= \tfrac{3}{2}a, \\ b^2 &= c^2 - a^2, \\ b &= \sqrt{c^2 - a^2} \\ &= \sqrt{\tfrac{9}{4}a^2 - a^2} \\ &= \frac{a}{2}\sqrt{5}. \end{aligned}$$

$$\sin A = \frac{a}{c} = \frac{a}{\frac{3}{2}a} = \frac{2}{3},$$

$$\cos A = \frac{b}{c} = \frac{\frac{a}{2}\sqrt{5}}{\frac{3}{2}a} = \tfrac{1}{3}\sqrt{5},$$

$$\tan A = \frac{a}{b} = \frac{a}{\frac{a}{2}\sqrt{5}} = \tfrac{2}{5}\sqrt{5},$$

$$\cot A = \frac{b}{a} = \frac{\sqrt{5}}{2},$$

$$\sec A = \frac{c}{b} = \frac{\frac{3}{2}a}{\frac{a}{2}\sqrt{5}} = \tfrac{3}{5}\sqrt{5},$$

$$\csc A = \frac{c}{a} = \frac{3}{2}.$$

16. Compute the functions of A when $a + b = \frac{5}{4}c$.

$$\begin{aligned} a + b &= \tfrac{5}{4}c, \\ a^2 + b^2 &= c^2, \\ a^2 + b^2 + 2ab &= \tfrac{25}{16}c^2, \\ 2ab &= \tfrac{9}{16}c^2, \\ a^2 - 2ab + b^2 &= \tfrac{7}{16}c^2, \\ a - b &= \frac{c}{4}\sqrt{7}, \\ a + b &= \tfrac{5}{4}c, \\ 2b &= \tfrac{5}{4}c - \frac{c}{4}\sqrt{7}, \\ b &= \tfrac{5}{8}c - \frac{c}{8}\sqrt{7}, \\ 2a &= \tfrac{5}{4}c + \frac{c}{4}\sqrt{7}, \\ a &= \tfrac{5}{8}c + \frac{c}{8}\sqrt{7}, \\ \frac{c}{8} &= \frac{a}{5 + \sqrt{7}}, \\ c &= \frac{8a}{5 + \sqrt{7}}. \end{aligned}$$

$$\sin A = \frac{a}{c} = \frac{a}{\frac{8a}{5+\sqrt{7}}} = \frac{5 + \sqrt{7}}{8},$$

$$\cos A = \frac{b}{c} = \frac{\frac{5}{8}c - \frac{c}{8}\sqrt{7}}{c} = \frac{5 - \sqrt{7}}{8},$$

$$\tan A = \frac{a}{b} = \frac{5 + \sqrt{7}}{5 - \sqrt{7}},$$

$$\cot A = \frac{b}{a} = \frac{5 - \sqrt{7}}{5 + \sqrt{7}},$$

$$\sec A = \frac{c}{b} = \frac{8}{5 - \sqrt{7}},$$

$$\csc A = \frac{c}{a} = \frac{8}{5 + \sqrt{7}}.$$

17. Compute the functions of A when

$$a - b = \frac{c}{4}.$$

$$a^2 - 2ab + b^2 = \frac{c^2}{16}$$
$$a^2 + b^2 = c^2$$

$$2ab = \frac{15c^2}{16}$$
$$a^2 + b^2 = c^2$$

$$a^2 + 2ab + b^2 = \frac{31c^2}{16}$$

$$a + b = \frac{c}{4}\sqrt{31},$$

$$a - b = \frac{c}{4},$$

$$2a = \frac{c}{4}\sqrt{31} + \frac{c}{4}.$$

$$\therefore a = \frac{c}{8}(\sqrt{31} + 1).$$

$$2b = \frac{c}{4}\sqrt{31} - \frac{c}{4}.$$

$$\therefore b = \frac{c}{8}(\sqrt{31} - 1).$$

$$\sin A = \frac{a}{c} = \frac{\frac{c}{8}(\sqrt{31}+1)}{c} = \frac{\sqrt{31}+1}{8},$$

$$\cos A = \frac{b}{c} = \frac{\frac{c}{8}(\sqrt{31}-1)}{c} = \frac{\sqrt{31}-1}{8},$$

$$\tan A = \frac{a}{b} = \frac{\sqrt{31}+1}{\sqrt{31}-1},$$

$$\cot A = \frac{b}{a} = \frac{\sqrt{31}-1}{\sqrt{31}+1},$$

$$\sec A = \frac{c}{b} = \frac{8}{\sqrt{31}-1},$$

$$\csc A = \frac{c}{a} = \frac{8}{\sqrt{31}+1}.$$

18. Find a if $\sin A = \frac{3}{5}$ and $c = 20.5$.

$$\sin A = \frac{a}{c} = \frac{3}{5},$$

$$\frac{a}{20.5} = \frac{3}{5},$$

$$5a = 61.5,$$

$$a = 12.3.$$

19. Find b if $\cos A = 0.44$ and $c = 3.5$.

$$\frac{b}{c} = 0.44,$$

$$\frac{b}{3.5} = 0.44.$$

$$\therefore b = 1.54.$$

20. Find a if $\tan A = \frac{11}{3}$ and $b = 2\frac{5}{11}$.

$$\frac{a}{b} = \frac{a}{2\frac{5}{11}} = \frac{11}{3}.$$

$$\therefore \frac{11a}{27} = \frac{11}{3}.$$

$$\therefore a = 9.$$

21. Find b if $\cot A = 4$ and $a = 17$.

$$\frac{b}{a} = \frac{b}{17} = 4.$$

$$\therefore b = 68.$$

22. Find c if $\sec A = 2$ and $b = 20$.

$$\frac{c}{b} = \frac{c}{20} = 2.$$

$$\therefore c = 40.$$

23. Find c if $\csc A = 6.45$ and $a = 35.6$

$$\csc A = \frac{c}{a} = \frac{c}{35.6} = 6.45.$$

$$\therefore\ c = 229.62.$$

24. Construct a right triangle; given $c = 6$, $\tan A = \frac{3}{2}$.

$$\tan A = \frac{a}{b}.$$

$$\therefore\ a = 3 \text{ and } b = 2.$$

Draw $AB = 2$, and $BC \perp$ to $AB = 3$; join C and A.

Prolong AC to D, making $AD = 6$.

Draw $DE \perp$ to AB produced.

Rt. $\triangle\ ADE$ will be similar to rt. $\triangle\ ACB$.

$\therefore\ ADE$ is the rt. $\triangle$ required.

25. Construct a right triangle; given $a = 3.5$, $\cos A = \frac{1}{2}$.

Construct $\triangle\ A'B'C'$ so that $b'=1$, $c'= 2$. Then $\cos A = \frac{1}{2}$.

Construct $\triangle\ ABC$ similar to $A'B'C'$, and having $a = 3.5$.

26. Construct a right triangle; given $b = 2$, $\sin A = 0.6$.

Construct rt. $\triangle\ A'B'C'$, making $a'= 6$, and $c = 10$.

Then $\sin A'= \frac{6}{10}$.

Construct $\triangle\ ABC$ similar to $A'B'C'$, and having $b = 2$.

27. Construct a right triangle; given $b = 4$, $\csc A = 4$.

Construct rt. $\triangle\ A'B'C'$, having $c'= 4$ and $a'= 1$.

Then construct $\triangle\ ABC$ similar to $\triangle\ A'B'C'$, and having $b = 4$.

28. In a right triangle, $c = 2.5$ miles, $\sin A = 0.6$, $\cos A = 0.8$; compute the legs.

$\sin A = \frac{a}{c}.$ $\qquad \cos A = \frac{b}{c}.$

$\therefore\ a = c \sin A.$ $\qquad \therefore\ b = c \cos A.$

$\therefore\ a = 1.5.$ $\qquad \therefore\ b = 2.$

30. Find, by means of the table, the legs of a right triangle if $A = 20°$, $c = 1$; also, if $A = 20°$, $c = 4$.

$$A = 20°,\ c = 1.$$

$\sin A = \frac{a}{c}.$ $\qquad \cos A = \frac{b}{c}.$

$\therefore\ a = c \sin A.$ $\qquad \therefore\ b = c \cos A.$

$\therefore\ a = 0.342.$ $\qquad \therefore\ b = 0.940.$

$$A = 20°,\ c = 4.$$

$\therefore\ a = 4 \times 0.342 = 1.368.$ $\qquad \therefore\ b = 4 \times 0.940 = 3.760.$

31. In a right triangle, given $a = 3$ and $c = 5$; find the hypotenuse of a similar triangle in which $a = 240{,}000$ miles.

$$a : c :: 240{,}000 : x,$$

$$3 : 5 :: 240{,}000 : x.$$

$$\therefore\ x = 400{,}000.$$

32. By dividing the length of a vertical rod by the length of its horizontal shadow, the tangent of the angle of elevation of the sun at the time of observation was found to be 0.82. How high is a tower, if the length of its horizontal shadow at the same time is 174.3 yards?

$$\tan A = \frac{a}{b} = 0.82.$$

$$\therefore\ a = 0.82\,b.$$

$$b = 174.3 \text{ yards}.$$

$$\therefore\ a = 0.82 \text{ of } 174.3 \text{ yards} = 142.926.$$

Exercise II. Page 8.

1. Represent by lines the functions of a larger angle than that shown in Fig. 3.

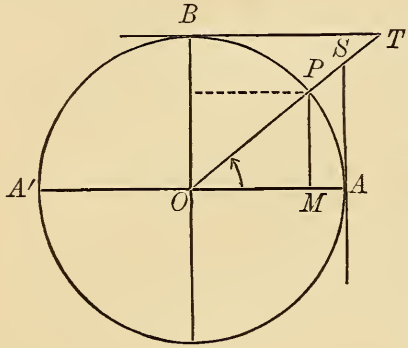


Fig. 1.

2. Show that $\sin x$ is less than $\tan x$.

In Fig. 1, $OM : PM :: OA : AS$,

but $OM < OA$.

$\therefore PM < AS$.

3. Show that $\sec x$ is greater than $\tan x$.

$OS = \sec$, $AS = \tan$.

In rt. $\triangle OAS$, Hyp. $OS >$ side AS.

$\therefore \sec > \tan$.

4. Show that $\csc x$ is greater than $\cot x$.

$OT = \csc$, $BT = \cot$.

In $\triangle BOT$, Hyp. $OT >$ side BT.

$\therefore \csc > \cot$.

5. Construct the angle x if $\tan x = 3$.

Let $\odot BAM$ be a unit circle, with centre O; then construct AT tangent to the circle at $A = 3\,OA$; then AOT is required angle.

6. Construct the angle x if $\csc x = 2$.

Let $\odot ABM$ be a unit circle, with centre O; construct BT tangent to the circle at $B = 2\,OA$; connect OT; then AOT is required angle.

7. Construct the angle x if $\cos x = \frac{1}{2}$.

Take $OM = \frac{1}{2}$ radius OA. At M erect a $\perp$ to meet the circumference at P. Draw OP.

Then is POM the angle required.

8. Construct the angle x if $\sin x = \cos x$.

Let $PM = \sin x$ and $OM = \cos x$.

But, by hypothesis, $PM = OM$.

$\therefore$ by Geometry, $x = 45°$.

Hence, construct an $\angle 45°$.

9. Construct the angle x if $\sin x = 2 \cos x$.

Construct rt. $\angle PMO$, making $PM = 2\,OM$. Draw OP.

Then POM is the angle required.

10. Construct the angle x if $4 \sin x = \tan x$.

Take $\frac{1}{4}$ of radius OA to M. At M erect a $\perp$ to meet the circumference at P. Draw OP.

Then POM is the required angle.

11. Show that the sine of an angle is equal to one-half the chord of twice the angle.

Have given $\angle POA$.

Construct $POB = 2\,POA$. Draw chord PB. Then it is $\perp$ to OA; and PM, its half, is the sine of POA.

$\therefore \sin x = \frac{1}{2}$ chord $2x$.

12. Find x if $\sin x$ is equal to one-half the side of a regular inscribed decagon.

Let AC be a side of a decagon.

Then $\frac{360°}{10} = 36°$ or AOC.

Draw OB bisecting AC. Then $\angle AOC$ will be bisected, and $\angle AOB = 18°$.

But the sine of $AOB = \frac{1}{2} AC$.

$\therefore$ x or $AOB = 18°$.

13. Given x and y ($x+y$ being less than 90°); construct the value of $\sin(x+y) - \sin x$.

Let $AB = \sin(x+y)$ in a circle whose centre is O, and $CD = \sin x$.

Then, with a radius equal to CD, describe an arc from B, as centre, cutting AB at E.

Then EA will be the constructed value of $\sin(x+y) - \sin x$.

14. Given x and y ($x+y$ being less than 90°); construct the value of $\tan(x+y) - \sin(x+y) + \tan x - \sin x$.

Let $AB = \sin(x+y)$,
and $CD = \sin x$;
also $EF = \tan(x+y)$,
and $GF = \tan x$.

From F with a radius $= AB$ take FH.

From H with a radius $= GF$ add HI.

From I with a radius $= CD$ take IK.

Then EK will be the constructed value of $\tan(x+y) - \sin(x+y) + \tan x - \sin x$.

15. Given an angle x; construct an angle y such that $\sin y = 2 \sin x$.

Let AB be the sine of the $\angle x$ in a circle whose centre is O.

Draw AC perpendicular to the vertical diameter.

Then $CO = AB$.

Take CF on vertical diameter $= CO$. Draw FD perpendicular to vertical diameter, and meeting circumference at D.

Draw DE perpendicular to OB and draw OD.

$OF = 2\,CO$ by construction.

$ED = FO$; FO being the projection of the radius OD.

$\therefore$ $DE = 2\,AB$, and DOB = angle required.

16. Given an angle x; construct an angle y such that $\cos y = \frac{1}{2} \cos x$.

Let $OB = \cos AOB$.

Erect a $\perp$ CD at C, the middle point of OB, and meeting the circumference at D. Draw DO.

Then DOB is the angle required.

17. Given an angle x; construct an angle y such that $\tan y = 3 \tan x$.

Let AB be the tangent of x.

Prolong AB to C, making $AC = 3\,AB$, and draw OC from O, the centre of the circle.

COA is the required angle.

18. Given an angle x; construct an angle y such that $\sec y = \csc x$.

Since sec = csc,

$$\frac{c}{b} = \frac{c}{a}.$$

$$\therefore\ a = b.$$

Hence, construct an isosceles right triangle.

The required angle will be 45°.

19. Show by construction that $2 \sin A > \sin 2A$.

Construct $\angle BOC$ and $\angle COA$ each equal to the given $\angle A$.

Then $AB = 2 \sin A$, and AD, the $\perp$ let fall from A to OB, $= \sin 2A$. But $AB > AD$.

Hence $2 \sin A > \sin 2A$.

20. Given two angles A and B ($A + B$ being less than 90°), show that $\sin(A + B) < \sin A + \sin B$.

Construct $HOK = \angle A$, and $COH = \angle B$.

Then $\sin(A + B) = CP$, $\sin A = HK$, $\sin B = CD$.

Now $CP < CD + DE$,

and $HK > DE$.

$\therefore CP < CD + HK$.

$\therefore \sin(A + B) < \sin A + \sin B$.

21. Given $\sin x$ in a unit circle; find the length of a line corresponding in position to $\sin x$ in a circle whose radius is r.

$1 : r :: \sin x :$ required line.

$\therefore$ length of line required $= r \sin x$.

22. In a right triangle, given the hypotenuse c, and also $\sin A = m$, $\cos A = n$; find the legs.

$$\sin A = \frac{a}{c} = m.$$

$$\therefore a = cm.$$

$$\cos A = \frac{b}{c} = n.$$

$$\therefore b = cn.$$

Exercise III. Page 11.

1. Express the following functions as functions of the complementary angle:

sin 30°.	csc 18° 10′.
cos 45°.	cos 37° 24′.
tan 89°.	cot 82° 19′.
cot 15°.	csc 54° 46′.

$\sin 30° = \cos(90° - 30°) = \cos 60°$.

$\cos 45° = \sin(90° - 45°) = \sin 45°$.

$\tan 89° = \cot(90° - 89°) = \cot 1°$.

$\cot 15° = \tan(90° - 15°) = \tan 75°$.

$\csc 18° 10' = \sec(90° - 18° 10')$
$= \sec 71° 50'$.

$\cos 37° 24' = \sin(90° - 37° 24')$
$= \sin 52° 36'$.

$\cot 82° 19' = \tan(90° - 82° 19')$
$= \tan 7° 41'$.

$\csc 54° 46' = \sec(90° - 54° 46')$
$= \sec 35° 14'$.

2. Express the following functions as functions of an angle less than 45°:

sin 60°.	csc 69° 2′.
cos 75°.	cos 85° 39′.
tan 57°.	cot 89° 59′.
cot 84°.	csc 45° 1′.

$\sin 60° = \cos(90° - 60°) = \cos 30°$.

$\cos 75° = \sin(90° - 75°) = \sin 15°$.

$\tan 57° = \cot(90° - 57°) = \cot 33°$.

$\cot 84° = \tan(90° - 84°) = \tan 6°$.

$\csc 69° 2' = \sec(90° - 69° 2')$
$= \sec 20° 58'$.

$\cos 85° 39' = \sin(90° - 85° 39')$
$= \sin 4° 21'$.

$\cot 89° 59' = \tan(90° - 89° 59')$
$= \tan 0° 1'$.

$\csc 45° 1' = \sec(90° - 45° 1')$
$= \sec 44° 59'$.

3. Given $\tan 30° = \frac{1}{3}\sqrt{3}$; find $\cot 60°$.

$$\tan 30° = \cot(90° - 30°)$$
$$= \cot 60°.$$
$$\therefore \cot 60° = \tfrac{1}{3}\sqrt{3}.$$

4. Given $\tan A = \cot A$; find A.

$$\tan A = \cot(90° - A),$$
$$90° - A = A,$$
$$2A = 90°.$$
$$\therefore A = 45°.$$

5. Given $\cos A = \sin 2A$; find A.

$$\cos A = \sin(90° - A),$$
$$90° - A = 2A,$$
$$3A = 90°.$$
$$\therefore A = 30°.$$

6. Given $\sin A = \cos 2A$; find A.

$$\sin A = \cos(90° - A),$$
$$90° - A = 2A,$$
$$3A = 90°.$$
$$\therefore A = 30°.$$

7. Given $\cos A = \sin(45° - \frac{1}{2}A$; find A.

$$\cos A = \sin(90° - A),$$
$$90° - A = 45° - \tfrac{1}{2}A,$$
$$180° - 2A = 90° - A.$$
$$\therefore A = 90°.$$

8. Given $\cot \frac{1}{2}A = \tan A$; find A.

$$\tan A = \cot(90° - A),$$
$$\tfrac{1}{2}A = 90° - A,$$
$$A = 180° - 2A,$$
$$3A = 180°.$$
$$\therefore A = 60°.$$

9. Given $\tan(45° + A) = \cot A$; find A.

$$\cot A = \tan(90° - A),$$
$$\tan(90° - A) = \tan(45° + A),$$
$$90° - A = 45° + A,$$
$$2A = 45°.$$
$$\therefore A = 22° 30'.$$

10. Find A if $\sin A = \cos 4A$.

$$\sin A = \cos(90° - A),$$
$$90° - A = 4A,$$
$$5A = 90°.$$
$$\therefore A = 18°.$$

11. Find A if $\cot A = \tan 8A$.

$$\cot A = \tan(90° - A),$$
$$8A = 90° - A,$$
$$9A = 90°.$$
$$\therefore A = 10°.$$

12. Find A if $\cot A = \tan nA$.

$$\cot A = \tan(90° - A),$$
$$90° - A = nA,$$
$$90° = A(n + 1).$$
$$\therefore A = \frac{90°}{n+1}.$$

Exercise IV. Page 12.

1. Prove Formulas [1]–[3], using for the functions the line values in unit circle given in § 3.

[1]. $\sin^2 A + \cos^2 A = 1.$

[2]. $\tan A = \dfrac{\sin A}{\cos A}.$

[3]. $\sin A \times \csc A = 1,$
$\cos A \times \sec A = 1,$
$\tan A \times \cot A = 1.$

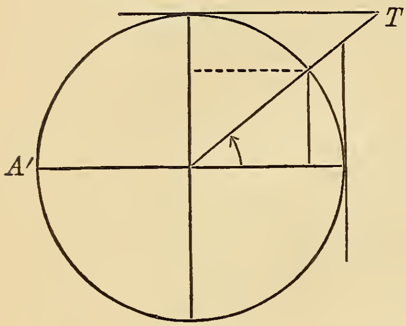


Fig. 2.

[1]. $DC = \sin A,$
$BD = \cos A,$
$DC^2 + BD^2 = CB^2;$
but $CB^2 = 1.$
$\therefore DC^2 + BD^2 = 1.$
$\therefore \sin^2 A + \cos^2 A = 1.$

[2]. $DC = \sin A,$
$BD = \cos A,$
$EF = \tan A.$

△ FBE and BCD are similar.

$\therefore FE : BE :: CD : BD.$

Or, $\frac{FE}{BE} = \frac{CD}{BD};$

but $BE = 1.$

$\therefore FE = \frac{CD}{BD}.$

$\therefore \tan A = \frac{\sin A}{\cos A}.$

[3]. $CD = \sin A,$
$BH = \csc A.$

In similar △ HGB and CBD,

$BH : GB :: BC : CD.$

$\therefore \frac{BH}{GB} = \frac{BC}{CD};$

but $GB = 1,$
$BC = 1.$
$BH = \frac{1}{CD}.$
$BH \times CD = 1,$
$\csc A \times \sin A = 1.$

$\cos A = BD,$
$\sec A = BF.$

In similar △ BFE and BCD,

$BF : BE :: BC : BD.$

$\frac{BF}{BE} = \frac{BC}{BD};$

but $BE = 1,$
$BC = 1.$
$\therefore BF = \frac{1}{BD}.$
$BF \times BD = 1,$
$\sec A \times \cos A = 1.$

$\tan A = EF,$
$\cot A = GH.$

In similar △ GHB and FEB,

$GH : GB :: BE : FE,$

$\frac{GH}{GB} = \frac{BE}{FE};$

but $GB = 1.$
$BE = 1.$
$GH = \frac{1}{FE},$
$GH \times FE = 1.$
$\cot A \times \tan A = 1.$

2. Prove that $1 + \tan^2 A = \sec^2 A.$

$\tan A = \frac{a}{b}, \qquad \sec A = \frac{b}{c}.$

$a^2 + b^2 = c^2.$

Dividing all the terms by b^2,

$$\frac{a^2}{b^2}+\frac{b^2}{b^2}=\frac{c^2}{b^2}.$$

Substituting for $\frac{a^2}{b^2}$ and $\frac{c^2}{b^2}$ their values $\tan^2 A$ and $\sec^2 A$, we have

$$\tan^2 A + 1 = \sec^2 A.$$

3. Prove that $1+\cot^2 A = \csc^2 A$.

$$\cot A = \frac{b}{a},$$

$$\csc A = \frac{c}{a}.$$

$$a^2+b^2=c^2.$$

Dividing all the terms by a^2,

$$\frac{a^2}{a^2}+\frac{b^2}{a^2}=\frac{c^2}{a^2}.$$

Substituting for $\frac{b^2}{a^2}$ and $\frac{c^2}{a^2}$ their values $\cot^2 A$ and $\csc^2 A$, we have

$$1+\cot^2 A = \csc^2 A.$$

4. Prove that $\cot A = \frac{\cos A}{\sin A}$.

$$\cot A = \frac{b}{a},$$

$$\sin A = \frac{a}{c},$$

$$\cos A = \frac{b}{c}.$$

Substituting, $\frac{b}{a}=\frac{b}{c}\div\frac{a}{c};$

$$\therefore \cot A = \frac{\cos A}{\sin A}.$$

Exercise V. Page 14.

1. Find the values of the other functions when $\sin A = \frac{12}{13}$.

$$\sin^2 A + \cos^2 A = 1,$$

$$\cos^2 A = 1-\left(\frac{12}{13}\right)^2,$$

$$\cos A = \sqrt{1-\left(\frac{12}{13}\right)^2}$$

$$=\sqrt{\frac{25}{169}}.$$

$$\therefore \cos A = \frac{5}{13}.$$

$$\tan A = \frac{\sin}{\cos}=\frac{12}{5}.$$

$\cot A$ is reciprocal of $\tan A$.

$$\therefore \cot A = \frac{5}{12}.$$

$\sec A$ is reciprocal of $\cos A$.

$$\therefore \sec A = \frac{13}{5}.$$

$\csc A$ is reciprocal of $\sin A$.

$$\therefore \csc A = \frac{13}{12}.$$

2. Find the values of the other functions when $\sin A = 0.8$.

$$\sin^2 A + \cos^2 A = 1,$$

$$\cos^2 A = 1-(0.8)^2,$$

$$\cos A = \sqrt{1-0.64}.$$

$$\therefore \cos A = 0.6.$$

$$\tan A = \frac{\sin}{\cos}=\frac{0.8}{0.6}.$$

$$\therefore \tan A = 1.3333.$$

$$\cot A = \frac{0.6}{0.8}.$$

$$\therefore \cot A = 0.75.$$

$$\sec A = \frac{1}{0.6}.$$

$$\therefore \sec A = 1.6667.$$

$$\csc A = \frac{1}{0.8}.$$

$$\therefore \csc A = 1.25.$$

3. Find the values of the other functions when $\cos A = \frac{60}{61}$.

$$\sin^2 + \cos^2 = 1,$$

$$\sin = \sqrt{1 - \frac{3600}{3721}} = \sqrt{\frac{121}{3721}} = \frac{11}{61}.$$

$$\tan = \frac{\sin}{\cos} = \frac{11}{60}.$$

$$\cot = \frac{1}{\tan} = \frac{60}{11}.$$

$$\sec = \frac{1}{\cos} = \frac{61}{60}.$$

$$\csc = \frac{1}{\sin} = \frac{61}{11}.$$

4. Find the values of the other functions when $\cos A = 0.28$.

$$\sin^2 + \cos^2 = 1.$$

$$\sin = \sqrt{1 - (0.28)^2} = \sqrt{0.9216}.$$

$$= 0.96.$$

$$\tan = \frac{\sin}{\cos} = \frac{0.96}{0.28} = 3.4285.$$

$$\cot = \frac{1}{\tan} = \frac{1}{3.4285} = 0.29167.$$

$$\sec = \frac{1}{\cos} = \frac{1}{0.28} = 3.5714.$$

$$\csc = \frac{1}{\sin} = \frac{1}{0.96} = 1.04167.$$

5. Find the values of the other functions when $\tan A = \frac{4}{3}$.

$$\tan A = \frac{4}{3},$$

$$\therefore \cot A = \frac{3}{4}.$$

$$\tan A = \frac{\sin A}{\cos A},$$

$$\frac{4}{3} = \frac{\sin A}{\sqrt{1 - \sin^2 A}}.$$

$$3 \sin A = 4\sqrt{1 - \sin^2 A},$$

$$9 \sin^2 A = 16 - 16 \sin^2 A,$$

$$25 \sin^2 A = 16,$$

$$5 \sin A = 4.$$

$$\therefore \sin A = \frac{4}{5}.$$

$$\cos A = \frac{\sin A}{\tan A} = \frac{3}{5}.$$

$$\sec A = \frac{1}{\cos A} = \frac{5}{3}.$$

$$\csc A = \frac{1}{\sin A} = \frac{5}{4}.$$

6. Find the values of the other functions when $\cot A = 1$.

$$\cot A = 1.$$

$$\therefore \tan A = 1.$$

$$\tan A = \frac{\sin A}{\cos A},$$

$$1 = \frac{\sin A}{\sqrt{1 - \sin^2 A}},$$

$$\sin A = \sqrt{1 - \sin^2 A},$$

$$\sin^2 A = 1 - \sin^2 A,$$

$$2 \sin^2 A = 1,$$

$$\sin^2 A = \frac{1}{2}.$$

$\therefore \sin A = \sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{2}.$

$\cos A = \frac{\sin A}{\tan A} = \frac{1}{2}\sqrt{2}.$

$\sec A = \frac{1}{\cos A} = \frac{1}{\frac{1}{2}\sqrt{2}} = \sqrt{2}.$

$\csc A = \frac{1}{\sin A} = \frac{1}{\frac{1}{2}\sqrt{2}} = \sqrt{2}.$

7. Find the values of the other functions when $\cot A = 0.5$.

$\tan A = \frac{1}{\cot A} = \frac{1}{0.5} = 2.$

$\tan A = \frac{\sin A}{\cos A}.$

$2 \cos A = \sin A.$

$$\begin{array}{ll} 4\cos^2 A - \sin^2 A = 0 & \text{(squaring)} \\ \underline{\cos^2 A + \sin^2 A = 1} & \\ 5\cos^2 A \qquad = 1 & \end{array}$$

$\cos A = \sqrt{\frac{1}{5}} = 0.45.$

$$\begin{array}{l} 4\cos^2 A + 4\sin^2 A = 4 \\ \underline{4\cos^2 A - \sin^2 A = 0} \\ \qquad 5\sin^2 A = 4 \end{array}$$

$\sin A = \sqrt{\frac{4}{5}} = 0.90.$

$\sec A = \frac{1}{\cos A} = 2.22.$

$\csc A = \frac{1}{\sin A} = 1.11.$

8. Find the values of the other functions when $\sec A = 2$.

$\cos A = \frac{1}{\sec A} = \frac{1}{2},$

$\sin A = \sqrt{1 - \cos^2 A}$

$= \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}}.$

$\therefore \sin A = \frac{1}{2}\sqrt{3}.$

$\tan A = \frac{\sin A}{\cos A} = \frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}} = \sqrt{3}.$

$\cot A = \frac{1}{\tan A} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}.$

$\csc A = \frac{1}{\sin A} = \frac{2}{3}\sqrt{3}.$

9. Find the values of the other functions when $\csc A = \sqrt{2}$.

$\sin A = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2},$

$\cos A = \sqrt{1 - (\frac{1}{2}\sqrt{2})^2} = \sqrt{1 - \frac{1}{2}}$

$= \sqrt{\frac{1}{2}} = \frac{1}{2}\sqrt{2},$

$\tan A = \frac{\frac{1}{2}\sqrt{2}}{\frac{1}{2}\sqrt{2}} = 1,$

$\cot A = \frac{1}{1} = 1,$

$\sec A = \frac{1}{\frac{1}{2}\sqrt{2}} = \sqrt{2}.$

10. Find the values of the other functions when $\sin A = m$.

$\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - m^2},$

$\tan A = \frac{\sin A}{\cos A} = \frac{m}{\sqrt{1 - m^2}}$

$= \frac{m\sqrt{1 - m^2}}{1 - m^2},$

$\cot A = \frac{1}{\tan A} = \frac{1 - m^2}{m\sqrt{1 - m^2}},$

$$\sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1-m^2}},$$

$$\csc A = \frac{1}{\sin A} = \frac{1}{m}.$$

11. Find the values of the other functions when $\sin A = \frac{2m}{1+m^2}$.

$$\cos A = \sqrt{1-\sin^2}.$$

$$\therefore \cos A = \sqrt{1 - \frac{4m^2}{1+2m^2+m^4}}$$

$$= \sqrt{\frac{1-2m^2+m^4}{1+2m^2+m^4}}$$

$$= \frac{1-m^2}{1+m^2}.$$

$$\tan A = \frac{\sin}{\cos} = \frac{2m}{1-m^2}.$$

$$\cot A = \frac{1}{\tan} = \frac{1-m^2}{2m}.$$

$$\sec A = \frac{1}{\cos} = \frac{1+m^2}{1-m^2}.$$

$$\csc A = \frac{1}{\sin} = \frac{1+m^2}{2m}.$$

12. Find the values of the other functions when $\cos A = \frac{2mn}{m^2+n^2}$.

$$\sin A = \sqrt{1-\cos^2}$$

$$= \sqrt{1 - \frac{4m^2n^2}{m^4+2m^2n^2+n^4}}$$

$$= \sqrt{\frac{m^4-2m^2n^2+n^4}{m^4+2m^2n^2+n^4}}$$

$$= \frac{m^2-n^2}{m^2+n^2}.$$

$$\tan A = \frac{\sin}{\cos} = \frac{m^2-n^2}{2mn}.$$

$$\cot A = \frac{1}{\tan} = \frac{2mn}{m^2-n^2}.$$

$$\sec A = \frac{1}{\cos} = \frac{m^2+n^2}{2mn}.$$

$$\csc A = \frac{1}{\sin} = \frac{m^2+n^2}{m^2-n^2}.$$

13. Given $\tan 45° = 1$; find the other functions of 45°.

$$\frac{\sin 45°}{\cos 45°} = \tan 45°.$$

(1) $\frac{\sin 45°}{\cos 45°} = 1.$

(2) $\sin^2 + \cos^2 = 1.$

By (1), $\sin 45° = \cos 45°$.

By (2), $\cos^2 45° + \cos^2 45° = 1.$

$$2\cos^2 45° = 1,$$

$$\cos^2 45° = \frac{1}{2},$$

$$\cos 45° = \sqrt{\frac{1}{2}} = \tfrac{1}{2}\sqrt{2}.$$

$$\sin 45° = \tfrac{1}{2}\sqrt{2}.$$

$$\cot 45° = \frac{1}{\tan 45°} = \frac{1}{1} = 1.$$

$$\sec 45° = \frac{1}{\frac{1}{2}\sqrt{2}} = \sqrt{2}.$$

$$\csc 45° = \frac{1}{\frac{1}{2}\sqrt{2}} = \sqrt{2}.$$

14. Given $\sin 30° = \frac{1}{2}$; find the other functions of 30°.

$$\sin^2 + \cos^2 = 1.$$

$$\cos 30° = \sqrt{1 - \frac{1}{4}}$$

$$= \sqrt{\frac{3}{4}} = \tfrac{1}{2}\sqrt{3}.$$

$$\tan 30° = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}} = \tfrac{1}{3}\sqrt{3}.$$

$$\cot 30° = \frac{1}{\frac{1}{3}\sqrt{3}} = \sqrt{3}.$$

$$\sec 30° = \frac{1}{\frac{1}{2}\sqrt{3}} = \tfrac{2}{3}\sqrt{3}.$$

$$\csc 30° = \frac{1}{\frac{1}{2}} = 2.$$

15. Given $\csc 60° = \frac{2}{3}\sqrt{3}$; find the other functions of 60°.

$$\sin = \frac{1}{\csc},$$

$$\sin 60° = \frac{1}{\frac{2}{3}\sqrt{3}} = \tfrac{1}{2}\sqrt{3}.$$

$$\cos 60° = \sqrt{1 - \sin^2},$$

$$\cos 60° = \sqrt{1 - (\tfrac{1}{2}\sqrt{3})^2}$$

$$= \sqrt{1 - \frac{3}{4}} = \frac{1}{2}.$$

$$\tan 60° = \frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}} = \sqrt{3}.$$

$$\cot 60° = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3}.$$

$$\sec 60° = \frac{1}{\frac{1}{2}} = 2.$$

16. Given $\tan 15° = 2 - \sqrt{3}$; find the other functions of 15°.

$$\frac{\sin 15°}{\cos 15°} = 2 - \sqrt{3}.$$

$$\sin^2 15° + \cos^2 15° = 1.$$

$$\sin 15° = \cos 15° (2 - \sqrt{3}).$$

$$[\cos (2 - \sqrt{3})]^2 + \cos^2 = 1,$$

$$\cos^2 (4 - 4\sqrt{3} + 3) + \cos^2 = 1,$$

$$\cos^2 (8 - 4\sqrt{3}) = 1.$$

$$\cos^2 15° = \frac{1}{4(2 - \sqrt{3})} = \frac{2 + \sqrt{3}}{4},$$

$$\cos\ 15° = \sqrt{\frac{2 + \sqrt{3}}{4}} = \tfrac{1}{2}\sqrt{2 + \sqrt{3}}.$$

$$\sin^2 = 1 - \cos^2.$$

$$\sin^2 15° = 1 - \frac{2 + \sqrt{3}}{4} = \frac{2 - \sqrt{3}}{4},$$

$$\sin\ 15° = \tfrac{1}{2}\sqrt{2 - \sqrt{3}}.$$

$$\cot\ 15° = \frac{1}{\tan 15°} = \frac{1}{2 - \sqrt{3}}$$

$$= 2 + \sqrt{3}.$$

17. Given $\cot 22°\ 30' = \sqrt{2} + 1$; find the other functions of 22° 30′.

$$\tan = \frac{1}{\cot} = \frac{1}{\sqrt{2} + 1} = \sqrt{2} - 1.$$

$$\frac{\sin}{\cos} = \tan, \qquad (1)$$

$$\cos^2 + \sin^2 = 1. \qquad (2)$$

From (1), $\cos \tan = \sin.$

Squaring, $\cos^2 \tan^2 = \sin^2$

From (2), $\cos^2 \quad = -\sin^2 + 1$

Add, $\cos^2 \tan^2 + \cos^2 = 1$

$$\cos^2 (\tan^2 + 1) = 1,$$

$$\cos^2 (4 - 2\sqrt{2}) = 1,$$

$$\cos \sqrt{4 - 2\sqrt{2}} = 1.$$

$$\therefore \cos = \frac{1}{\sqrt{4-2\sqrt{2}}} = \sqrt{\frac{4+2\sqrt{2}}{8}}$$

$$= \tfrac{1}{2}\sqrt{2+\sqrt{2}}.$$

$$\sin = \sqrt{1 - \frac{2+\sqrt{2}}{4}}$$

$$= \sqrt{\frac{4-2-\sqrt{2}}{4}}$$

$$= \sqrt{\frac{2-\sqrt{2}}{4}} = \tfrac{1}{2}\sqrt{2-\sqrt{2}}.$$

18. Given $\sin 0° = 0$; find the other functions of $0°$.

$$\cos = \sqrt{1-\sin^2} = \sqrt{1-0}.$$

$$\therefore \cos = 1.$$

$$\tan = \frac{\sin}{\cos} = \frac{0}{1} = 0.$$

$$\cot = \frac{1}{\tan} = \frac{1}{0} = \infty.$$

$$\sec = \frac{1}{\cos} = \frac{1}{1} = 1.$$

$$\csc = \frac{1}{\sin} = \frac{1}{0} = \infty.$$

19. Given $\sin 90° = 1$; find the other functions of $90°$.

$$\sin 90° = 1.$$

$$\cos = \sqrt{1-\sin^2} = 0.$$

$$\tan = \frac{\sin}{\cos} = \frac{1}{0} = \infty.$$

$$\cot = \frac{1}{\tan} = \frac{1}{\infty} = 0.$$

$$\sec = \frac{1}{\cos} = \frac{1}{0} = \infty.$$

$$\csc = \frac{1}{\sin} = \frac{1}{1} = 1.$$

20. Given $\tan 90° = \infty$; find the other functions of $90°$.

$$\tan 90° = \infty.$$

$$\cot = \frac{1}{\tan} = \frac{1}{\infty} = 0.$$

$$\frac{\sin}{\cos} = \infty.$$

$$\sin^2 = \infty \cos^2$$

$$\sin^2 + \cos^2 = 1$$

$$\overline{\quad - \cos^2 = \infty \cos^2 - 1}$$

$$\infty \cos^2 = 1.$$

$$\cos = \sqrt{\frac{1}{\infty}} = 0.$$

$$\frac{\sin}{0} = \infty.$$

$$\sin = 1.$$

$$\sec = \frac{1}{0} = \infty.$$

$$\csc = 1.$$

21. Express the values of all the other functions in terms of $\sin A$.

By formulæ on pages 11 and 12,

$$\sin A = \sin A,$$

$$\cos A = \sqrt{1-\sin^2 A},$$

$$\tan A = \frac{\sin A}{\sqrt{1-\sin^2 A}},$$

$$\cot A = \frac{\sqrt{1-\sin^2 A}}{\sin A},$$

$$\sec A = \frac{1}{\sqrt{1-\sin^2 A}},$$

$$\csc A = \frac{1}{\sin A}.$$

22. Express the values of all the other functions in terms of $\cos A$.

By formulæ on pages 11 and 12,

$$\sin A = \sqrt{1-\cos^2 A},$$

$$\cos A = \cos A,$$

$$\tan A = \frac{\sqrt{1-\cos^2 A}}{\cos A},$$

$$\cot A = \frac{\cos A}{\sqrt{1-\cos^2 A}},$$

$$\sec A = \frac{1}{\cos A},$$

$$\csc A = \frac{1}{\sqrt{1-\cos^2 A}}.$$

23. Express the values of all the other functions in terms of $\tan A$.

$$\cot A = \frac{1}{\tan A}.$$

$$\frac{a}{b} = \tan A.$$

$$a^2 + b^2 = 1.$$

$$a = b \tan A.$$

$$a^2 = b^2 \tan^2 A.$$

$$\begin{array}{l} a^2 - b^2 \tan^2 A = 0 \\ a^2 + b^2 \qquad\quad = 1 \\ \hline b^2(1+\tan^2 A) = 1 \end{array}$$

$$b^2 = \frac{1}{1+\tan^2 A}.$$

$$\sin A = \sqrt{1-\cos^2 A}$$

$$= \sqrt{1 - \frac{1}{1+\tan^2 A}}$$

$$= \sqrt{\frac{1-1+\tan^2 A}{1+\tan^2 A}}$$

$$= \frac{\tan A}{\sqrt{1+\tan^2 A}}.$$

$$\sec A = \frac{1}{\cos} = \sqrt{1+\tan^2 A}.$$

$$\csc A = \frac{1}{\sin} = \frac{\sqrt{1+\tan^2 A}}{\tan A}.$$

24. Express the values of all the other functions in terms of $\cot A$.

$$\frac{1}{\cot A} = \tan A.$$

$$\frac{\sin A}{\cos A} = \tan A.$$

Let $\qquad x = \sin, \quad y = \cos.$

$$\frac{x}{y} = \frac{1}{\cot A}.$$

$$x \cot A = y,$$

$$x^2 \cot^2 A = y^2.$$

$$\begin{array}{l} x^2 \cot^2 A - y^2 = 0 \\ x^2 \qquad\quad + y^2 = 1 \\ \hline x^2(1+\cot^2 A) = 1 \end{array}$$

$$x^2 = \frac{1}{1+\cot^2 A}.$$

$$\sin A = \frac{1}{\sqrt{1+\cot^2 A}}.$$

$$\cos A = \sqrt{1-\sin^2 A}$$

$$= \sqrt{1 - \frac{1}{1+\cot^2 A}}$$

$$= \sqrt{\frac{1+\cot^2 A - 1}{1+\cot^2 A}}$$

$$= \frac{\cot A}{\sqrt{1+\cot^2 A}}.$$

$$\sec A = \frac{1}{\cos A} = \frac{\sqrt{1+\cot^2 A}}{\cot A}.$$

$$\csc A = \frac{1}{\sin A} = \sqrt{1+\cot^2 A}.$$

25. Given $2 \sin A = \cos A$; find $\sin A$ and $\cos A$.

$$\sin^2 A + \cos^2 A = 1.$$
$$\sin^2 A + 4\sin^2 A = 1.$$
$$5\sin^2 A = 1,$$
$$\sin^2 A = \frac{1}{5},$$
$$\sin A = \sqrt{\frac{1}{5}} = \tfrac{1}{5}\sqrt{5}.$$
$$\therefore \cos A = \tfrac{2}{5}\sqrt{5}.$$

26. Given $4\sin A = \tan A$; find $\sin A$ and $\tan A$.

$$\tan A = \frac{\sin A}{\cos A}.$$

But $\tan A = 4\sin A.$

$$\therefore 4\sin A = \frac{\sin A}{\cos A}.$$
$$4\sin A \times \cos A = \sin A.$$
$$\therefore \quad \cos A = \frac{\sin A}{4\sin A} = \frac{1}{4}.$$
$$\sin^2 A + \cos^2 A = 1.$$
$$\therefore \quad \sin A = \sqrt{1 - \frac{1}{16}} = \sqrt{\frac{15}{16}}$$
$$= \tfrac{1}{4}\sqrt{15}.$$
$$\tan A = \frac{\sin A}{\cos A}$$
$$= \frac{\tfrac{1}{4}\sqrt{15}}{\tfrac{1}{4}} = \sqrt{15}.$$

27. If $\sin A : \cos A = 9 : 40$, find $\sin A$ and $\cos A$.

$$40\sin A = 9\cos A.$$
$$\text{(sq.)}\ 1600\sin^2 A = 81\cos^2 A.$$
$$1600\sin^2 A - 81\cos^2 A = 0.$$

But $\sin^2 A + \cos^2 A = 1.$

Multiplying by 81 and adding,

$$1681\sin^2 A = 81,$$
$$\therefore 41\sin A = 9.$$
$$\sin A = \frac{9}{41}.$$
$$\sin^2 A + \cos^2 A = 1.$$
$$\cos A = \sqrt{1 - \sin^2 A}.$$
$$\therefore \cos A = \sqrt{1 - \left(\frac{9}{41}\right)^2} = \frac{40}{41}.$$

28. Transform the quantity $\tan^2 A + \cot^2 A - \sin^2 A - \cos^2 A$ into a form containing only $\cos A$.

$$\tan^2 A = \frac{\sin^2 A}{\cos^2 A} = \frac{1 - \cos^2 A}{\cos^2 A}.$$
$$\cot^2 A = \frac{\cos^2 A}{\sin^2 A} = \frac{\cos^2 A}{1 - \cos^2 A}.$$
$$\frac{1 - \cos^2 A}{\cos^2 A} + \frac{\cos^2 A}{1 - \cos^2 A}$$
$$-1 + \cos^2 A - \cos^2 A$$
$$= \frac{1 - 2\cos^2 A + 2\cos^4 A - \cos^2 A + \cos^4 A}{\cos^2 A - \cos^4 A}$$
$$= \frac{1 - 3\cos^2 A + 3\cos^4 A}{\cos^2 A - \cos^4 A}.$$

29. Prove that $\sin A + \cos A = (1 + \tan A)\cos A$.

$$\frac{\sin A}{\cos A} = \tan A.$$
$$\sin A = \tan A \cos A.$$
$$\sin A + \cos A = \tan A \cos A + \cos A$$
$$= (1 + \tan A)\cos A.$$

30. Prove that $\tan A + \cot A = \sec A \times \csc A$.

$$\tan A = \frac{\sin A}{\cos A}.$$

$$\cot A = \frac{\cos A}{\sin A}.$$

$$\tan A + \cot A = \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}$$

$$= \frac{\sin^2 A + \cos^2 A}{\cos A \sin A}.$$

$$\sin^2 A + \cos^2 A = 1.$$

$$\therefore \tan A + \cot A = \frac{1}{\cos A \sin A}$$

$$= \sec A \times \csc A.$$

Exercise VI. Page 20.

1. In Case II. give another way of finding c, after b has been found.

$$\cos A = \frac{b}{c},$$

$$b = c \cos A,$$

$$c = \frac{b}{\cos A}.$$

2. In Case III. give another way of finding c, after a has been found.

$$\sin A = \frac{a}{c},$$

$$c \sin A = a,$$

$$c = \frac{a}{\sin A}.$$

3. In Case IV. give another way of finding b, after the angles have been found.

$$\cos A = \frac{b}{c},$$

$$b = c \cos A.$$

4. In Case V. give another way of finding c, after the angles have been found.

$$\cos A = \frac{b}{c},$$

$$c \cos A = b,$$

$$c = \frac{b}{\cos A}.$$

5. Given B and c; find A, a, b.

$$A = (90° - B),$$

$$\sin A = \frac{a}{c},$$

$$a = c \sin A.$$

$$\cos A = \frac{b}{c},$$

$$b = c \cos A.$$

6. Given B and b; find A, a, c.

$$A = (90° - B),$$

$$\tan A = \frac{a}{b},$$

$$a = b \tan A.$$

$$\sin B = \frac{b}{c},$$

$$b = c \sin B,$$

$$c = \frac{b}{\sin B}.$$

7. Given B and a; find A, b, c.

$$A = (90° - B),$$

$$\tan A = \frac{a}{b},$$

$$b \tan A = a,$$

$$b = \frac{a}{\tan A}.$$

$$\sin A = \frac{a}{c},$$
$$c \sin A = a,$$
$$c = \frac{a}{\sin A}.$$

8. Given b and c; find A, b, a.

$$\cos A = \frac{b}{c},$$
$$B = (90° - A).$$
$$\sin A = \frac{a}{c},$$
$$a = c \sin A.$$

9. Given $a = 6$, $c = 12$; required $A = 30°$, $B = 60°$, $b = 10.392$.

$$\sin A = \frac{a}{c} = \frac{1}{2} = \sin 30°,$$
$$A = 30°.$$
$$B = (90° - A) = 60°.$$
$$\cos A = \frac{b}{c}.$$
$$\therefore\ b = c \cos A.$$

$$\begin{aligned} \log \cos A &= 9.93753 \\ \log 12 &= \underline{1.07918} \\ \log b &= 1.01671 \\ b &= 10.392. \end{aligned}$$

10. Given $A = 60°$, $b = 4$; required $B = 30°$, $c = 8$, $a = 6.9282$.

Since $A = 60°$ and $b = 4$,

$$B = (90° - 60°) = 30°,$$

and $c = 8$. (By Geometry.)

$$c^2 = a^2 + b^2.$$
$$\therefore c^2 - b^2 = a^2 = 48.$$
$$\log 48 = \log a^2 = 1.68124,$$
$$\log a = 0.84062,$$
$$a = 6.9282.$$

11. Given $A = 30°$, $a = 3$; required $B = 60°$, $c = 6$, $b = 5.1961$.

Since $A = 30°$ and $a = 3$,

$$B = (90° - 30°) = 60°,$$

and $c = 6$.

$$c^2 = a^2 + b^2.$$
$$\therefore c^2 - a^2 = b^2 = 27.$$
$$\log 27 = \log b^2 = 1.43136,$$
$$\log b = 0.71568,$$
$$b = 5.1961.$$

12. Given $a = 4$, $b = 4$; required $A = B = 45°$, $c = 5.6568$.

Since a and b each $= 4$, the $\triangle$ is an isosceles $\triangle$, and the $\angle$s A and B are equal.

$$\therefore A = \tfrac{1}{2} \text{ of } 90° = 45°,$$
$$B = \tfrac{1}{2} \text{ of } 90° = 45°.$$
$$c^2 = a^2 + b^2 = 32.$$
$$\log 32 = \log c^2 = 1.50515,$$
$$\log c = 0.75257,$$
$$c = 5.6568.$$

13. Given $a = 2$, $c = 2.82843$; required $A = B = 45°$, $b = 2$.

$$b = \sqrt{c^2 - a^2}$$
$$= \sqrt{(c+a)(c-a)},$$
$$\log b^2 = \log (c + a) + \log (c - a).$$

$$\begin{aligned} \log (c + a) &= 0.68381 \\ \log (c - a) &= \underline{9.91826 - 10} \\ \log b^2 &= 0.60207 \end{aligned}$$

$\log b \quad = 0.30103,$

$b \quad = 2.$

$\therefore$ the $\triangle$ is an isosceles rt. $\triangle$.

$\therefore A = B = 45°.$

14. Given $c = 627$, $A = 23° 30'$; required $B = 66° 30'$, $a = 250.02$, $b = 575.0$.

$$B = (90° - A) = 66° 30'.$$

$$a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\begin{aligned} \log c &= 2.79727 \\ \log \sin A &= \underline{9.60070} \\ \log a &= 2.39797 \\ a &= 250.02. \end{aligned}$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\begin{aligned} \log c &= 2.79727 \\ \log \cos A &= \underline{9.96240} \\ \log b &= 2.75967 \\ b &= 575. \end{aligned}$$

15. Given $c = 2280$, $A = 28° 5'$; required $B = 61° 55'$, $a = 1073.3$, $b = 2011.6$.

$$B = 61° 55'.$$

$$a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\begin{aligned} \log c &= 3.35793 \\ \log \sin A &= \underline{9.67280} \\ \log a &= 3.03073 \\ a &= 1073.3. \end{aligned}$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\begin{aligned} \log c &= 3.35793 \\ \log \cos A &= \underline{9.94560} \\ \log b &= 3.30353 \\ b &= 2011.6 \end{aligned}$$

16. Given $c = 72.15$, $A = 39° 34'$; required $B = 50° 26'$, $a = 45.958$, $b = 55.620$.

$$B = 50° 26'.$$

$$a = c \sin A.$$

$$\log a = \log c + \log \sin A,$$

$$\begin{aligned} \log c &= 1.85824 \\ \log \sin A &= \underline{9.80412} \\ \log a &= 1.66236 \\ a &= 45.958. \end{aligned}$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\begin{aligned} \log c &= 1.85824 \\ \log \cos A &= \underline{9.88699} \\ \log b &= 1.74523 \\ b &= 55.620. \end{aligned}$$

17. Given $c = 1$, $A = 36°$; required $B = 54°$, $a = 0.58779$, $b = 0.80902$.

$$B = 54°.$$

$$\sin A = \frac{a}{c},$$

$$a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\begin{aligned} \log c &= 0.00000 \\ \log \sin A &= \underline{9.76922} \\ \log a &= 9.76922 \\ a &= 0.58779. \end{aligned}$$

$$\cos A = \frac{b}{c},$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\begin{aligned} \log c &= 0.00000 \\ \log \cos A &= \underline{9.90796} \\ \log b &= 9.90796 - 10 \\ b &= 0.80902. \end{aligned}$$

18. Given $c = 200$, $B = 21° 47'$; required $A = 68° 13'$, $a = 185.72$, $b = 74.219$.

$$A = 68° 13'.$$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\begin{aligned} \log c &= 2.30103 \\ \log \sin A &= \underline{9.96783} \\ \log a &= 2.26886 \\ a &= 185.73. \end{aligned}$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\begin{aligned} \log c &= 2.30103 \\ \log \cos A &= \underline{9.56949} \\ \log b &= 1.87052 \\ b &= 74.22. \end{aligned}$$

19. Given $c = 93.4$, $B = 76° 25'$; required $A = 13° 35'$, $a = 21.936$, $b = 90.788$.

$$A = 13° 35'.$$

$$a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\begin{aligned} \log c &= 1.97035 \\ \log \sin A &= \underline{9.37081} \\ \log a &= 1.34116 \\ a &= 21.936. \end{aligned}$$

$$b = a \cot A.$$

$$\log b = \log a + \log \cot A.$$

$$\begin{aligned} \log a &= 1.34116 \\ \log \cot A &= \underline{0.61687} \\ \log b &= 1.95803 \\ b &= 90.788. \end{aligned}$$

20. Given $a = 637$, $A = 4° 35'$; required $B = 85° 25'$, $b = 7946$, $c = 7971.5$.

$$B = 85° 25'.$$

$$b = a \cot A.$$

$$\log b = \log a + \log \cot A.$$

$$\begin{aligned} \log a &= 2.80414 \\ \log \cot A &= \underline{1.09601} \\ \log b &= 3.90015 \\ b &= 7946. \end{aligned}$$

$$\begin{aligned} \log c &= \log a + \text{colog} \sin A. \\ \log a &= 2.80414 \\ \text{colog} \sin A &= \underline{1.09740} \\ \log c &= 3.90154 \\ c &= 7971.5. \end{aligned}$$

21. Given $a = 48.532$, $A = 36° 44'$; required $B = 53° 16'$, $b = 65.033$, $c = 81.144$.

$$\begin{aligned} B &= 90° - A \\ &= 90° - 36° 44' \\ &= 53° 16'. \end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log a + \text{colog} \sin A.$$

$$\begin{aligned}
\log a \quad &= 1.68603 \\
\text{colog} \sin A &= 0.22323 \\
\log c \quad &= 1.90926 \\
c \quad &= 81.144.
\end{aligned}$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\begin{aligned}
\log c \quad &= 1.90926 \\
\log \cos A &= 9.90386 \\
\log b \quad &= 1.81312 \\
b \quad &= 65.031.
\end{aligned}$$

22. Given $a = 0.0008$, $A = 86°$; required $B = 4°$, $b = 0.0000559$, $c = 0.000802$.

$$\begin{aligned}
B &= 90° - A \\
&= 90° - 86° \\
&= 4°.
\end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log a + \text{colog} \sin A.$$

$$\begin{aligned}
\log a \quad &= 6.90309 - 10 \\
\text{colog} \sin A &= 0.00106 \\
\log c \quad &= 6.90415 - 10 \\
c \quad &= 0.000802.
\end{aligned}$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\begin{aligned}
\log c \quad &= 6.90415 - 10 \\
\log \cos A &= 8.84358 \\
\log b \quad &= 5.74773 - 10 \\
b \quad &= 0.0000559.
\end{aligned}$$

23. Given $b = 50.937$, $B = 43° 48'$; required $A = 46° 12'$, $a = 53.116$, $c = 73.59$.

$$A = 46° 12'.$$

$$\tan A = \frac{a}{b}.$$

$$a = b \tan A.$$

$$\begin{aligned}
\log b \quad &= 1.70703 \\
\log \tan A &= 0.01820 \\
\log a \quad &= 1.72523 \\
a \quad &= 53.116.
\end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin A}.$$

$$\begin{aligned}
\log a \quad &= 1.72523 \\
\text{colog} \sin A &= 0.14161 \\
\log c \quad &= 1.86684 \\
c \quad &= 73.593.
\end{aligned}$$

24. Given $b = 2$, $B = 3° 38'$; required $A = 86° 22'$, $a = 31.497$, $c = 31.560$.

$$A = 86° 22'.$$

$$\tan A = \frac{a}{b}.$$

$$a = b \tan A.$$

$$\begin{aligned}
\log b \quad &= 0.30103 \\
\log \tan A &= 1.19723 \\
\log a \quad &= 1.49826 \\
a \quad &= 31.496.
\end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin A}.$$

$$\begin{array}{ll} \log a & = 1.49826 \\ \text{colog} \sin A & = \underline{0.00087} \\ \log c & = 1.49913 \\ c & = 31.560. \end{array}$$

25. Given $a = 992$, $B = 76° 19'$; $A = 13° 41'$, $b = 4074.45$, $c = 4193.55$.

$$\begin{aligned} A &= 90° - 76° 19' \\ &= 13° 41'. \end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$\log c = \log a + \text{colog} \sin A.$$

$$\begin{array}{ll} \log a & = 2.99651 \\ \text{colog} \sin A & = \underline{0.62607} \\ \log c & = 3.62258 \\ c & = 4193.6. \end{array}$$

$$\sin B = \frac{b}{c}.$$

$$\log b = \log c + \log \sin B.$$

$$\begin{array}{ll} \log c & = 3.62258 \\ \log \sin B & = \underline{9.98750} \\ \log b & = 3.61008 \\ b & = 4074.5. \end{array}$$

26. Given $a = 73$, $B = 68° 52'$; required $A = 21° 8'$, $b = 188.86$, $c = 202.47$.

$$A = 90° - B = 21° 8'.$$

$$\sin A = \frac{a}{c}.$$

$$\log c = \log a + \text{colog} \sin A.$$

$$\begin{array}{ll} \log a & = 1.86332 \\ \text{colog} \sin A & = \underline{0.44305} \\ \log c & = 2.30637 \\ c & = 202.47. \end{array}$$

$$\sin B = \frac{b}{c}.$$

$$\log b = \log c + \log \sin B.$$

$$\begin{array}{ll} \log c & = 2.30637 \\ \log \sin B & = \underline{9.96976} \\ \log b & = 2.27613 \\ b & = 188.86. \end{array}$$

27. Given $a = 2.189$, $B = 45° 25'$; required $A = 44° 35'$, $b = 2.2211$, $c = 3.1185$.

$$\begin{aligned} A &= 90° - 45° 25' \\ &= 44° 35'. \end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log a + \text{colog} \sin A.$$

$$\begin{array}{ll} \log a & = 0.34025 \\ \text{colog} \sin A & = \underline{0.15370} \\ \log c & = 0.49395 \\ c & = 3.1185. \end{array}$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\begin{array}{ll} \log c & = 0.49395 \\ \log \cos A & = \underline{9.85262} \\ \log b & = 0.34657 \\ b & = 2.2211. \end{array}$$

28. Given $b = 4$, $A = 37° 56'$; required $B = 52° 4'$, $a = 3.1176$, $c = 5.0714$.

$B = 52° 4'.$

$\cos A = \frac{b}{c}.$

$b = c \cos A.$

$c = \frac{b}{\cos A}.$

$\log c = \log b + \text{colog} \cos A.$

$\log b = 0.60206$
$\text{colog} \cos A = 0.10307$
$\log c = 0.70513$
$c = 5.0714.$

$\tan A = \frac{a}{b}.$

$a = b \tan A.$

$\log a = \log b + \log \tan A.$

$\log b = 0.60206$
$\log \tan A = 9.89177$
$\log a = 0.49383$
$a = 3.1176.$

29. Given $c = 8590$, $a = 4476$; required $A = 31° 24'$, $B = 58° 36'$, $b = 7332.8$.

$\sin A = \frac{a}{c}.$

$\log \sin A = \log a + \text{colog}\, c.$

$\log a = 3.65089$
$\text{colog}\, c = 6.06601$
$\log \sin A = 9.71690$
$A = 31° 24'.$
$B = 58° 36'.$

$\cot A = \frac{b}{a}.$

$\log b = \log a + \log \cot A.$

$\log a = 3.65089$
$\log \cot A = 10.21438$
$\log b = 3.86527$
$b = 7332.8.$

30. Given $c = 86.53$, $a = 71.78$; required $A = 56° 3'$, $B = 33° 57'$, $b = 48.324$.

$\log \sin A = \log a + \text{colog}\, c.$

$\log a = 1.85600$
$\text{colog}\, c = 8.06283$
$\log \sin A = 9.91883$
$A = 56° 3'.$
$B = 33° 57'.$

$\log b = \log a + \log \cot A.$

$\log a = 1.85600$
$\log \cot A = 9.82817$
$\log b = 1.68417$
$b = 48.324.$

31. Given $c = 9.35$, $a = 8.49$; required $A = 65° 14'$, $B = 24° 46'$, $b = 3.917$.

$\sin A = \frac{a}{c}.$

$B = 90° - A.$

$\text{colog}\, c = 9.02919$
$\log a = 0.92891$
$\log \sin A = 9.95810$
$A = 65° 14'.$
$B = 24° 46'.$

$\cos A = \frac{b}{c}.$

$b = c \cos A.$

$\log c = 0.97081$
$\log \cos A = 9.62214$
$\log b = 0.59295$
$b = 3.917.$

32. Given $c = 2194$, $b = 1312.7$; required $A = 53° 15'$, $B = 36° 45'$, $a = 1758$.

$$\cos A = \frac{b}{c}.$$

$$\begin{aligned} \log b &= 3.11816 \\ \text{colog } c &= 6.65876 \\ \hline \log \cos A &= 9.77692 \\ A &= 53^\circ\ 15'. \\ B &= (90^\circ - A) \\ &= 36^\circ\ 45'. \end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A.$$

$$\begin{aligned} \log c &= 3.34124 \\ \log \sin A &= 9.90377 \\ \hline \log a &= 3.24501 \\ a &= 1758. \end{aligned}$$

33. Given $c = 30.69$, $b = 18.256$; required $A = 53^\circ\ 30'$, $B = 36^\circ\ 30'$, $a = 24.67$.

$$\cos A = \frac{b}{c}.$$

$$\log \cos A = \log b + \text{colog } c.$$

$$\begin{aligned} \log b &= 1.26140 \\ \text{colog } c &= 8.51300 \\ \hline \log \cos A &= 9.77440 \\ A &= 53^\circ\ 30'. \\ B &= 36^\circ\ 30'. \end{aligned}$$

$$\tan A = \frac{a}{b}.$$

$$\log a = \log \tan A + \log b.$$

$$\begin{aligned} \log \tan A &= 10.13079 \\ \log b &= 1.26140 \\ \hline \log a &= 1.39219 \\ a &= 24.671. \end{aligned}$$

34. Given $a = 38.313$, $b = 19.522$; required $A = 63^\circ$, $B = 27^\circ$, $c = 43$.

$$\tan A = \frac{a}{b}.$$

$$\log \tan A = \log a + \text{colog } b.$$

$$\begin{aligned} \log a &= 1.58335 \\ \text{colog } b &= 8.70948 \\ \hline \log \tan A &= 10.29283 \\ A &= 63^\circ. \\ B &= 27^\circ. \end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$\log c = \log a + \text{colog} \sin A.$$

$$\begin{aligned} \log a &= 1.58335 \\ \text{colog } \sin A &= 0.05012 \\ \hline \log c &= 1.63347 \\ c &= 43. \end{aligned}$$

35. Given $a = 1.2291$, $b = 14.950$; required $A = 4^\circ\ 42'$, $B = 85^\circ\ 18'$, $c = 15$.

$$\tan A = \frac{a}{b}.$$

$$\begin{aligned} \log a &= 0.08959 \\ \text{colog } b &= 8.82536 - 10 \\ \hline \log \tan A &= 8.91495 \\ A &= 4^\circ\ 42'. \\ B &= 85^\circ\ 18'. \end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A.$$

$$c = \frac{a}{\sin A}.$$

$$\begin{aligned} \log a &= 0.08959 \\ \text{colog } \sin A &= 1.08651 \\ \hline \log c &= 1.17610 \\ c &= 15. \end{aligned}$$

36. Given $a = 415.38$, $b = 62.080$; required $A = 81° 30'$, $B = 8° 30'$, $c = 420$.

$$\tan A = \frac{a}{b}.$$

$$\begin{aligned}
\log a &= 2.61845 \\
\text{colog } b &= \underline{8.20705 - 10} \\
\log \tan A &= 10.82550 \\
A &= 81° 30'. \\
B &= 8° 30'.
\end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A.$$

$$c = \frac{a}{\sin A}.$$

$$\begin{aligned}
\log a &= 2.61845 \\
\text{colog } \sin A &= \underline{0.00480} \\
\log c &= 2.62325 \\
c &= 420.
\end{aligned}$$

37. Given $a = 13.690$, $b = 16.926$; required $A = 38° 58'$, $B = 51° 2'$, $c = 21.77$.

$$\tan A = \frac{a}{b}.$$

$$\log \tan A = \log a + \text{colog } b.$$

$$\begin{aligned}
\log a &= 1.13640 \\
\text{colog } b &= \underline{8.77144 - 10} \\
\log \tan A &= 9.90784 \\
A &= 38° 58'. \\
B &= 51° 2'.
\end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log a + \text{colog } \sin A.$$

$$\begin{aligned}
\log a &= 1.13640 \\
\text{colog } \sin A &= \underline{0.20144} \\
\log c &= 1.33784 \\
c &= 21.769.
\end{aligned}$$

38. Given $c = 91.92$, $a = 2.19$; required $A = 1° 21' 55''$, $B = 88° 38' 5''$, $b = 91.894$.

$$\sin A = \frac{a}{c}.$$

$$\log \sin A = \log a + \text{colog } c.$$

$$\begin{aligned}
\log a &= 0.34044 \\
\text{colog } c &= \underline{8.03659 - 10} \\
\log \sin A &= 8.37703 \\
A &= 1° 21' 55''. \\
B &= 88° 38' 5''.
\end{aligned}$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\begin{aligned}
\log c &= 1.96341 \\
\log \cos A &= \underline{9.99988} \\
\log b &= 1.96329 \\
b &= 91.894.
\end{aligned}$$

39. Compute the unknown parts and also the area, having given $a = 5$, $b = 6$.

$$\tan A = \frac{a}{b}.$$

$$\log \tan A = \log a + \text{colog } b.$$

$$\begin{aligned}
\log a &= 0.69897 \\
\text{colog } b &= \underline{9.22185 - 10} \\
\log \tan A &= 9.92082 \\
A &= 39° 48'. \\
B &= 50° 12'.
\end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log a + \text{colog} \sin A.$$

$$\begin{aligned} \log a &= 0.69897 \\ \text{colog} \sin A &= \underline{0.19375} \\ \log c &= 0.89272 \\ c &= 7.8112. \\ F &= \frac{a \times b}{2} = \frac{30}{2} = 15. \end{aligned}$$

40. Compute the unknown parts and also the area, having given $a = 0.615$, $c = 70$.

$$\sin A = \frac{a}{c}.$$

$$\log \sin A = \log a + \text{colog } c.$$

$$\begin{aligned} \log a &= 9.78888 - 10 \\ \text{colog } c &= \underline{8.15490 - 10} \\ \log \sin A &= 7.94378 \\ A &= 30' \, 12''. \\ B &= 89° \, 29' \, 48''. \end{aligned}$$

$$\tan A = \frac{a}{b}.$$

$$\log b = \log a + \text{colog} \tan A.$$

$$\begin{aligned} \log a &= 9.78888 - 10 \\ \text{colog} \tan A &= \underline{2.05626} \\ \log b &= 1.84514 \\ b &= 70.007. \end{aligned}$$

$$F = \tfrac{1}{2} ab.$$

$$\begin{aligned} \log a &= 9.78888 - 10 \\ \log b &= 1.84514 \\ \text{colog } 2 &= \underline{9.69897 - 10} \\ \log F &= 1.33291 \\ F &= 21.528. \end{aligned}$$

41. Compute the unknown parts and also the area, having given $b = \sqrt[3]{2}$, $c = \sqrt{3}$.

$$\sqrt[3]{2} = 1.25991.$$

$$\sqrt{3} = 1.73205.$$

$$\cos A = \frac{b}{c}.$$

$$\log \cos A = \log b + \text{colog } c.$$

$$\begin{aligned} \log b &= 0.10034 \\ \text{colog } c &= \underline{9.76144 - 10} \\ \log \cos A &= 9.86178 \\ A &= 43° \, 20'. \\ B &= 46° \, 40'. \end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\begin{aligned} \log c &= 0.23856 \\ \log \sin A &= \underline{9.83648} \\ \log a &= 0.07504 \\ a &= 1.1886. \end{aligned}$$

$$F = \tfrac{1}{2} ab.$$

$$\begin{aligned} \log a &= 0.07504 \\ \log b &= 0.10034 \\ \text{colog } 2 &= \underline{9.69897 - 10} \\ \log F &= 9.87435 - 10 \\ F &= 0.74876. \end{aligned}$$

42. Compute the unknown parts and also the area, having given $a = 7$, $A = 18° \, 14'$.

$$B = 71° \, 46'.$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log a + \operatorname{colog} \sin A.$$

$$\begin{aligned}
\log a &= 0.84510 \\
\operatorname{colog} \sin A &= \underline{0.50461} \\
\log c &= 1.34971 \\
c &= 22.372.
\end{aligned}$$

$$\tan A = \frac{a}{b}.$$

$$b = \frac{a}{\tan A}.$$

$$\log b = \log a + \operatorname{colog} \tan A.$$

$$\begin{aligned}
\log a &= 0.84510 \\
\operatorname{colog} \tan A &= \underline{0.48224 - 10} \\
\log b &= 1.32734 \\
b &= 21.249.
\end{aligned}$$

$$F = \tfrac{1}{2} ab.$$

$$\begin{aligned}
\log a &= 0.84510 \\
\log b &= 1.32734 \\
\operatorname{colog} 2 &= \underline{9.69897 - 10} \\
\log F &= 1.87141 \\
F &= 74.371.
\end{aligned}$$

43. Compute the unknown parts and also the area, having given $b = 12$, $A = 29° 8'$.

$$A = 29° 8'.$$

$$B = 60° 52'.$$

$$\cos A = \frac{b}{c}.$$

$$c = \frac{b}{\cos A}.$$

$$\log c = \log b + \operatorname{colog} \cos A.$$

$$\begin{aligned}
\log b &= 1.07918 \\
\operatorname{colog} \cos A &= \underline{0.05874} \\
\log c &= 1.13792 \\
c &= 13.738.
\end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\begin{aligned}
\log c &= 1.13792 \\
\log \sin A &= \underline{9.68739} \\
\log a &= 0.82531 \\
a &= 6.6882.
\end{aligned}$$

$$F = \tfrac{1}{2} ab.$$

$$\log F = \log a + \log b + \operatorname{colog} 2.$$

$$\begin{aligned}
\log a &= 0.82531 \\
\log b &= 1.07918 \\
\operatorname{colog} 2 &= \underline{9.69897 - 10} \\
\log F &= 1.60346 \\
F &= 40.129.
\end{aligned}$$

44. Compute the unknown parts and also the area, having given $c = 68$, $A = 69° 54'$.

$$A = 69° 54'.$$

$$B = 20° 6'.$$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\begin{aligned}
\log c &= 1.83251 \\
\log \sin A &= \underline{9.97271} \\
\log a &= 1.80522 \\
a &= 63.859.
\end{aligned}$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\begin{aligned} \log c &= 1.83251 \\ \log \cos A &= \underline{9.53613} \\ \log b &= 1.36864 \\ b &= 23.369. \end{aligned}$$

$$F = \tfrac{1}{2}\, ab.$$

$$\begin{aligned} \log a &= 1.80522 \\ \log b &= 1.36864 \\ \text{colog } 2 &= \underline{9.69897 - 10} \\ \log F &= 2.87283 \\ F &= 746.15. \end{aligned}$$

45. Compute the unknown parts and also the area, having given $c = 27$, $B = 44° 4'$.

$$A = 45° 56'.$$

$$a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\begin{aligned} \log c &= 1.43136 \\ \log \sin A &= \underline{9.85645} \\ \log a &= 1.28781 \\ a &= 19.40. \end{aligned}$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\begin{aligned} \log c &= 1.43136 \\ \log \cos A &= \underline{9.84229} \\ \log b &= 1.27365 \\ b &= 18.778. \end{aligned}$$

$$F = \tfrac{1}{2}\, ab.$$

$$\begin{aligned} \log a &= 1.28781 \\ \log b &= 1.27365 \\ \text{colog } 2 &= \underline{9.69897 - 10} \\ \log F &= 2.26043 \\ F &= 182.15. \end{aligned}$$

46. Compute the unknown parts and also the area, having given $a = 47$, $B = 48° 49'$.

$$A = 41° 11'.$$

$$b = a \cot A.$$

$$\log b = \log a + \log \cot A.$$

$$\begin{aligned} \log a &= 1.67210 \\ \log \cot A &= \underline{10.05803} \\ \log b &= 1.73013 \\ b &= 53.719. \end{aligned}$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log a + \text{colog} \sin A.$$

$$\begin{aligned} \log a &= 1.67210 \\ \text{colog} \sin A &= \underline{0.18146} \\ \log c &= 1.85356 \\ c &= 71.377. \end{aligned}$$

$$F = \tfrac{1}{2}\, ab.$$

$$\begin{aligned} \log a &= 1.67210 \\ \log b &= 1.73013 \\ \text{colog } 2 &= \underline{9.69897 - 10} \\ \log F &= 3.10120 \\ F &= 1262.4. \end{aligned}$$

47. Compute the unknown parts and also the area, having given $b = 9$, $B = 34° 44'$.

$$A = 55° 16'.$$

$$a = b \tan A.$$

$$\log a = \log b + \log \tan A.$$

$$\begin{aligned} \log b &= 0.95424 \\ \log \tan A &= \underline{10.15908} \\ \log a &= 1.11332 \\ a &= 12.981. \end{aligned}$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log a + \text{colog} \sin A.$$

$$\begin{aligned}
\log a &= 1.11332 \\
\text{colog} \sin A &= \underline{0.08523} \\
\log c &= 1.19855 \\
c &= 15.7960
\end{aligned}$$

$$F = \tfrac{1}{2}\, ab.$$

$$\begin{aligned}
\log a &= 1.11332 \\
\log b &= 0.95424 \\
\text{colog } 2 &= \underline{9.69897 - 10} \\
\log F &= 1.76653 \\
F &= 58.416.
\end{aligned}$$

48. Compute the unknown parts and also the area, having given $c = 8.462$, $B = 86° 4'$.

$$A = 3° 56'.$$

$$a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\begin{aligned}
\log c &= 0.92747 \\
\log \sin A &= \underline{8.83630} \\
\log a &= 9.76377 - 10 \\
a &= 0.58046.
\end{aligned}$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\begin{aligned}
\log c &= 0.92747 \\
\log \cos A &= \underline{9.99898} \\
\log b &= 0.92645 \\
b &= 8.442.
\end{aligned}$$

$$F = \tfrac{1}{2}\, ab.$$

$$\begin{aligned}
\log a &= 9.76377 - 10 \\
\log b &= 0.92645 \\
\text{colog } 2 &= \underline{9.69897 - 10} \\
\log F &= 0.38919 \\
F &= 2.4501.
\end{aligned}$$

49. Find the value of F in terms of c and A.

$$F = \tfrac{1}{2}\, ab.$$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A.$$

$$\cos A = \frac{b}{c}.$$

$$b = c \cos A.$$

Substitute,

$$\begin{aligned}
F &= \tfrac{1}{2}\, ab \\
&= \tfrac{1}{2}\, (c^2 \sin A \cos A).
\end{aligned}$$

50. Find the value of F in terms of a and A.

$$F = \tfrac{1}{2}\, ab.$$

$$\cot A = \frac{b}{a}.$$

$$b = a \cot A.$$

Substitute,

$$\begin{aligned}
F &= \tfrac{1}{2}\, ab \\
&= \tfrac{1}{2}\, (a^2 \cot A).
\end{aligned}$$

51. Find the value of F in terms of b and A.

$$F = \tfrac{1}{2}\, ab.$$

$$\tan A = \frac{a}{b}.$$

$$a = b \tan A.$$

Substitute,

$$\begin{aligned}
F &= \tfrac{1}{2}\, ab \\
&= \tfrac{1}{2}\, (b^2 \tan A).
\end{aligned}$$

52. Find the value of F in terms of a and c.

$$F = \tfrac{1}{2}\, ab.$$

$$c^2 = a^2 + b^2.$$

$$b^2 = c^2 - a^2.$$

$$b = \sqrt{c^2 - a^2}.$$

Substitute,

$$F = \tfrac{1}{2}\, (a \sqrt{c^2 - a^2}).$$

53. Given $F = 58$, $a = 10$; solve the triangle.

$$F = \tfrac{1}{2}ab.$$

$$b = \frac{2F}{a}.$$

$$\log b = \log 2F + \text{colog}\, a.$$

$$\begin{aligned} \log 2F &= 2.06446 \\ \text{colog}\, a &= 9.00000 - 10 \\ \hline \log b &= 1.06446 \\ b &= 11.6. \end{aligned}$$

$$\tan A = \frac{a}{b}.$$

$$\log \tan A = \log a + \text{colog}\, b.$$

$$\begin{aligned} \log a &= 1.00000 \\ \text{colog}\, b &= 8.93554 - 10 \\ \hline \log \tan A &= 9.93554 \\ A &= 40°\ 45'\ 48''. \\ B &= 49°\ 14'\ 12''. \end{aligned}$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log a + \text{colog} \sin A.$$

$$\begin{aligned} \log a &= 1.00000 \\ \text{colog} \sin A &= 0.18513 \\ \hline \log c &= 1.18513 \\ c &= 15.315. \end{aligned}$$

54. Given $F = 18$, $b = 5$; solve the triangle.

$$F = \tfrac{1}{2}ab.$$

$$a = \frac{2F}{b}.$$

$$\log a = \log 2F + \text{colog}\, b.$$

$$\begin{aligned} \log 2F &= 1.55630 \\ \text{colog}\, b &= 9.30103 - 10 \\ \hline \log a &= 0.85733 \\ a &= 7.2. \end{aligned}$$

$$\tan A = \frac{a}{b}.$$

$$\log \tan A = \log a + \text{colog}\, b.$$

$$\begin{aligned} \log a &= 0.85733 \\ \text{colog}\, b &= 9.30103 - 10 \\ \hline \log \tan A &= 10.15836 \\ A &= 55°\ 13'\ 20''. \\ B &= 34°\ 46'\ 40''. \end{aligned}$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log a + \text{colog} \sin A.$$

$$\begin{aligned} \log a &= 0.85733 \\ \text{colog} \sin A &= 0.08546 \\ \hline \log c &= 0.94279 \\ c &= 8.7658. \end{aligned}$$

55. Given $F = 12$, $A = 29°$; solve the triangle.

$$B = 61°.$$

$$F = \tfrac{1}{2}ab = 12.$$

$$ab = 24.$$

$$a = \frac{24}{b}.$$

$$\tan A = \frac{a}{b}.$$

$$\tan 29° = \frac{24}{b^2}.$$

$$b^2 = \frac{24}{\tan 29°}.$$

$$\log b = \tfrac{1}{2}(\log 24 + \text{colog} \tan 29°).$$

$$\begin{aligned} \log 24 &= 1.38021 \\ \text{colog} \tan 29° &= 0.25625 \\ \hline &2)\ 1.63646 \\ \log b &= 0.81823 \\ b &= 6.58. \end{aligned}$$

$$\tan 29° = \frac{a}{b}.$$

$$a = b \tan 29°.$$

$$\log a = \log b + \log \tan 29°.$$

$$\begin{aligned} \log b &= 0.81823 \\ \log \tan 29° &= \underline{9.74375} \\ \log a &= 0.56198 \\ a &= 3.6474. \end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$c = \frac{a}{\sin 29°}.$$

$$\log c = \log a + \operatorname{colog} \sin 29°.$$

$$\begin{aligned} \log a &= 0.56198 \\ \operatorname{colog} \sin 29° &= \underline{0.31443} \\ \log c &= 0.87641 \\ c &= 7.5233. \end{aligned}$$

56. Given $F = 100$, $c = 22$; solve the triangle.

$$F = \tfrac{1}{2}ab = 100.$$

$$ab = 200.$$

$$a = \frac{200}{b}.$$

$$a^2 = \frac{40000}{b^2}.$$

$$a^2 + b^2 = c^2 = 484.$$

Substitute,

$$\frac{40000}{b^2} + b^2 = 484.$$

$$40000 + b^4 = 484 b^2.$$

$$b^4 - 484 b^2 = -40000.$$

$$b^4 - (\ \ ) + (242)^2 = 18564.$$

$$\log \sqrt{18564} = 2\underline{)4.26867}$$

$$= 2.13434;$$

but $2.13434 = \log 136.25.$

$$\therefore b^2 - 242 = 136.25.$$

$$b^2 = 378.25.$$

$$\log b = \tfrac{1}{2}(\log 378.25).$$

$$= 1.28889.$$

$$b = 19.449.$$

$$\cos A = \frac{b}{c}.$$

$$\log \cos A = \log b + \operatorname{colog} c.$$

$$\begin{aligned} \log b &= 1.28889 \\ \operatorname{colog} c &= \underline{8.65758} \\ \log \cos A &= 9.94647 \\ A &= 27° \ 52'. \\ B &= 62° \ \ 8'. \end{aligned}$$

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\begin{aligned} \log c &= 1.34242 \\ \log \sin A &= \underline{9.66970} \\ \log a &= 1.01212 \\ a &= 10.283. \end{aligned}$$

57. Find the angles of a right triangle if the hypotenuse is equal to three times one of the legs.

Let c = hypotenuse,

and let c = three times a, one of the legs.

$$\sin A = \frac{a}{c}.$$

$$\log \sin A = \log a + \operatorname{colog} c.$$

$$\begin{aligned}
\log a &= 0.00000\\
\text{colog } c &= \underline{9.52288 - 10}\\
\log \sin A &= 9.52288\\
A &= 19^\circ\ 28'\ 17''.\\
B &= 70^\circ\ 31'\ 43''.
\end{aligned}$$

58. Find the legs of a right triangle if the hypotenuse $= 6$, and one angle is twice the other.

Let $c = \text{hypotenuse} = 6$,

and let $B = \text{twice } A$;

then $B = 60^\circ$,

$A = 30^\circ$.

$$\sin A = \frac{a}{c}.$$

$$a = c \sin A.$$

$$\log a = \log c + \log \sin A.$$

$$\begin{aligned}
\log c &= 0.77815\\
\log \sin A &= \underline{9.69897}\\
\log a &= 0.47712\\
a &= 3.
\end{aligned}$$

$$\sin B = \frac{b}{c}.$$

$$b = c \sin B.$$

$$\log b = \log c + \log \sin B.$$

$$\begin{aligned}
\log c &= 0.77815\\
\log \sin B &= \underline{9.93753}\\
\log b &= 0.71568\\
b &= 5.1961.
\end{aligned}$$

59. In a right triangle given c, and $A = nB$; find a and b.

$$\begin{aligned}
B &= 90^\circ - A\\
&= 90^\circ - nB.
\end{aligned}$$

$$B(n+1) = 90^\circ.$$

$$B = \frac{90^\circ}{n+1}.$$

$$\cos B = \frac{a}{c}.$$

$$\cos \frac{90^\circ}{n+1} = \frac{a}{c}.$$

$$a = c \cos \frac{90^\circ}{n+1}.$$

$$\sin B = \frac{b}{c}.$$

$$\sin \frac{90^\circ}{n+1} = \frac{b}{c}.$$

$$b = c \sin \frac{90^\circ}{n+1}.$$

60. In a right triangle the difference between the hypotenuse and the greater leg is equal to the difference between the two legs; find the angles.

$$c - a = a - b.$$

$$2a - b = c. \qquad (1)$$

$$a^2 + b^2 = c^2. \qquad (2)$$

Squaring (1),

$$\begin{aligned}
4a^2 - 4ab + b^2 &= c^2\\
a^2 \qquad\quad + b^2 &= c^2\\
\hline
3a^2 - 4ab \qquad\quad &= 0
\end{aligned}$$

$$3a^2 = 4ab.$$

$$3a = 4b.$$

$$a = \frac{4b}{3}.$$

$$\tan A = \frac{a}{b} = \frac{4}{3}.$$

$$\log \tan A = \log 4 + \text{colog } 3.$$

$$\begin{aligned}
\log 4 &= 0.60206\\
\text{colog } 3 &= \underline{9.52288 - 10}\\
\log \tan A &= 10.12494\\
A &= 53^\circ\ \ 7'\ 48''.\\
B &= 36^\circ\ 52'\ 12''.
\end{aligned}$$

61. At a horizontal distance of 120 feet from the foot of a steeple, the angle of elevation of the top was found to be 60° 30′; find the height of the steeple.

$$\tan A = \frac{a}{b}.$$

$$a = b \tan A.$$

$$\log a = \log b + \log \tan A.$$

$$\begin{aligned} \log b &= 2.07918 \\ \log \tan A &= \underline{10.24736} \\ \log a &= 2.32654 \\ a &= 212.1. \end{aligned}$$

62. From the top of a rock that rises vertically 326 feet out of the water, the angle of depression of a boat was found to be 24°; find the distance of the boat from the foot of the rock.

$$\cot A = \frac{b}{a}.$$

$$b = a + \cot A.$$

$$\log b = \log a + \log \cot a.$$

$$\begin{aligned} \log a &= 2.51322 \\ \log \cot A &= \underline{10.35142} \\ \log b &= 2.86464 \\ b &= 732.22. \end{aligned}$$

63. How far is a monument, in a level plain, from the eye, if the height of the monument is 200 feet and the angle of elevation of the top 3° 30′?

$$\cot A = \frac{b}{a}.$$

$$b = a \cot A.$$

$$\log b = \log a + \log \cot A.$$

$$\begin{aligned} \log a &= 2.30103 \\ \log \cot A &= \underline{1.21351} \\ \log b &= 3.51454 \\ b &= 3270. \end{aligned}$$

64. In order to find the breadth of a river a distance AB was measured along the bank, the point A being directly opposite a tree C on the other side. The angle ABC was also measured. If $AB = 96$ feet, and $ABC = 21°\ 14'$, find the breadth of the river.

If $ABC = 45°$, what would be the breadth of the river?

$$\tan B = AC \div AB.$$

$$AC = AB \times \tan B.$$

$$\log AC = \log AB + \log \tan B.$$

$$\begin{aligned} \log AB &= 1.98227 \\ \log \tan B &= \underline{9.58944} \\ \log AC &= 1.57171 \\ AC &= 37.3 \text{ feet.} \end{aligned}$$

$$\log AC = \log AB + \log \tan B.$$

$$\begin{aligned} \log AB &= 1.98227 \\ \log \tan B &= \underline{10.00000} \\ \log AC &= 1.98227 \\ AC &= 96 \text{ feet.} \end{aligned}$$

65. Find the angle of elevation of the sun when a tower a feet high casts a horizontal shadow b feet long. Find the angle when $a = 120$, $b = 70$.

$$\tan A = \frac{a}{b}.$$

$$\tan A = \frac{120}{70}.$$

$$\log \tan A = \log 120 + \text{colog } 70.$$

$$\begin{aligned}
\log 120 &= 2.07918 \\
\text{colog } 70 &= \underline{8.15490 - 10} \\
\log \tan A &= 10.23408 \\
A &= 59^\circ\ 44'\ 35''.
\end{aligned}$$

66. How high is a tree that casts a horizontal shadow b feet in length when the angle of elevation of the sun is $A°$? Find the height of the tree when $b = 80°$, $A = 50°$.

$$\tan A = \frac{a}{b}.$$

$$a = b \tan A.$$

$$\log a = \log b + \log \tan A.$$

$$\begin{aligned}
\log b &= 1.90309 \\
\log \tan A &= \underline{10.07619} \\
\log a &= 1.97928 \\
a &= 95.34.
\end{aligned}$$

67. What is the angle of elevation of an inclined plane if it rises 1 foot in a horizontal distance of 40 feet?

$$\tan A = \frac{a}{b}.$$

$$\log \tan A = \log a + \text{colog } b.$$

$$\begin{aligned}
\log a &= 0.00000 \\
\text{colog } b &= \underline{8.39794 - 10} \\
\log \tan A &= 8.39794 \\
A &= 1^\circ\ 25'\ 56''.
\end{aligned}$$

68. A ship is sailing due north-east with a velocity of 10 miles an hour. Find the rate at which she is moving due north and also due east.

Let AB be the direction of the vessel, and equal one hour's progress = 10 miles.

AC = distance due east passed over in one hour.

As the direction of the ship is north-east,

$$A = 45°.$$

$$b = c \cos A.$$

$$\log b = \log c + \log \cos A.$$

$$\begin{aligned}
\log 10 &= 1.00000 \\
\log \cos A &= \underline{9.84949} \\
\log b &= 0.84949 \\
b &= 7.0712 \text{ miles due east,}
\end{aligned}$$

and also due north, since

$$AP = AC.$$

69. In front of a window 20 feet high is a flower-bed 6 feet wide. How long must a ladder be to reach from the edge of the bed to the window?

$$\tan A = \frac{a}{b}.$$

$$\log \tan A = \log 20 + \text{colog } 6.$$

$$\begin{aligned}
\log 20 &= 1.30103 \\
\text{colog } 6 &= \underline{9.22185 - 10} \\
\log \tan A &= 10.52288 \\
A &= 73^\circ\ 18'.
\end{aligned}$$

$$c = \frac{a}{\sin A}.$$

$$\log c = \log 20 + \text{colog} \sin A.$$

$$\begin{aligned}
\log a &= 1.30103 \\
\text{colog} \sin A &= \underline{0.01871} \\
\log c &= 1.31974 \\
c &= 20.88.
\end{aligned}$$

70. A ladder 40 feet long may be so placed that it will reach a window 33 feet high on one side of the street, and by turning it over without moving its foot it will reach a window 21 feet high on the other side. Find the breadth of the street.

$$\cos B = \frac{33}{40}.$$

$$\begin{aligned} \log 33 &= 1.51851 \\ \text{colog } 40 &= \underline{8.39794 - 10} \\ \log \cos B &= 9.91645 \\ B &= 34° \; 24' \; 45''. \end{aligned}$$

$$\tan B = \frac{b}{33}.$$

$$b = 33 \tan B.$$

$$\begin{aligned} \log 33 &= 1.51851 \\ \log \tan B &= \underline{9.83571} \\ \log b &= 1.35422 \\ b &= 22.605. \end{aligned}$$

$$\cos B' = \frac{21}{40}.$$

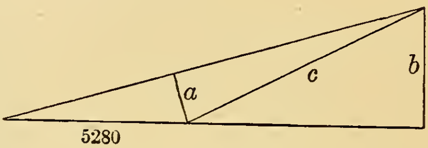
$$\begin{aligned} \log 21 &= 1.32222 \\ \text{colog } 40 &= \underline{8.39794 - 10} \\ \log \cos B' &= 9.72016 \\ B' &= 58° \; 19' \; 54''. \end{aligned}$$

$$\tan B' = \frac{b'}{21}.$$

$$b' = 21 \tan B'.$$

$$\begin{aligned} \log 21 &= 1.32222 \\ \log \tan B' &= \underline{0.20982} \\ \log b' &= 1.53204 \\ b' &= 34.044 \\ b &= \underline{22.605} \\ b + b' & \quad 56.649 \end{aligned}$$

71. From the top of a hill the angles of depression of two successive milestones, on a straight level road leading to the hill, are observed to be 5° and 15°. Find the height of the hill.



$$\sin 5° = \frac{a}{5280}.$$

$$a = 5280 \sin 5°.$$

$$\begin{aligned} \log 5280 &= 3.72263 \\ \log \sin 5° &= \underline{8.94030} \\ \log a &= 2.66293 \end{aligned}$$

$$\sin 10° = \frac{a}{c}.$$

$$a = c \sin 10°.$$

$$c = \frac{a}{\sin 10°}.$$

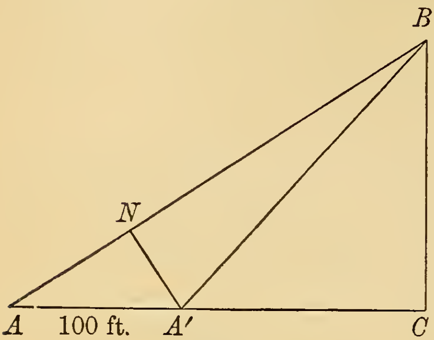
$$\begin{aligned} \log a &= 2.66293 \\ \text{colog} \sin 10° &= \underline{0.76033} \\ \log c &= 3.42326 \end{aligned}$$

$$\cos 75° = \frac{b}{c}.$$

$$b = c \cos 75°.$$

$$\begin{aligned} \log c &= 3.42326 \\ \log \cos 75° &= \underline{9.41300} \\ \log b &= 2.83626 \\ b &= 685.9 \text{ feet} \\ &= 228.63 \text{ yards} \end{aligned}$$

72. A fort stands on a horizontal plane. The angle of elevation at a certain point on the plane is 30°, and at a point 100 feet nearer the fort it is 45°. How high is the fort?



Let B represent the fort, AC the horizontal plane, BC a $\perp$ from fort to plane.

$BAC=$ angle made by line from eye of observer $= 30°$.

$BA'C = 45° =$ angle of elevation 100 feet nearer.

From A' draw $A'N \perp$ to AB.

In rt. $\triangle AA'N$,

$$\angle NAA' = 30°,$$

and $$\angle NA'A = 60°.$$

$$\therefore NA' = 50 \text{ feet}.$$

$$\therefore AN = \sqrt{(100)^2 - (50)^2}$$
$$= \sqrt{7500} = 50\sqrt{3}$$
$$= 86.602.$$

In rt. $\triangle BNA'$

$$\frac{BN}{NA'} = \cot NBA' = \cot 15°,$$

and $$BN = NA' \cot 15°.$$

$\log NA'$	$= 1.69897$
$\log \cot 15°$	$= 0.57195$
$\log BN$	$= 2.27092$
BN	$= 186.60$
AN	$= 86.60$
AB	$= 273.20$

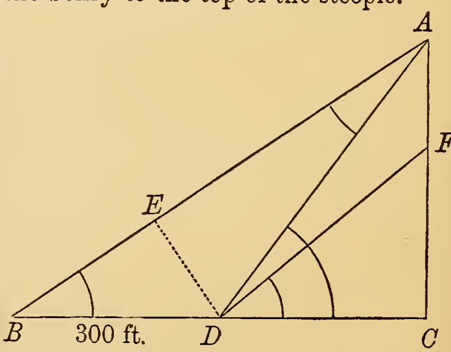
In rt. $\triangle ABC$,

$$\angle BAC = 30°,$$

and $$\angle ABC = 60°.$$

$$\therefore BC = \tfrac{1}{2}AB = \tfrac{1}{2} \times 273.20$$
$$= 136.60 \text{ feet}.$$

73. From a certain point on the ground the angles of elevation of the belfry of a church and of the top of a steeple were found to be 40° and 51° respectively. From a point 300 feet farther off, on a horizontal line, the angle of elevation of the top of the steeple is found to be 33° 45′. Find the distance from the belfry to the top of the steeple.



Draw $DE \perp$ to AB from D.

In $\triangle BED$

$$\frac{ED}{BD} = \sin 33° 45'.$$

$$ED = 300 \times \sin 33° 45'.$$

$\log 300$	$= 2.47712$
$\log \sin 33° 45'$	$= 9.74474$
$\log ED$	$= 2.22186$

$\angle EAD = 180° - 33°\ 45' - (180° - 51°) = 17°\ 15'$.

In $\triangle ADE$

$$\frac{ED}{AD} = \sin 17° 15'.$$

$$AD = \frac{ED}{\sin 17° \, 15'}.$$

$$\begin{aligned}
\log ED &= 2.22186 \\
\text{colog} \sin 17° \, 15' &= \underline{0.52791} \\
\log AD &= 2.74977
\end{aligned}$$

In $\triangle ADC$

$$\frac{DC}{AD} = \cos 51°.$$

$$DC = AD \cos 51°.$$

$$\begin{aligned}
\log AD &= 2.74977 \\
\log \cos 51° &= \underline{9.79887} \\
\log DC &= 2.54864
\end{aligned}$$

In $\triangle ADC$

$$\frac{AC}{DC} = \tan 51°.$$

$$AC = DC \tan 51°.$$

$$\begin{aligned}
\log DC &= 2.54864 \\
\log \tan 51° &= \underline{10.09163} \\
\log AC &= 2.64027 \\
AC &= 436.79
\end{aligned}$$

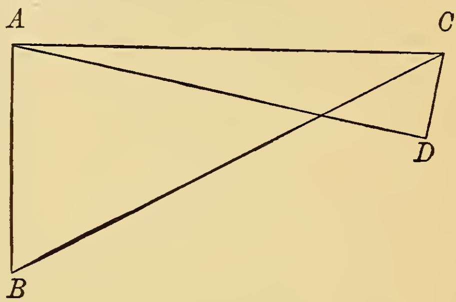
In $\triangle FDC$

$$\frac{FC}{DC} = \tan 40°.$$

$$FC = DC \tan 40°.$$

$$\begin{aligned}
\log DC &= 2.54864 \\
\log \tan 40° &= \underline{9.92381} \\
\log FC &= 2.47245 \\
FC &= 296.79. \\
AC - FC &= 140.
\end{aligned}$$

74. The angle of elevation of the top of an inaccessible fort C, observed from a point A, is 12°. At a point B, 219 feet from A and on a line AB perpendicular to AC, the angle ABC is 61° 45′. Find the height of the fort.



In rt. $\triangle CAB$

$$\frac{AB}{AC} = \cot ABC.$$

$$\therefore \; AC = \frac{AB}{\cot ABC}.$$

$$\log AC = \log AB + \text{colog} \cot ABC.$$

$$\begin{aligned}
\log AB &= 2.34044 \\
\text{colog} \cot ABC &= \underline{0.26977} \\
\log AC &= 2.61021
\end{aligned}$$

In rt. $\triangle ADC$

$$\frac{CD}{AC} = \sin CAD.$$

$$CD = AC \sin CAD.$$

$$\log CD = \log AC + \log \sin CAD.$$

$$\begin{aligned}
\log AC &= 2.61021 \\
\log \sin CAD &= \underline{9.31788} \\
\log CD &= 1.92809 \\
CD &= 84.74 \text{ feet.}
\end{aligned}$$

Exercise VII. Page 25.

1. In an isosceles triangle, given a and A; find C, c, h.

$$C = 180^\circ - 2A$$
$$= 2(90^\circ - A).$$
$$\frac{\frac{1}{2}c}{a} = \cos A.$$
$$c = 2a\cos A.$$
$$\frac{h}{a} = \sin A.$$
$$h = a\sin A.$$

2. In an isosceles triangle, given a and C; find A, c, h.

$$C + 2A = 180^\circ.$$
$$A = 90^\circ - \tfrac{1}{2}C.$$
$$\frac{\frac{1}{2}c}{a} = \cos A.$$
$$c = 2a\cos A.$$
$$\frac{h}{a} = \sin A.$$
$$h = a\sin A.$$

3. In an isosceles triangle, given c and A; find C, a, h.

$$C = 180^\circ - 2A$$
$$= 2(90^\circ - A).$$
$$\frac{\frac{1}{2}c}{a} = \cos A.$$
$$2a = \frac{c}{\cos A}.$$
$$a = \frac{c}{2\cos A}.$$
$$\frac{h}{a} = \sin A.$$
$$h = a\sin A.$$

4. In an isosceles triangle, given c and C; find A, a, h.

$$A = 90^\circ - \tfrac{1}{2}C.$$
$$\frac{\frac{1}{2}c}{a} = \cos A.$$
$$a = \frac{c}{2\cos A}.$$
$$\frac{h}{a} = \sin A.$$
$$h = a\sin A.$$

5. In an isosceles triangle, given h and A; find C, a, c.

$$C = 2(90^\circ - A).$$
$$\sin A = \frac{h}{a}.$$
$$\therefore\ a = h \div \sin A.$$
$$\cos A = \frac{\frac{1}{2}c}{a} = \frac{c}{2a}.$$
$$\therefore\ c = 2a\cos A.$$

6. In an isosceles triangle, given h and C; find A, a, c.

$$A = 90^\circ - \tfrac{1}{2}C.$$
$$\sin A = \frac{h}{a}.$$
$$a = h \div \sin A.$$
$$\cos A = \frac{\frac{1}{2}c}{a} = \frac{c}{2a}.$$
$$c = 2a\cos A.$$

7. In an isosceles triangle, given a and h; find A, C, c.

$$\sin A = h \div a.$$
$$C = 180^\circ - 2A.$$

$$= 2(90° - A).$$

$$\cos A = \frac{\frac{1}{2}c}{a} = \frac{c}{2a}.$$

$$c = 2a \cos A.$$

8. In an isosceles triangle, given c and h; find A, C, a.

$$\tan A = \frac{h}{\frac{1}{2}c}.$$

$$C = 180° - 2A$$
$$= 2(90° - A).$$

$$\sin A = \frac{h}{a}.$$

$$a = h \div \sin A.$$

9. In an isosceles triangle, given $a = 14.3$, $c = 11$; find A, C, h.

$$\cos A = \frac{\frac{1}{2}c}{a}.$$

$$\log \cos A = \log \tfrac{1}{2}c + \text{colog } a.$$

$$\begin{aligned}
\log \tfrac{1}{2}c &= 0.74036 \\
\text{colog } a &= 8.84466 - 10 \\
\log \cos A &= 9.58502 \\
A &= 67° \; 22' \; 50''. \\
C &= 2(90° - A) \\
&= 45° \; 14' \; 20''.
\end{aligned}$$

$$\sin A = \frac{h}{a}.$$

$$h = a \sin A.$$

$$\log h = \log a + \log \sin A.$$

$$\begin{aligned}
\log a &= 1.15534 \\
\log \sin A &= 9.96524 \\
\log h &= 1.12058 \\
h &= 13.2.
\end{aligned}$$

10. In an isosceles triangle, given $a = 0.295$, $A = 68° \; 10'$; find c, h, F.

$$\sin A = \frac{h}{a}.$$

$$h = a \sin A.$$

$$\log h = \log a + \log \sin A.$$

$$\begin{aligned}
\log a &= 9.46982 - 10 \\
\log \sin A &= 9.96767 \\
\log h &= 9.43749 - 10 \\
h &= 0.27384.
\end{aligned}$$

$$\cos A = \frac{\frac{1}{2}c}{a}.$$

$$\tfrac{1}{2}c = a \cos A.$$

$$\log \tfrac{1}{2}c = \log a + \log \cos A.$$

$$\begin{aligned}
\log a &= 9.46982 - 10 \\
\log \cos A &= 9.57044 \\
\log \tfrac{1}{2}c &= 9.04026 - 10 \\
\tfrac{1}{2}c &= 0.109713. \\
c &= 0.21943.
\end{aligned}$$

$$F = \tfrac{1}{2}ch.$$

$$2F = ch.$$

$$\log 2F = \log c + \log h.$$

$$\begin{aligned}
\log c &= 9.34130 - 10 \\
\log h &= 9.43749 - 10 \\
\log 2F &= 8.77879 - 10 \\
2F &= 0.060089. \\
F &= 0.03004.
\end{aligned}$$

11. In an isosceles triangle, given $c = 2.352$, $C = 69° \; 49'$; find a, h, F.

$$\tfrac{1}{2}C = 34° \; 54' \; 30''.$$

$$\sin \tfrac{1}{2}C = \frac{\frac{1}{2}c}{a}.$$

$$a = \frac{\frac{1}{2}c}{\sin \frac{1}{2}C}.$$

$$\log a = \log \tfrac{1}{2}c + \text{colog} \sin \tfrac{1}{2}C.$$

$$\log \tfrac{1}{2}c = 0.07041$$
$$\text{colog} \sin \tfrac{1}{2}C = 0.24240$$
$$\log a = 0.31281$$
$$a = 2.0555.$$

$$\cos \tfrac{1}{2}C = \frac{h}{a}.$$
$$h = a \cos \tfrac{1}{2}C.$$
$$\log h = \log a + \log \cos \tfrac{1}{2}C.$$

$$\log a = 0.31281$$
$$\log \cos \tfrac{1}{2}C = 9.91385$$
$$\log h = 0.22666$$
$$h = 1.6852.$$

$$F = \tfrac{1}{2}ch.$$
$$2F = ch.$$
$$\log 2F = \log c + \log h.$$

$$\log c = 0.37144$$
$$\log h = 0.22666$$
$$\log 2F = 0.59810$$
$$2F = 3.9637.$$
$$F = 1.9819.$$

12. In an isosceles triangle, given $h = 7.4847$, $A = 76° 14'$; find a, c, F.

$$\sin A = \frac{h}{a}.$$
$$a = \frac{h}{\sin A}.$$
$$\log a = \log h + \text{colog} \sin A.$$

$$\log h = 0.87417$$
$$\text{colog} \sin A = 0.01266$$
$$\log a = 0.88683$$
$$a = 7.706.$$

$$\tan A = \frac{h}{\tfrac{1}{2}c}.$$
$$\tfrac{1}{2}c = \frac{h}{\tan A}.$$
$$\log \tfrac{1}{2}c = \log h + \text{colog} \tan A$$

$$\log h = 0.87417$$
$$\text{colog} \tan A = 9.38918 - 10$$
$$\log \tfrac{1}{2}c = 0.26335$$
$$\tfrac{1}{2}c = 1.8338.$$
$$c = 3.6676.$$

$$F = \tfrac{1}{2}ch.$$
$$\log F = \log \tfrac{1}{2}c + \log h.$$

$$\log \tfrac{1}{2}c = 0.26335$$
$$\log h = 0.87417$$
$$\log F = 1.13752$$
$$F = 13.725.$$

13. In an isosceles triangle, given $a = 6.71$, $h = 6.60$; find A, C, c.

$$\sin A = \frac{h}{a}.$$
$$\log \sin A = \log h + \text{colog}\, a.$$

$$\log h = 0.81954$$
$$\text{colog}\, a = 9.17328 - 10$$
$$\log \sin A = 9.99282$$
$$A = 79° 36' 30''.$$
$$C = 20° 47'.$$

$$\cos A = \frac{\tfrac{1}{2}c}{a}.$$
$$\tfrac{1}{2}c = a \cos A.$$
$$\log \tfrac{1}{2}c = \log a + \log \cos A.$$

$$\log a = 0.82672$$
$$\log \cos A = 9.25617$$
$$\log \tfrac{1}{2}c = 0.08289$$
$$\tfrac{1}{2}c = 1.2103.$$
$$c = 2.4206.$$

14. In an isosceles triangle, given $c = 9$, $h = 20$; find A, c, a.

$$\tan \tfrac{1}{2} C = \frac{\frac{1}{2} c}{h}.$$

$$\log \tan \tfrac{1}{2} c = \log \tfrac{1}{2} c + \operatorname{colog} h.$$

$$\begin{aligned}
\log \tfrac{1}{2} c &= 0.65321 \\
\operatorname{colog} h &= \underline{8.69897 - 10} \\
\log \tan \tfrac{1}{2} C &= 9.35218 \\
\tfrac{1}{2} C &= 12° \, 40' \, 49''. \\
C &= 25° \, 21' \, 38''.
\end{aligned}$$

$$2A = 180° - C.$$

$$A = 77° \, 19' \, 11''.$$

$$\sin A = \frac{h}{a}.$$

$$a = \frac{h}{\sin A}.$$

$$\log a = \log h + \operatorname{colog} \sin A.$$

$$\begin{aligned}
\log h &= 1.30103 \\
\operatorname{colog} \sin A &= \underline{0.01072} \\
\log a &= 1.31175 \\
a &= 20.5.
\end{aligned}$$

15. In an isosceles triangle, given $c = 147$, $F = 2572.5$; find A, C, a, h.

$$F = \tfrac{1}{2} ch.$$

$$h = \frac{2F}{c}.$$

$$\log h = \log 2F + \operatorname{colog} c.$$

$$\begin{aligned}
\log 2F &= 3.71139 \\
\operatorname{colog} c &= \underline{7.83268 - 10} \\
\log h &= 1.54407 \\
h &= 35.
\end{aligned}$$

$$\tan A = \frac{h}{\frac{1}{2} c}.$$

$$\log \tan A = \log h + \operatorname{colog} \tfrac{1}{2} c.$$

$$\begin{aligned}
\log h &= 1.54407 \\
\operatorname{colog} \tfrac{1}{2} c &= \underline{8.13371 - 10} \\
\log \tan A &= 9.67778 \\
A &= 25° \, 28'. \\
C &= 2(90° - A) \\
&= 129° \, 4'.
\end{aligned}$$

$$a = \frac{h}{\sin A}.$$

$$\begin{aligned}
\log a &= \log h + \operatorname{colog} \sin A. \\
\log h &= 1.54407 \\
\operatorname{colog} \sin A &= \underline{0.36655} \\
\log a &= 1.91062 \\
a &= 81.41.
\end{aligned}$$

16. In an isosceles triangle, given $h = 16.8$, $F = 43.68$; find A, C, a, c.

$$F = \tfrac{1}{2} ch.$$

$$\tfrac{1}{2} c = \frac{F}{h}.$$

$$\begin{aligned}
\log \tfrac{1}{2} c &= \log F + \operatorname{colog} h. \\
\log F &= 1.64028 \\
\operatorname{colog} h &= \underline{8.77469 - 10} \\
\log \tfrac{1}{2} c &= 0.41497 \\
\tfrac{1}{2} c &= 2.60. \\
c &= 5.2.
\end{aligned}$$

$$\tan A = \frac{h}{\frac{1}{2} c}.$$

$$\log \tan A = \log h + \operatorname{colog} \tfrac{1}{2} c.$$

$$\begin{aligned}
\log h &= 1.22531 \\
\operatorname{colog} \tfrac{1}{2} c &= \underline{9.58503 - 10} \\
\log \tan A &= 10.81034 \\
A &= 81° \, 12' \, 9''. \\
\tfrac{1}{2} C &= 8° \, 47' \, 51''. \\
C &= 17° \, 35' \, 42''.
\end{aligned}$$

$$\cos A = \frac{\frac{1}{2}c}{a}.$$

$$\begin{aligned}
\log a &= \log \tfrac{1}{2}c + \operatorname{colog}\cos A.\\
\log \tfrac{1}{2}c &= 0.41497\\
\operatorname{colog}\cos A &= \underline{0.81547}\\
\log a &= 1.23044\\
a &= 17.
\end{aligned}$$

17. In an isosceles triangle, find the value of F in terms of a and c.

$$F = \tfrac{1}{2}ch.$$

$$h = \sqrt{a^2 - \frac{c^2}{4}}$$

$$= \sqrt{\frac{4a^2 - c^2}{4}}$$

$$= \tfrac{1}{2}\sqrt{4a^2 - c^2}.$$

$$F = \tfrac{1}{2}c\left(\tfrac{1}{2}\sqrt{4a^2 - c^2}\right)$$

$$= \tfrac{1}{4}c\sqrt{4a^2 - c^2}.$$

18. In an isosceles triangle, find the value of F in terms of a and C.

$$F = \tfrac{1}{2}ch.$$

$$\tfrac{1}{2}c = a \sin \tfrac{1}{2}C.$$

$$h = a \cos \tfrac{1}{2}C.$$

$$F = a \sin \tfrac{1}{2}C \times a \cos \tfrac{1}{2}C.$$

$$= a^2 \sin \tfrac{1}{2}C \cos \tfrac{1}{2}C.$$

19. In an isosceles triangle, find the value of F in terms of a and A.

$$F = \tfrac{1}{2}ch.$$

$$\tfrac{1}{2}c = a \cos A.$$

$$h = a \sin A.$$

$$F = a \cos A \times a \sin A$$

$$= a^2 \sin A \cos A.$$

20. In an isosceles triangle, find the value of F in terms of h and C.

$$F = \tfrac{1}{2}ch.$$

$$\tfrac{1}{2}c = h \tan \tfrac{1}{2}C.$$

$$F = h\,(h \tan \tfrac{1}{2}C)$$

$$= h^2 \tan \tfrac{1}{2}C.$$

21. A barn is 40×80 feet, the pitch of the roof is 45°; find the length of the rafters and the area of both sides of the roof.

$$40 \div 2 = 20 = \tfrac{1}{2}c.$$

$$\cos A = \tfrac{1}{2}c \div a$$

$$= 20 \div a.$$

$$20 = a \cos A.$$

$$a = \frac{20}{\cos A}.$$

$$\log a = \log 20 + \operatorname{colog}\cos A.$$

$$\begin{aligned}
\log 20 &= 1.30103\\
\operatorname{colog}\cos A &= \underline{0.15051}\\
\log a &= 1.45154\\
a &= 28.284.
\end{aligned}$$

$$28.284 \times 80 = 2262.72.$$

$$2262.72 \times 2 = 4525.44.$$

22. In a unit circle what is the length of the chord corresponding to the angle 45° at the centre?

$$\sin \tfrac{1}{2}C = \frac{\frac{1}{2}c}{a}.$$

$$\begin{aligned}
\log \tfrac{1}{2}c &= \log a + \log \sin \tfrac{1}{2}C.\\
\log a &= 0.00000\\
\log \sin \tfrac{1}{2}C &= \underline{9.58284}\\
\log \tfrac{1}{2}c &= 9.58284 - 10\\
\tfrac{1}{2}c &= 0.382683.\\
c &= 0.76537.
\end{aligned}$$

23. If the radius of a circle $= 30$, and the length of a chord $= 44$, find the angle at the centre.

$$\sin \tfrac{1}{2} C = \frac{\tfrac{1}{2} c}{a}.$$

$$\log \sin \tfrac{1}{2} C = \log \tfrac{1}{2} c + \text{colog}\, a.$$

$$\begin{aligned}
\log \tfrac{1}{2} c &= 1.34242 \\
\text{colog}\, a &= 8.52288 - 10 \\
\log \sin \tfrac{1}{2} C &= 9.86530 \\
\tfrac{1}{2} C &= 47^\circ\ 10'. \\
C &= 94^\circ\ 20'.
\end{aligned}$$

24. Find the radius of a circle if a chord whose length is 5 subtends at the centre an angle of 133°.

$$\sin \tfrac{1}{2} C = \frac{\tfrac{1}{2} c}{a}.$$

$$\log a = \log \tfrac{1}{2} c + \text{colog} \sin \tfrac{1}{2} C.$$

$$\begin{aligned}
\log \tfrac{1}{2} c &= 0.39794 \\
\text{colog} \sin \tfrac{1}{2} C &= 0.03760 \\
\log a &= 0.43554 \\
a &= 2.7261.
\end{aligned}$$

25. What is the angle at the centre of a circle if the corresponding chord is equal to $\tfrac{2}{3}$ of the radius?

Let $a = 3$, then $c = 2$, and $\tfrac{1}{2} c = 1$.

$$\sin \tfrac{1}{2} C = \frac{1}{3}.$$

$$\log \sin \tfrac{1}{2} C = \log 1 + \text{colog}\, 3.$$

$$\begin{aligned}
\log 1 &= 0.00000 \\
\text{colog}\, 3 &= 9.52288 - 10 \\
\log \sin \tfrac{1}{2} C &= 9.52288 \\
\tfrac{1}{2} C &= 19^\circ\ 28'\ 16\tfrac{2}{3}''. \\
C &= 38^\circ\ 56'\ 33\tfrac{1}{3}''.
\end{aligned}$$

26. Find the area of a circular sector if the radius of the circle $= 12$ and the angle of the sector $= 30°$.

Area $\odot = \pi R^2$.

$$\text{Area sector} = \frac{30\, \pi R^2}{360}.$$

log area sector = log 30 + colog 360 + log π + 2 log R.

$$\begin{aligned}
\log 30 &= 1.47712 \\
\text{colog}\, 360 &= 7.44370 - 10. \\
\log \pi &= 0.49715 \\
2 \log R &= 2.15836 \\
\log \text{area} &= 1.57633 \\
\text{Area} &= 37.699.
\end{aligned}$$

Exercise VIII. Page 26.

1. In a regular polygon given $n = 10$, $c = 1$; find r, h, F.

$$\tfrac{1}{2} C = \frac{180^\circ}{10} = 18^\circ.$$

$$\tfrac{1}{2} c = 0.5.$$

$$A = 72^\circ.$$

$$h = \tfrac{1}{2} c \tan A.$$

$$\log h = \log \tfrac{1}{2} c + \log \tan A.$$

$$\begin{aligned}
\log \tfrac{1}{2} c &= 9.69897 - 10 \\
\log \tan A &= 10.48822 \\
\log h &= 0.18719 \\
h &= 1.5388.
\end{aligned}$$

$$\log r = \log \tfrac{1}{2} c + \text{colog} \cos A.$$

$$\begin{aligned}
\log \tfrac{1}{2} c &= 9.69897 - 10 \\
\text{colog} \cos A &= 0.51002 \\
\log r &= 0.20899 \\
r &= 1.618.
\end{aligned}$$

$$F = \tfrac{1}{2} hp.$$

$$\begin{aligned}
\log h &= 0.18719 \\
\log p &= \underline{1.00000} \\
\log 2F &= 1.18719 \\
2F &= 15.388 \\
F &= 7.694.
\end{aligned}$$

2. In a regular polygon given $n = 12$, $p = 70$; find r, h, F.

$$\tfrac{1}{2} C = 15^\circ.$$

$$A = 75^\circ.$$

$$c = 70 \div 12 = 5.833.$$

$$\tfrac{1}{2} c = 2.917.$$

$$h = \tfrac{1}{2} c \tan A.$$

$$\begin{aligned}
\log \tfrac{1}{2} c &= 0.46494 \\
\log \tan A &= \underline{10.57195} \\
\log h &= 1.03689 \\
h &= 10.886.
\end{aligned}$$

$$r = \tfrac{1}{2} c \cos A.$$

$$\begin{aligned}
\log \tfrac{1}{2} c &= 0.46494 \\
\text{colog} \cos A &= \underline{0.58700} \\
\log r &= 1.05194 \\
r &= 11.27.
\end{aligned}$$

$$F = \tfrac{1}{2} hp.$$

$$\begin{aligned}
\log h &= 1.03689 \\
\log p &= \underline{1.84510} \\
\log 2F &= 2.88199 \\
2F &= 762.07. \\
F &= 381.04.
\end{aligned}$$

3. In a regular polygon given $n = 18$, $r = 1$; find h, p, F.

$$\tfrac{1}{2} C = 10^\circ.$$

$$A = 80^\circ.$$

$$h = r \sin A.$$

$$\begin{aligned}
\log r &= 0.00000 \\
\log \sin A &= \underline{9.99335} \\
\log h &= 9.99335 \\
h &= 0.9848.
\end{aligned}$$

$$\tfrac{1}{2} c = r \cos A.$$

$$\begin{aligned}
\log r &= 0.00000 \\
\log \cos A &= \underline{9.23967} \\
\log \tfrac{1}{2} c &= 9.23967 - 10 \\
\tfrac{1}{2} c &= 0.17365. \\
p &= 6.2514.
\end{aligned}$$

$$F = \tfrac{1}{2} hp.$$

$$\begin{aligned}
\log h &= 9.99335 - 10 \\
\log p &= \underline{0.79598} \\
\log 2F &= 0.78933 \\
2F &= 6.1564. \\
F &= 3.0782.
\end{aligned}$$

4. In a regular polygon given $n = 20$, $r = 20$; find h, c, F.

$$\tfrac{1}{2} C = 9^\circ.$$

$$A = 81^\circ.$$

$$h = r \sin A.$$

$$\begin{aligned}
\log r &= 1.30103 \\
\log \sin A &= \underline{9.99462} \\
\log h &= 1.29565 \\
h &= 19.754.
\end{aligned}$$

$$\tfrac{1}{2} c = r \cos A.$$

$$\begin{aligned}
\log r &= 1.30103 \\
\log \cos A &= \underline{9.19433} \\
\log \tfrac{1}{2} c &= 0.49536 \\
\tfrac{1}{2} c &= 3.1286. \\
c &= 6.257. \\
p &= 125.14.
\end{aligned}$$

$$F = \tfrac{1}{2} hp.$$

$$\begin{aligned}
\log h &= 1.29565 \\
\log p &= 2.09740 \\
\log 2F &= 3.39305 \\
2F &= 2472. \\
F &= 1236.
\end{aligned}$$

5. In a regular polygon, given $n = 8$, $h = 1$; find r, c, F.

$$\tfrac{1}{2} C = 22° \; 30'.$$

$$\tan \tfrac{1}{2} C = \frac{\frac{1}{2}c}{h}.$$

$$\log \tfrac{1}{2} c = \log h + \log \tan \tfrac{1}{2} C.$$

$$\begin{aligned}
\log h &= 0.00000 \\
\log \tan \tfrac{1}{2} C &= 9.61722 \\
\log \tfrac{1}{2} c &= 9.61722 - 10 \\
\tfrac{1}{2} c &= 0.41421. \\
c &= 0.82842.
\end{aligned}$$

$$\cos \tfrac{1}{2} C = \frac{h}{r}.$$

$$\log r = \log h + \operatorname{colog} \cos \tfrac{1}{2} C.$$

$$\begin{aligned}
\log h &= 0.00000 \\
\operatorname{colog} \cos \tfrac{1}{2} C &= 0.03438 \\
\log r &= 0.03438 \\
r &= 1.0824.
\end{aligned}$$

$$\begin{aligned}
F &= \tfrac{1}{2} hp. \\
&= 3.3137.
\end{aligned}$$

6. In a regular polygon, given $n = 11$, $F = 20$; find r, h, c.

$$2F = ph.$$

$$40 = ph.$$

$$c = \frac{p}{11}, \quad h = \frac{40}{p}.$$

$$\tfrac{1}{2} C = 16° \; 22'.$$

$$\tan \tfrac{1}{2} C = \frac{\frac{1}{2}c}{h}.$$

Substituting values of h and c,

$$\tan \tfrac{1}{2} C = \frac{p}{22} \div \frac{40}{p} = \frac{p^2}{880}.$$

$$\log p = \tfrac{1}{2} (\log 880 + \log \tan \tfrac{1}{2} C).$$

$$\begin{aligned}
\log 880 &= 2.94448 \\
\log \tan \tfrac{1}{2} C &= 9.46788 \\
&\quad 2 \overline{)2.41236} \\
\log p &= 1.20618 \\
p &= 16.076. \\
c &= 1.4615.
\end{aligned}$$

$$\sin \tfrac{1}{2} C = \frac{\frac{1}{2}c}{r}.$$

$$\log r = \log \tfrac{1}{2} c + \operatorname{colog} \sin \tfrac{1}{2} C.$$

$$\begin{aligned}
\log \tfrac{1}{2} c &= 9.86376 - 10 \\
\operatorname{colog} \sin \tfrac{1}{2} C &= 0.55008 \\
\log r &= 0.41384 \\
r &= 2.592.
\end{aligned}$$

$$\cos \tfrac{1}{2} C = \frac{h}{r}.$$

$$\log h = \log r + \log \cos \tfrac{1}{2} C.$$

$$\begin{aligned}
\log r &= 0.41384 \\
\log \cos \tfrac{1}{2} C &= 9.98204 \\
\log h &= 0.39588 \\
h &= 2.4882.
\end{aligned}$$

7. In a regular polygon, given $n = 7$, $F = 7$; find r, h, p.

$$14 = ph.$$

$$h = \frac{14}{p}.$$

$$c = \frac{p}{7}.$$

$$\tfrac{1}{2} C = 25° \; 43'.$$

$$\tan \tfrac{1}{2} C = \frac{\frac{1}{2}c}{h}.$$

$$\tan \tfrac{1}{2} C = \frac{p}{14} \div \frac{14}{p} = \frac{p^2}{196}.$$

$$\log p = \tfrac{1}{2} (\log 196 + \log \tan \tfrac{1}{2} C).$$

$\log 196$	$= 2.29226$
$\log \tan \frac{1}{2} C$	$= 9.68271$
	2) 1.97497
$\log p$	$= 0.98749$
p	$= 9.716.$
$\frac{1}{2} c$	$= 0.694.$

$$\tan \tfrac{1}{2} C = \frac{\frac{1}{2}c}{h}.$$

$$\log h = \log \tfrac{1}{2} c + \operatorname{colog} \tan \tfrac{1}{2} C.$$

$\log \frac{1}{2} c$	$= 9.84136 - 10$
$\operatorname{colog} \tan \frac{1}{2} C$	$= 0.31729$
$\log h$	$= 0.15865$
h	$= 1.441.$

$$\sin \tfrac{1}{2} C = \frac{\frac{1}{2}c}{r}.$$

$$\log r = \log \tfrac{1}{2} c + \operatorname{colog} \sin \tfrac{1}{2} C.$$

$\log \frac{1}{2} c$	$= 9.84136 - 10$
$\operatorname{colog} \sin \frac{1}{2} C$	$= 0.36259$
$\log r$	$= 0.20395$
r	$= 1.5994.$

8. Find the side of a regular decagon inscribed in a unit circle.

$$\tfrac{1}{2} C = 18°.$$

$$\sin \tfrac{1}{2} C = \frac{\frac{1}{2}c}{r}.$$

$$\log c = \log 2 + \log \sin \tfrac{1}{2} C.$$

$\log 2$	$= 0.30103$
$\log \sin \frac{1}{2} C$	$= 9.48998$
$\log c$	$= 9.79101 - 10$
c	$= 0.6181.$

9. Find the side of a regular decagon circumscribed about a unit circle.

$$\tfrac{1}{2} C = 18°.$$

$$\tan \tfrac{1}{2} C = \frac{\frac{1}{2}c}{h}.$$

$$\log \tfrac{1}{2} c = \log h + \log \tan \tfrac{1}{2} C.$$

$\log h$	$= 0.00000$
$\log \tan \frac{1}{2} C$	$= 9.51178$
$\log \frac{1}{2} c$	$= 9.51178 - 10$
$\frac{1}{2} c$	$= 0.32492.$
c	$= 0.64984.$

10. If the sides of an inscribed regular hexagon is equal to 1, find the side of an inscribed regular dodecagon.

Let O be the centre of the circle, BC a side of the hexagon, and BA a side of the dodecagon. Also let OD be $\perp$ to BA.

Then $OB = BC = 1.$

$\therefore \angle BOD = 15°.$

In rt. $\triangle ODB$

$$\sin BOD = \tfrac{1}{2} AB \div OB$$

$$AB = 2\,OB \times \sin BOD.$$

$$\log AB = \log 2\,OB + \log \sin BOD.$$

$\log 2\,OB$	$= 0.30103$
$\log \sin 15°$	$= 9.41300$
$\log AB$	$= 9.71403 - 10$
AB	$= 0.51764.$

11. Given n and c, and let b denote the side of the inscribed regular polygon having $2n$ sides; find b in terms of n and c.

Let O be the centre of the circle, BC the side of the polygon having n sides, BA the side of the polygon having $2n$ sides. Then OA is $\perp$ to BC at its middle point D.

$$\angle BOA = \frac{360°}{2n} = \frac{180°}{n}.$$

$$\angle OBC = 90° - \frac{180°}{n}.$$

The $\triangle$ BOA is isosceles.

$$\therefore \angle OBA = \frac{1}{2}\left(180° - \frac{180°}{n}\right)$$

$$= 90° - \frac{90°}{n}.$$

$$\angle ABC = \angle OBA - \angle OBC$$

$$= \left(90° - \frac{90°}{n}\right) - \left(90° - \frac{180°}{n}\right)$$

$$= \frac{90°}{n}.$$

$$\frac{\frac{1}{2}c}{b} = \cos\frac{90°}{n}. \quad \therefore \tfrac{1}{2}c = b\cos\frac{90°}{n}.$$

Whence,

$$b = \frac{\frac{1}{2}c}{\cos\frac{90°}{n}} = \frac{c}{2\cos\frac{90°}{n}}.$$

12. Compute the difference between the areas of a regular octagon and a regular nonagon if the perimeter of each is 16.

$$\tfrac{1}{2}c = \frac{p}{2n} = \frac{16}{16} = 1.$$

$$A = \frac{180°}{n} = 22°\ 30'.$$

$$\log h = \log \tfrac{1}{2}c + \log \cot A.$$

$\log \frac{1}{2}c$	$= 0.00000$
$\log \cot A$	$= 10.38278$
$\log h$	$= 0.38278$

$$\log F = \log h + \log \tfrac{1}{2}p.$$

$\log h$	$= 0.38278$
$\log \frac{1}{2}p$	$= 0.90309$
$\log F$	$= 1.28587$
F	$= 19.3139.$

$$\tfrac{1}{2}c' = \frac{p}{2n'} = \frac{16}{18} = 0.8889.$$

$$A' = \frac{180°}{n'} = 20°.$$

$$\log h' = \log \tfrac{1}{2}c' + \log \cot A'.$$

$\log \frac{1}{2}c'$	$= 9.94885 - 10$
$\log \cot A'$	$= 10.43893$
$\log h'$	$= 0.38778$

$$\log F' = \log h' + \log \tfrac{1}{2}p.$$

$\log h'$	$= 0.38778$
$\log \frac{1}{2}p$	$= 0.90309$
$\log F'$	$= 1.29087$
F'	$= 19.5377.$
$F' - F$	$= 19.5377 - 19.3139$
	$= 0.2238.$

13. Compute the difference between the perimeters of a regular pentagon and a regular hexagon if the area of each is 12.

$$F = 12, \quad n = 5.$$

$$\tfrac{1}{2}C = \frac{180°}{5} = 36°.$$

$$F = \tfrac{1}{2}hp.$$

$$h = \frac{24}{p}.$$

$$\tfrac{1}{2}c = \frac{p}{2n} = \frac{p}{10}.$$

$$\tan \tfrac{1}{2}C = \frac{\frac{1}{2}c}{h} = \frac{\frac{p}{10}}{\frac{24}{p}} = \frac{p^2}{240}.$$

$$p^2 = 240 \tan \tfrac{1}{2} C.$$

$$\begin{aligned}
\log 240 &= 2.38021 \\
\log \tan \tfrac{1}{2} C &= 9.86126 \\
&\quad 2)\underline{2.24147} \\
\log p &= 1.12074 \\
p &= 13.205.
\end{aligned}$$

$$n = 6, \quad \tfrac{1}{2} C' = 30°.$$

$$\tan \tfrac{1}{2} C' = \frac{\dfrac{p'}{12}}{\dfrac{24}{p'}} = \frac{p'^2}{288}.$$

$$p'^2 = 288 \tan \tfrac{1}{2} C'.$$

$$\begin{aligned}
\log 288 &= 2.45939 \\
\log \tan \tfrac{1}{2} C' &= 9.76144 \\
&\quad 2)\underline{2.22083} \\
\log p' &= 1.11042 \\
p' &= 12.895. \\
p - p' &= 0.310.
\end{aligned}$$

14. From a square whose side is equal to 1 the corners are cut away so that a regular octagon is left. Find the area of this octagon.

$$h = \tfrac{1}{2}.$$

$$\tfrac{1}{2} C = \frac{1}{2}\left(\frac{360°}{8}\right) = 22° \, 30'.$$

$$\begin{aligned}
A &= 90° - 22° \, 30' \\
&= 67° \, 30'.
\end{aligned}$$

$$\tan A = \frac{h}{\tfrac{1}{2} c}.$$

$$\tfrac{1}{2} c = \frac{h}{\tan A}.$$

$$\log \tfrac{1}{2} c = \log h + \operatorname{colog} \tan A.$$

$$\begin{aligned}
\log h &= 9.69897 - 10 \\
\operatorname{colog} \tan A &= 9.61722 - 10 \\
\log \tfrac{1}{2} c &= 9.31619 - 10
\end{aligned}$$

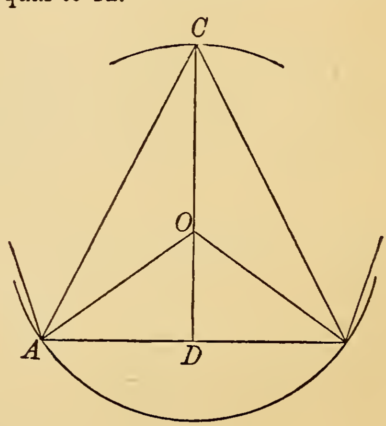
$$p = \tfrac{1}{2} c \times 2n = nc.$$

$$F = \tfrac{1}{2} ph = \tfrac{1}{2} c \times \tfrac{1}{2} n.$$

$$\log F = \log \tfrac{1}{2} c + \log \tfrac{1}{2} n.$$

$$\begin{aligned}
\log \tfrac{1}{2} c &= 9.31619 - 10 \\
\log \tfrac{1}{2} n &= 0.60206 \\
\log F &= 9.91825 - 10 \\
F &= 0.82842.
\end{aligned}$$

15. Find the area of a regular pentagon if its diagonals are each equal to 12.



$$\angle AOD = \frac{180°}{n} = 36°.$$

$$\angle AOC = 180° - 36° = 144°.$$

$$\begin{aligned}
\angle ACD &= \tfrac{1}{2}(180° - 144°) \\
&= 18° = \angle CAO.
\end{aligned}$$

$$\angle OAD = 90° - \angle AOD = 54°.$$

$$\angle DAC = 54° + 18° = 72°.$$

$$\cos DAC = \frac{AD}{AC} = \frac{\tfrac{1}{2} c}{12}.$$

$$\log \tfrac{1}{2} c = \log 12 + \log \cos 72°.$$

$$\begin{aligned}
\log \cos 72° &= 9.48998 \\
\log 12 &= 1.07918 \\
\log \tfrac{1}{2} c &= 0.56916
\end{aligned}$$

$$\tan DAO = \frac{h}{\frac{1}{2}c}.$$

$$\log h = \log \tfrac{1}{2}c + \log \tan 54°.$$

$$\begin{aligned}
\log \tfrac{1}{2}c &= 0.56916 \\
\log \tan 54° &= \underline{10.13874} \\
\log h &= 0.70790
\end{aligned}$$

$$p = \tfrac{1}{2}c \times 2n.$$

$$F = \tfrac{1}{2}ph = \tfrac{1}{2}c \times nh.$$

$$\log F = \log \tfrac{1}{2}c + \log n + \log h.$$

$$\begin{aligned}
\log \tfrac{1}{2}c &= 0.56916 \\
\log n &= 0.69897 \\
\log h &= \underline{0.70790} \\
\log F &= 1.97603 \\
F &= 94.63.
\end{aligned}$$

16. The area of an inscribed regular pentagon is 331.8; find the area of a regular polygon of 11 sides inscribed in the same circle.

Let AB be a side of a regular inscribed pentagon, and AD the side of a regular inscribed polygon of 11 sides.

Let R be the radius of the circle whose centre is O, and h and h' the apothems of the 2 polygons, respectively.

Given F the area of pentagon $= 331.8$. Find F', the area of 11-sided polygon.

Let p and p' and c and c' represent the perimeters and sides of the pentagon and 11-sided polygon, respectively.

$$F = \tfrac{1}{2}ph.$$

$$331.8 = \tfrac{1}{2}ph.$$

$$ph = 663.6.$$

$$h = \frac{663.6}{p}.$$

$$c = \frac{p}{5}.$$

$$\tfrac{1}{2}c = \frac{p}{10}.$$

$$\angle AOE = 36°.$$

$$\tan 36° = \frac{\frac{1}{2}c}{h} = \frac{p}{10} \times \frac{p}{663.6}$$

$$= \frac{p^2}{6636}.$$

$$\log p^2 = \log \tan 36° + \log 6636.$$

$$\begin{aligned}
\log 6636 &= 3.82191 \\
\log \tan 36° &= \underline{9.86126} \\
\log p^2 &= 3.68317 \\
\log p &= 1.84159
\end{aligned}$$

Since $\tfrac{1}{2}c = \tfrac{1}{10}$ of p,

$$\log \tfrac{1}{2}c = 0.84159.$$

$$\sin \angle AOE = \frac{\frac{1}{2}c}{R}.$$

$$\log R = \log \tfrac{1}{2}c + \operatorname{colog} \sin 36°.$$

$$\begin{aligned}
\log \tfrac{1}{2}c &= 0.84159 \\
\operatorname{colog} \sin 36° &= \underline{0.23078} \\
\log R &= 1.07237
\end{aligned}$$

$$\angle AOC = \frac{360°}{22} = 16°21'49''.$$

$$\sin \angle AOC = \tfrac{1}{2}c' \div R.$$

$$\begin{aligned}
\log R &= 1.07237 \\
\log \sin AOC &= \underline{9.44985} \\
\log \tfrac{1}{2}c' &= 0.52222
\end{aligned}$$

$$\tan AOC = \frac{\frac{1}{2}c'}{h'}.$$

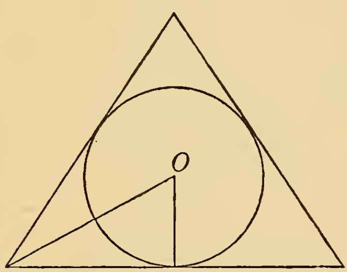
$$\log h' = \log \tfrac{1}{2}c' + \operatorname{colog} \tan AOC.$$

$$\begin{aligned}
\log \tfrac{1}{2}c' &= 0.52222 \\
\operatorname{colog} \tan AOC &= \underline{0.53220} \\
\log h' &= 1.05442
\end{aligned}$$

$$F = \tfrac{1}{2}p'h'$$
$$= \tfrac{1}{2}c' \times 11 \times h'.$$

$\log \tfrac{1}{2}c'$	$= 0.52222$
$\log 11$	$= 1.04139$
$\log h'$	$= 1.05442$
$\log F$	$= 2.61803$
F	$= 414.99.$

17. The perimeter of an equilateral triangle is 20; find the area of the inscribed circle.



$$\text{Perimeter} = 20.$$
$$AB = \tfrac{1}{6} \times 20 = \tfrac{10}{3}.$$
$$\angle OAB = \tfrac{1}{6} \text{ of } 180^\circ = 30^\circ.$$
$$\tan 30^\circ = \frac{r}{AB}.$$

$\log AB$	$= 0.52288$
$\log \tan 30^\circ$	$= 9.76144$
$\log r$	$= 0.28432$

$$\text{Area} = \pi r^2.$$

$\log \pi$	$= 0.49715$
$\log r^2$	$= 0.56864$
$\log$ area	$= 1.06579$
Area	$= 11.636.$

18. The area of a regular polygon of 16 sides, inscribed in a circle, is 100; find the area of a regular polygon of 15 sides, inscribed in the same circle.

$$\tfrac{1}{2}C = \frac{360^\circ}{32} = 11^\circ\,15'.$$
$$\tfrac{1}{2}C' = \frac{360^\circ}{30} = 12^\circ.$$

Let $AC = h,$

$$AB = r,$$
$$BC = \tfrac{1}{2}c.$$
$$F = \tfrac{1}{2}hp.$$
$$100 = \tfrac{1}{2}hp.$$
$$h = \frac{200}{p}.$$
$$\tan \tfrac{1}{2}C = \frac{\frac{p}{32}}{\frac{200}{p}}.$$
$$p^2 = 6400 \tan \tfrac{1}{2}C.$$

$\log 6400$	$= 3.80618$
$\log \tan \tfrac{1}{2}C$	$= 9.29866$
	$2)\,3.10484$
$\log p$	$= 1.55242$
p	$= 35.68.$

$$\tfrac{1}{2}c = 35.68 \div 32$$
$$= 1.115.$$
$$\sin \tfrac{1}{2}C = \frac{\frac{1}{2}c}{r} = \frac{1.115}{r}.$$

$\log 1.115$	$= 0.04727$
$\operatorname{colog} \sin \tfrac{1}{2}C$	$= 0.70976$
$\log r$	$= 0.75703$

$$\frac{h'}{r} = \cos \tfrac{1}{2}C' \; (12^\circ).$$
$$h' = r \times \cos \tfrac{1}{2}C'.$$

$\log r \quad = 0.75703$
$\log \cos \frac{1}{2} C' = 9.99040$
$\log h' \quad = 0.74743$

$$\frac{\frac{1}{2}c'}{r} = \sin \frac{1}{2} C'.$$

$$\frac{1}{2}c' = r \times \sin \frac{1}{2} C'.$$

$\log r \quad = 0.75703$
$\log \sin \frac{1}{2} C' = 9.31788$
$\log \frac{1}{2} c' \quad = 0.07491$

$$F = \frac{1}{2}\left(\frac{c'}{2} \times 2\,nh'\right).$$

$$\log F = \log \tfrac{1}{2} c' + \log n + \log h'.$$

$\log \frac{1}{2} c' \quad = 0.07491$
$\log 15 \quad = 1.17609$
$\log h' \quad = 0.74743$
$\quad\quad 1.99843$
$F \quad = 99.640.$

19. A regular dodecagon is circumscribed about a circle, the circumference of which is equal to 1; find the perimeter of the dodecagon.

Given circumference of inscribed $\odot = 1$, $n = 12$; find p.

$$2\pi r = \text{circumference}.$$

$$r = \frac{\text{circ.}}{2\pi}.$$

$$\tfrac{1}{2} C = \frac{360°}{24} = 15°.$$

$$\tan 15° = \frac{\frac{1}{2}c}{r} = \pi c.$$

$$c = \frac{\tan 15°}{3.1416}.$$

$\log \tan 15° = 9.42805$
$\text{colog } 3.1416 = 9.50284 - 10$
$\log c \quad = 8.93089 - 10$
$\log 12 \quad = 1.07918$
$\log p \quad = 0.01007$
$p \quad = 1.0235.$

20. The area of a regular polygon of 25 sides is equal to 40; find the area of the ring comprised between the circumferences of the inscribed and the circumscribed circles.

$$\tfrac{1}{2} ch = \frac{40}{25} = 1.6.$$

$$\tfrac{1}{2} C = 7° \, 12'.$$

$$A = \frac{360°}{2n} = 7° \, 12'.$$

$$\frac{\frac{1}{2}c}{h} = \tan \tfrac{1}{2} C,$$

or
$$\frac{\frac{1}{2}ch}{h^2} = \tan \tfrac{1}{2} C.$$

$$h^2 = \frac{1.6}{\tan \frac{1}{2} C}.$$

$\log 1.6 \quad = 0.20412$
$\text{colog } \tan \frac{1}{2} C = 0.89850$
$\log h^2 \quad = 1.10262$
$\log h \quad = 0.55131.$

$$\frac{h}{r} = \cos \tfrac{1}{2} C.$$

$$r = \frac{h}{\cos \frac{1}{2} C}.$$

$\log h \quad = 0.55131$
$\text{colog } \cos \frac{1}{2} C = 0.00344$
$\log r \quad = 0.55475$
$\log r^2 \quad = 1.10950.$

πr^2 = area of circumscribed $\odot$.

$\log \pi \quad = 0.49715$
$\log r^2 \quad = 1.10950$
$\log F \quad = 1.60665$
$F \quad = 40.425.$
$\log \pi \quad = 0.49715$
$\log h^2 \quad = 1.10262$
$\log \pi h^2 \quad = 1.59977$
Area $= 39.790$, (inscrib'd $\odot$)
$40.425 - 39.790 = 0.635.$

Exercise IX. Page 36.

1. Construct the functions of an angle in Quadrant II. What are their signs?

Sines and tangents extending upwards from horizontal diameter are positive; downwards, negative. Cosines and cotangents extending from vertical diameter towards the right are positive; towards the left, negative. Signs of secant and cosecant are made to agree with cosine and sine, respectively. Hence,

sin and csc are +

cos and sec are −

tan and cot are −

2. Construct the functions of an angle in Quadrant III. What are their signs?

sin and csc are −

cos and sec are −

tan and cot are +

3. Construct the functions of an angle in Quadrant IV. What are their signs?

sin and csc are −

cos and sec are +

tan and cot are −

4. What are the signs of the functions of the following angles: 340°, 239°, 145°, 400°, 700°, 1200°, 3800°?

340° is in Quadrant IV.

sin = −	tan = −	sec = +
cos = +	cot = −	csc = −

239° is in Quadrant III.

sin = −	tan = +	sec = −
cos = −	cot = +	csc = −

145° is in Quadrant II.

sin = +	tan = −	sec = −
cos = −	cot = −	csc = +

$400 = 360° + 40°$ = signs of functions of 40°.

40° is in Quadrant I.

sin = +	tan = +	sec = +
cos = +	cot = +	csc = +

$700° = 360° + 340°$ = signs of the functions of 340°.

340° is in Quadrant IV.

sin = −	tan = −	sec = +
cos = +	cot = −	csc = −

$1200° = 3 \times 360° + 120°$ = signs of the functions of 120°.

120° is in Quadrant II.

sin = +	tan = −	sec = −
cos = −	cot = −	csc = +

$3800° = 10 \times 360° + 200°$ = signs of the functions of 200°.

200° is in Quadrant III.

sin = −	tan = +	sec = −
cos = −	cot = +	csc = −

5. How many angles less than 360° have the value of the sine equal to $+\frac{5}{7}$, and in what quadrants do they lie?

Since the sine is +, by § 24, the angles can lie in but two quadrants, the first and second.

In the first quadrant, by § 3, the sine increases from 0 to 1, and in the second, decreases from 1 to 0. This is a continually increasing and decreasing quantity.

Therefore there can be but one angle whose sine is equal to $+\frac{5}{7}$ in each quadrant, the first and second.

6. How many values less than 720° can the angle x have if $\cos x = +\frac{3}{5}$, and in what quadrants do they lie?

720° is twice 360°; hence the moving radius will make exactly 2 complete revolutions.

The cosine has the + sign in the first and fourth quadrants, hence it will have four values: two in Quadrant I. and two in Quadrant IV.

7. If we take into account only angles less than 180°, how many values can x have if $\sin x = \frac{5}{7}$? if $\cos x = \frac{1}{5}$? if $\cos x = -\frac{4}{5}$? if $\tan x = \frac{2}{3}$? if $\cot x = -7$?

(i.) Sign being +, the angle can be in Quadrant I. or II.

∴ two values, one in Quadrant I. and one in Quadrant II.

(ii.) Sign being +, the angle is in Quadrant I. or IV.

∴ two values, one in Quadrant I. and one in Quadrant IV.

(iii.) Sign being −, the angle can be in Quadrant II. or III.

∴ two values, one in Quadrant II. and one in Quadrant III.

(iv.) Sign being +, the angle can be in Quadrant I. or III.

∴ two values, one in Quadrant I. and one in Quadrant III.

(v.) Sign being −, the angle can be in Quadrant II. or IV.

∴ two values, one in Quadrant II. and one in Quadrant IV.

8. Within what limits must the angle x lie if $\cos x = -\frac{2}{9}$? if $\cot x = 4$? if $\sec x = 80$? if $\csc x = -3$? (x to be less than 360°.)

If $\cos x = -\frac{2}{9}$, x must lie in the second or third quadrant, or between 90° and 270°.

If $\cot x = 4$, x is between 0° and 90°, or 180° and 270°.

If $\sec x = 80$, x is between 0° and 90° or 270° and 360°.

If $\csc x = -3$, x is between 180° and 360°.

9. In what quadrant does an angle lie if sine and cosine are both negative? if cosine and tangent are both negative? if the cotangent is positive and the sine negative?

(i.) Sine is negative in Quadrants II. and III.; cosine is negative in Quadrants III. and IV.

∴ angles having both sine and cosine negative are in Quadrant III.

(ii.) Cosine is negative in Quadrants II. and III.; tangent is negative in quadrants II. and IV.

∴ angles having both cosine and tangent negative are in Quadrant II.

(iii.) Cotangent is positive in Quadrants I. and III.; sine is negative in Quadrants III. and IV.

∴ angles having cotangent positive and sine negative are in Quadrant III.

10. Between 0° and 3600° how many angles are there whose sines have the absolute value $\frac{3}{5}$? Of these sines how many are positive and how many negative?

Between 0° and 3600° there are 10 revolutions, and in each there are 4 angles whose sines have the absolute value $\frac{3}{5}$. $\therefore$ there are 40 angles. The sine is positive in Quadrants I. and II., and negative ing Quadrants III. and IV. $\therefore$ there are 20 angles with the sine positive, and 20 with the sine negative.

11. In finding $\cos x$ by means of the equation $\cos x = \pm\sqrt{1-\sin^2 x}$, when must we choose the positive sign and when the negative sign?

Since cosines only of angles in Quadrants I. or IV. are positive, we use the sign + only when angle x lies within these limits.

Also, since cosines of angles in Quadrants II. and III. are negative, we use the sign $-$, when x is known to lie in either of these.

12. Given $\cos x = -\sqrt{\frac{1}{2}}$; find the other functions when x is an angle in Quadrant II.

$$\sin^2 x + \cos^2 x = 1.$$

$$\sin x = \sqrt{1-\cos^2 x}$$

$$= \sqrt{1-(-\sqrt{\tfrac{1}{2}})^2} = \sqrt{\tfrac{1}{2}}.$$

$$\csc x = \frac{1}{\sin x} = \frac{1}{\sqrt{\frac{1}{2}}} = \sqrt{2}.$$

$$\sec x = \frac{1}{\cos x} = \frac{1}{-\sqrt{\frac{1}{2}}} = -\sqrt{2}.$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\sqrt{\frac{1}{2}}}{-\sqrt{\frac{1}{2}}} = -1.$$

$$\cot x = \frac{1}{\tan x} = \frac{1}{-1} = -1.$$

13. Given $\tan x = \sqrt{3}$; find the other functions when x is an angle in Quadrant III.

$$\tan x = \sqrt{3}.$$

$$\cot x = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3}.$$

$$\tan x = \frac{\sin x}{\cos x}.$$

$$\tan x \times \cos x = \sin x.$$

$$\sqrt{3}\cos x = \sin x.$$

$$\begin{aligned} 3\cos^2 x - \sin^2 x &= 0 \\ \cos^2 x + \sin^2 x &= 1 \\ \hline 4\cos^2 x &= 1 \\ \cos^2 x &= \tfrac{1}{4} \\ \cos x &= \pm\tfrac{1}{2}. \end{aligned}$$

The angle being in Quadrant III. the cosine is negative.

$$\therefore \cos x = -\tfrac{1}{2}.$$

$$\sin x = \sqrt{1-(-\tfrac{1}{2})^2}$$

$$= \sqrt{\tfrac{3}{4}} = \pm\tfrac{1}{2}\sqrt{3}.$$

Sine is negative.

$$\therefore \sin x = -\tfrac{1}{2}\sqrt{3}.$$

$$\sec x = \frac{1}{-\frac{1}{2}} = -2.$$

$$\csc x = \frac{1}{-\frac{1}{2}\sqrt{3}} = -\tfrac{2}{3}\sqrt{3}.$$

14. Given $\sec x = +7$, and $\tan x$ negative; find the other functions of x.

x must be in Quadrant IV.

$\therefore$ sine, cosine, tangent, and cotangent will be negative, and cosine positive.

$$\cos x = \frac{1}{\sec x} = \frac{1}{7}.$$

$$\sin x = \pm\sqrt{1 - \frac{1}{49}} = \pm\sqrt{\frac{48}{49}}$$

$$= -\tfrac{4}{7}\sqrt{3}.$$

$$\csc x = \frac{1}{\sin x} = \frac{1}{-\frac{4}{7}\sqrt{3}}$$

$$= -\tfrac{7}{12}\sqrt{3}.$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{-\frac{4}{7}\sqrt{3}}{\frac{1}{7}}$$

$$= -4\sqrt{3}.$$

$$\cot x = \frac{1}{\tan x} = -\frac{1}{4\sqrt{3}}$$

$$= \tfrac{1}{12}\sqrt{3}.$$

15. Given $\cot x = -3$; find all the possible values of the other functions.

By [3] $\tan x = -\frac{1}{3}$, and may be in Quadrant II. or IV.

By [1],

$$\sin^2 x = 1 - \cos^2 x.$$

$$\sin x = \sqrt{1 - \cos^2 x}.$$

By [2],

$$\frac{1}{3} = \frac{\sqrt{1 - \cos^2 x}}{\cos x}.$$

$$\frac{1}{9} = \frac{1 - \cos^2 x}{\cos^2 x}.$$

$$\cos^2 x = 9 - 9\cos^2 x.$$

$$\cos^2 x = \frac{9}{10}.$$

$$\cos x = \frac{3}{\sqrt{10}} = \tfrac{3}{10}\sqrt{10},$$

and is $-$ in Quadrant II., $+$ in IV.

By [1],

$$\sin x = \sqrt{1 - \frac{9}{10}} = \sqrt{\frac{1}{10}}$$

$$= \tfrac{1}{10}\sqrt{10},$$

and is $+$ in Quadrant II., $-$ in IV.

$$\sec x = \frac{\sqrt{10}}{3} = \tfrac{1}{3}\sqrt{10}.$$

$$\csc x = \sqrt{10}.$$

16. What functions of an angle of a triangle may be negative? In what case are they negative?

When an angle of a triangle is acute, its functions are all positive. When an angle is obtuse, its functions are those of an angle in Quadrant II.

∴ sine and cosecant are positive, and cosine, tangent, cotangent, and secant are negative.

17. What functions of an angle of a triangle determine the angle, and what functions fail to do so?

The sine and cosecant being positive in the first and second quadrant, leave it doubtful whether the angle is obtuse or acute; but the other functions, if positive, determine an angle in the first quadrant, that is to say, an acute angle; if negative, an angle in the second quadrant, an obtuse angle.

18. Why may $\cot 360°$ be considered equal either to $+\infty$ or to $-\infty$?

The nearer an acute angle is to 0°, the greater the positive value of its cotangent; and the nearer an angle is to 360°, the greater the negative value of its cotangent. When the angle is 0° or 360°, cotangent is parallel to the horizontal diameter and cannot meet it. But cotangent 360° may be regarded as extending either in the positive or in the negative direction; and hence either $+\infty$ or $-\infty$.

19. Obtain by means of Formulas [1]-[3] the other functions of the angles given:

(i.) $\tan 90° = \infty$.

(ii.) $\cos 180° = -1$.

(iii.) $\cot 270° = 0$.

(iv.) $\csc 360° = -\infty$.

(i.)

$$\tan 90° = \infty = \frac{1}{0}.$$

$$\cot 90° = \frac{1}{\infty} = 0.$$

$$\frac{\sin 90°}{\cos 90°} = \frac{1}{0}.$$

$$\cos 90° = 0 \sin 90° = 0.$$

$$\cos^2 90° + \sin^2 90° = 1.$$

$$\sin^2 90° = 1.$$

$$\sin 90° = 1.$$

(ii.)

$$\cos 180° = -1.$$

$$\sin^2 180° + \cos^2 180° = 1.$$

$$\sin^2 180° + 1 = 1.$$

$$\sin 180° = 0.$$

$$\tan 180° = \frac{\sin 180°}{\cos 180°} = \frac{0}{-1}$$

$$= -0.$$

$$\cot 180° = \frac{\cos 180°}{\sin 180°} = \frac{-1}{0}$$

$$= -\infty.$$

(iii.)

$$\cot 270° = 0.$$

$$\tan 270° = \frac{1}{0} = \infty.$$

$$\frac{\cos 270°}{\sin 270°} = 0.$$

$$\cos 270° = 0 \sin 270° = 0.$$

$$\sin^2 270° + \cos^2 270° = 1.$$

$$\sin^2 270° + 0 = 1.$$

$$\sin^2 270° = 1.$$

$$\sin 270° = -1.$$

(iv.)

$$\csc 360° = -\infty.$$

$$\sin 360° = \frac{1}{-\infty} = -0.$$

$$\sin^2 360° + \cos^2 360° = 1.$$

$$\cos^2 360° = 1.$$

$$\cos 360° = 1.$$

$$\tan 360° = \frac{-0}{1} = -0.$$

$$\cot 360° = \frac{1}{-0} = -\infty.$$

20. Find the values of $\sin 450°$, $\tan 540°$, $\cos 630°$, $\cot 720°$, $\sin 810°$, $\csc 900°$.

$$\begin{aligned}\sin 450° &= \sin(360° + 90°)\\ &= \sin 90°\\ &= 1.\end{aligned}$$

$$\begin{aligned}\tan 540° &= \tan(360° + 180°)\\ &= \tan 180°\\ &= 0.\end{aligned}$$

$$\begin{aligned}\cos 630° &= \cos(360° + 270°)\\ &= \cos 270°\\ &= 0.\end{aligned}$$

$$\cot 720° = \cot(360° + 360°)$$
$$= \cot 360°$$
$$= \infty.$$

$$\sin 810° = \sin(2 \times 360° + 90°)$$
$$= \sin 90°$$
$$= 1.$$

$$\csc 900° = \csc(2 \times 360° + 180°)$$
$$= \csc 180°$$
$$= \infty.$$

21. For what angle in each quadrant are the absolute values of the sine and cosine alike?

The sine and cosine of 45° are equal in absolute value. Corresponding to the angle of 45° in the first quadrant are the angles (90° + 45°), (180° + 45°), (270° + 45°) in the second, third, and fourth quadrants. Hence the sines and cosines of 45°, 135°, 225°, 315°, etc., are all equal in absolute value.

22. Compute the value of

$a \sin 0° + b \cos 90° - c \tan 180°$.

$$\sin 0° = 0.$$
$$\cos 90° = 0.$$
$$\tan 180° = 0.$$

Substituting,

$$a \times 0 + b \times 0 - c \times 0 = 0.$$

23. Compute the value of

$a \cos 90° - b \tan 180° + c \cot 90°$.

$$\cos 90° = 0.$$
$$\tan 180° = 0.$$
$$\cot 90° = 0.$$

Substituting,

$$a \times 0 - b \times 0 + c \times 0 = 0.$$

24. Compute the value of

$a \sin 90° - b \cos 360° + (a - b) \cos 180°$.

$$\sin 90° = 1.$$
$$\cos 360° = 1.$$
$$\cos 180° = -1.$$

Substituting,

$$a \times 1 - b \times 1 + (a - b) \times -1 = 0.$$

25. Compute the value of

$(a^2 - b^2) \cos 360° - 4ab \sin 270°$.

$$\cos 360° = 1.$$
$$\sin 270° = -1.$$

Substituting,

$$(a^2 - b^2) \times 1 - 4ab \times -1$$
$$= a^2 - b^2 + 4ab.$$

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2. Express sin 172° in terms of the functions of angles less than 45°.

$$\sin 172° = \sin(180° - 8°)$$
$$= \sin 8°.$$

3. Express cos 100° in terms of the functions of angles less than 45°.

$$\cos 100° = \cos(90° + 10°)$$
$$= -\sin 10°.$$

4. Express tan 125° in terms of the functions of angles less than 45°.

$$\tan 125° = \tan(90° - 35°)$$
$$= -\cot 35°.$$

5. Express cot 91° in terms of the functions of angles less than 45°.

$$\cot 91^\circ = \cot(90^\circ + 1^\circ) = -\tan 1^\circ.$$

6. Express sec 110° in terms of the functions of angles less than 45°.

$$\sec 110^\circ = \sec(90^\circ + 20^\circ) = -\csc 20^\circ.$$

7. Express csc 157° in terms of the functions of angles less than 45°.

$$\csc 157^\circ = \csc(180^\circ - 23^\circ) = \csc 23^\circ.$$

8. Express sin 204° in terms of the functions of angles less than 45°.

$$\sin 204^\circ = \sin(180^\circ + 24^\circ) = -\sin 24^\circ.$$

9. Express cos 359° in terms of the functions of angles less than 45°.

$$\cos 359^\circ = \cos(360^\circ - 1^\circ) = \cos 1^\circ.$$

10. Express tan 300° in terms of the functions of angles less than 45°.

$$\tan 300^\circ = \tan(270^\circ + 30^\circ) = -\cot 30^\circ.$$

11. Express cot 264° in terms of the functions of angles less than 45°.

$$\cot 264^\circ = \cot(270^\circ - 6^\circ) = \tan 6^\circ.$$

12. Express sec 244° in terms of the functions of angles less than 45°.

$$\sec 244^\circ = \sec(270^\circ - 26^\circ) = -\csc 26^\circ.$$

13. Express csc 271° in terms of the functions of angles less than 45°.

$$\csc 271^\circ = \csc(270^\circ - 1^\circ) = -\sec 1^\circ.$$

14. Express sin 163° 49′ in terms of the functions of angles less than 45°.

$$\sin 163^\circ 49' = \sin(180^\circ - 16^\circ 11') = \sin 16^\circ 11'.$$

15. Express cos 195° 33′ in terms of the functions of angles less than 45°.

$$\cos 195^\circ 33' = \cos(180^\circ + 15^\circ 33') = -\cos 15^\circ 33'.$$

16. Express tan 269° 15′ in terms of the functions of angles less than 45°.

$$\tan 269^\circ 15' = \tan(270^\circ - 45') = \cot 45'.$$

17. Express cot 139° 17′ in terms of the functions of angles less than 45°.

$$\cot 139^\circ 17' = \cot(180^\circ - 40^\circ 43') = -\cot 40^\circ 43'.$$

18. Express sec 299° 45′ in terms of the functions of angles less than 45°.

$$\sec 299^\circ 45' = \sec(270^\circ + 29^\circ 45') = \csc 29^\circ 45'.$$

19. Express csc 92° 25′ in terms of the functions of angles less than 45°.

$$\csc 92^\circ 25' = \csc(90^\circ + 2^\circ 25') = \sec 2^\circ 25'.$$

20. Express all the functions of -75° in terms of those of positive angles less than 45°.

$\sin(-75°) = \sin(270° + 15°)$
$= -\cos 15°.$
$\cos(-75°) = \cos(270° + 15°)$
$= \sin 15°.$
$\tan(-75°) = \tan(270° + 15°)$
$= -\cot 15°.$
$\cot(-75°) = \cot(270° + 15°)$
$= -\tan 15°.$

21. Express all the functions of $-127°$ in terms of those of positive angles less than 45°.

$\sin(-127°) = \sin(270° - 37°)$
$= -\cos 37°.$
$\cos(-127°) = \cos(270° - 37°)$
$= -\sin 37°.$
$\tan(-127°) = \tan(270° - 37°)$
$= \cot 37°.$
$\cot(-127°) = \cot(270° - 37°)$
$= \tan 37°.$

22. Express all the functions of $-200°$ in terms of those of positive angles less than 45°.

$\sin(-200°) = \sin(180° - 20°)$
$= \sin 20°.$
$\cos(-200°) = \cos(180° - 20°)$
$= -\cos 20°.$
$\tan(-200°) = \tan(180° - 20°)$
$= -\tan 20°.$
$\cot(-200°) = \cot(180° - 20°)$
$= -\cot 20°.$

23. Express all the functions of $-345°$ in terms of those of positive angles less than 45°.

$\sin(-345°) = \sin 15°$, etc.

24. Express all the functions of $-52° 37'$ in terms of those of positive angles less than 45°.

$\sin(-52° 37') = \sin(270° - 37° 23')$
$= -\cos 37° 23'.$
$\cos(-52° 37') = \cos(270° + 37° 23')$
$= \sin 37° 23'.$
$\tan(-52° 37') = \tan(270° + 37° 23')$
$= -\cot 37° 23'.$
$\cot(-52° 37') = \cot(270° + 37° 23')$
$= -\tan 37° 23'.$

25. Express all the functions of $-196° 54'$ in terms of those of positive angles less than 45°.

$\sin(-196° 54') = \sin(180° - 16° 54')$
$= \sin 16° 54'.$
$\cos(-196° 54') = \cos(180° - 16° 54')$
$= -\cos 16° 54'.$
$\tan(-196° 54') = \tan(180° - 16° 54')$
$= -\tan 16° 54'.$
$\cot(-196° 54') = \cot(180° - 16° 54')$
$= -\cot 16° 54'.$

26. Find the functions of 120°.

$\sin 120° = \sin(90° + 30°) = \cos 30°$
$= \frac{1}{2}\sqrt{3}.$
$\cos 120° = \cos(90° + 30°)$
$= -\sin 30° = -\frac{1}{2}.$
$\tan 120° = \tan(90° + 30°)$
$= -\tan 30° = -\sqrt{3}.$
$\cot 120° = \cot(90° + 30°)$
$= -\cot 30° = -\frac{1}{3}\sqrt{3}.$
$\sec 120° = -2.$
$\csc 120° = \frac{2}{3}\sqrt{3}.$

27. Find the functions of 135°.

$$\sin 135° = \sin(90° + 45°) = \cos 45° = \tfrac{1}{2}\sqrt{2}.$$

$$\cos 135° = \cos(90° + 45°) = -\sin 45° = -\tfrac{1}{2}\sqrt{2}.$$

$$\tan 135° = \frac{\sin 135°}{\cos 135°} = -1.$$

$$\cot 135° = \frac{\cos 135°}{\sin 135°} = -1.$$

$$\sec 135° = \frac{1}{\cos 135°} = -\sqrt{2}.$$

$$\csc 135° = \frac{1}{\sin 135°} = \sqrt{2}.$$

28. Find the functions of 150°.

$$\sin 150° = \sin(180° - 30°) = \sin 30° = \tfrac{1}{2}.$$

$$\cos 150° = \cos(180° - 30°) = -\cos 30° = -\tfrac{1}{2}\sqrt{3}.$$

$$\tan 150° = \tan(180° - 30°) = -\tan 30° = \frac{\sin 30°}{-\cos 30°} = -\tfrac{1}{3}\sqrt{3}.$$

$$\cot 150° = \cot(180° - 30°) = -\cot 30° = \frac{-\cos 30°}{\sin 30°} = -\sqrt{3}.$$

$$\sec 150° = \sec(180° - 30°) = -\sec 30° = \frac{1}{-\cos 30°} = -\tfrac{2}{3}\sqrt{3}.$$

$$\csc 150° = \csc(180° - 30°) = \csc 30° = \frac{1}{\sin 30°} = 2.$$

29. Find the functions of 210°.

$$\sin 210° = \sin(180° + 30°) = -\sin 30° = -\tfrac{1}{2}.$$

$$\cos 210° = \cos(180° + 30°) = -\cos 30° = -\tfrac{1}{2}\sqrt{3}.$$

$$\tan 210° = \tan(180° + 30°) = \tan 30° = \tfrac{1}{3}\sqrt{3}.$$

$$\cot 210° = \cot(180° + 30°) = \cot 30° = \sqrt{3}.$$

30. Find the functions of 225°.

$$\sin 225° = \sin(180° + 45°) = -\sin 45° = -\tfrac{1}{2}\sqrt{2}.$$

$$\cos 225° = \cos(180° + 45°) = -\cos 45° = -\tfrac{1}{2}\sqrt{2}.$$

$$\tan 225° = \tan(180° + 45°) = \tan 45° = 1.$$

$$\cot 225° = \cot(180° + 45°) = \cot 45° = 1.$$

31. Find the functions of 240°.

$$\sin 240° = \sin(270° - 30°) = -\cos 30° = -\tfrac{1}{2}\sqrt{3}.$$

$$\cos 240° = \cos(270° - 30°) = -\sin 30° = -\tfrac{1}{2}.$$

$$\tan 240° = \tan(270° - 30°) = \cot 30° = \sqrt{3}.$$

$$\cot 240° = \cot(270° - 30°) = \tan 30° = \tfrac{1}{3}\sqrt{3}.$$

32. Find the functions of 300°.

$$\sin 300° = \sin(270° + 30°) = -\cos 30° = -\tfrac{1}{2}\sqrt{3}.$$

$$\cos 300° = \cos(270° + 30°) = \sin 30° = \tfrac{1}{2}.$$

$$\tan 300° = \tan(270° + 30°)$$
$$= -\cot 30° = -\sqrt{3}.$$
$$\cot 300° = \cot(270° + 30°)$$
$$= -\tan 30° = -\tfrac{1}{3}\sqrt{3}.$$

33. Find the functions of $-30°$.

$\sin -30° = -\sin 30° = -\frac{1}{2}$.

$\cos -30° = \cos 30° = \frac{1}{2}\sqrt{3}$.

$\tan -30° = -\tan 30° = -\frac{1}{3}\sqrt{3}$.

$\cot -30° = -\cot 30° = -\sqrt{3}$.

$\sec -30° = \sec 30° = \frac{2}{3}\sqrt{3}$.

$\csc -30° = -\csc 30° = -2$.

34. Find the functions of $-225°$.

$-225° = 90° + 45°$.

$$\sin -225° = \sin(90° + 45°)$$
$$= \cos 45° = \tfrac{1}{2}\sqrt{2}.$$
$$\cos -225° = \cos(90° + 45°)$$
$$= -\sin 45° = -\tfrac{1}{2}\sqrt{2}.$$
$$\tan -225° = \tan(90° + 45°)$$
$$= -\cot 45° = -1.$$
$$\cot -225° = \cot(90° + 45°)$$
$$= -\tan 45° = -1.$$
$$\sec -225° = \frac{1}{\cos(90° + 45°)}$$
$$= -\sqrt{2}.$$
$$\csc -225° = \frac{1}{\sin(90° + 45°)} = \sqrt{2}.$$

35. Given $\sin x = -\sqrt{\frac{1}{2}}$, and $\cos x$ negative; find the other functions of x, and the value of x.

Since $\sin 45° = \sqrt{\frac{1}{2}}$, and the signs of both the sine and cosine are negative, the angle must be in Quadrant III., and must be, therefore,

$180° + 45° = 225°$.

Then $\cos 45° = \sqrt{\frac{1}{2}}$.

Hence $\cos(180° + 45°) = -\sqrt{\frac{1}{2}}$.

$$\tan(180° + 45°) = \frac{\sin 225°}{\cos 225°}$$
$$= \frac{-\sqrt{\frac{1}{2}}}{-\sqrt{\frac{1}{2}}} = 1.$$
$$\cot(180° + 45°) = \frac{1}{\tan 225°} = 1.$$
$$\sec 225° = \frac{1}{\cos 225°} = \frac{1}{-\sqrt{\frac{1}{2}}}$$
$$= -\sqrt{2}.$$
$$\csc 225° = \frac{1}{\sin 225°} = \frac{1}{-\sqrt{\frac{1}{2}}}$$
$$= -\sqrt{2}.$$

36. Given $\cot x = -\sqrt{3}$, and x in Quadrant II.; find the other functions of x, and the value of x.

Since $\cot 30° = \sqrt{3}$, and the sign is negative, the angle is in Quadrant II.

$$\tan x = \frac{1}{\cot x} = \frac{1}{-\sqrt{3}} = -\tfrac{1}{3}\sqrt{3}.$$
$$\frac{\sin x}{\cos x} = -\tfrac{1}{3}\sqrt{3}.$$
$$\sin x = -\tfrac{1}{3}\sqrt{3}\cos x.$$
$$\sin^2 x = \tfrac{1}{3}\cos^2 x.$$

But $\sin^2 x + \cos^2 x = 1$,

$\therefore\ \frac{1}{3}\cos^2 x + \cos^2 x = 1$;

and $\cos^2 x = \dfrac{3}{4}$,

$\therefore\ \cos x = \frac{1}{2}\sqrt{3}$;

and $\sin^2 x = \dfrac{1}{4}$,

$\therefore\ \sin x = \dfrac{1}{2}$.

$$\sec x = \frac{1}{\cos x} = \tfrac{2}{3}\sqrt{3}.$$
$$\csc x = \frac{1}{\sin x} = 2.$$

37. Find the functions of 3540°.

$3540° = 9 \times 360° + 300°.$

$$\sin 300° = \sin(360° - 60°)$$
$$= \sin 60° = -\tfrac{1}{2}\sqrt{3}.$$
$$\cos 300° = \cos(360° - 60°)$$
$$= \cos 60° = \frac{1}{2}.$$
$$\tan 300° = \frac{\sin 300°}{\cos 300°} = \frac{-\frac{1}{2}\sqrt{3}}{\frac{1}{2}}$$
$$= -\sqrt{3}.$$
$$\cot 300° = \frac{1}{\tan 300°} = \frac{1}{-\sqrt{3}}$$
$$= -\tfrac{1}{3}\sqrt{3}.$$
$$\sec 300° = \frac{1}{\cos 300°} = \frac{1}{\frac{1}{2}} = 2.$$
$$\csc 300° = \frac{1}{\sin 300°} = \frac{1}{-\frac{1}{2}\sqrt{3}}$$
$$= -\tfrac{2}{3}\sqrt{3}.$$

38. What angles less than 360° have a sine equal to $-\frac{1}{2}$? a tangent equal to $-\sqrt{3}$?

(i.) Since $\sin 30° = \frac{1}{2}$ and the sign is negative, the angle must be in Quadrant III. or IV., and must be therefore $180° + 30° = 210°$, or $360° - 30° = 330°$.

(ii.) Since $\tan 60° = \sqrt{3}$ and the sign is negative, the angle must be in Quadrant II. or IV., and must be therefore $180° - 60° = 120°$, or $360° - 60° = 300°$.

39. Which of the angles mentioned in Examples 27–34 have a cosine equal to $-\sqrt{\frac{1}{2}}$? a cotangent equal to $-\sqrt{3}$?

(i.) Since $\cos 45° = \sqrt{\frac{1}{2}}$ and the sign is negative, the angle must be in Quadrant II. or III., and must be therefore $180° - 45° = 135°$, or $180° + 45° = 225°$. Also, the functions of $-225°$ are the same as the functions of $360° - 225° = 135°$. Hence the angles are 135°, 225°, or $-225°$.

(ii.) Since $\cot 30° = \sqrt{3}$ and the sign is negative, the angle must be in Quadrant II. or IV., and must be therefore $180° - 30° = 150°$, or $360° - 30° = 330°$, or $-30°$. Hence the angles are 150° or $-30°$.

40. What values of x between 0° and 720° will satisfy the equation $\sin x = +\frac{1}{2}$?

Since $\sin 30° = \frac{1}{2}$ and the sign is positive, the angle must be in Quadrant I. or II., and must be therefore 30°, or $180° - 30° = 150°$, the first revolution. In the second revolution, these angles must be increased by 360°. Hence the angles are 30°, 150°, 390°, and 510°.

41. In each of the following cases find the other angle between 0° and 360° for which the corresponding function (sign included) has the same value: sin 12°, cos 26°, tan 45°, cot 72°; sin 191°, cos 120°, tan 244°, cot 357°.

In order that the sign shall be the same,

sin 12° must be in Quadrant II.

$$= \sin(180° - 12°) = \sin 168°.$$

cos 26° must be in Quadrant IV.

$$= \cos(360° - 26°) = \cos 334°.$$

tan 45° must be in Quadrant III.

$$= \tan(180° + 45°) = \tan 225°.$$

cot 72° must be in Quadrant III.

$= \cot(180° + 72°) = \cot 252°.$

sin 191° must be in Quadrant IV.

$= \sin(360° - 11°) = \sin 349°.$

cos 120° must be in Quadrant III.

$= \cos(180° + 60°) = \cos 240°.$

tan 244° must be in Quadrant I.

$= \tan(244° - 180°) = \tan 64°.$

cot 357° must be in Quadrant II.

$= \cot(357° - 180°) = \cot 177°.$

42. Given $\tan 238° = 1.6$; find $\sin 122°$.

$$\tan 238° = \tan(180° + 58°) = \tan 58°.$$

$$\sin 122° = \sin(180° - 58°) = \sin 58°.$$

But $\tan 238° = 1.6$.

$\therefore \quad \tan 58° = 1.6.$

$$\tan 58° = \frac{\sin 58°}{\cos 58°}.$$

$$1.6 = \frac{\sin 58°}{\sqrt{1 - \sin^2 58°}}.$$

$$2.56 - 2.56 \sin^2 58° = \sin^2 58°.$$

$$3.56 \sin^2 58° = 2.56.$$

$$\sin 58° = \sqrt{\frac{2.56}{3.56}} = 0.848.$$

43. Given $\cos 333° = 0.89$; find $\tan 117°$.

$$\cos 333° = 0.89.$$

$$= \cos(270° + 63°)$$
$$= \sin 63°$$
$$= \tan(180° - 63°)$$
$$= -\tan 63°.$$

$$\sin^2 63° + \cos^2 63° = 1.$$

$$(0.89)^2 + \cos^2 63° = 1.$$

$$\cos^2 63° = 0.2079.$$

$$\cos 63° = 0.456.$$

$$-\tan 63° = -\frac{\sin 63°}{\cos 63°} = -\frac{0.89}{0.456} = -1.952.$$

44. Simplify the expression

$a \cos(90° - x) + b \cos(90° + x)$

$= a \sin x + b(-\sin x)$

$= \sin x(a - b).$

45. Simplify the expression

$m(\cos 90° - x) \sin(90° - x).$

$\cos(90° - x) = \sin x.$

$\sin(90° - x) = \cos x.$

$\therefore$ the expression $= m \sin x \cos x.$

46. Simplify the expression

$(a - b)\tan(90° - x) + (a + b)\cot(90° + x).$

$\tan(90° - x) = \cot x.$

$\cot(90° + x) = -\tan x.$

$\therefore$ the expression equals

$(a - b)\cot x - (a + b)\tan x.$

47. Simplify the expression

$a^2 + b^2 - 2ab\cos(180° - x)$

$= a^2 + b^2 - 2ab(-\cos x)$

$= a^2 + b^2 + 2ab\cos x.$

48. Simplify the expression

$\sin(90° + x)\sin(180° + x) + \cos(90° + x)\cos(180° - x)$

$= (\cos x)(-\sin x) + (-\sin x)(-\cos x)$

$= -\sin\cos x + \sin\cos x$

$= 0.$

49. Simplify the expression

$\cos(180° + x)\cos(270° - y) - \sin(180° + x)\sin(270° - y).$

$\cos(180° + x) = -\cos x.$
$\cos(270° - y) = -\sin y.$
$\sin(180° + x) = -\sin x.$
$\sin(270° - y) = -\cos y.$
Hence the expression
$= \cos x \sin y - \sin x \cos y.$

50. Simplify the expression
$\tan x + \tan(-y) - \tan(180° - y).$
$\tan(-y) = -\tan y.$
$-\tan(180° - y) = \tan y.$
Hence the expression $= \tan x.$

51. For what values of x is the expression $\sin x + \cos x$ positive, and for what values negative? Represent the result by a drawing in which the sectors corresponding to the negative values are shaded.

If x be any angle in Quadrant I., $\sin x + \cos x$ must be positive since both the sine and cosine are positive. In Quadrant II. the sine is positive and cosine negative; hence, so long as the sine is greater than, or equal to, the cosine, the expression $\sin x + \cos x$ is positive; but after passing the middle of Quadrant II., viz., 135°, the cosine of $\angle x$ is greater than sine, and the expression is negative. In Quadrant III. both sine and cosine are negative, and hence their sum must be negative. In Quadrant IV. the sine is negative and cosine positive. The sine and cosine are equal at 315°, after which the cosine is greater than sine. Hence the expression $\sin x + \cos x$ is negative from 135° to 315°, and positive between 0° and 135°, and 315° and 360°.

52. Answer the question of last example for $\sin x - \cos x$.

As x increases from 0° to 45°, the sine increases in value, and cosine decreases, until at 45° sine = cosine. Hence up to this point $\sin x - \cos x$ is negative. For the remainder of Quadrant I. sine is greater than cosine, and consequently the expression $\sin x - \cos x$ is positive. In Quadrant II. sine is positive and cosine negative, so the expression $\sin x - \cos x$ is uniformly positive. In Quadrant III. sine is negative and cosine negative; hence, so long as sine is less than cosine, the expression is positive, viz., to 225°; after that point, sine is greater than cosine, and $\sin x - \cos x$ is negative. In Quadrant IV. sine is negative and cosine positive: therefore $\sin x - \cos x$ is uniformly negative. The expression is, then, negative between 0° and 45°, and 225° and 360°; positive between 45° and 225°.

53. Find the functions of $(x - 90°)$ in terms of the functions of x.

$$x - 90° = 360° - (90° - x)$$
$$= 270° + x.$$
$$\sin(x - 90°) = \sin(270° + x)$$
$$= -\cos x.$$
$$\cos(x - 90°) = \cos(270° + x)$$
$$= \sin x.$$
$$\tan(x - 90°) = \tan(270° + x)$$
$$= -\cot x.$$
$$\cot(x - 90°) = \cot(270° + x)$$
$$= -\tan x.$$

54. Find the functions of $(x - 180°)$ in terms of the functions of x.

$$x - 180° = 360° - (180° - x)$$
$$= 180° + x.$$
$$\sin(x - 180°) = \sin(180° + x)$$
$$= -\sin x.$$
$$\cos(x - 180°) = \cos(180° + x)$$
$$= -\cos x.$$
$$\tan(x - 180°) = \tan(180° - x)$$
$$= \tan x.$$
$$\cot(x - 180°) = \cot(180° + x)$$
$$= \cot x.$$

Exercise XI. Page 48.

1. Find the value of $\sin(x+y)$ and $\cos(x+y)$ when $\sin x = \frac{3}{5}$, $\cos x = \frac{4}{5}$, $\sin y = \frac{5}{13}$, $\cos y = \frac{12}{13}$.

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$= \left(\frac{3}{5} \times \frac{12}{13}\right) + \left(\frac{4}{5} \times \frac{5}{13}\right)$$
$$= \frac{36}{65} + \frac{20}{65} = \frac{56}{65}.$$
$$\cos(x+y) = \cos x \cos x - \sin x \sin y$$
$$= \left(\frac{4}{5} \times \frac{12}{13}\right) - \left(\frac{3}{5} \times \frac{5}{13}\right)$$
$$= \frac{48}{65} - \frac{15}{65} = \frac{33}{65}.$$

2. Find $\sin(90° - y)$ and $\cos(90° - y)$ by making $x = 90°$ in Formulas [8] and [9].

$\sin(90° - y)$
$$= \sin 90° \cos y - \cos 90° \sin y.$$
$\sin 90° = 1. \quad \cos 90° = 0.$

$\therefore \sin(90° - y)$
$$= (1 \times \cos y) - (0 \times \sin y)$$
$$= \cos y.$$
$\cos(90° - y)$
$$= \cos 90° \cos y + \sin 90° \sin y$$
$$= (0 \times \cos y) + (1 \times \sin y)$$
$$= \sin y.$$

3. Find, by Formulas [4]–[11], the first four functions of $90° + y$.

$\sin(90° + y)$
$$= \sin 90° \cos y + \cos 90° \sin y$$
$$= (1 \times \cos y) + (0 \times \sin y)$$
$$= \cos y.$$
$\cos(90° + y)$
$$= \cos 90° \cos y - \sin 90° \sin y$$
$$= (0 \times \cos y) - (1 \times \sin y)$$
$$= -\sin y.$$
$(\tan 90° + y)$
$$= -\frac{\cos y}{\sin y} = -\cot y.$$
$\cot(90° + y)$
$$= -\frac{\sin y}{\cos y} = -\tan y.$$

4. Find, by Formulas [4]–[11], the first four functions of $180° - y$.

$\sin(180° - y)$
$$= \sin 180° \cos y - \cos 180° \sin y$$
$$= (0 \times \cos y) - (-1 \times \sin y)$$
$$= \sin y.$$
$\cos(180° - y)$
$$= \cos 180° \cos y + \sin 180° \sin y$$
$$= (-1 \times \cos y) + (0 \times \sin y)$$
$$= -\cos y.$$
$\tan(180° - y)$
$$= -\frac{\sin y}{\cos y} = -\tan y.$$
$\cot(180° - y)$
$$= -\frac{\cos y}{\sin y} = -\cot y.$$

5. Find, by Formulas [4]–[11], the first four functions of $180° + y$.

$\sin(180° + y)$
$= \sin 180° \cos y + \cos 180° \sin y$
$= (0 \times \cos y) + (-1 \times \sin y)$
$= -\sin y.$

$\cos(180° + y)$
$= \cos 180° \cos y - \sin 180° \sin y$
$= (-1 \times \cos y) - (0 \times \sin y)$
$= -\cos y.$

$\tan(180° + y)$
$= \frac{-\sin y}{-\cos y} = \tan y.$

$\cot(180° + y)$
$= \frac{-\cos y}{-\sin y} = \cot y.$

6. Find, by Formulas [4]–[11], the first four functions of $270° - y$.

$\sin(270° - y)$
$= \sin 270° \cos y - \cos 270° \sin y$
$= (-1 \times \cos y) - (0 \times \sin y)$
$= -\cos y.$

$\cos(270° - y)$
$= \cos 270° \cos y + \sin 270° \sin y$
$= (0 \times \cos y) + (-1 \times \sin y)$
$= -\sin y.$

$\tan(270° - y)$
$= \frac{-\cos y}{-\sin y} = \cot y.$

$\cot(270° - y)$
$= \frac{-\sin y}{-\cos y} = \tan y.$

7. Find, by Formulas [4]–[11], the first four functions of $270° + y$.

$\sin(270° + y)$
$= \sin 270° \cos y + \cos 270° \sin y$
$= (-1 \times \cos y) + (0 \times \sin y)$
$= -\cos y.$

$\cos(270° + y)$
$= \cos 270° \cos y - \sin 270° \sin y$
$= (0 \times \cos y) - (-1 \times \sin y)$
$= \sin y.$

$\tan(270° + y)$
$= \frac{-\cos y}{\sin y} = -\cot y.$

$\cot(270° + y)$
$= \frac{\sin y}{-\cos y} = -\tan y.$

8. Find, by Formulas [4]–[11], the first four functions of $360° - y$.

$\sin(360° - y)$
$= \sin 360° \cos y - \cos 360° \sin y$
$= (0 \times \cos y) - (1 \times \sin y)$
$= -\sin y.$

$\cos(360° - y)$
$= \cos 360° \cos y + \sin 360° \sin y$
$= (1 \times \cos y) + (0 \times \sin y)$
$= \cos y.$

$\tan(360° - y)$
$= \frac{-\sin y}{\cos y} = -\tan y.$

$\cot(360° - y)$
$= \frac{\cos y}{-\sin y} = -\cot y.$

9. Find, by Formulas [4]–[11], the first four functions of $360° + y$.

$\sin(360° + y)$
$= \sin 360° \cos y + \cos 360° \sin y$
$= (0 \times \cos y) + (1 \times \sin y)$
$= \sin y.$

$\cos(360° + y)$
$= \cos 360° \cos y - \sin 360° \sin y$
$= (1 \times \cos y) - (0 \times \sin y)$
$= \cos y.$

$\tan(360° + y)$
$= \frac{\sin y}{\cos y} = \tan y.$

$\cot(360° + y)$

$$= \frac{\cos y}{\sin y} = \cot y.$$

10. Find, by Formulas [4]-[11], the first four functions of $x - 90°$.

$\sin(x - 90°)$

$= \sin x \cos 90° - \cos x \sin 90°$

$= (0 \times \sin x) - (1 \times \cos x)$

$= - \cos x.$

$\cos(x - 90°)$

$= \cos x \cos 90° + \sin x \sin 90°$

$= (0 \times \cos x) + (1 \times \sin x)$

$= \sin x.$

$\tan(x - 90°)$

$$= \frac{-\cos x}{\sin x} = - \cot x.$$

$\cot(x - 90°)$

$$= \frac{\sin x}{-\cos x} = - \tan x.$$

11. Find, by Formulas [4]-[11], the first four functions of $x - 180°$.

$\sin(x - 180°)$

$= \sin x \cos 180° - \cos x \sin 180°$

$= \sin x (-1) - \cos x \times 0$

$= - \sin x.$

$\cos(x - 180°)$

$= \cos x \cos 180° + \sin x \sin 180°$

$= \cos x (-1) + \sin x \times 0$

$= - \cos x.$

$\tan(x - 180°)$

$$= \frac{-\sin x}{-\cos x} = \tan x.$$

$\cot(x - 180°)$

$$= \frac{-\cos x}{-\sin x} = \cot x.$$

12. Find, by Formulas [4]-[11], the first four functions of $x - 270°$.

$\sin(x - 270°)$

$= \sin x \cos 270° - \cos x \sin 270°$

$= \sin x \times 0 - \cos x \times (-1)$

$= \cos x.$

$\cos(x - 270°)$

$= \cos x \cos 270° + \sin x \sin 270°$

$= \cos x \times 0 + \sin x (-1)$

$= - \sin x.$

$\tan(x - 270°)$

$$= \frac{\cos x}{-\sin x} = - \cot x.$$

$\cot(x - 270°)$

$$= \frac{-\sin x}{\cos x} = - \tan x.$$

13. Find, by Formulas [4]-[11], the first four functions of $-y$.

$\sin(0° - y)$

$= \sin 0° \cos y - \cos 0° \sin y$

$= (0 \times \cos y) - (1 \times \sin y)$

$= - \sin y.$

$\cos(0° - y)$

$= \cos 0° \cos y + \sin 0° \sin y$

$= (1 \times \cos y) + (0 \times \sin y)$

$= \cos y.$

$\tan(0° - y)$

$$= \frac{-\sin y}{\cos y} = - \tan y.$$

$\cot(0° - y)$

$$= \frac{\cos y}{-\sin y} = - \cot y.$$

14. Find, by Formulas [4]-[11], the first four functions of $45° - y$.

$\sin(45° - y)$

$= \sin 45° \cos y - \cos 45° \sin y$

$= \frac{1}{2}\sqrt{2} \cos y - \frac{1}{2}\sqrt{2} \sin y$

$= \frac{1}{2}\sqrt{2} (\cos y - \sin y).$

$\cos(45^\circ - y)$

$= \cos 45^\circ \cos y + \sin 45^\circ \sin y$

$= \frac{1}{2}\sqrt{2}\cos y + \frac{1}{2}\sqrt{2}\sin y$

$= \frac{1}{2}\sqrt{2}(\cos y + \sin y).$

$\tan(45^\circ - y)$

$$= \frac{\cos y - \sin y}{\cos y + \sin y} = \frac{1 - \tan y}{1 + \tan y}.$$

$\cot(45^\circ - y)$

$$= \frac{\cos y + \sin y}{\cos y - \sin y} = \frac{\cot y + 1}{\cot y - 1}.$$

15. Find, by Formulas [4]–[11], the first four functions of $45^\circ + y$.

$\sin(45^\circ + y)$

$= \sin 45^\circ \cos y + \cos 45^\circ \sin y$

$= \frac{1}{2}\sqrt{2}\cos y + \frac{1}{2}\sqrt{2}\sin y$

$= \frac{1}{2}\sqrt{2}(\cos y + \sin y).$

$\cos(45^\circ + y)$

$= \cos 45^\circ \cos y - \sin 45^\circ \sin y$

$= \frac{1}{2}\sqrt{2}\cos y - \frac{1}{2}\sqrt{2}\sin y$

$= \frac{1}{2}\sqrt{2}(\cos y - \sin y).$

$\tan(45^\circ + y)$

$$= \frac{\cos y + \sin y}{\cos y - \sin y} = \frac{1 + \tan y}{1 - \tan y}.$$

$\cot(45^\circ + y)$

$$= \frac{\cos y - \sin y}{\cos y + \sin y} = \frac{\cot y - 1}{\cot y + 1}.$$

16. Find, by Formulas [4]-[11], the first four functions of $30^\circ + y$.

$\sin(30^\circ + y)$

$= \sin 30^\circ \cos y + \cos 30^\circ \sin y$

$= \frac{1}{2}(\cos y + \sqrt{3}\sin y).$

$\cos(30^\circ + y)$

$= \cos 30^\circ \cos y - \sin 30^\circ \sin y$

$= \frac{1}{2}(\sqrt{3}\cos y - \sin y).$

$\tan(30^\circ + y)$

$$= \frac{\cos y + \sqrt{3}\sin y}{\sqrt{3}\cos y - \sin y};$$

divide each term by $\sqrt{3}\cos y$,

$$= \frac{\frac{1}{3}\sqrt{3} + \tan y}{1 - \frac{1}{3}\sqrt{3}\tan y}.$$

$\cot(30^\circ + y)$

$$= \frac{\sqrt{3}\cos y - \sin y}{\cos y + \sqrt{3}\sin y};$$

divide each term by $\sin y$,

$$= \frac{\sqrt{3}\cot y - 1}{\cot y + \sqrt{3}}.$$

17. Find, by Formulas [4]–[11], the first four functions of $60^\circ - y$.

$\sin(60^\circ - y)$

$= \sin 60^\circ \cos y - \cos 60^\circ \sin y$

$= \frac{1}{2}(\sqrt{3}\cos y - \sin y).$

$\cos(60^\circ - y)$

$= \cos 60^\circ \cos y + \sin 60^\circ \sin y$

$= \frac{1}{2}(\cos y + \sqrt{3}\sin y).$

$\tan(60^\circ - y)$

$$= \frac{\sqrt{3}\cos y - \sin y}{\cos y + \sqrt{3}\sin y}$$

$$= \frac{\sqrt{3} - \tan y}{1 + \sqrt{3}\tan y}.$$

$\cot(60^\circ - y)$

$$= \frac{\cos y + \sqrt{3}\sin y}{\sqrt{3}\cos y - \sin y}$$

$$= \frac{\frac{1}{3}\sqrt{3}\cot y + 1}{\cot y - \frac{1}{3}\sqrt{3}}.$$

18. Find $\sin 3x$ in terms of $\sin x$.

$\sin 3x = \sin(2x + x)$

$= \sin 2x \cos x + \cos 2x \sin x.$

$\sin 2x = 2 \sin x \cos x.$

$\cos 2x = \cos^2 x - \sin^2 x.$

Substituting,

$$\sin 3x = 2 \sin x \cos^2 x + \sin x \cos^2 x - \sin^3 x$$
$$= 3 \sin x \cos^2 x - \sin^3 x.$$

But $\cos^2 x = 1 - \sin^2 x.$

Substituting,

$$\sin 3x = 3 \sin x - 3 \sin^3 x - \sin^2 x$$
$$= 3 \sin x - 4 \sin^3 x.$$

19. Find $\cos 3x$ in terms of $\cos x$.

$$\cos 3x = \cos (2x + x)$$
$$= \cos 2x \cos x - \sin 2x \sin x.$$

$\sin 2x = 2 \sin x \cos x.$

$\cos 2x = \cos^2 x - \sin^2 x.$

Substituting,

$$\cos 3x = \cos^3 x - \sin^2 x \cos x - 2 \sin^2 x \cos x$$
$$= \cos^3 x - 3 \sin^2 x \cos x.$$

But $\sin^2 x = 1 - \cos^2 x.$

Substituting,

$$\cos 3x = \cos^3 x - 3 \cos x + 3 \cos^3 x$$
$$= 4 \cos^3 x - 3 \cos x.$$

20. Given $\tan \frac{1}{2} x = 1$; find $\cos x$.

$$\tan \tfrac{1}{2} x = \sqrt{\frac{1 - \cos x}{1 + \cos x}}.$$

$$1 = \sqrt{\frac{1 - \cos x}{1 + \cos x}}.$$

$$1 = \frac{1 - \cos x}{1 + \cos x}.$$

$$1 + \cos x = 1 - \cos x.$$

$$2 \cos x = 0.$$

$$\cos x = 0.$$

21. Given $\cot \frac{1}{2} x = \sqrt{3}$; find $\sin x$.

$$\cot \tfrac{1}{2} x = \sqrt{\frac{1 + \cos x}{1 - \cos x}}.$$

$$\sqrt{3} = \sqrt{\frac{1 + \cos x}{1 - \cos x}}.$$

$$3 = \frac{1 + \cos x}{1 - \cos x}.$$

$$3 - 3 \cos x = 1 + \cos x.$$

$$-4 \cos x = -2.$$

$$\cos x = \frac{1}{2}.$$

$$\sin^2 x = 1 - \cos^2 x$$
$$= 1 - \frac{1}{4} = \frac{3}{4}.$$

$$\sin x = \sqrt{\frac{3}{4}} = \tfrac{1}{2} \sqrt{3}.$$

22. Given $\sin x = 0.2$; find $\sin \frac{1}{2} x$ and $\cos \frac{1}{2} x$.

$$\sin x = 0.2.$$

$$\cos^2 x = 1 - \sin^2 x$$
$$= 1 - 0.04.$$

$$\cos x = \sqrt{0.96}.$$

$$\sin \tfrac{1}{2} x = \sqrt{\frac{1 - \cos x}{2}}$$
$$= \sqrt{\frac{1 - \sqrt{0.96}}{2}}$$
$$= \sqrt{\frac{1 - 0.4\sqrt{6}}{2}}$$
$$= 0.10051.$$

$$\cos \tfrac{1}{2} x = \sqrt{\frac{1 + \cos x}{2}}$$
$$= \sqrt{\frac{1 + 0.4\sqrt{6}}{2}}$$
$$= 0.99494.$$

23. Given $\cos x = 0.5$; find $\cos 2x$ and $\tan 2x$.

$$\cos 2x = \cos^2 x - \sin^2 x.$$

$$\sin x = \sqrt{1-\left(\frac{1}{2}\right)^2} = \tfrac{1}{2}\sqrt{3}.$$

$$\therefore \cos 2x = 0.25 - 0.75$$

$$= -0.50 = -\frac{1}{2}.$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\frac{1}{2}\sqrt{3}}{\frac{1}{2}} = \sqrt{3}.$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} = \frac{2\sqrt{3}}{1-3}$$

$$= -\sqrt{3}.$$

24. Given $\tan 45° = 1$; find the functions of $22° 30'$.

$$\tan x = \frac{\sin x}{\cos x}.$$

$$\therefore \sin x = \cos x.$$

$$\sin^2 x + \cos^2 x = 1.$$

$$\sin^2 x + \sin^2 x = 1.$$

$$2\sin^2 x = 1.$$

$$\sin^2 x = \tfrac{1}{2}.$$

$$\sin x = \tfrac{1}{2}\sqrt{2} = \cos x.$$

$\sin \frac{1}{2}x$ or $\sin 22° 30'$

$$= \sqrt{\frac{1-\frac{1}{2}\sqrt{2}}{2}}$$

$$= \tfrac{1}{2}\sqrt{2-\sqrt{2}}$$

$$= 0.3827.$$

$\cos \frac{1}{2}x$ or $\cos 22° 30'$

$$= \sqrt{\frac{1+\frac{1}{2}\sqrt{2}}{2}}$$

$$= \tfrac{1}{2}\sqrt{2+\sqrt{2}}$$

$$= 0.9239.$$

$$\tan \tfrac{1}{2}x = \frac{\sin \frac{1}{2}x}{\cos \frac{1}{2}x} = \frac{\frac{1}{2}\sqrt{2-\sqrt{2}}}{\frac{1}{2}\sqrt{2+\sqrt{2}}}$$

$$= \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}};$$

multiply by $\dfrac{2-\sqrt{2}}{2-\sqrt{2}}$,

$$= \sqrt{\frac{(2-\sqrt{2})^2}{4-2}}$$

$$= \tfrac{1}{2}\sqrt{(2-\sqrt{2})^2} \times \sqrt{2}$$

$$= (1-\tfrac{1}{2}\sqrt{2}) \times \sqrt{2}$$

$$= \sqrt{2} - 1 = 0.4142.$$

$$\cot \tfrac{1}{2}x = \frac{\cos \frac{1}{2}x}{\sin \frac{1}{2}x}$$

$$= \frac{\frac{1}{2}\sqrt{2+\sqrt{2}}}{\frac{1}{2}\sqrt{2-\sqrt{2}}}$$

$$= \sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}}$$

$$= \sqrt{2} + 1 = 2.4142.$$

25. Given $\sin 30° = 0.5$; find the functions of $15°$.

$$\sin 30° = 0.5 = \frac{1}{2}.$$

$$\therefore \cos 30° = \sqrt{1-\frac{1}{4}} = \sqrt{\frac{3}{4}}$$

$$= \tfrac{1}{2}\sqrt{3}.$$

$$\sin \tfrac{1}{2}x = \sqrt{\frac{1-\cos x}{2}}.$$

$$\therefore \sin 15° = \sqrt{\frac{1-\frac{1}{2}\sqrt{3}}{2}}$$

$$= \tfrac{1}{2}\sqrt{2-\sqrt{3}} = 0.25885.$$

$$\cos 15° = \sqrt{\frac{1+\frac{1}{2}\sqrt{3}}{2}}$$

$$= \tfrac{1}{2}\sqrt{2+\sqrt{3}} = 0.96592.$$

$$\tan 15^\circ = \sqrt{\frac{1-\frac{1}{2}\sqrt{3}}{1+\frac{1}{2}\sqrt{3}}}$$

$$= \sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}$$

$$= \sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2+\sqrt{3}}}$$

$$= \sqrt{\frac{(2-\sqrt{3})^2}{4-3}}$$

$$= 2-\sqrt{3} = 0.26799.$$

$$\cot 15^\circ = \sqrt{\frac{1+\frac{1}{2}\sqrt{3}}{1-\frac{1}{2}\sqrt{3}}}$$

$$= 2+\sqrt{3} = 3.7321.$$

26. Prove that

$$\tan 18^\circ = \frac{\sin 33^\circ + \sin 3^\circ}{\cos 33^\circ + \cos 3^\circ}.$$

Let $x = 18^\circ$,

$y = 15^\circ$.

Then

(1) $2 \sin x \cos y$

$$= \sin(x+y) + \sin(x-y).$$

(2) $2 \cos x \cos y$

$$= \cos(x+y) + \cos(x-y).$$

Divide (1) by (2).

$$\tan x = \frac{\sin(x+y) + \sin(x-y)}{\cos(x+y) + \cos(x-y)}.$$

Substitute values of x and y.

$$\tan 18^\circ = \frac{\sin 33^\circ + \sin 3^\circ}{\cos 33^\circ + \cos 3^\circ}.$$

27. Prove the formula

$$\sin 2x = \frac{2 \tan x}{1 + \tan^2 x}.$$

$$\sin 2x = 2 \sin x \cos x.$$

$$2 \tan x = \frac{2 \sin x}{\cos x}.$$

$$1 + \tan^2 x = 1 + \frac{\sin^2 x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}.$$

But $\cos^2 x + \sin^2 x = 1$.

$$\therefore 1 + \tan^2 x = \frac{1}{\cos^2 x}.$$

$$2 \sin x \cos x = \frac{2 \sin x}{\cos x} \times \frac{\cos^2 x}{1}.$$

$$\therefore 2 \sin x \cos x = 2 \sin x \cos x.$$

28. Prove the formula

$$\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}.$$

$$\cos 2x = \frac{1 - \frac{\sin^2 x}{\cos^2 x}}{1 + \frac{\sin^2 x}{\cos^2 x}}.$$

$$\cos 2x = \cos^2 x - \sin^2 x.$$

$$\cos^2 x - \sin^2 x = \frac{1 - \frac{\sin^2 x}{\cos^2 x}}{1 + \frac{\sin^2 x}{\cos^2 x}}$$

$$= \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x}$$

$$= \frac{\cos^2 x - \sin^2 x}{1}$$

$$= \cos^2 x - \sin^2 x.$$

29. Prove the formula

$$\tan \tfrac{1}{2} x = \frac{\sin x}{1 + \cos x}.$$

$$\tan \tfrac{1}{2} x = \frac{\sqrt{1 - \cos x}}{\sqrt{1 + \cos x}}.$$

$$\frac{\sin x}{1+\cos x}=\frac{\sqrt{1-\cos^2 x}}{1+\cos x}.$$

$$\therefore \frac{\sqrt{1-\cos x}}{\sqrt{1+\cos x}}=\frac{\sqrt{1-\cos^2 x}}{1+\cos x}.$$

$$\frac{1-\cos x}{1+\cos x}=\frac{1-\cos^2 x}{(1+\cos x)^2}$$

$$=\frac{1-\cos x}{1+\cos x}.$$

30. Prove the formula

$$\cot \tfrac{1}{2}x=\frac{\sin x}{1-\cos x}.$$

$$\sin x=\sqrt{1-\cos^2 x}.$$

$$\cot \tfrac{1}{2}x=\sqrt{\frac{1+\cos x}{1-\cos x}}.$$

By substituting,

$$\sqrt{\frac{1+\cos x}{1-\cos x}}=\frac{\sqrt{1-\cos^2 x}}{1-\cos x}.$$

$$\frac{1+\cos x}{1-\cos x}=\frac{1-\cos^2 x}{(1-\cos x)^2}$$

$$=\frac{1+\cos x}{1-\cos x}.$$

31. Prove the formula

$$\sin \tfrac{1}{2}x \pm \cos \tfrac{1}{2}x=\sqrt{1\pm \sin x}.$$

By squaring,

$$\sin^2 \tfrac{1}{2}x \pm 2\sin \tfrac{1}{2}x\cos \tfrac{1}{2}x+\cos^2 \tfrac{1}{2}x$$

$$=1\pm \sin x.$$

But $\sin \tfrac{1}{2}x=\sqrt{\dfrac{1-\cos x}{2}}$,

and $\cos \tfrac{1}{2}x=\sqrt{\dfrac{1+\cos x}{2}}$.

Substitute values of $\sin \frac{1}{2}x$ and $\cos \frac{1}{2}x$.

$$\frac{1-\cos x}{2}\pm 2\sin \tfrac{1}{2}x\cos \tfrac{1}{2}x+\frac{1+\cos x}{2}$$

$$=1\pm \sin x.$$

$$1\pm 2\sin \tfrac{1}{2}x\cos \tfrac{1}{2}x=1\pm \sin x.$$

$$\pm 2\sin \tfrac{1}{2}x\cos \tfrac{1}{2}x=\pm \sin x.$$

$$2\sin \tfrac{1}{2}x\cos \tfrac{1}{2}x$$

$$=\pm 2\sqrt{\frac{1-\cos^2 x}{4}}.$$

$$\pm \sin x=\pm\sqrt{\frac{1-\cos 2x}{2}}.$$

$$\therefore \pm 2\sqrt{\frac{1-\cos^2 x}{4}}=\pm\sqrt{\frac{1-\cos 2x}{2}}.$$

$$1-\cos^2 x=\frac{1-\cos 2x}{2}.$$

$$\sqrt{1-\cos^2 x}=\sqrt{\frac{1-\cos 2x}{2}}.$$

$$\therefore \sin x=\sin x.$$

32. Prove the formula

$$\frac{\tan x\pm\tan y}{\cot x\pm\cot y}=\pm\tan x\tan y.$$

$$\tan x\pm\tan y$$

$$=\pm\tan x\cot x\tan y$$

$$+\cot y\tan y\tan x$$

But $\tan x\cot x=1$,

and $\tan y\cot y=1$.

$$\therefore \tan x\pm\tan y=\tan x\pm\tan y.$$

33. Prove the formula

$$\tan(45^\circ-x)=\frac{1-\tan x}{1+\tan x}.$$

$$\tan(45^\circ-x)=\frac{\sin(45^\circ-x)}{\cos(45^\circ-x)}.$$

$$\sin(45^\circ-x)$$

$$=\sin 45^\circ\cos x-\cos 45^\circ\sin x$$

$$=\tfrac{1}{2}\sqrt{2}\cos x-\tfrac{1}{2}\sqrt{2}\sin x$$

$$=\tfrac{1}{2}\sqrt{2}(\cos x-\sin x).$$

$$\cos(45^\circ-x)$$

$$=\cos 45^\circ\cos x+\sin 45^\circ\sin x$$

$= \frac{1}{2}\sqrt{2}\cos x + \frac{1}{2}\sqrt{2}\sin x$

$= \frac{1}{2}\sqrt{2}(\cos x + \sin x).$

$\tan(45° - x) = \dfrac{\cos x - \sin x}{\cos x + \sin x}.$

Dividing numerator and denominator by $\cos x$,

$\tan(45° - x) = \dfrac{1 - \tan x}{1 + \tan x}.$

34. If A, B, C are the angles of a triangle, prove that

$\sin A + \sin B + \sin C$

$= 4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C.$

$\sin A + \sin B + \sin C$

$= \sin A + \sin B + \sin[180° - (A+B)]$

$= \sin A + \sin B + \sin(A+B)$

$= 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$
$+ 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A+B)$

$= 2\sin\frac{1}{2}(A+B)[\cos\frac{1}{2}(A-B)$
$+ \cos\frac{1}{2}(A+B)].$

But $\frac{1}{2}(A+B) = A'$,
$\frac{1}{2}(A-B) = B'$.

Then, by [22],

$\cos A' + \cos B'$

$= 2\cos\frac{1}{2}A\cos\frac{1}{2}B.$

$\therefore = 4\sin\frac{1}{2}(A+B)\cos\frac{1}{2}A\cos\frac{1}{2}B.$

But $\cos\frac{1}{2}C = \cos[90° - \frac{1}{2}(A+B)]$

$= \sin\frac{1}{2}(A+B).$

$\therefore \sin A + \sin B + \sin C$

$= 4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C.$

35. If A, B, C are the angles of a triangle, prove that

$\cos A + \cos B + \cos C$

$= 1 + 4\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C.$

$\cos C = \cos[180° - (A+B)]$

$= -\cos(A+B).$

$\therefore \cos A + \cos B + \cos C$

$= \cos A + \cos B - \cos(A+B).$

By [22],

$= 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$
$-\cos(A+B).$

By [17],

$= 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$
$- 2\cos^2\frac{1}{2}(A+B) + 1$

$= [2\cos\frac{1}{2}(A+B)]$
$\times[\cos\frac{1}{2}(A-B) - \cos\frac{1}{2}(A+B)] + 1.$

By [23],

$= [2\cos\frac{1}{2}(A+B)]$
$\times[2\sin\frac{1}{2}A\sin\frac{1}{2}B] + 1$

$= (2\sin\frac{1}{2}C)(2\sin\frac{1}{2}A\sin\frac{1}{2}B) + 1$

$= 1 + 4\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C.$

36. If A, B, C are the angles of a triangle, prove that

$\tan A + \tan B + \tan C$

$= \tan A \times \tan B \times \tan C.$

Since $A + B + C = 180°$,

$C = 180° - (A+B).$

$\therefore \tan C = \tan[180° - (A+B)]$

$= -\tan(A+B).$

Again,

$\tan A + \tan B$

$= \tan(A+B)(1 - \tan A\tan B)$

$= \tan(A+B)$
$- \tan(A+B)\tan A\tan B.$

$\therefore \tan A + \tan B + \tan C$

$= \tan(A+B) - \tan(A+B)$
$- \tan(A+B)\tan A\tan B$

$= -\tan(A+B)\tan A\tan B$

$= \tan A\tan B\tan C.$

37. If A, B, C are the angles of a triangle, prove that

$$\cot \tfrac{1}{2} A + \cot \tfrac{1}{2} B + \cot \tfrac{1}{2} C = \cot \tfrac{1}{2} A \times \cot \tfrac{1}{2} B \times \cot \tfrac{1}{2} C.$$

Since $\tfrac{1}{2} A + \tfrac{1}{2} B + \tfrac{1}{2} C = 90°$,

$$\tfrac{1}{2} C = 90° - \tfrac{1}{2}(A + B).$$

$$\therefore \cot \tfrac{1}{2} C = \tan \tfrac{1}{2}(A + B),$$

and $\cot \tfrac{1}{2} B = \tan \tfrac{1}{2}(A + C)$,

and $\cot \tfrac{1}{2} A = \tan \tfrac{1}{2}(B + C)$.

$$\therefore \cot \tfrac{1}{2} A + \cot \tfrac{1}{2} B + \cot \tfrac{1}{2} C = \tan \tfrac{1}{2}(A + B) + \tan \tfrac{1}{2}(A + C) + \tan \tfrac{1}{2}(B + C)$$

$$= \tan \tfrac{1}{2}(A + B) \times \tan \tfrac{1}{2}(A + C) \times \tan \tfrac{1}{2}(B + C).$$

By substitution,

$$\cot \tfrac{1}{2} A + \cot \tfrac{1}{2} B + \cot \tfrac{1}{2} C = \cot \tfrac{1}{2} A \times \cot \tfrac{1}{2} B \times \cot \tfrac{1}{2} C.$$

38. Change to a form more convenient for logarithmic computation $\cot x + \tan x$.

$$\cot x + \tan x = \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\sin x \cos x}$$

$$= \frac{2(\cos^2 x + \sin^2 x)}{2 \sin x \cos x}$$

$$= \frac{2}{\sin 2x}.$$

39. Change to a form more convenient for logarithmic computation $\cot x - \tan x$.

$$\cot x - \tan x = \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}$$

$$= \frac{\cos^2 x - \sin^2 x}{\sin x \cos x}$$

$$= \frac{\cos 2x}{\sin x \cos x} \quad [13]$$

$$= \frac{2 \cos 2x}{2 \sin x \cos x}$$

$$= \frac{2 \cos 2x}{\sin 2x} \quad [12]$$

$$= 2 \cot 2x.$$

40. Change to a form more convenient for logarithmic computation $\cot x + \tan y$.

$$\cot x = \frac{\cos x}{\sin x}, \quad \tan y = \frac{\sin y}{\cos y}. \quad [2]$$

Adding,

$$= \frac{\cos x \cos y + \sin x \sin y}{\sin x \cos y}.$$

Substitute for $\cos x \cos y + \sin x \sin y$ its equal $\cos(x - y)$, [9]

$$= \frac{\cos(x - y)}{\sin x \cos y}.$$

41. Change to a form more convenient for logarithmic computation $\cot x - \tan y$.

$$\tan y = \frac{\sin y}{\cos y}.$$

$$\cot x = \frac{\cos x}{\sin x}.$$

$$\cot x - \tan y = \frac{\cos x}{\sin x} - \frac{\sin y}{\cos y}$$

$$= \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y}$$

$$= \frac{\cos(x + y)}{\sin x \cos y}.$$

42. Change to a form more convenient for logarithmic computation

$$\frac{1 - \cos 2x}{1 + \cos 2x}.$$

$$\frac{1-\cos 2x}{1+\cos 2x}=\frac{\dfrac{1-\cos 2x}{2}}{\dfrac{1+\cos 2x}{2}}$$

$$=\frac{\sin^2 x}{\cos^2 x}$$

$$=\tan^2 x.$$

43. Change to a form more convenient for logarithmic computation $1+\tan x \tan y$.

$$1+\tan x \tan y$$

$$=1+\frac{\sin x}{\cos x}\times\frac{\sin y}{\cos y}$$

$$=\frac{\cos x \cos y+\sin x \sin y}{\cos x \cos y}$$

$$=\frac{\cos(x-y)}{\cos x \cos y}.$$

44. Change to a form more convenient for logarithmic computation $1-\tan x \tan y$.

$$1-\tan x \tan y$$

$$=1-\frac{\sin x \sin y}{\cos x \cos y}$$

$$=\frac{\cos x \cos y-\sin x \sin y}{\cos x \cos y}$$

$$=\frac{\cos(x+y)}{\cos x \cos y}.$$

45. Change to a form more convenient for logarithmic computation $\cot x \cot y+1$.

$$\cot x \cot y+1$$

$$=\frac{\cos x}{\sin x}\times\frac{\cos y}{\sin y}+1$$

$$=\frac{\cos x \cos y+\sin x \sin y}{\sin x \sin y}.$$

$$\text{By [9]} =\frac{\cos(x-y)}{\sin x \sin y}.$$

46. Change to a form more convenient for logarithmic computation $\cot x \cot y-1$.

$$\cot x \cot y-1$$

$$=\frac{\cos x \cos y}{\sin x \sin y}-1$$

$$=\frac{\cos x \cos y-\sin x \sin y}{\sin x \sin y}$$

$$=\frac{\cos(x+y)}{\sin x \sin y}.$$

47. Change to a form more convenient for logarithmic computation $\dfrac{\tan x+\tan y}{\cot x+\cot y}$.

$$\frac{\tan x+\tan y}{\cot x+\cot y}$$

$$=\frac{\dfrac{\sin x}{\cos x}+\dfrac{\sin y}{\cos y}}{\dfrac{\cos x}{\sin x}+\dfrac{\cos y}{\sin y}}$$

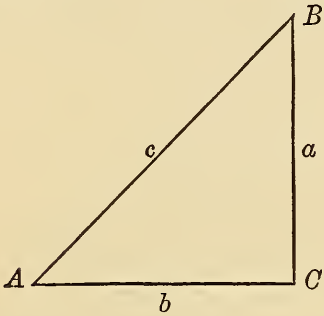
$$=\frac{\dfrac{\sin x \cos y+\cos x \sin y}{\cos x \cos y}}{\dfrac{\sin x \cos y+\cos x \sin y}{\sin x \sin y}}$$

$$=\frac{\sin x \sin y}{\cos x \cos y}$$

$$=\tan x \tan y.$$

EXERCISE XII. PAGE 53.

1. What do the formulas of § 36 become when one of the angles is a right angle?



If angle C is a right angle,

$$\frac{a}{c} = \frac{\sin A}{\sin C} = \sin A;$$

$$\frac{c}{b} = \frac{\sin C}{\sin B} = \frac{1}{\sin B};$$

$$\frac{a}{b} = \frac{\sin A}{\sin B} = \tan A;$$

$$\frac{a}{\sin A} = \frac{c}{\sin C} = c;$$

$$\frac{b}{\sin B} = \frac{c}{\sin C} = c.$$

2. Prove by means of the Law of Sines that the bisector of an angle of a triangle divides the opposite side into parts proportional to the adjacent sides.

Let CD bisect angle C.

Then $$\frac{AD}{CD} = \frac{\sin \frac{1}{2} C}{\sin A},$$

and $$\frac{DB}{CD} = \frac{\sin \frac{1}{2} C}{\sin B}.$$

By division,

$$\frac{AD}{DB} = \frac{\sin B}{\sin A}.$$

But $$\frac{\sin B}{\sin A} = \frac{b}{a}.$$

$$\therefore \frac{AD}{BD} = \frac{b}{a}.$$

3. What does Formula [26] become when $A = 90°$? when $A = 0°$? when $A = 180°$? What does the triangle become in each of these cases?

Formula [26] is

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

When $A = 90°$, $\cos A = 0°$.

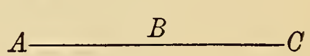
$\therefore a^2 = b^2 + c^2.$

When $A = 0°$, $\cos A = 1.$

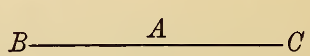
$\therefore a^2 = b^2 + c^2 - 2bc.$

When $A = 180°$, $\cos A = -1.$

$\therefore a^2 = b^2 + c^2 + 2bc.$



$a = BC.$	$c = AB.$
$b = AC.$	$a = b - c.$



$a = BC.$	$c = BA.$
$b = AC.$	$a = b + c.$

4. Prove that whether the angle B is acute or obtuse $c = a \cos B + b \cos A$. What are the two symmetrical formulas obtained by changing the letters? What does the formula become when $B = 90°$?

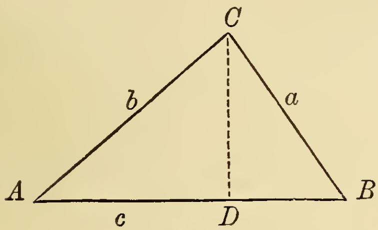


Fig. 1.

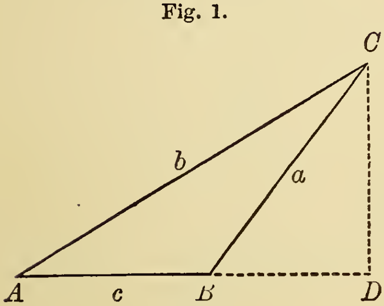


Fig. 2.

Case I. When angle B is acute (Fig. 1).

(1) $$\cos B = \frac{DB}{a}.$$

$$\cos A = \frac{AD}{b}.$$

$$\therefore DB = a \cos B,$$

and $$AD = b \cos A.$$

Add, $DB + AD = a \cos B + b \cos A.$

But $DB + AD = c.$

$$\therefore c = a \cos B + b \cos A.$$

Case II. When angle B is obtuse (Fig. 2).

$$\frac{AD}{b} = \cos A.$$

$$\frac{BD}{a} = \cos (180° - B)$$

$$= - \cos B.$$

$$\therefore AD = b \cos A,$$

and $$BD = - a \cos B.$$

Subtract, observing that the sign of $\cos B$ is minus.

$$AD - BD = b \cos A + a \cos B.$$

But $AD - BD = c.$

$$\therefore c = a \cos B + b \cos A.$$

The symmetrical formulas are

$$b = a \cos C + c \cos A,$$

$$a = b \cos C + c \cos B.$$

When $B = 90°.$

(3) $$\cos A = \frac{c}{b}.$$

$$\therefore c = b \cos A.$$

5. From the three following equations (found in the last exercise) prove the theorem of § 37:

$$c = a \cos B + b \cos A,$$

$$b = a \cos C + c \cos A,$$

$$a = b \cos C + c \cos B.$$

$$c^2 = ac \cos B + bc \cos A. \quad (1)$$

$$b^2 = ab \cos C + bc \cos A. \quad (2)$$

$$a^2 = ab \cos C + ac \cos B. \quad (3)$$

Add (2) and (3),

$$a^2 + b^2 = 2ab \cos C + bc \cos A + ac \cos B. \quad (4)$$

Subtract (4) from (1),

$$c^2 - a^2 - b^2 = - 2ab \cos C.$$

$$\therefore c^2 = a^2 + b^2 - 2ab \cos C. \quad \text{§ 37}$$

6. In Formula [27] what is the maximum value of $\frac{1}{2}(A - B)$? of $\frac{1}{2}(A + B)$?

$$\frac{a - b}{a + b} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}.$$

The limit of $A - B$ is $180°$.

$\therefore$ the limit of the maximum value of $\frac{1}{2}(A - B)$

$$= \frac{180°}{2} = 90°.$$

The limit of $A + B$ is $180°$.

$\therefore$ the limit of the maximum value of $\frac{1}{2}(A + B)$

$$= \frac{180°}{2} = 90°.$$

7. Find the form to which Formula [27] reduces, and describe the nature of the triangle when

(i.) $C = 90°$;

(ii.) $A - B = 90°$, and $B = C$.

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}.$$

(i.) When $C = 90°$.

$$A + B = 90°.$$

$$B = 90° - A.$$

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}[A-(90°-A)]}{\tan 45°}$$

$$= \frac{\tan(A-45°)}{1}$$

$$= \tan(A - 45°).$$

Since C is a right angle, the triangle is a right triangle.

(ii.) When $A - B = 90°$, and $B = C$.

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}.$$

$$A + B + C = 180°,$$

or $$A + 2B = 180°$$

$$A - B = 90°$$

$$\therefore 3B = 90°$$

$$B = 30°,$$

$$C = 30°,$$

and $$A = 120°.$$

$$\frac{a-b}{a+b} = \frac{\tan 45°}{\tan 75°}$$

$$= \frac{\tan 45°}{\cot 15°}$$

$$= \frac{1}{2+\sqrt{3}}.$$

$$\therefore a + b = (a - b)(2 + \sqrt{3}).$$

Since $A = B$, the triangle is isosceles.

Exercise XIII. Page 55.

1. Given	find
$a = 500$,	$C = 123°\ 12'$,
$A = 10°\ 12'$,	$b = 2051.48$,
$B = 46°\ 36'$;	$c = 2362.61$.

$$a = 500.$$

$$A = 10°\ 12'$$

$$B = 46°\ 36'$$

$$A + B = 56°\ 48'$$

$$\therefore C = 123°\ 12'.$$

$$\log a = 2.69897$$
$$\text{colog} \sin A = 0.75182$$
$$\log \sin B = 9.86128$$
$$\log b = 3.31207$$
$$b = 2051.48.$$

$$\log a = 2.69897$$
$$\text{colog} \sin A = 0.75182$$
$$\log \sin C = 9.92260$$
$$\log c = 3.37339$$
$$c = 2362.61.$$

2. Given $a = 795$, $A = 79° 59'$, $B = 44° 41'$; find $C = 55° 20'$, $b = 567.688$, $c = 663.986$.

$$a = 795.$$

$$\begin{aligned} A &= 79° 59' \\ B &= \underline{44° 41'} \\ A + B &= 124° 40' \\ \therefore C &= 55° 20'. \end{aligned}$$

$$\begin{aligned} \log a &= 2.90037 \\ \text{colog} \sin A &= 0.00667 \\ \log \sin B &= \underline{9.84707} \\ \log b &= 2.75411 \\ b &= 567.688. \end{aligned}$$

$$\begin{aligned} \log a &= 2.90037 \\ \text{colog} \sin A &= 0.00667 \\ \log \sin C &= \underline{9.91512} \\ \log c &= 2.82216 \\ c &= 663.986. \end{aligned}$$

3. Given $a = 804$, $A = 99° 55'$, $B = 45° 1'$; find $C = 35° 4'$, $b = 577.31$, $c = 468.93$.

$$a = 804.$$

$$\begin{aligned} A &= 99° 55' \\ B &= \underline{45° 1'} \\ A + B &= 144° 56' \\ \therefore C &= 35° 4'. \end{aligned}$$

$$\begin{aligned} \log a &= 2.90526 \\ \text{colog} \sin A &= 0.00654 \\ \log \sin B &= \underline{9.84961} \\ \log b &= 2.76141 \\ b &= 577.31. \end{aligned}$$

$$\begin{aligned} \log a &= 2.90526 \\ \text{colog} \sin A &= 0.00654 \\ \log \sin C &= \underline{9.75931} \\ \log c &= 2.67111 \\ c &= 468.93. \end{aligned}$$

4. Given $a = 820$, $A = 12° 49'$, $B = 141° 59'$; find $C = 25° 12'$, $b = 2276.63$, $c = 1573.89$.

$$a = 820.$$

$$\begin{aligned} A &= 12° 49' \\ B &= \underline{141° 59'} \\ A + B &= 154° 48' \\ \therefore C &= 25° 12'. \end{aligned}$$

$$\begin{aligned} \log a &= 2.91381 \\ \text{colog} \sin A &= 0.65398 \\ \log \sin B &= \underline{9.78950} \\ \log b &= 3.35729 \\ b &= 2276.63. \end{aligned}$$

$$\begin{aligned} \log a &= 2.91381 \\ \text{colog} \sin A &= 0.65398 \\ \log \sin C &= \underline{9.62918} \\ \log c &= 3.19697 \\ c &= 1573.89. \end{aligned}$$

5. Given $c = 1005$, $A = 78° 19'$, $B = 54° 27'$; find $C = 47° 14'$, $a = 1340.6$, $b = 1113.8$.

$$c = 1005.$$

$$\begin{aligned} A &= 78° 19' \\ B &= \underline{54° 27'} \\ A + B &= 132° 46' \\ \therefore C &= 47° 14'. \end{aligned}$$

$$\begin{aligned}
\log c &= 3.00217\\
\text{colog} \sin C &= 0.13423\\
\log \sin A &= 9.99091\\
\hline
\log a &= 3.12731\\
a &= 1340.6.
\end{aligned}$$

$$\begin{aligned}
\log c &= 3.00217\\
\text{colog} \sin C &= 0.13423\\
\log \sin B &= 9.91042\\
\hline
\log b &= 3.04682\\
b &= 1113.8.
\end{aligned}$$

6. Given $b = 13.57$, $B = 13° 57'$, $C = 57° 13'$; find $A = 108° 50'$, $a = 53.276$, $c = 47.324$.

$$\begin{aligned}
b &= 13.57.\\
B &= 13° 57'\\
C &= 57° 13'\\
\hline
B + C &= 71° 10'\\
\therefore A &= 108° 50'.
\end{aligned}$$

$$\begin{aligned}
\log b &= 1.13258\\
\text{colog} \sin B &= 0.61785\\
\log \sin A &= 9.97610\\
\hline
\log a &= 1.72653\\
a &= 53.276.
\end{aligned}$$

$$\begin{aligned}
\log a &= 1.72653\\
\text{colog} \sin A &= 0.02390\\
\log \sin C &= 9.92465\\
\hline
\log c &= 1.67508\\
c &= 47.324.
\end{aligned}$$

7. Given $a = 6412$, $A = 70° 55'$, $C = 52° 9'$; find $B = 56° 56'$, $b = 5685.9$, $c = 5357.5$.

$$\begin{aligned}
a &= 6412.\\
A &= 70° 55'\\
C &= 52° 9'\\
\hline
A + C &= 123° 4'\\
\therefore B &= 56° 56'.
\end{aligned}$$

$$\begin{aligned}
\log a &= 3.80699\\
\log \sin B &= 9.92326\\
\text{colog} \sin A &= 0.02455\\
\hline
\log b &= 3.75480\\
b &= 5685.9.
\end{aligned}$$

$$\begin{aligned}
\log a &= 3.80699\\
\log \sin C &= 9.89742\\
\text{colog} \sin A &= 0.02455\\
\hline
\log c &= 3.72896\\
c &= 5357.5.
\end{aligned}$$

8. Given $b = 999$, $A = 37° 58'$, $C = 65° 2'$; find $B = 77°$, $a = 630.77$, $c = 929.48$.

$$\begin{aligned}
b &= 999.\\
A &= 37° 58'\\
C &= 65° 2'\\
\hline
A + C &= 103°\\
\therefore B &= 77°.
\end{aligned}$$

$$\begin{aligned}
\log b &= 2.99957\\
\text{colog} \sin B &= 0.01128\\
\log \sin A &= 9.78902\\
\hline
\log a &= 2.79987\\
a &= 630.77.
\end{aligned}$$

$$\begin{aligned}
\log b &= 2.99957\\
\text{colog} \sin B &= 0.01128\\
\log \sin C &= 9.95739\\
\hline
\log c &= 2.96824\\
c &= 929.48.
\end{aligned}$$

9. In order to determine the distance of a hostile fort A from a place B, a line BC and the angles ABC and BCA were measured, and found to be 322.55 yards, 60° 34′, and 56° 10′, respectively. Find the distance AB.

$$\begin{aligned} a &= 322.55. \\ B &= \ \ 60^\circ\ 34' \\ C &= \ \ 56^\circ\ 10' \\ \hline B + C &= 116^\circ\ 44' \\ \therefore\ A &= \ \ 63^\circ\ 16'. \end{aligned}$$

$$\begin{aligned} \log a &= 2.50860 \\ \text{colog} \sin A &= 0.04910 \\ \log \sin C &= 9.91942 \\ \hline \log c &= 2.47712 \\ c &= 300. \end{aligned}$$

10. In making a survey by triangulation, the angles B and C of a triangle ABC were found to be 50° 30′ and 122° 9′, respectively, and the length BC is known to be 9 miles. Find AB and AC.

$$\begin{aligned} C &= 122^\circ\ \ 9' \\ B &= \ \ 50^\circ\ 30' \\ \hline B + C &= 172^\circ\ 39' \\ \therefore\ A &= \ \ \ \ 7^\circ\ 21'. \end{aligned}$$

$$\begin{aligned} \log BC &= 0.95424 \\ \text{colog} \sin A &= 0.89303 \\ \log \sin B &= 9.88741 \\ \hline \log b &= 1.73468 \\ b = AC &= 54.285. \end{aligned}$$

$$\begin{aligned} \log BC &= 0.95424 \\ \text{colog} \sin A &= 0.89303 \\ \log \sin C &= 9.92771 \\ \hline \log c &= 1.77498 \\ c = AB &= 59.564. \end{aligned}$$

11. Two observers 5 miles apart on a plain, and facing each other, find that the angles of elevation of a balloon in the same vertical plane with themselves are 55° and 58°, respectively. Find the distance from the balloon to each observer, and also the height of the balloon above the plain.

$$\begin{aligned} B &= \ \ 58^\circ \\ A &= \ \ 55^\circ \\ \hline A + B &= 113^\circ \\ \therefore\ C &= \ \ 67^\circ. \end{aligned}$$

$$\begin{aligned} \log c &= 0.69897 \\ \text{colog} \sin C &= 0.03597 \\ \log \sin A &= 9.91336 \\ \hline \log a &= 0.64830 \\ a = BC &= 4.4494. \end{aligned}$$

$$\begin{aligned} \log c &= 0.69897 \\ \text{colog} \sin C &= 0.03597 \\ \log \sin B &= 9.92842 \\ \hline \log b &= 0.66336 \\ b = AC &= 4.6064. \end{aligned}$$

To find h.

$$\frac{h}{a} = \sin B.$$

$$\therefore h = a \sin B.$$

$$\begin{aligned} \log a &= 0.64830 \\ \log \sin B &= 9.92842 \\ \hline \log h &= 0.57672 \\ h &= 3.7733. \end{aligned}$$

12. In a parallelogram, given a diagonal d and the angles x and y which this diagonal makes with the sides. Find the sides. Compute the results when $d = 11.237$, $x = 19° 1'$, and $y = 42° 54'$.

$$d = 11.237.$$

$$x = 19° \ 1'$$
$$y = 42° \ 54'$$
$$x + y = 61° \ 55'$$
$$\therefore z = 118° \ 5'$$

$$\begin{aligned} \log d &= 1.05065 \\ \text{colog} \sin z &= 0.05440 \\ \log \sin x &= 9.51301 \\ \hline \log a &= 0.61806 \\ a &= 4.1501. \end{aligned}$$

$$\begin{aligned} \log d &= 1.05065 \\ \text{colog} \sin z &= 0.05440 \\ \log \sin y &= 9.84297 \\ \hline \log c &= 0.93802 \\ c &= 8.67. \end{aligned}$$

13. A lighthouse was observed from a ship to bear N. 34° E.; after sailing due south 3 miles, it bore N. 23° E. Find the distance from the lighthouse to the ship in both positions.

$$c = 3.$$

$$A = 23°$$
$$B = (180° - 34°) = 146°$$
$$A + B = 169°$$
$$\therefore C = 11°.$$

$$\begin{aligned} \log c &= 0.47712 \\ \text{colog} \sin C &= 0.71940 \\ \log \sin A &= 9.59188 \\ \hline \log a &= 0.78840 \\ a &= 6.1433. \end{aligned}$$

$$\begin{aligned} \log c &= 0.47712 \\ \text{colog} \sin C &= 0.71940 \\ \log \sin B &= 9.74756 \\ \hline \log b &= 0.94408 \\ b &= 8.7918. \end{aligned}$$

14. In a trapezoid, given the parallel sides a and b, and the angles x and y at the ends of one of the parallel sides. Find the non-parallel sides. Compute the results when $a = 15$, $b = 7$, $x = 70°$, $y = 40°$.

Given parallel sides,

$$AB = 7 \text{ and } \quad DC = 15;$$

also, $ADC = 40°$ and $BCD = 70°$;

required AD and BC.

Draw $AE \parallel BC$;

then $\quad AB = EC \quad$ (||s comp. bet. ||s),

and $\quad DE = DC - AB$

$$= 15 - 7 = 8.$$

Also $AED = BCD = 70°$(ext. int. ∠s).

Now

$$DAE = 180° - (40° + 70°)$$
$$= 70°.$$

But since

$$AED = DAE = 70°,$$

the △ is isosceles, and side

$$DA = DE = 8.$$

Now $AE = BC$, and we are to find BC.

$$\frac{AE}{DE} = \frac{\sin ADE}{\sin DAE}.$$

$$\begin{aligned} \log DE &= 0.90309 \\ \log \sin ADE &= 9.80807 \\ \text{colog} \sin DAE &= 0.02701 \\ \hline \log AE &= 0.73817 \\ AE = BC &= 5.4723. \end{aligned}$$

15. Given $b = 7.07107$, $A = 30°$, $C = 105°$; find a and c without using logarithms.

Let p and q denote the segments of c made by the $\perp$ dropped from C.

$$B = 45°.$$

$$\sin A = \frac{1}{2}.$$

$$\sin B = \tfrac{1}{2}\sqrt{2}.$$

$$\therefore \frac{a}{b} = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{2}}.$$

$$a = \frac{b}{\sqrt{2}}$$

$$= \frac{7.07107}{1.41421} = 5.$$

$$\frac{p}{b} = \cos A = \tfrac{1}{2}\sqrt{3} = 0.86602.$$

$$p = b \times 0.86602$$
$$= 7.07107 \times 0.86602$$
$$= 6.12369.$$

$$\frac{q}{a} = \sin B = \tfrac{1}{2}\sqrt{2} = 0.70711.$$

$$q = a \times 0.70711$$
$$= 5 \times 0.70711 = 3.53555.$$

$$c = p + q$$
$$= 6.12369 + 3.53555$$
$$= 9.65924.$$

16. Given $c = 9.562$, $A = 45°$, $B = 60°$; find a and b without using logarithms.

$$C = 75°.$$

$$a = \frac{c \sin A}{\sin C}.$$

$$\sin C = \sin (45° + 30°)$$
$$= \sin 45° \cos 30° + \cos 45° \sin 30°.$$
$$= \tfrac{1}{2}\sqrt{2} \times \tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{2} \times \tfrac{1}{2}$$
$$= \tfrac{1}{4}(\sqrt{6} + \sqrt{2}).$$

$$\therefore a = \frac{9.562 \times \frac{1}{2}\sqrt{2}}{\frac{1}{4}(\sqrt{6} + \sqrt{2})}$$

$$= \frac{19.124 \times \sqrt{2}}{\sqrt{6} + \sqrt{2}}$$

$$= \frac{(19.124 \times \sqrt{2})(\sqrt{6} - \sqrt{2})}{6 - 2}$$

$$= 9.562(\sqrt{3} - 1)$$
$$= 6.999 = 7.$$

$$b = \frac{a \sin B}{\sin A} = \frac{7 \times \frac{1}{2}\sqrt{3}}{\frac{1}{2}\sqrt{2}}$$

$$= \frac{7\sqrt{3}}{\sqrt{2}} = \frac{7\sqrt{6}}{2}$$

$$= 3.5\sqrt{6} = 8.573.$$

17. The base of a triangle is 600 feet, and the angles at the base are 30° and 120°. Find the other sides and the altitude without using logarithms.

$$AB = 600.$$
$$A = 30°.$$
$$B = 120°.$$
$$\therefore C = 30°.$$
$$A = C.$$
$$a = c = 600 \text{ feet}.$$

$$b = \frac{a \sin B}{\sin A}$$

$$= \frac{600 \times \sin(180° - 60°)}{\sin 30°}$$

$$= \frac{600 \times \frac{1}{2}\sqrt{3}}{\frac{1}{2}}$$

$$= 600 \times 1.732051$$
$$= 1039.2.$$

$$h = b \sin A = 1039.2 \times \tfrac{1}{2}$$
$$= 519.6 \text{ feet}.$$

18. Two angles of a triangle are, the one 20°, the other 40°. Find the ratio of the opposite sides without using logarithms.

Let $x = 20°$,

$y = 40°$,

and a and b be opposite sides.

Then $\frac{\sin x}{\sin y} = \frac{a}{b}$.

$\text{nat} \sin x = 0.3420.$

$\text{nat} \sin y = 0.6428.$

$\therefore a : b :: 3420 : 6428.$

$:: 855 : 1607.$

19. The angles of a triangle are as 5 : 10 : 21, and the side opposite the smallest angle is equal to 3. Find the other sides without using logarithms.

Since the angles A, B, C are as 5 : 10 : 21,

$A = \frac{5}{36}$ of $180° = 25°.$

$B = \frac{10}{36}$ of $180° = 50°.$

$C = \frac{21}{36}$ of $180° = 105°.$

$$b = \frac{a \sin B}{\sin A} = \frac{3 \times 0.766}{0.4226}$$

$$= 5.43775.$$

$$c = \frac{a \sin C}{\sin A} = \frac{3 \times 0.9659}{0.4226}$$

$$= 6.857.$$

20. Given one side of a triangle equal to 27, the adjacent angles equal each to 30°. Find the radius of the circumscribed circle without using logarithms.

$$2R = \frac{a}{\sin A}.$$

$$\sin A = \sin 120°$$

$$= \sin (180° - 60°)$$

$$= \sin 60°.$$

$$\sin 60° = \tfrac{1}{2}\sqrt{3}.$$

$$\therefore 2R = \frac{27}{\frac{1}{2}\sqrt{3}} = \frac{54}{\sqrt{3}} = \frac{54 \times \sqrt{3}}{3}$$

$$= 18\sqrt{3}.$$

$$\therefore \quad R = 9\sqrt{3} = 15.588.$$

Exercise XIV. Page 59.

1. Determine the number of solutions in each of the following cases:

(i.) $a = 80$, $b = 100$, $A = 30°$.

$a = 80 < b = 100,$

but $> b \sin A = 100 \times \frac{1}{2},$

and $A < 90°.$

$\therefore$ two solutions.

(ii.) $a = 50$, $b = 100$, $A = 30°$.

$a = 50 = b \sin A = 100 \times \frac{1}{2}.$

$\therefore$ one solution.

(iii.) $a = 40$, $b = 100$, $A = 30°$.

$a = 40 < b \sin A = 100 \times \frac{1}{2},$

and $A < 90°.$

$\therefore$ no solution.

(iv.) $a = 13.4$, $b = 11.46$, $A = 77° 20'$.

$a = 13.4 > b = 11.46.$

$\therefore$ one solution.

(v.) $a = 70$, $b = 75$, $A = 60°$.

$a = 70 < b = 75,$

but $> b \sin A = 75 \times \frac{1}{2}\sqrt{3},$

and $A = 60° < 90°$.

$\therefore$ two solutions.

(vi.) $a = 134.16$, $b = 84.54$,

$B = 52°\ 9'\ 11''$.

$b < a$, $B < 90°$,

nat sin $B = 0.7897$.

$\therefore$ $b < a \sin B$.

$84.54 < 134.16 \times 0.7897$.

$\therefore$ no solution.

2. Given find

$a = 840$, $B = 12°\ 13'\ 34''$,

$b = 485$, $C = 146°\ 15'\ 26''$,

$A = 21°\ 31'$; $c = 1272.15$.

Here $a > b$, and $\log \sin B < 0$.

$\therefore$ one solution.

$$\begin{aligned}
\text{colog } a &= 7.07572 - 10 \\
\log b &= 2.68574 \\
\log \sin A &= 9.56440 \\
\hline
\log \sin B &= 9.32586 \\
B &= 12°\ 13'\ 34''. \\
\therefore C &= 146°\ 15'\ 26''.
\end{aligned}$$

$$\begin{aligned}
\log a &= 2.92428 \\
\log \sin C &= 9.74466 \\
\text{colog } \sin A &= 0.43560 \\
\hline
\log c &= 3.10454 \\
c &= 1272.15.
\end{aligned}$$

3. Given find

$a = 9.399$, $B = 57°\ 23'\ 40''$,

$b = 9.197$, $C = 2°\ 1'\ 20''$,

$A = 120°\ 35'$; $c = 0.38525$.

$$\begin{aligned}
\text{colog } a &= 9.02692 - 10 \\
\log b &= 0.96365 \\
\log \sin A &= 9.93495 \\
\hline
\log \sin B &= 9.92552 \\
B &= 57°\ 23'\ 40''. \\
\therefore C &= 2°\ 1'\ 20''.
\end{aligned}$$

$$\begin{aligned}
\log a &= 0.97308 \\
\log \sin C &= 8.54761 \\
\text{colog } \sin A &= 0.06505 \\
\hline
\log c &= 9.58574 - 10 \\
c &= 0.38525.
\end{aligned}$$

4. Given find

$a = 91.06$, $B = 41°\ 13'$,

$b = 77.04$, $C = 87°\ 37'\ 54''$,

$A = 51°\ 9'\ 6''$; $c = 116.82$.

$$\begin{aligned}
\text{colog } a &= 8.04067 - 10 \\
\log b &= 1.88672 \\
\log \sin A &= 9.89143 \\
\hline
\log \sin B &= 9.81882 \\
B &= 41°\ 13'. \\
\therefore C &= 87°\ 37'\ 54''.
\end{aligned}$$

$$\begin{aligned}
\log a &= 1.95933 \\
\log \sin C &= 9.99963 \\
\text{colog } \sin A &= 0.10857 \\
\hline
\log c &= 2.06753 \\
c &= 116.82.
\end{aligned}$$

5. Given find

$a = 55.55$, $A = 54°\ 31'\ 13''$,

$b = 66.66$, $C = 47°\ 44'\ 7''$,

$B = 77°\ 44'\ 40''$; $c = 50.481$.

Here $b > a$, and $\log \sin A < 0$.

$\therefore$ one solution.

$$\begin{aligned}
\log a &= 1.74468 \\
\log \sin B &= 9.98999 \\
\text{colog } b &= 8.17613 - 10 \\
\hline
\log \sin A &= 9.91080 \\
A &= 54°\ 31'\ 13''. \\
\therefore C &= 47°\ 44'\ 7''.
\end{aligned}$$

$$\begin{aligned}
\log a &= 1.74468\\
\log\sin C &= 9.86925\\
\text{colog}\sin A &= \underline{0.08920}\\
\log c &= 1.70313\\
c &= 50.481.
\end{aligned}$$

6. Given

$a = 309,\ b = 360,\ A = 21°\ 14'\ 25''$;

find
$$\begin{aligned}
B &= 24°\ 57'\ 54'',\\
B' &= 155°\ \ 2'\ \ 6'',\\
C &= 133°\ 47'\ 41'',\\
C' &= 3°\ 43'\ 29'',\\
c &= 615.67.\\
c' &= 55.41.
\end{aligned}$$

$$\begin{aligned}
\log b &= 2.55630\\
\log\sin A &= 9.55904\\
\text{colog}\ a &= \underline{7.51004 - 10}\\
\log\sin B &= 9.62538\\
B &= 24°\ 57'\ 54''.\\
\therefore C &= 133°\ 47'\ 41''.
\end{aligned}$$

$$\begin{aligned}
\log a &= 2.48996\\
\log\sin C &= 9.85843\\
\text{colog}\sin A &= \underline{0.44096}\\
\log c &= 2.78935\\
c &= 615.67.
\end{aligned}$$

Second Solution.

$$\begin{aligned}
B' &= 180° - B\\
&= 155°\ 2'\ 6''.
\end{aligned}$$

$$\begin{aligned}
C' &= B - A\\
&= 3°\ 43'\ 29''.
\end{aligned}$$

$$\begin{aligned}
\log a &= 2.48996\\
\log\sin C' &= 8.81267\\
\text{colog}\sin A &= \underline{0.44096}\\
\log c' &= 1.74359\\
c' &= 55.41.
\end{aligned}$$

7. Given
$$\begin{aligned}
a &= 8.716,\\
b &= 9.787,\\
A &= 38°\ 14'\ 12'';
\end{aligned}$$

find
$$\begin{aligned}
B &= 44°\ \ 1'\ 28'',\\
B' &= 135°\ 58'\ 32'',\\
C &= 97°\ 44'\ 20'',\\
C' &= 5°\ 47'\ 16'',\\
c &= 13.954,\\
c' &= 1.4203.
\end{aligned}$$

Here $a < b$, and $\log\sin B < 0$.

$\therefore$ two solutions.

$$\begin{aligned}
\text{colog}\ a &= 9.05968 - 10\\
\log b &= 0.99065\\
\log\sin A &= \underline{9.79163}\\
\log\sin B &= 9.84196
\end{aligned}$$

$$\begin{aligned}
B &= 44°\ \ 1'\ 28'',\\
B' &= 135°\ 58'\ 32''.\\
C &= 97°\ 44'\ 20'',\\
C' &= 5°\ 47'\ 16''.
\end{aligned}$$

$$\begin{aligned}
\log a &= 0.94032\\
\log\sin C &= 9.99602\\
\text{colog}\sin A &= \underline{0.20837}\\
\log c &= 1.14471\\
c &= 13.954.
\end{aligned}$$

$$\begin{aligned}
\log a &= 0.94032\\
\log\sin C' &= 9.00365\\
\text{colog}\sin A &= \underline{0.20837}\\
\log c' &= 0.15234\\
c' &= 1.4203.
\end{aligned}$$

8. Given find

Given	find
$a = 4.4,$	$B = 90°,$
$b = 5.21,$	$C = 32°\ 22'\ 43'',$
$A = 57°\ 37'\ 17''$;	$c = 2.79.$

$$\begin{aligned}
\log \sin A &= 9.92661 \\
\log b &= 0.71684 \\
\text{colog } a &= \underline{9.35655 - 10} \\
\log \sin B &= 10.00000 \\
B &= 90°. \\
\therefore C &= 32° \, 22' \, 43''.
\end{aligned}$$

$$\begin{aligned}
\log b &= 0.71684 \\
\log \cos A &= \underline{9.72877} \\
\log c &= 0.44561 \\
c &= 2.791.
\end{aligned}$$

9. Given $a = 34$,

$b = 22$,

$B = 30° \, 20'$;

find $A = 51° \, 18' \, 27''$,

$A' = 128° \, 41' \, 33''$,

$C = 98° \, 21' \, 33''$,

$C' = 20° \, 58' \, 27''$,

$c = 43.098$,

$c' = 15.593$.

Here $b < a$, but $> a \sin B$, and $B < 90°$.

$\therefore$ two solutions

$$\begin{aligned}
\log a &= 1.53148 \\
\log \sin B &= 9.70332 \\
\text{colog } b &= \underline{8.65758 - 10} \\
\log \sin A &= 9.89238
\end{aligned}$$

$$\begin{aligned}
A &= 51° \, 18' \, 27''. \\
A' &= 128° \, 41' \, 33''. \\
\therefore C &= 98° \, 21° \, 33''. \\
\therefore C' &= 20° \, 58' \, 27''.
\end{aligned}$$

$$\begin{aligned}
\log a &= 1.53148 \\
\log \sin C &= 9.99536 \\
\text{colog } \sin A &= \underline{0.10762} \\
\log c &= 1.63446 \\
c &= 43.098.
\end{aligned}$$

$$\begin{aligned}
\log a &= 1.53148 \\
\log \sin C' &= 9.55382 \\
\text{colog } \sin A &= \underline{0.10762} \\
\log c' &= 1.19292 \\
c' &= 15.593.
\end{aligned}$$

10. Given $b = 19$,

$c = 18$,

$C = 15° \, 49'$;

find $B = 16° \, 43' \, 13''$,

$B' = 163° \, 16' \, 47''$,

$A = 147° \, 27' \, 47''$,

$A' = 0° \, 54' \, 13''$,

$a = 35.52$,

$a' = 1.0415$.

$$\begin{aligned}
\log b &= 1.27875 \\
\log \sin C &= 9.43546 \\
\text{colog } c &= \underline{8.74473 - 10} \\
\log \sin B &= 9.45894 \\
B &= 16° \, 43' \, 13''. \\
B' &= 163° \, 16' \, 47''.
\end{aligned}$$

$$\begin{aligned}
A &= 147° \, 27' \, 47''. \\
A' &= 0° \, 54' \, 13''.
\end{aligned}$$

$$\begin{aligned}
\log b &= 1.27875 \\
\text{colog } \sin B &= 0.54106 \\
\log \sin A &= \underline{9.73065} \\
\log a &= 1.55046 \\
a &= 35.519.
\end{aligned}$$

$$\begin{aligned}
\log b &= 1.27875 \\
\text{colog } \sin B' &= 0.54106 \\
\log \sin A' &= \underline{8.19784} \\
\log a' &= 0.01765 \\
a' &= 1.0415.
\end{aligned}$$

11. Given $a = 75$, $b = 29$, $B = 16° \, 15' \, 36''$; find the difference between the areas of the two corresponding triangles.

$$\begin{aligned}
\log a \quad &= 1.87506 \\
\text{colog } b \quad &= 8.53760 - 10 \\
\log \sin B &= 9.44715 \\
\hline
\log \sin A &= 9.85981 \\
A &= 46^\circ\ 23'\ 45'', \\
A' &= 133^\circ\ 36'\ 15''.
\end{aligned}$$

$$\begin{aligned}
C &= 117^\circ\ 20'\ 39'' \\
C' &= 30^\circ\ 8'\ 9''.
\end{aligned}$$

$$\begin{aligned}
\log a \quad &= 1.87506 \\
\text{colog} \sin A &= 0.14019 \\
\log \sin C &= 9.94854 \\
\hline
\log c \quad &= 1.96379
\end{aligned}$$

$$\begin{aligned}
\log a \quad &= 1.87506 \\
\text{colog} \sin A &= 0.14019 \\
\log \sin C' &= 9.70075 \\
\hline
\log c' \quad &= 1.71600
\end{aligned}$$

$$\begin{aligned}
F &= \tfrac{1}{2} ch. \\
h &= b \sin A.
\end{aligned}$$

$$\begin{aligned}
\log b \quad &= 1.46240 \\
\log \sin A &= 9.85981 \\
\hline
\log h \quad &= 1.32221
\end{aligned}$$

$$\begin{aligned}
\log c \quad &= 1.96379 \\
\log h \quad &= 1.32221 \\
\text{colog } 2 \quad &= 9.69897 \\
\hline
\log F \quad &= 2.98497 \\
F \quad &= 965.98.
\end{aligned}$$

$$\begin{aligned}
\log c' \quad &= 1.71600 \\
\log h \quad &= 1.32221 \\
\text{colog } 2 \quad &= 9.69897 \\
\hline
\log F' \quad &= 2.73718 \\
F' \quad &= 545.99. \\
F - F' \quad &= 419.99.
\end{aligned}$$

12. Given in a parallelogram the side a, a diagonal d, and the angle A made by the two diagonals; find the other diagonal.

Special case: $a = 35$, $d = 63$, $A = ^\circ\ 36'\ 30''$.

$$\begin{aligned}
a &= 35. \\
\tfrac{1}{2} d &= 31.5. \\
A &= 21^\circ\ 36'\ 30''.
\end{aligned}$$

$$\begin{aligned}
\text{colog } a \quad &= 8.45593 - 10 \\
\log \tfrac{1}{2} d \quad &= 1.49831 \\
\log \sin A &= 9.56615 \\
\hline
\log \sin B &= 9.52039 \\
B \quad &= 19^\circ\ 21'\ 20''. \\
C \quad &= 139^\circ\ 2'\ 10''.
\end{aligned}$$

$$\begin{aligned}
\log a \quad &= 1.54407 \\
\log \sin C &= 9.81663 \\
\text{colog} \sin A &= 0.43385 \\
\hline
\log \tfrac{1}{2} d' &= 1.79455 \\
\tfrac{1}{2} d' \quad &= 62.3085. \\
d' \quad &= 124.617.
\end{aligned}$$

Exercise XV. Page 62.

1. Given $a = 77.99$, $b = 83.39$, $C = 72^\circ\ 15'$; find $A = 51^\circ\ 15'$, $B = 56^\circ\ 30'$, $c = 95.24$.

$$\begin{aligned}
a + b &= 161.38 \\
b - a &= 5.4
\end{aligned}$$

$$\begin{aligned}
A + B &= 107^\circ\ 45' \\
\tfrac{1}{2}(A + B) &= 53^\circ\ 52'\ 30'' \\
\hline
\tfrac{1}{2}(B - A) &= 2^\circ\ 37'\ 30'' \\
A &= 51^\circ\ 15'. \\
B &= 56^\circ\ 30'.
\end{aligned}$$

$$\begin{aligned}
\log(b-a) &= 0.73239\\
\operatorname{colog}(a+b) &= 7.79215-10\\
\log\tan\tfrac12(A+B) &= 0.13675\\
\hline
\log\tan\tfrac12(B-A) &= 8.66129\\
\tfrac12(B-A) &= 2^\circ\ 37'\ 30''.
\end{aligned}$$

$$\begin{aligned}
\log b &= 1.92111\\
\log\sin C &= 9.97882\\
\operatorname{colog}\sin B &= 0.07889\\
\hline
\log c &= 1.97882\\
c &= 95.24.
\end{aligned}$$

2. Given $b = 872.5$, $c = 632.7$, $A = 80^\circ$; find $B = 60^\circ\ 45'$, $C = 39^\circ\ 15'$, $a = 984.8$.

$$\begin{aligned}
b-c &= 239.8\\
b+c &= 1505.2\\
B+C &= 100^\circ\\
\tfrac12(B+C) &= 50^\circ\\
\hline
\tfrac12(B-C) &= 10^\circ\ 45'\\
B &= 60^\circ\ 45'.\\
C &= 39^\circ\ 15'.
\end{aligned}$$

$$\begin{aligned}
\log(b-c) &= 2.37985\\
\log\tan\tfrac12(B+C) &= 0.07619\\
\operatorname{colog}(b+c) &= 6.82240-10\\
\hline
\log\tan\tfrac12(B-C) &= 9.27844\\
\tfrac12(B-C) &= 10^\circ\ 45'.
\end{aligned}$$

$$\begin{aligned}
\log b &= 2.94077\\
\log\sin A &= 9.99335\\
\operatorname{colog}\sin B &= 0.05924\\
\hline
\log a &= 2.99336\\
a &= 984.83.
\end{aligned}$$

3. Given $a = 17$, $b = 12$, $C = 59^\circ\ 17'$; find $A = 77^\circ\ 12'\ 53''$, $B = 43^\circ\ 30'\ 7''$, $c = 14.987$.

$$\begin{aligned}
a+b &= 29\\
a-b &= 5\\
A+B &= 120^\circ\ 43'\\
\tfrac12(A+B) &= 60^\circ\ 21'\ 30''\\
\hline
\tfrac12(A-B) &= 16^\circ\ 51'\ 23''\\
A &= 77^\circ\ 12'\ 53''.\\
B &= 43^\circ\ 30'\ 7''.
\end{aligned}$$

$$\begin{aligned}
\log(a-b) &= 0.69897\\
\operatorname{colog}(a+b) &= 8.53760-10\\
\log\tan\tfrac12(A+B) &= 10.24486\\
\hline
\log\tan\tfrac12(A-B) &= 9.48143\\
\tfrac12(A-B) &= 16^\circ\ 51'\ 23''.
\end{aligned}$$

$$\begin{aligned}
\log b &= 1.07918\\
\log\sin C &= 9.93435\\
\operatorname{colog}\sin B &= 0.16218\\
\hline
\log c &= 1.17571\\
c &= 14.987.
\end{aligned}$$

4. Given $b = \sqrt5$, $c = \sqrt3$, $A = 35^\circ\ 53'$; find $B = 93^\circ\ 28'\ 36''$, $C = 50^\circ\ 38'\ 24''$, $a = 1.313$.

$$\begin{aligned}
\sqrt5 &= 2.2361\\
\sqrt3 &= 1.7321\\
b+c &= 3.9681\\
b-c &= 0.5040\\
B+C &= 144^\circ\ 7'\\
\tfrac12(B+C) &= 72^\circ\ 3'\ 30''\\
\hline
\tfrac12(B-C) &= 21^\circ\ 25'\ 6''\\
B &= 93^\circ\ 28'\ 36''.\\
C &= 50^\circ\ 38'\ 24''.
\end{aligned}$$

$$\begin{aligned}
\log(b-c) &= 9.70243-10\\
\operatorname{colog}(b+c) &= 9.40142-10\\
\log\tan\tfrac12(B+C) &= 10.48973\\
\hline
\log\tan\tfrac12(B-C) &= 9.59358\\
\tfrac12(B-C) &= 21^\circ\ 25'\ 6''.
\end{aligned}$$

$\log c$	$= 0.23856$
$\log \sin A$	$= 9.76800$
$\text{colog} \sin C$	$= 0.11172$
$\log a$	$= 0.11828$
a	$= 1.313.$

5. Given $a = 0.917$, $b = 0.312$, $C = 33° 7' 9''$; find $A = 132° 18' 27''$, $B = 14° 34' 24''$, $c = 0.67748$.

$$a + b = 1.229$$
$$a - b = 0.605$$
$$A + B = 146° 52' 51''$$
$$\tfrac{1}{2}(A + B) = 73° 26' 25''$$
$$\tfrac{1}{2}(A - B) = 58° 52' 1''$$
$$A = 132° 18' 27''.$$
$$B = 14° 34' 24''.$$

$\log (a - b)$	$= 9.78176 - 10$
$\log \tan \frac{1}{2}(A+B)$	$= 10.52674$
$\text{colog} (a + b)$	$= 9.91045 - 10$
$\log \tan \frac{1}{2}(A-B)$	$= 10.21895$
$\frac{1}{2}(A-B)$	$= 58° 52' 1''.$

$\log b$	$= 9.49415 - 10$
$\log \sin C$	$= 9.73750$
$\text{colog} \sin B$	$= 0.59925$
$\log c$	$= 9.83090 - 10$
c	$= 0.67748.$

6. Given $a = 13.715$, $c = 11.214$, $B = 15° 22' 36''$; find $A = 118° 55' 49''$, $C = 45° 41' 35''$, $b = 4.1554$.

$$a - c = 2.501.$$
$$a + c = 24.929.$$
$$A + C = 164° 37' 24''$$
$$\tfrac{1}{2}(A+C) = 82° 18' 42''$$
$$\tfrac{1}{2}(A-C) = 36° 37' 7''$$
$$A = 118° 55' 49''.$$
$$C = 45° 41' 35''.$$

$\log (a - c)$	$= 0.39811$
$\log \tan \frac{1}{2}(A + C)$	$= 0.86968$
$\text{colog} (a + c)$	$= 8.60330 - 10$
$\log \tan \frac{1}{2}(A - C)$	$= 9.87109$
$\frac{1}{2}(A - C)$	$= 36° 37' 7''.$

$\log \sin B$	$= 9.42352$
$\log a$	$= 1.13720$
$\text{colog} \sin A$	$= 0.05789$
$\log b$	$= 0.61861$
b	$= 4.1554.$

7. Given $b = 3000.9$, $c = 1587.2$, $A = 86° 4' 4''$; find $B = 65° 13' 51''$, $C = 28° 42' 5''$, $a = 3297.2$.

$$b + c = 4588.1$$
$$b - c = 1413.7$$
$$B + C = 93° 55' 56''$$
$$\tfrac{1}{2}(B + C) = 46° 57' 58''$$
$$\tfrac{1}{2}(B - C) = 18° 15' 53''$$
$$C = 28° 42' 5''.$$
$$B = 65° 13' 51''.$$

$\log (b - c)$	$= 3.15036$
$\text{colog} (b + c)$	$= 6.33837 - 10$
$\log \tan \frac{1}{2}(B + C)$	$= 0.02983$
$\log \tan \frac{1}{2}(B - C)$	$= 9.51856$
$\frac{1}{2}(B - C)$	$= 18° 15' 53''.$

$\log b$	$= 3.47726$
$\log \sin A$	$= 9.99898$
$\text{colog} \sin B$	$= 0.04191$
$\log a$	$= 3.51815$
a	$= 3297.2.$

8. Given $a = 4527$, $b = 3465$, $C = 66° 6' 27''$; find $A = 68° 29' 15''$, $B = 45° 24' 18''$, $c = 4449$.

$$a + b = 7992.$$
$$a - b = 1062.$$
$$A + B = 113° 53' 33''$$
$$\tfrac{1}{2}(A + B) = 56° 56' 47''$$
$$\tfrac{1}{2}(A - B) = 11° 32' 28''$$
$$A = 68° 29' 15''.$$
$$B = 45° 24' 18''.$$

$\log (a - b)$	$= 3.02612$
$\text{colog}\ (a + b)$	$= 6.09734 - 10$
$\log \tan \frac{1}{2}(A + B)$	$= 10.18659$
$\log \tan \frac{1}{2}(A - B)$	$= 9.31005$
$\frac{1}{2}(A - B)$	$= 11° 32' 28''.$

$\log \sin C$	$= 9.96109$
$\text{colog} \sin A$	$= 0.03136$
$\log a$	$= 3.65581$
$\log c$	$= 3.64826$
c	$= 4449.$

9. Given $a = 55.14$, $b = 33.09$, $C = 30° 24'$; find $A = 117° 24' 33''$, $B = 32° 11' 27''$, $c = 31.431$.

$$a + b = 88.23$$
$$a - b = 22.05$$
$$A + B = 149° 36'$$
$$\tfrac{1}{2}(A + B) = 74° 48'$$
$$\tfrac{1}{2}(A - B) = 42° 36' 33''$$
$$A = 117° 24' 33''.$$
$$B = 32° 11' 27''.$$

$\log (a - b)$	$= 1.34341$
$\text{colog}\ (a + b)$	$= 8.05438 - 10$
$\log \tan \frac{1}{2}(A + B)$	$= 10.56592$
$\log \tan \frac{1}{2}(A - B)$	$= 9.96371$
$\frac{1}{2}(A - B)$	$= 42° 36' 33''.$

$\log b$	$= 1.51970$
$\log \sin C$	$= 9.70418$
$\text{colog} \sin B$	$= 0.27348$
$\log c$	$= 1.49736$
c	$= 31.431.$

10. Given $a = 47.99$, $b = 33.14$, $C = 175° 19' 10''$; find $A = 2° 46' 8''$, $B = 1° 54' 42''$, $c = 81.066$.

$$a + b = 81.13$$
$$a - b = 14.85$$
$$A + B = 4° 40' 50''$$
$$\tfrac{1}{2}(A + B) = 2° 20' 25''$$
$$\tfrac{1}{2}(A - B) = 0° 25' 43''$$
$$A = 2° 46' 8''.$$
$$B = 1° 54' 42''.$$

$\log (a - b)$	$= 1.17173$
$\text{colog}\ (a + b)$	$= 8.09082 - 10$
$\log \tan \frac{1}{2}(A + B)$	$= 8.61138$
$\log \tan \frac{1}{2}(A - B)$	$= 7.87393$
$\frac{1}{2}(A - B)$	$= 0° 25' 43''.$

$\log b$	$= 1.52035$
$\log \sin C$	$= 8.91169$
$\text{colog} \sin B$	$= 1.47680$
$\log c$	$= 1.90884$
c	$= 81.066.$

11. If two sides of a triangle are each equal to 6, and the included angle is 60°, find the third side.

Since

$$a = b,$$
$$A = B,$$
$$A + B = 120°.$$
$$\therefore A = B = 60°;$$

$$\begin{aligned} \log a &= 0.77815 \\ \log \sin C &= 9.93753 \\ \operatorname{colog} \sin B &= 0.06247 \\ \hline \log c &= 0.77815 \\ c &= 6. \end{aligned}$$

12. If two sides of a triangle are each equal to 6, and the included angle is 120°, find the third side.

$$A + B = 60°,$$
$$\therefore A = B = 30°,$$
$$a = 6 = b.$$

$$\begin{aligned} \log a &= 0.77815 \\ \log \sin C &= 9.93753 \\ \operatorname{colog} \sin A &= 0.30103 \\ \hline \log c &= 1.01671 \\ c &= 10.392. \end{aligned}$$

13. Apply Solution I. to the case in which $a = b$ or the triangle is isosceles.

If $a = b$ and the Δ is isosceles, the angles A and B are equal, being opposite the equal sides.

Now as $a = b$ and $A = B$, the formula

$$\tan \tfrac{1}{2}(A - B) = \frac{a - b}{a + b} \times \tan \tfrac{1}{2}(A + B)$$

will become

$$\tan \tfrac{1}{2}(0) = 0 \times \tan \tfrac{1}{2}(2A).$$
$$0 = 0.$$

14. If two sides of a triangle are 10 and 11, and the included angle is 50°, find the third side.

$$\begin{aligned} a + b &= 21 \\ a - b &= 1 \\ A + B &= 130° \\ \tfrac{1}{2}(A + B) &= 65° \\ \hline \tfrac{1}{2}(A - B) &= 5°\ 49'\ 51'' \\ A &= 70°\ 49'\ 51''. \\ B &= 59°\ 10'\ 9''. \end{aligned}$$

$$\begin{aligned} \log (a - b) &= 0.00000 \\ \operatorname{colog} (a + b) &= 8.67778 - 10 \\ \log \tan \tfrac{1}{2}(A + B) &= 10.33133 \\ \hline \log \tan \tfrac{1}{2}(A - B) &= 9.00911 \\ \tfrac{1}{2}(A - B) &= 5°\ 49'\ 51''. \end{aligned}$$

$$\begin{aligned} \log b &= 1.00000 \\ \log \sin C &= 9.88425 \\ \operatorname{colog} \sin B &= 0.06617 \\ \hline \log c &= 0.95042 \\ c &= 8.9212. \end{aligned}$$

15. If two sides of a triangle are 43.301 and 25, and the included angle is 30°, find the third side.

$$\begin{aligned} a + b &= 68.301. \\ a - b &= 18.301. \\ A + B &= 150° \\ \tfrac{1}{2}(A + B) &= 75° \\ \hline \tfrac{1}{2}(A - B) &= 45° \\ A &= 120°. \\ B &= 30°. \end{aligned}$$

$\therefore$ in isos. triangle ABC

$$c = b = 25.$$

$$\begin{aligned} \log (a - b) &= 1.26247 \\ \operatorname{colog} (a + b) &= 8.16557 - 10 \\ \log \tan \tfrac{1}{2}(A + B) &= 10.57195 \\ \hline \log \tan \tfrac{1}{2}(A - B) &= 9.99999 \\ \tfrac{1}{2}(A - B) &= 45°. \end{aligned}$$

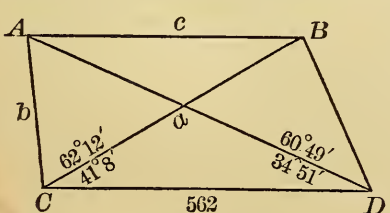
16. In order to find the distance between two objects A and B separated by a swamp, a station C was chosen, and the distances $CA = 3825$ yards, $CB = 3475.6$ yards, together with the angle $ACB = 62° 31'$, were measured. Find the distance from A to B.

$$\begin{aligned} b + a &= 7300.6 \\ b - a &= 349.4 \\ B + A &= 117° 29' \\ \tfrac{1}{2}(B + A) &= 58° 44' 30'' \\ \hline \tfrac{1}{2}(B - A) &= 4° 30' 30'' \\ B &= 63° 15'. \\ A &= 54° 14'. \end{aligned}$$

$\log (b - a)$	$= 2.54332$
$\text{colog } (b + a)$	$= 6.13664 - 10$
$\log \tan \frac{1}{2}(B + A)$	$= 10.21680$
$\log \tan \frac{1}{2}(B - A)$	$= 8.89676$
$\frac{1}{2}(B - A)$	$= 4° 30' 30''.$

$\log b$	$= 3.58263$
$\log \sin C$	$= 9.94799$
$\text{colog} \sin B$	$= 0.04916$
$\log c$	$= 3.57978$
c	$= 3800.$

17. Two inaccessible objects A and B are each viewed from two stations C and D 562 yards apart. The angle ACB is $62° 12'$, BCD $41° 8'$, ADB $60° 49'$, and ADC $34° 51'$; required the distance AB.



In triangle ACD

$$\begin{aligned} A &= 180° - (C + D) \\ &= 41° 49'. \end{aligned}$$

$$\frac{b}{562} = \frac{\sin 34° 51'}{\sin 41° 49'}.$$

$$\therefore\ b = \frac{562 \sin 34° 51'}{\sin 41° 49'}.$$

$\log 562$	$= 2.74974$
$\log \sin 34° 51'$	$= 9.75696$
$\text{colog} \sin 41° 49'$	$= 0.17604$
$\log b$	$= 2.68274$
b	$= 481.65.$

In triangle CBD

$$\begin{aligned} B &= 180° - (C + D) \\ &= 43° 12'. \end{aligned}$$

$$\frac{a}{562} = \frac{\sin 95° 40'}{\sin 43° 12'}.$$

$$\therefore\ a = \frac{562 \cos 5° 40'}{\sin 43° 12'}.$$

$\log 562$	$= 2.74974$
$\log \cos 5° 40'$	$= 9.99787$
$\text{colog} \sin 43° 12'$	$= 0.16460$
$\log a$	$= 2.91221$
a	$= 816.98.$

In triangle ACB

$$\tan \tfrac{1}{2}(A - B) = \frac{a - b}{a + b} \times \tan \tfrac{1}{2}(A + B)$$

$$\begin{aligned} \tfrac{1}{2}(A + B) &= \tfrac{1}{2}(180° - C) \\ &= 58° 54'. \\ a - b &= 816.98 - 481.65 \\ &= 335.33. \\ a + b &= 816.98 + 481.65 \\ &= 1298.63. \end{aligned}$$

$$\begin{aligned}
\log (a-b) &= 2.52547\\
\operatorname{colog} (a+b) &= 6.88651-10\\
\log \tan \tfrac{1}{2}(A+B) &= 10.21951\\
\hline
\log \tan \tfrac{1}{2}(A-B) &= 9.63149\\
\tfrac{1}{2}(A-B) &= 23^\circ\ 10'\ 26''.\\
A &= 82^\circ\ 4'\ 26''.
\end{aligned}$$

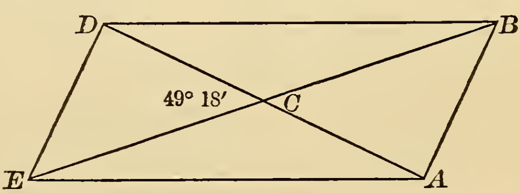
$$\begin{aligned}
\log a &= 2.91221\\
\log \sin C &= 9.94674\\
\operatorname{colog} \sin A &= 0.00418\\
\hline
\log c &= 2.86313\\
c &= 729.68.
\end{aligned}$$

18. Two trains start at the same time from the same station, and move along straight tracks that form an angle of 30°, one train at the rate of 30 miles an hour, the other at the rate of 40 miles an hour. How far apart are the trains at the end of half an hour?

$$\begin{aligned}
a+b &= 35\\
a-b &= 5\\
A+B &= 150^\circ\\
\tfrac{1}{2}(A+B) &= 75^\circ\\
\hline
\tfrac{1}{2}(A-B) &= 28^\circ\ 4'\\
B &= 46^\circ\ 56'\\
A &= 103^\circ\ 4'.
\end{aligned}$$

$$\begin{aligned}
\log (a-b) &= 0.69897\\
\operatorname{colog} (a+b) &= 8.45593-10\\
\log \tan \tfrac{1}{2}(A+B) &= 10.57197\\
\hline
\log \tan \tfrac{1}{2}(A-B) &= 9.72687\\
\tfrac{1}{2}(A-B) &= 28^\circ\ 4'.
\end{aligned}$$

$$\begin{aligned}
\log b &= 1.17609\\
\log \sin C &= 9.69897\\
\operatorname{colog} \sin B &= 0.13634\\
\hline
\log c &= 1.01140\\
c &= 10.266.
\end{aligned}$$



19. In a parallelogram given the two diagonals 5 and 6, and the angle that they form 49° 18′. Find the sides.

In the parallelogram $ABDE$

let $EB = 6$,

$AD = 5$,

and $\angle BCA = 49^\circ\ 18'$.

In triangle ACB

let $BC = a = 3$.

$AC = b = 2.5$.

Find $AB = c$.

$$\begin{aligned}
a-b &= 0.5\\
a+b &= 5.5\\
A+B &= 130^\circ\ 42'\\
\tfrac{1}{2}(A+B) &= 65^\circ\ 21'\\
\hline
\tfrac{1}{2}(A-B) &= 11^\circ\ 12'\ 20''\\
A &= 76^\circ\ 33'\ 20''.\\
B &= 54^\circ\ 8'\ 40''.
\end{aligned}$$

$$\begin{aligned}
\log (a-b) &= 9.69897-10\\
\operatorname{colog} (a+b) &= 9.25964-10\\
\log \tan \tfrac{1}{2}(A+B) &= 10.33829\\
\hline
\log \tan \tfrac{1}{2}(A-B) &= 9.29690\\
\tfrac{1}{2}(A-B) &= 11^\circ\ 12'\ 20''
\end{aligned}$$

$$\begin{aligned}
\log a &= 0.47712\\
\operatorname{colog}\sin A &= 0.01207\\
\log\sin C &= 9.87975\\
\log c &= 0.36894\\
c = AB &= 2.3385.
\end{aligned}$$

In triangle AEC

$$\begin{aligned}
EC &= a = 3,\\
AC &= b = 2.5,\\
\angle ACE &= 130^\circ\ 42'.\\
A + E &= 49^\circ\ 18'\\
\tfrac{1}{2}(A+E) &= 24^\circ\ 39'\\
\tfrac{1}{2}(A-E) &= \ \ 2^\circ\ 23'\ 20''\\
A &= 27^\circ\ \ 2'\ 20''.
\end{aligned}$$

$$\begin{aligned}
\log(a-b) &= 9.69897-10\\
\operatorname{colog}(a+b) &= 9.25964-10\\
\log\tan\tfrac{1}{2}(A+E) &= 9.66171\\
\log\tan\tfrac{1}{2}(A-E) &= 8.62032\\
\tfrac{1}{2}(A-E) &= \ \ 2^\circ\ 23'\ 20''\\
A &= 27^\circ\ \ 2'\ 20''
\end{aligned}$$

$$\begin{aligned}
\log a &= 0.47712\\
\operatorname{colog}\sin A &= 0.34238\\
\log\sin c &= 9.87975-10\\
\log c &= 0.69925\\
c = EA &= 5.0032.
\end{aligned}$$

20. In a triangle one angle equals $139^\circ\ 54'$, and the sides forming the angle have the ratio $5:9$. Find the other two angles.

$$\begin{aligned}
a &= 9\\
b &= 5\\
a+b &= 14\\
a-b &= 4\\
A+B &= 40^\circ\ \ 6'\\
\tfrac{1}{2}(A+B) &= 20^\circ\ \ 3'\\
\tfrac{1}{2}(A-B) &= \ \ 5^\circ\ 57'\ 10''\\
A &= 26^\circ\ \ 0'\ 10''.\\
B &= 14^\circ\ \ 5'\ 50''.
\end{aligned}$$

$$\begin{aligned}
\log(a-b) &= 0.60206\\
\operatorname{colog}(a+b) &= 8.85387-10\\
\log\tan\tfrac{1}{2}(A+B) &= 9.56224\\
\log\tan\tfrac{1}{2}(A-B) &= 9.01817\\
\tfrac{1}{2}(A-B) &= 5^\circ\ 57'\ 10''.
\end{aligned}$$

Exercise XVI. Page 67.

1. Given $a = 51$, $b = 65$, $c = 20$; find the angles.

$$\begin{aligned}
a &= 51\\
b &= 65\\
c &= 20\\
2s &= 136\\
s &= 68.\\
s-a &= 17.\\
s-b &= 3.\\
s-c &= 48.
\end{aligned}$$

$$\begin{aligned}
\operatorname{colog} s &= 8.16749-10\\
\operatorname{colog}(s-a) &= 8.76955-10\\
\log(s-b) &= 0.47712\\
\log(s-c) &= 1.68124\\
2)\underline{19.09540}&-20\\
\log\tan\tfrac{1}{2}A &= 9.54770\\
\tfrac{1}{2}A &= 19^\circ\ 26'\ 24''.\\
A &= 38^\circ\ 52'\ 48''.
\end{aligned}$$

$$\begin{aligned}
\text{colog } s &= 8.16749 - 10\\
\text{colog } (s-b) &= 9.52288 - 10\\
\log (s-a) &= 1.23045\\
\log (s-c) &= \underline{1.68124}\\
&2)\underline{20.60206 - 20}\\
\log \tan \tfrac{1}{2} B &= 10.30103\\
\tfrac{1}{2} B &= 63° \ 26' \ 6''.\\
B &= 126° \ 52' \ 12''.\\
A + B &= 165° \ 45'.\\
\therefore C &= 14° \ 15'.
\end{aligned}$$

2. Given $a = 78$, $b = 101$, $c = 29$; find the angles.

$$\begin{aligned}
a &= 78\\
b &= 101\\
c &= \underline{29}\\
2s &= 208\\
s &= 104.\\
s-a &= 26.\\
s-b &= 3.\\
s-c &= 75.
\end{aligned}$$

$$\begin{aligned}
\text{colog } s &= 7.98297 - 10\\
\text{colog } (s-a) &= 8.58503 - 10\\
\log (s-b) &= 0.47712\\
\log (s-c) &= \underline{1.87506}\\
&2)\underline{18.92018 - 20}\\
\log \tan \tfrac{1}{2} A &= 9.46009\\
\tfrac{1}{2} A &= 16° \ 5' \ 27''.\\
A &= 32° \ 10' \ 54''.
\end{aligned}$$

$$\begin{aligned}
\text{colog } s &= 7.98297 - 10\\
\text{colog } (s-b) &= 9.52288 - 10\\
\log (s-a) &= 1.41497\\
\log (s-c) &= \underline{1.87506}\\
&2)\underline{20.79588 - 20}\\
\log \tan \tfrac{1}{2} B &= 10.39794\\
\tfrac{1}{2} B &= 68° \ 11' \ 55''.\\
B &= 136° \ 23' \ 50''.\\
A + B &= 168° \ 34' \ 44''.\\
\therefore C &= 11° \ 25' \ 16''.
\end{aligned}$$

3. Given $a = 111$, $b = 145$, $c = 40$; find the angles.

$$\begin{aligned}
a &= 111\\
b &= 145\\
c &= \underline{40}\\
2s &= 296\\
s &= 148.\\
s-a &= 37.\\
s-b &= 3.\\
s-c &= 108.
\end{aligned}$$

$$\begin{aligned}
\text{colog } s &= 7.82974 - 10\\
\text{colog } (s-a) &= 8.43180 - 10\\
\log (s-b) &= 0.47712\\
\log (s-c) &= \underline{2.03342}\\
&2)\underline{18.77208 - 20}\\
\log \tan \tfrac{1}{2} A &= 9.38604\\
\tfrac{1}{2} A &= 13° \ 40' \ 16''.\\
A &= 27° \ 20' \ 32''.
\end{aligned}$$

$$\begin{aligned}
\text{colog } s &= 7.82974 - 10\\
\log (s-a) &= 1.56820\\
\text{colog } (s-b) &= 9.52288 - 10\\
\log (s-c) &= \underline{2.03342}\\
&2)\underline{20.95424 - 20}\\
\log \tan \tfrac{1}{2} B &= 10.47712\\
\tfrac{1}{2} B &= 71° \ 33' \ 54''.\\
B &= 143° \ 7' \ 48''.\\
B + A &= 170° \ 28' \ 20''.\\
\therefore C &= 9° \ 31' \ 40''.
\end{aligned}$$

4. Given $a = 21$, $b = 26$, $c = 31$; find the angles.

$$\begin{aligned} a &= 21 \\ b &= 26 \\ c &= 31 \\ \hline 2s &= 78 \\ s &= 39. \\ s-a &= 18. \\ s-b &= 13. \\ s-c &= 8. \end{aligned}$$

$$\begin{aligned} \text{colog } s &= 8.40894-10 \\ \text{colog } (s-a) &= 8.74473-10 \\ \log (s-b) &= 1.11394 \\ \log (s-c) &= 0.90309 \\ 2)\ &\underline{19.17070-20} \\ \log \tan \tfrac{1}{2} A &= 9.58535 \\ \tfrac{1}{2} A &= 21^\circ\ 3'\ 6.3''. \\ \therefore\ A &= 42^\circ\ 6'\ 13''. \end{aligned}$$

$$\begin{aligned} \text{colog } s &= 8.40894-10 \\ \log (s-a) &= 1.25527 \\ \text{colog } (s-b) &= 8.88606-10 \\ \log (s-c) &= 0.90309 \\ 2)\ &\underline{19.45336-20} \\ \log \tan \tfrac{1}{2} B &= 9.72668 \\ \tfrac{1}{2} B &= 28^\circ\ 3'\ 18''. \\ \therefore\ B &= 56^\circ\ 6'\ 36''. \\ A+B &= 98^\circ\ 12'\ 49''. \\ \therefore\ C &= 81^\circ\ 47'\ 11''. \end{aligned}$$

5. Given $a=19$, $b=34$, $c=49$; find the angles.

$$\begin{aligned} a &= 19 \\ b &= 34 \\ c &= 49 \\ \hline 2s &= 102 \\ s &= 51. \\ s-a &= 32. \\ s-b &= 17. \\ s-c &= 2. \end{aligned}$$

$$\begin{aligned} \text{colog } s &= 8.29243-10 \\ \text{colog } (s-a) &= 8.49485-10 \\ \log (s-b) &= 1.23045 \\ \log (s-c) &= 0.30103 \\ 2)\ &\underline{18.31876-20} \\ \log \tan \tfrac{1}{2} A &= 9.15938 \\ \tfrac{1}{2} A &= 8^\circ\ 12'\ 48''. \\ A &= 16^\circ\ 25'\ 36''. \end{aligned}$$

$$\begin{aligned} \text{colog } s &= 8.29243-10 \\ \text{colog } (s-b) &= 8.76955-10 \\ \log (s-c) &= 0.30103 \\ \log (s-a) &= 1.50515 \\ 2)\ &\underline{18.86816-20} \\ \log \tan \tfrac{1}{2} B &= 9.43408 \\ \tfrac{1}{2} B &= 15^\circ\ 12'. \\ B &= 30^\circ\ 24'. \\ \therefore\ C &= 133^\circ\ 10'\ 24''. \end{aligned}$$

6. Given $a=43$, $b=50$, $c=57$; find the angles.

$$\begin{aligned} a &= 43 \\ b &= 50 \\ c &= 57 \\ \hline 2s &= 150 \\ s &= 75. \\ s-a &= 32. \\ s-b &= 25. \\ s-c &= 18. \end{aligned}$$

$$\begin{aligned} \text{colog } s &= 8.12494-10 \\ \text{colog } (s-a) &= 8.49485-10 \\ \log (s-b) &= 1.39794 \\ \log (s-c) &= 1.25527 \\ 2)\ &\underline{19.27300-20} \\ \log \tan \tfrac{1}{2} A &= 9.63650 \\ \tfrac{1}{2} A &= 23^\circ\ 24'\ 47''. \\ A &= 46^\circ\ 49'\ 35''. \end{aligned}$$

$$\begin{aligned}
\text{colog } s &= 8.12494 - 10\\
\log(s-a) &= 1.50515\\
\text{colog }(s-b) &= 8.60206 - 10\\
\log(s-c) &= 1.25527\\
&\ 2)\underline{19.48742 - 20}\\
\log\tan\tfrac{1}{2}B &= 9.74371\\
\tfrac{1}{2}B &= 28^\circ\ 59'\ 52''.\\
B &= 57^\circ\ 59'\ 44''.\\
\therefore C &= 75^\circ\ 10'\ 41''.
\end{aligned}$$

7. Given $a = 37$, $b = 58$, $c = 79$; find the angles.

$$\begin{aligned}
a &= 37\\
b &= 58\\
c &= 79\\
2s &= 174\\
s &= 87.\\
s-a &= 50.\\
s-b &= 29.\\
s-c &= 8.
\end{aligned}$$

$$\begin{aligned}
\text{colog } s &= 8.06048 - 10\\
\text{colog }(s-a) &= 8.30103 - 10\\
\log(s-b) &= 1.46240\\
\log(s-c) &= 0.90309\\
&\ 2)\underline{18.72700 - 20}\\
\log\tan\tfrac{1}{2}A &= 9.36350\\
\tfrac{1}{2}A &= 13^\circ\ 0'\ 14''.\\
A &= 26^\circ\ 0'\ 29''.
\end{aligned}$$

$$\begin{aligned}
\text{colog } s &= 8.06048 - 10\\
\log(s-a) &= 1.69897\\
\text{colog }(s-b) &= 8.53760 - 10\\
\log(s-c) &= 0.90309\\
&\ 2)\underline{19.20014 - 20}\\
\log\tan\tfrac{1}{2}B &= 9.60007\\
\tfrac{1}{2}B &= 21^\circ\ 42'\ 40''.\\
B &= 43^\circ\ 25'\ 20''.\\
\therefore C &= 110^\circ\ 34'\ 11''.
\end{aligned}$$

8. Given $a = 73$, $b = 82$, $c = 91$; find the angles.

$$\begin{aligned}
a &= 73\\
b &= 82\\
c &= 91\\
2s &= 246\\
s &= 123.\\
s-a &= 50.\\
s-b &= 41.\\
s-c &= 32.
\end{aligned}$$

$$\begin{aligned}
\text{colog } s &= 7.91009 - 10\\
\text{colog }(s-a) &= 8.30103 - 10\\
\log(s-b) &= 1.61278\\
\log(s-c) &= 1.50515\\
&\ 2)\underline{19.32905 - 20}\\
\log\tan\tfrac{1}{2}A &= 9.66453\\
\tfrac{1}{2}A &= 24^\circ\ 47'\ 29''.\\
A &= 49^\circ\ 34'\ 58''.
\end{aligned}$$

$$\begin{aligned}
\text{colog } s &= 7.91009 - 10\\
\log(s-a) &= 1.69897\\
\text{colog }(s-b) &= 8.38722 - 10\\
\log(s-c) &= 1.50515\\
&\ 2)\underline{19.50143 - 20}\\
\log\tan\tfrac{1}{2}B &= 9.75072\\
\tfrac{1}{2}B &= 29^\circ\ 23'\ 29''.\\
B &= 58^\circ\ 46'\ 58''.\\
\therefore C &= 71^\circ\ 38'\ 4''.
\end{aligned}$$

9. Given $a = 14.493$, $b = 55.4363$, $c = 66.9129$; find the angles.

$$\begin{aligned}
a &= 14.493\\
b &= 55.4363\\
c &= 66.9129\\
2s &= 136.8422\\
s &= 68.4211.\\
s-a &= 53.9281.\\
s-b &= 12.9848.\\
s-c &= 1.5082.
\end{aligned}$$

$$\begin{aligned}
\text{colog } s &= 8.16481 - 10\\
\text{colog } (s-a) &= 8.26819 - 10\\
\log (s-b) &= 1.11344\\
\log (s-c) &= \underline{0.17846}\\
&\ 2)\overline{17.72490 - 20}\\
\log \tan \tfrac{1}{2} A &= 8.86245\\
\tfrac{1}{2} A &= 4^\circ\ 10'.\\
A &= 8^\circ\ 20'.
\end{aligned}$$

$$\begin{aligned}
\text{colog } s &= 8.16481 - 10\\
\log (s-a) &= 1.73181\\
\text{colog } (s-b) &= 8.88656 - 10\\
\log (s-c) &= \underline{0.17846}\\
&\ 2)\overline{18.96164 - 20}\\
\log \tan \tfrac{1}{2} B &= 9.48082\\
\tfrac{1}{2} B &= 16^\circ\ 50'.\\
B &= 33^\circ\ 40'.\\
\therefore C &= 138^\circ.
\end{aligned}$$

10. Given $a = \sqrt{5}$, $b = \sqrt{6}$, $c = \sqrt{7}$; find the angles.

$$\begin{aligned}
a &= \sqrt{5} = 2.2361\\
b &= \sqrt{6} = 2.4495\\
c &= \sqrt{7} = \underline{2.6458}\\
2s &= 7.3314\\
s &= 3.6657.\\
s-a &= 1.4296.\\
s-b &= 1.2162.\\
s-c &= 1.0199.
\end{aligned}$$

$$\begin{aligned}
\log (s-b) &= 0.08500\\
\log (s-c) &= 0.00856\\
\text{colog } s &= 9.43585 - 10\\
\text{colog } (s-a) &= \underline{9.84478 - 10}\\
&\ 2)\overline{19.37419 - 20}\\
\log \tan \tfrac{1}{2} A &= 9.68709\\
\tfrac{1}{2} A &= 25^\circ\ 56'\ 36''.\\
A &= 51^\circ\ 53'\ 12''.
\end{aligned}$$

$$\begin{aligned}
\text{colog } (s-b) &= 9.91500 - 10\\
\log (s-c) &= 0.00856\\
\text{colog } s &= 9.43585 - 10\\
\log (s-a) &= \underline{0.15522}\\
&\ 2)\overline{19.51463 - 20}\\
\log \tan \tfrac{1}{2} B &= 9.75732\\
\tfrac{1}{2} B &= 29^\circ\ 45'\ 54''.\\
B &= 59^\circ\ 31'\ 48''.\\
\therefore C &= 68^\circ\ 35'.
\end{aligned}$$

11. Given $a = 6$, $b = 8$, $c = 10$; find the angles.

$$\begin{aligned}
a &= 6.\\
b &= 8.\\
c &= 10.\\
s &= 12.\\
s-a &= 6.\\
s-b &= 4.\\
s-c &= 2.
\end{aligned}$$

$$\begin{aligned}
\text{colog } s &= 8.92082 - 10\\
\text{colog } (s-a) &= 9.22185 - 10\\
\log (s-b) &= 0.60206\\
\log (s-c) &= \underline{0.30103}\\
&\ 2)\overline{19.04576 - 20}\\
\log \tan \tfrac{1}{2} A &= 9.52288\\
\tfrac{1}{2} A &= 18^\circ\ 26'\ 6''.\\
A &= 36^\circ\ 52'\ 12''.
\end{aligned}$$

Since this is a right triangle,

$$\begin{aligned}
C &= 90^\circ.\\
B &= 90^\circ - A\\
&= 53^\circ\ 7'\ 48''.
\end{aligned}$$

12. Given $a = 6$, $b = 6$, $c = 10$; find the angles.

$$\begin{aligned}
a &= 6\\
b &= 6\\
c &= \underline{10}\\
2s &= 22
\end{aligned}$$

$$s = 11.$$
$$s - a = 5.$$
$$s - b = 5.$$
$$s - c = 1.$$

$$\begin{aligned}
\text{colog } s &= 8.95861 - 10\\
\text{colog } (s-c) &= 0.00000\\
\log (s-b) &= 0.69897\\
\log (s-a) &= 0.69897\\
2)\,&\underline{20.35655 - 20}\\
\log \tan \tfrac{1}{2} C &= 10.17828\\
\tfrac{1}{2} C &= 56^\circ\ 26'\ 33''.\\
C &= 112^\circ\ 53'\ 6''.
\end{aligned}$$

Since this is an isosceles triangle,

$$A = B = \tfrac{1}{2}(180^\circ - C)$$
$$= 33^\circ\ 33'\ 27''.$$

13. Given $a = 6$, $b = 6$, $c = 6$; find the angles.

The triangle is equilateral and also equiangular.

$\therefore\ A = B = C = \frac{1}{3}$ of $180^\circ = 60^\circ$.

14. Given $a = 6$, $b = 5$, $c = 12$; find the angles.

The sum of the two sides a and b is less than the side c.

$\therefore$ the triangle is impossible.

15. Given $a = 2$, $b = \sqrt{6}$, $c = \sqrt{3} - 1$; find the angles.

$$\begin{aligned}
a &= 2\\
b &= \sqrt{6} = 2.4495\\
c &= \sqrt{3} - 1 = 0.732\\
2s &= 5.1815\\
s &= 2.5908.\\
s - a &= 0.5908.\\
s - b &= 0.1413.\\
s - c &= 1.8588.
\end{aligned}$$

$$\begin{aligned}
\log (s-a) &= 9.77144 - 10\\
\log (s-b) &= 9.15014 - 10\\
\log (s-c) &= 0.26923\\
\text{colog } s &= 9.58656 - 10\\
\log r^2 &= 18.77737 - 20\\
\log r &= 9.38869 - 10
\end{aligned}$$

$$\begin{aligned}
\log \tan \tfrac{1}{2} A &= 9.61725\\
\log \tan \tfrac{1}{2} B &= 10.23855\\
\log \tan \tfrac{1}{2} C &= 9.11946\\
\tfrac{1}{2} A &= 22^\circ\ 30'.\\
\tfrac{1}{2} B &= 60^\circ.\\
\tfrac{1}{2} C &= 7^\circ\ 30'.\\
A &= 45^\circ.\\
B &= 120^\circ.\\
C &= 15^\circ.
\end{aligned}$$

16. Given $a = 2$, $b = \sqrt{6}$, $c = \sqrt{3} + 1$; find the angles.

$$\begin{aligned}
a &= 2\\
b &= \sqrt{6} = 2.4495\\
c &= \sqrt{3} + 1 = 2.732\\
2s &= 7.1815\\
s &= 3.5908\\
s - a &= 1.5908\\
s - b &= 1.1412\\
s - c &= 0.8588
\end{aligned}$$

$$\begin{aligned}
\log (s-a) &= 0.20162\\
\log (s-b) &= 0.05740\\
\log (s-c) &= 9.93385 - 10\\
\text{colog } s &= 9.44481 - 10\\
\log r^2 &= 19.63768 - 20\\
\log r &= 9.81884 - 10
\end{aligned}$$

$$\begin{aligned}
\log \tan \tfrac{1}{2} A &= 9.61721.\\
\log \tan \tfrac{1}{2} B &= 9.76146.\\
\log \tan \tfrac{1}{2} C &= 9.88494.
\end{aligned}$$

$$\tfrac{1}{2}A = 22° \ 30'.$$
$$\tfrac{1}{2}B = 30°.$$
$$\tfrac{1}{2}C = 37° \ 30'.$$

$$A = 45°$$
$$B = 60°$$
$$C = 75°$$

17. The distances between three cities A, B, and C are as follows: $AB = 165$ miles, $AC = 72$ miles, and $BC = 185$ miles. B is due east from A. In what direction is C from A? What two answers are admissible?

$$\begin{aligned} a &= 185 \\ b &= 72 \\ c &= 165 \\ \hline 2s &= 422 \\ s &= 211. \\ s-a &= 26. \\ s-b &= 139. \\ s-c &= 46. \end{aligned}$$

$$\begin{aligned} \text{colog } s &= 7.67572 - 10 \\ \text{colog } (s-a) &= 8.58503 - 10 \\ \log (s-b) &= 2.14301 \\ \log (s-c) &= 1.66276 \\ \hline &2)\ 20.06652 - 20 \\ \log \tan \tfrac{1}{2}A &= 10.03326 \\ \tfrac{1}{2}A &= 47° \ 11' \ 30''. \\ A &= 94° \ 23'. \end{aligned}$$

Angle $BAC = 94° \ 23'$. Subtract 90° of the quadrant E to N, and we obtain 4° 23′ W. of N.

But C may be to the southward of A. Hence two answers are admissible: W. of N. or W. of S.

18. Under what visual angle is an object 7 feet long seen by an observer whose eye is 5 feet from one end of the object and 8 feet from the other end?

$$\begin{aligned} a &= 5 \\ b &= 8 \\ c &= 7 \\ \hline 2s &= 20 \\ s &= 10. \\ s-a &= 5. \\ s-b &= 2. \\ s-c &= 3. \end{aligned}$$

$$\begin{aligned} \text{colog } s &= 9.00000 - 10 \\ \text{colog } (s-a) &= 9.30103 - 10 \\ \log (s-b) &= 0.30103 \\ \log (s-c) &= 0.47712 \\ \hline &2)\ 19.07918 - 20 \\ \log \tan \tfrac{1}{2}A &= 9.53959 \\ \tfrac{1}{2}A &= 19° \ 6' \ 24''. \\ A &= 38° \ 12' \ 48''. \end{aligned}$$

$$\begin{aligned} \text{colog } s &= 9.00000 - 10 \\ \log (s-a) &= 0.69897 \\ \text{colog } (s-b) &= 9.69897 - 10 \\ \log (s-c) &= 0.47712 \\ \hline &2)\ 19.87506 - 20 \\ \log \tan \tfrac{1}{2}B &= 9.93753 \\ \tfrac{1}{2}B &= 40° \ 53' \ 36''. \\ B &= 81° \ 47' \ 12''. \\ \therefore C &= 60°. \end{aligned}$$

19. When Formula [28] is used for finding the value of an angle, why does the ambiguity that occurs in Case II. not exist?

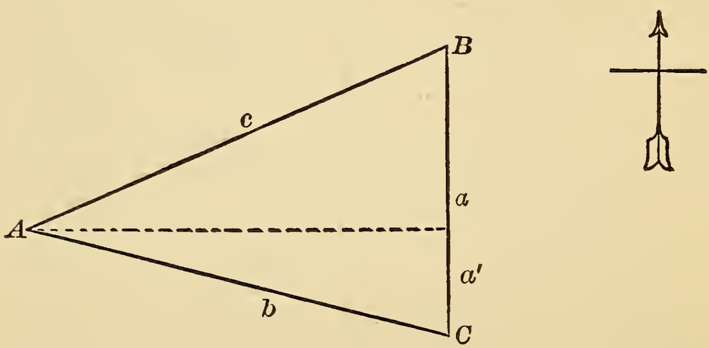
When Formula [28] is used for finding the value of an angle, the ambiguity that occurs in Case II. does not exist because the sides are all known and the angle can have but one value; while in Case II. the side opposite the angle is not known, and may have two values; therefore the angle also may have two values.

20. If the sides of a triangle are 3, 4, and 6, find the sine of the largest angle.

$$\begin{aligned} a &= 3 \\ b &= 4 \\ c &= \underline{6} \\ 2s &= 13 \\ s &= 6.5. \end{aligned}$$

$$\begin{aligned} s-a &= 3.5. \\ s-b &= 2.5. \\ s(s-c) &= 3.25. \end{aligned}$$

$$\begin{aligned} \log(s-a) &= 0.54407 \\ \log(s-b) &= 0.39794 \\ \text{colog}\, s(s-c) &= \underline{9.48812-10} \\ &\ 2)\underline{20.43013-20} \\ \log\tan\tfrac{1}{2}C &= 10.21507 \\ \tfrac{1}{2}C &= 58^\circ\ 38'\ 25''. \\ C &= 117^\circ\ 16'\ 50''. \\ \log\sin C &= 9.94879. \\ \sin C &= 0.88877. \end{aligned}$$



21. Of three towns A, B, and C, A is 200 miles from B and 184 miles from C, B is 150 miles due north from C; how far is A north of C?

$$\begin{aligned} a &= 150 & s &= 267. \\ b &= 184 & s-a &= 117. \\ c &= \underline{200} & s-b &= 83. \\ 2s &= 534 & s-c &= 67. \end{aligned}$$

$$\begin{aligned} \text{colog}\, s &= 7.57349-10 \\ \text{colog}\,(s-c) &= 8.17393-10 \\ \log(s-a) &= 2.06819 \\ \log(s-b) &= \underline{1.91908} \\ &\ 2)\underline{19.73469-20} \\ \log\tan\tfrac{1}{2}A &= 9.86735 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}A &= 36^\circ\ 22'\ 58''. \\ A &= 72^\circ\ 45'\ 56''. \end{aligned}$$

Draw $\perp$ from A to BC. To find a' (part cut off by $\perp$ on BC from c).

$$a' = b\cos C.$$

$$\begin{aligned} \log b &= 2.26482 \\ \log\cos C &= \underline{9.47171} \\ \log a' &= 1.73653 \\ a' &= 54.516. \end{aligned}$$

Exercise XVII. Page 69.

1. Given $a = 4474.5$, $b = 2164.5$, $C = 116° 30' 20''$; find the area.

$$F = \tfrac{1}{2} ab \sin C.$$

$$\begin{aligned} \log a &= 3.65075 \\ \log b &= 3.33536 \\ \text{colog } 2 &= 9.69897 - 10 \\ \log \sin C &= \underline{9.95177} \\ \log F &= 6.63685 \\ F &= 4333600. \end{aligned}$$

2. Given $b = 21.66$, $c = 36.94$, $A = 66° 4' 19''$; find the area.

$$F = \tfrac{1}{2} bc \sin A.$$

$$\begin{aligned} \log b &= 1.33566 \\ \log c &= 1.56750 \\ \log \sin A &= \underline{9.96097} \\ \log 2F &= 2.86413 \\ 2F &= 731.36. \\ F &= 365.68. \end{aligned}$$

3. Given $a = 510$, $c = 173$, $B = 162° 30' 28''$; find the area.

$$\begin{aligned} \log a &= 2.70757 \\ \log c &= 2.23805 \\ \log \sin B &= 9.47795 \\ \text{colog } 2 &= \underline{9.69897 - 10} \\ \log F &= 4.12254 \\ F &= 13260. \end{aligned}$$

4. Given $a = 408$, $b = 41$, $c = 401$; find the area.

$$\begin{aligned} a &= 408 \\ b &= 41 \\ c &= \underline{401} \\ 2s &= 850 \\ s &= 425. \\ s - a &= 17. \\ s - b &= 384. \\ s - c &= 24. \end{aligned}$$

$$\begin{aligned} \log s &= 2.62839 \\ \log (s - a) &= 1.23045 \\ \log (s - b) &= 2.58433 \\ \log (s - c) &= \underline{1.38021} \\ & 2)\,\underline{7.82338} \\ \log F &= 3.91169 \\ F &= 8160. \end{aligned}$$

5. Given $a = 40$, $b = 13$, $c = 37$; find the area.

$$\begin{aligned} a &= 40 \\ b &= 13 \\ c &= \underline{37} \\ 2s &= 90 \\ s &= 45. \\ s - a &= 5. \\ s - b &= 32. \\ s - c &= 8. \end{aligned}$$

$$\begin{aligned} \log s &= 1.65321 \\ \log (s - a) &= 0.69897 \\ \log (s - b) &= 1.50515 \\ \log (s - c) &= \underline{0.90309} \\ & 2)\,\underline{4.76042} \\ \log F &= 2.38021 \\ F &= 240. \end{aligned}$$

6. Given $a = 624$, $b = 205$, $c = 445$; find the area.

$$\begin{aligned} a &= 624 \\ b &= 205 \\ c &= \underline{445} \\ 2s &= 1274 \end{aligned}$$

$$\begin{aligned} s &= 637. \\ s-a &= 13. \\ s-b &= 432. \\ s-c &= 192. \end{aligned}$$

$$\begin{aligned} \log s &= 2.80414 \\ \log(s-a) &= 1.11394 \\ \log(s-b) &= 2.63548 \\ \log(s-c) &= 2.28330 \\ \hline 2\log F &= 8.83686 \\ \log F &= 4.41843. \\ F &= 26208. \end{aligned}$$

7. Given $b=149$, $A=70°\ 42'\ 30''$, $B=39°\ 18'\ 28''$; find the area.

$$\begin{aligned} A &= 70°\ 42'\ 30'' \\ B &= 39°\ 18'\ 28'' \\ \therefore C &= 69°\ 59'\ \ 2'' \end{aligned}$$

$$\begin{aligned} \log b &= 2.17319 \\ \text{colog} \sin B &= 0.19827 \\ \log \sin A &= 9.97490 \\ \hline \log a &= 2.34636 \end{aligned}$$

$$\begin{aligned} \log a &= 2.34636 \\ \text{colog} \sin A &= 0.02510 \\ \log \sin C &= 9.97294 \\ \hline \log c &= 2.34440 \end{aligned}$$

$$\begin{aligned} \text{colog } 2 &= 9.69897-10 \\ \log a &= 2.34636 \\ \log b &= 2.17319 \\ \log \sin C &= 9.97294 \\ \hline \log F &= 4.19146 \\ F &= 15540. \end{aligned}$$

8. Given $a=215.9$, $c=307.7$, $A=25°\ 9'\ 31''$; find the area.

$a < c$ and $> c \sin A$.

$A < 90°$. $\therefore$ two solutions.

$$\begin{aligned} \log c &= 2.48813 \\ \log \sin A &= 9.62852 \\ \text{colog } a &= 7.66575-10 \\ \hline \log \sin C &= 9.78240 \\ C &= 37°\ 17'\ 38''. \\ \therefore B &= 117°\ 32'\ 51''. \\ \text{Or,}\quad C' &= 142°\ 42'\ 22''. \\ \therefore B' &= 12°\ \ 8'\ \ 7''. \end{aligned}$$

$$\begin{aligned} \text{colog } 2 &= 9.69897-10 \\ \log a &= 2.33425 \\ \log c &= 2.48813 \\ \log \sin B &= 9.94774 \\ \hline \log F &= 4.46909 \\ F &= 29450. \end{aligned}$$

$$\begin{aligned} \text{colog } 2 &= 9.69897-10 \\ \log a &= 2.33425 \\ \log c &= 2.48813 \\ \log \sin B' &= 9.32269 \\ \hline \log F' &= 3.84404 \\ F' &= 6983. \end{aligned}$$

9. Given $b=8$, $c=5$, $A=60°$; find the area.

$$\begin{aligned} F &= \tfrac{1}{2} bc \sin A \\ &= \tfrac{1}{2}(8\times 5)(0.86602) \\ &= 20 \times 0.86602 \\ &= 17.3204. \end{aligned}$$

10. Given $a=7$, $c=3$, $A=60°$; find the area.

$$\begin{aligned} \text{colog } a &= 9.15490-10 \\ \log c &= 0.47712 \\ \log \sin A &= 9.93753 \\ \hline \log \sin C &= 9.56955 \\ C &= 21°\ 47'\ 12''. \\ \therefore B &= 98°\ 12'\ 48''. \end{aligned}$$

$$\begin{aligned}
\log a &= 0.84510\\
\log \sin B &= 9.99552\\
\text{colog} \sin A &= 0.06247\\
\hline
\log b &= 0.90309\\
b &= 8.
\end{aligned}$$

$$\begin{aligned}
F &= \tfrac{1}{2} bc \sin A\\
&= \tfrac{1}{2} \times 24 \times \tfrac{1}{2}\sqrt{3}\\
&= 6\sqrt{3}, \text{ or } 10.3923.
\end{aligned}$$

11. Given $a = 60$, $B = 40° \, 35' \, 12''$, area $= 12$; find the radius of the inscribed circle.

$$\tfrac{1}{2} ac \sin B = 12.$$

$$c = \frac{24}{a \sin B}.$$

$$\begin{aligned}
\log 24 &= 1.38021\\
\text{colog}\, a &= 8.22185 - 10\\
\text{colog} \sin B &= 0.18665\\
\hline
\log c &= 9.78875 - 10\\
c &= 0.61483.
\end{aligned}$$

$\tan \frac{1}{2}(A - C)$

$$= \frac{a - c}{a + c} \times \tan \tfrac{1}{2}(A + C)$$

$$= \frac{59.38517}{60.61483} \times \tan (69° \, 42' \, 24'').$$

$$\begin{aligned}
\log (a - c) &= 1.77368\\
\text{colog}\, (a + c) &= 8.21742 - 10\\
\log \tan \tfrac{1}{2}(A + C) &= 0.43206\\
\hline
\log \tan \tfrac{1}{2}(A - C) &= 0.42316
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}(A - C) &= 69° \, 19' \, 19''\\
\tfrac{1}{2}(A + C) &= 69° \, 42' \, 24''\\
\hline
\therefore A &= 139° \, 1' \, 43''
\end{aligned}$$

$$\frac{b}{a} = \frac{\sin B}{\sin A}, \qquad \therefore b = \frac{a \sin B}{\sin A}.$$

$$\begin{aligned}
\log a &= 1.77815\\
\log \sin B &= 9.81331\\
\text{colog} \sin A &= 0.18331\\
\hline
\log b &= 1.77477\\
b &= 59.534.
\end{aligned}$$

$$\begin{aligned}
a &= 60\\
b &= 59.534\\
c &= 0.61483\\
\hline
2s &= 120.14883\\
s &= 60.07442.\\
s - a &= 0.07442.\\
s - b &= 0.54042.\\
s - c &= 59.45959.
\end{aligned}$$

$$\begin{aligned}
\log (s - a) &= 8.87169 - 10\\
\log (s - b) &= 9.73274 - 10\\
\log (s - c) &= 1.77422\\
\text{colog}\, c &= 8.22131 - 10\\
\hline
&2)\underline{18.59996 - 20}\\
\log r &= 9.29998 - 10\\
r &= 0.19952.
\end{aligned}$$

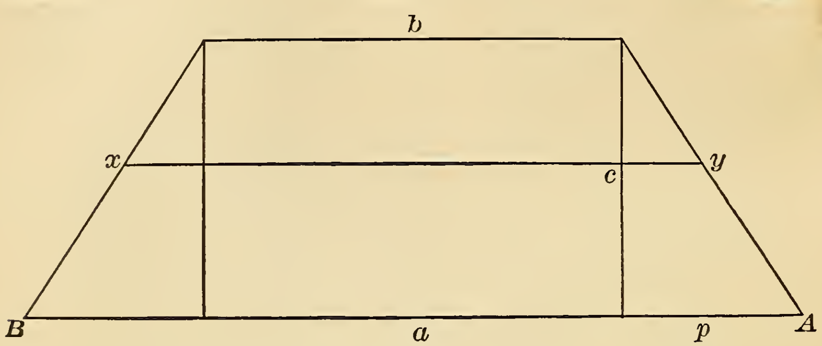
12. Obtain a formula for the area of a parallelogram in terms of two adjacent sides and the included angle.

By Geometry, area of parallelogram = base × height.

In this case $= bh$.

But $h = a \sin A$.

$\therefore$ area of $\square = ab \sin A$.



13. Obtain a formula for the area of an isosceles trapezoid in terms of the two parallel sides and an acute angle.

Let $AB = a$.

$$F = \tfrac{1}{2}(a + b)\,c.$$

$$\frac{c}{p} = \tan A$$

$$c = p \tan A.$$

$$p = \tfrac{1}{2}(a - b).$$

$$\therefore\ F = \tfrac{1}{2}(a + b) \times \tfrac{1}{2}(a - b) \tan A$$
$$= \tfrac{1}{4}(a^2 - b^2) \tan A.$$

14. Two sides and included angle of a triangle are 2416, 1712, and 30°; and two sides and included angle of another triangle are 1948, 2848, and 150°; find the sum of their areas.

Let $a = 2416$, $c = 1712$, $B = 30°$.

$F = \tfrac{1}{2}\,ac \sin B$.

$\log a$	$= 3.38310$
$\log c$	$= 3.23350$
colog 2	$= 9.69897 - 10$
$\log \sin B$	$= 9.69897$
$\log F$	$= 6.01454$
F	$= 1034000.$

Let $a' = 1948$, $c' = 2848$, $B' = 150°$.

$F' = \tfrac{1}{2}\,a'c' \sin B'$.

$\log a'$	$= 3.28959$
$\log c'$	$= 3.45454$
colog 2	$= 9.69897 - 10$
$\log \sin B'$	$= 9.69897$
$\log F'$	$= 6.14207$
F'	$= 1387000.$
$F + F'$	$= 2421000.$

15. The base of an isosceles triangle is 20, and its area is $100 \div \sqrt{3}$; find its angles.

$$a = b.$$

$$c = 20.$$

$$F = 100 \div \sqrt{3}.$$

$$\tfrac{1}{2}\,ch = \frac{100}{\sqrt{3}}.$$

$$10\,h = \frac{100}{\sqrt{3}}.$$

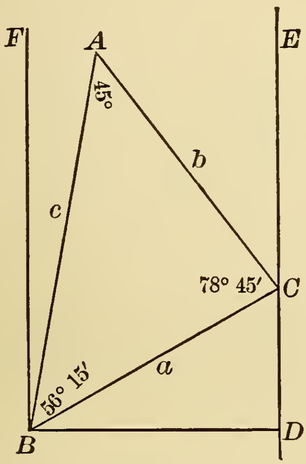
$$h = \frac{10}{\sqrt{3}}.$$

$$\frac{h}{\tfrac{1}{2}c} = \tan A.$$

$\log h$	$= 0.76144$
colog $\tfrac{1}{2}c$	$= 9.00000 - 10$
$\log \tan A$	$= 9.76144$
A	$= 30°.$
B	$= 30°.$
C	$= 120°.$

Exercise XVIII. Page 70.

1. From a ship sailing down the English Channel the Eddystone was observed to bear N. 33° 45′ W.; and after the ship had sailed 18 miles S. 67° 30′ W. it bore N. 11° 15′ E. Find its distance from each position of the ship.



$a = 18$ miles.

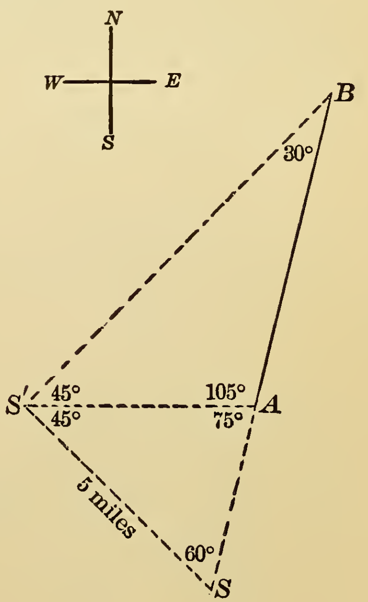
$$\begin{aligned} ACE &= 33° \; 45'. \\ DCB &= 67° \; 30°. \\ ABF &= 11° \; 15'. \\ ACB &= 180° - (ACE + DCB) \\ &= 78° \; 45'. \\ CBD &= 90° - DCB \\ &= 22° \; 30'. \\ ABC &= 90° - (CBD + ABF) \\ &= 56° \; 15'. \\ \therefore BAC &= 45°. \end{aligned}$$

$$\frac{b}{a} = \frac{\sin B}{\sin A}. \qquad \frac{c}{a} = \frac{\sin C}{\sin A}.$$

$$\begin{aligned} \log a &= 1.25527 \\ \log \sin B &= 9.91985 \\ \text{colog} \sin A &= 0.15051 \\ \hline \log b &= 1.32563 \\ b &= 21.166. \end{aligned}$$

$$\begin{aligned} \log a &= 1.25527 \\ \log \sin C &= 9.99157 \\ \text{colog} \sin A &= 0.15051 \\ \hline \log c &= 1.39735 \\ c &= 24.966. \end{aligned}$$

2. Two objects, A and B, were observed from a ship to be at the same instant in a line bearing N. 15° E. The ship then sailed north-west 5 miles, when it was found that A bore due east and B bore north-east. Find the distance from A to B.



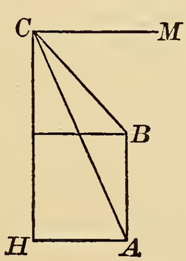
$$\frac{S'A}{SS'} = \frac{\sin ASS'}{\sin S'AS}.$$

$$\begin{aligned}
\log SS' &= 0.69897\\
\text{colog} \sin SAS' &= 0.01506\\
\log \sin ASS' &= 9.93753\\
\hline
\log S'A &= 0.65156
\end{aligned}$$

$$\frac{AB}{S'A} = \frac{\sin BS'A}{\sin S'BA}.$$

$$\begin{aligned}
\log S'A &= 0.65156\\
\text{colog} \sin S'BA &= 0.30103\\
\log \sin BS'A &= 9.84949\\
\hline
\log AB &= 0.80208\\
AB &= 6.3399.
\end{aligned}$$

3. A castle and a monument stand on the same horizontal plane. The angles of depression of the top and the bottom of the monument viewed from the top of the castle are 40° and 80°; the height of the castle is 140 feet. Find the height of the monument.



HC = height of castle.

AB = height of monument.

$MCB = 40°$. $HAC = 80°$.

$HCA = 10°$. $HC = 140$ ft.

$$AC = \frac{140}{\sin A}.$$

$$\begin{aligned}
\log 140 &= 2.14613\\
\text{colog} \sin A &= 0.00665\\
\hline
\log AC &= 2.15278
\end{aligned}$$

$$\begin{aligned}
HCA &= 10°,\\
MCB &= 40°,\\
\therefore ACB &= 40°;\\
CAB &= 10°,\\
\therefore ABC &= 130°.
\end{aligned}$$

$$AB = \frac{AC \sin C}{\sin B}.$$

$$\begin{aligned}
\log AC &= 2.15278\\
\log \sin C &= 9.80807\\
\text{colog} \sin B &= 0.11575\\
\hline
\log AB &= 2.07660\\
AB &= 119.29.
\end{aligned}$$

4. If the sun's altitude is 60°, what angle must a stick make with the horizon in order that its shadow in a horizontal plane may be the longest possible?

The shadow of the stick will be the longest when the stick is perpendicular to the rays of the sun.

Let BC represent the stick, and AC the horizontal plane.

$$\begin{aligned}
B &= 90°.\\
A &= 60°.\\
\therefore C &= 30°.
\end{aligned}$$

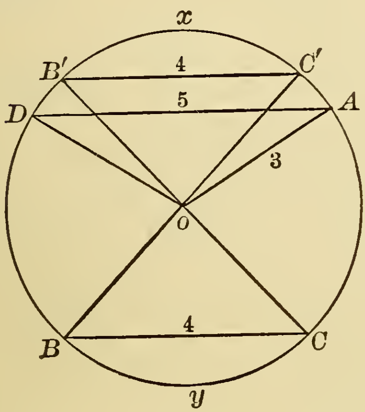
5. If the sun's altitude is 30°, find the length of the longest shadow cast on a horizontal plane by a stick 10 feet in length.

Let a be a stick $\perp$ to rays of sun, and c be the longest shadow.

$$\frac{a}{c} = \sin A.$$

$$\begin{aligned} \log a &= 1.00000 \\ \operatorname{colog} \sin A &= 0.30103 \\ \log c &= 1.30103 \\ c &= 20. \end{aligned}$$

6. In a circle with the radius 3 find the area of the part comprised between parallel chords whose lengths are 4 and 5. (Two solutions.)



$$\begin{aligned} o &= 4 \\ b &= 3 \\ c &= 3 \\ 2s &= 10 \\ s &= 5. \\ s-b &= 2. \\ s-c &= 2. \\ s(s-o) &= 5. \end{aligned}$$

$$\begin{aligned} \log (s-b) &= 0.30103 \\ \log (s-c) &= 0.30103 \\ \operatorname{colog} s(s-a) &= 9.30103-10 \\ &\ 2)\,19.90309-20 \\ \log \tan \tfrac{1}{2} BOC &= 9.95155-10 \\ \tfrac{1}{2} BOC &= 41^\circ\ 48'\ 38''. \\ BOC &= 83^\circ\ 37'\ 16''. \end{aligned}$$

By Table VI.

$$R = 3.$$

$$\therefore \text{ area } \odot = 28.274.$$

$$BOC = \frac{301036}{1296000} \times 360^\circ = \frac{75259}{324000} \times 360^\circ.$$

$\therefore$ area sector BOC

$$= \frac{75259}{324000} \times 28.274.$$

$$\begin{aligned} \log 75259 &= 4.87656 \\ \log 28.274 &= 1.45139 \\ \operatorname{colog} 324000 &= 4.48945-10 \\ \log \text{area} &= 0.81740 \end{aligned}$$

Area sector $BOC = 6.5675$.

In triangle BOC

$$F = \sqrt{s(s-a)(s-b)(s-c)}.$$

$$\begin{aligned} \log s(s-a) &= 0.69897 \\ \log (s-b) &= 0.30103 \\ \log (s-c) &= 0.30103 \\ &\ 2)\,1.30103 \\ \log F &= 0.65052 \\ F &= 4.4722. \end{aligned}$$

Area segment $ByC = 2.0953$.

In triangle DOA

$$\tan \tfrac{1}{2} DOA = \sqrt{\frac{(s-d)(s-a)}{s(s-o)}}.$$

$$\begin{aligned} o &= 5 \\ a &= 3 \\ d &= 3 \\ 2s &= 11 \\ s &= 5.5. \\ s-d &= 2.5. \\ s-a &= 2.5. \\ s(s-o) &= 2.75. \end{aligned}$$

$$\begin{aligned}
\log(s-d) &= 0.39794\\
\log(s-a) &= 0.39794\\
\operatorname{colog} s(s-o) &= 9.56067-10\\
&2)\underline{0.35655}\\
\log\tan\tfrac{1}{2}DOA &= 0.17828\\
\tfrac{1}{2}DOA &= 56°\ 26'\ 35.5''.\\
DOA &= 112°\ 53'\ 11''.
\end{aligned}$$

In triangle DOA

$$F=\sqrt{s(s-o)(s-a)(s-d)}.$$

$$\begin{aligned}
\log s(s-o) &= 0.43933\\
\log(s-d) &= 0.39794\\
\log(s-a) &= 0.39794\\
&2)\underline{1.23521}\\
\log F &= 0.61761\\
F &= 4.1458.
\end{aligned}$$

$$DOA=\frac{406391}{1296000}\times 360°.$$

$$\therefore \text{sector } DOA=\frac{406391}{1296000}\times 28.274.$$

$$\begin{aligned}
\log 406391 &= 5.60894\\
\log 28.274 &= 1.45139\\
\operatorname{colog} 1296000 &= 3.88739-10\\
\log \text{area} &= 0.94772\\
\text{Area sector} &= 8.8658.
\end{aligned}$$

Area segment DxA

$= 4.72.$

Area segment $DACB$

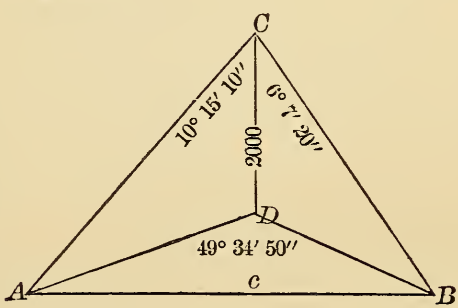
$= \text{area } \odot - [ByC + DxA]$

$= 21.4587.$

Area segment $DAC'B'$

$= DxA - B'xC'$

$= 2.6247.$



7. A and B, two inaccessible objects in the same horizontal plane, are observed from a balloon at C and from a point D directly under the balloon, and in the same horizontal plane with A and B. If $CD = 2000$ yards, $\angle ACD = 10°\,15'\,10''$, $\angle BCD = 6°\,7'\,20''$, $\angle ADB = 49°\,34'\,50''$, find AB.

$$AD = DC\times\tan ACD.$$

$$\begin{aligned}
\log\tan ACD &= 9.25739\\
\log DC &= 3.30103\\
\log AD &= 2.55342\\
AD &= 361.76.
\end{aligned}$$

$$DB = DC\times\tan BCD.$$

$$\begin{aligned}
\log DC &= 3.30103\\
\log\tan BCD &= 9.03045\\
\log DB &= 2.33148\\
DB &= 214.53.
\end{aligned}$$

$$\tan\tfrac{1}{2}(B-A)$$

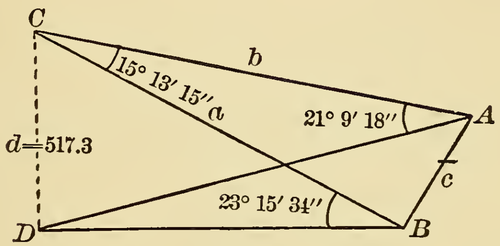
$$=\frac{b-a}{b+a}\times\tan\tfrac{1}{2}(B+A).$$

$\frac{1}{2}(B+A) = 65° \ 12' \ 35''.$

$\log(b-a) = 2.16800$
$\text{colog}(b+a) = 7.23936 - 10$
$\log\tan\frac{1}{2}(B+A) = 0.33549$
$\log\tan\frac{1}{2}(B-A) = 9.74285$
$\frac{1}{2}(B-A) = 28° \ 56' \ 58''.$

$B = 94° \ 9' \ 33''.$

$\log AD = 2.55842$
$\text{colog}\sin B = 0.00115$
$\log\sin C = 9.88156$
$\log c = 2.44113$
$c = AB = 276.14.$



8. A and B are two objects whose distance, on account of intervening obstacles, cannot be directly measured. At the summit C of a hill, whose height above the common horizontal plane of the objects is known to be 517.3 yards, $\angle ACB$ is found to be 15° 13′ 15″. The angles of elevation of C viewed from A and B are 21° 9′ 18″ and 23° 15′ 34″ respectively. Find the distance from A to B.

In triangle DCA, being a rt. △,

$$\frac{d}{b} = \sin A. \qquad b = \frac{d}{\sin A}.$$

$\log d = 2.71374$
$\text{colog}\sin A = 0.44262$
$\log b = 3.15636$
$b = 1433.4.$

In right triangle CDB

$$\frac{d}{a} = \sin B. \qquad a = \frac{d}{\sin B}.$$

$\log d = 2.71374$
$\text{colog}\sin B = 0.40352$
$\log a = 3.11726$
$a = 1310.$

$$\tan\tfrac{1}{2}(B-A) = \frac{b-a}{b+a} \times \tan\tfrac{1}{2}(B+A).$$

$\frac{1}{2}(B+A) = 82° \ 23' \ 22.5''.$

$\log(b-a) = 2.09132$
$\text{colog}(b+a) = 6.56171 - 10$
$\log\tan\frac{1}{2}(B+A) = 10.87415$
$\log\tan\frac{1}{2}(B-A) = 9.52718$
$\frac{1}{2}(B-A) = 18° \ 36' \ 21''.$
$B = 100° \ 59' \ 43.5''.$
$A = 63° \ 47' \ 1.5''$

$$c = \frac{a\sin C}{\sin A}.$$

$\log a = 3.11726$
$\log\sin C = 9.41920$
$\text{colog}\sin A = 0.04714$
$\log c = 2.58360$
$c = 383.35.$

SPHERICAL TRIGONOMETRY.

EXERCISE XIX. PAGE 104.

1. The angles of a triangle are 70°, 80°, and 100°; find the sides of the polar triangle.

Given $A = 70°$, $B = 80°$, $C = 100°$; to find a', b', c'.

$$a' = 180° - 70° = 110°.$$

$$b' = 180° - 80° = 100°.$$

$$c' = 180° - 100° = 80°.$$

2. The sides of a triangle are 40°, 90°, and 125°; find the angles of the polar triangle.

Given $a = 40°$, $b = 90°$, $c = 125°$; required A', B', C'.

$$A' = 180° - 40° = 140°.$$

$$B' = 180° - 90° = 90°.$$

$$C' = 180° - 125° = 55°.$$

3. Prove that the polar of a quadrantal triangle is a right triangle.

Let the triangle ABC be a quadrantal triangle.

Then $\qquad b = 90°.$

Let $A'B'C'$ be the polar triangle.

$$B' + b = 180°.$$

But $\qquad b = 90°.$

$$\therefore B' = 90°.$$

$\therefore$ triangle $A'B'C'$ is a right triangle.

4. Prove that, if a triangle have three right angles, the sides of the triangle are quadrants.

If angles A, B, C, respectively, are right angles, the side b the measure of angle B, c the measure of angle C, and a the measure of angle A, are each $= 90°$;

$\therefore$ sides of triangle ABC are quadrants.

5. Prove that, if a triangle have two right angles, the sides opposite these angles are quadrants, and the third angle is measured by the number of degrees in the opposite side.

In spherical triangle ABC

let $\qquad B = C =$ rt. angle.

We are to prove AC and AB quadrants, also that A is measured by the number of degrees in BC.

Let $A'B'C'$ be the polar triangle of ABC.

Now $\qquad B + b' = 180°,$

$$\therefore b' = 90°;$$

$$C + c' = 180°,$$

$$c' = 90°.$$

$\therefore$ triangle $A'B'C'$ is isosceles, and $B' = 90°$, $C' = 90°$.

But $\qquad B' + b = 180°,$

$\therefore\ b = 90°.$

$C' + c = 180°,$

$\therefore\ c = 90°.$

$\therefore\ b$ and c are quadrants.

Now ABC is bi-rectangular.

$\therefore\ A$ is the pole of side BC.

$\therefore\ A$ is measured by side BC.

6. How can the sides of a spherical triangle be found in units of length, when the length of the radius of the sphere is known.

By using the formula $2\pi R = C$.

For instance, if the sides of a triangle were 40°, 90°, 125°, the sides in terms of R would be

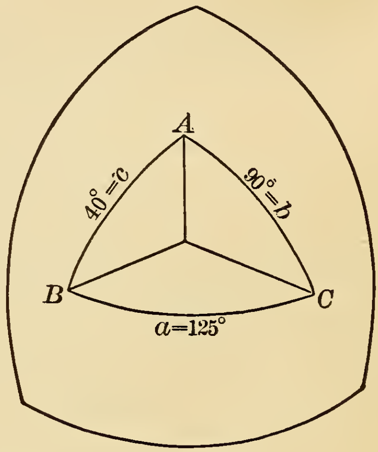
$$\frac{40°}{360°} \times 2\pi R,$$

$$\frac{90°}{360°} \times 2\pi R,$$

$$\frac{125°}{360°} \times 2\pi R.$$

If in this case $R = 4$, the sides would become respectively $\frac{8}{9}\pi$, 2π, $\frac{25}{9}\pi$.

7. Find the lengths of the sides of the triangle in Example 2, if the radius of the sphere is 4 feet.



$a = 125°.$

$b = 90°.$

$c = 40°.$

$\text{circ.} = 2\pi R$

$= 2\pi \times 4$

$= 8\pi.$

$a = \frac{125}{360}$ of $8\pi = \frac{25}{9}\pi.$

$b = \frac{90}{360}$ of $8\pi = 2\pi.$

$c = \frac{40}{360}$ of $8\pi = \frac{8}{9}\pi.$

Exercise XX. Page 108.

1. Prove, by aid of Formula [37], that the hypotenuse of a right triangle is *less than* or *greater than* 90°, according as the two legs are *alike* or *unlike* in kind.

When the legs are alike in kind.

By Formula [37]

$$\cos c = \cos a \times \cos b.$$

If a and b are less than 90°, the cosines will be positive, and the product a positive quantity; that is, greater than 0.

If a and b are greater than 90°, their cosines will both be negative, but their product will be a positive quantity.

In either case, $\cos c =$ a positive quantity.

$\therefore$ its value is less than 90°.

When the legs are unlike in kind.

If a is greater than 90° and b less

than 90°, their cosines have opposite signs, and their product will be negative.

If $\cos c =$ a negative quantity, then c is greater than 90°.

2. Prove, by aid of Formula [40], that in a right spherical triangle each leg and the opposite angle are always alike in kind.

Formula [40],

$$\cos A = \cos a \times \sin B.$$

$B < 180°$. $\therefore$ $\sin B$ is positive.

Hence sign of $\cos A$ is same as sign of $\cos a$, and both must be either greater or less than 90°; that is, alike in kind.

3. What inferences may be drawn respecting the values of the other parts: (i.) if $c = 90°$; (ii.) if $a = 90°$; (iii.) if $c = 90°$ and $a = 90°$; (iv.) if $a = 90°$ and $b = 90°$?

(i.) If $c = 90°$.

$$0 = \cos a \times \cos b.$$

$$\therefore \cos a \text{ or } \cos b = 0.$$

Hence a or $b = 90°$.

(ii.) If $a = 90°$.

$$\cos A = 0 \times \sin B.$$

$$\therefore \cos A = 0.$$

$$\therefore A = 90°.$$

$$\cos B = \cos b \times 1$$

$$= \cos b.$$

$$\therefore \quad B = b.$$

If $\quad b = 90°$,

$$B = 90° \text{ and } A = a.$$

(iii.) If $c = 90°$ and $a = 90°$.

$$\cos A = \cos a \sin B.$$

$$\therefore \quad A = a = 90°.$$

Also $\quad C = c = 90°$.

$$\cos B = \cos b.$$

$$\therefore \quad B = b.$$

(iv.) If $\quad a = 90°,\ b = 90°$.

Then $\quad A = 90°,\ B = 90°$.

$$\cos c = 0 \times 0.$$

$$\therefore c = 90°.$$

$$\therefore C = 90°.$$

4. Deduce from [37]–[42] the formula

$$\tan^2 \tfrac{1}{2} b = \tan \tfrac{1}{2}(c - a) \tan \tfrac{1}{2}(c + a).$$

From [37], page 74, we have

$$\cos b = \cos c \sec a;$$

whence

$$1 - \cos b = \frac{\cos a - \cos c}{\cos a},$$

$$1 + \cos b = \frac{\cos a + \cos c}{\cos a}.$$

$$\therefore \frac{1 - \cos b}{1 + \cos b} = \frac{\cos a - \cos c}{\cos a + \cos c}.$$

But by [18], page 47,

$$\frac{1 - \cos b}{1 + \cos b} = \tan^2 \tfrac{1}{2} b.$$

And if in [23] and [22], page 48, we write a and c in place of A and B and divide [23] by [22], we get

$$\frac{\cos a - \cos c}{\cos a + \cos c}$$

$$= -\tan \tfrac{1}{2}(a + c) \tan \tfrac{1}{2}(a - c)$$

$$= \tan \tfrac{1}{2}(c + a) \tan \tfrac{1}{2}(c - a).$$

$$\therefore \tan^2 \tfrac{1}{2} b$$

$$= \tan \tfrac{1}{2}(c + a) \tan \tfrac{1}{2}(c - a).$$

5. Deduce from [37]–[42] the formula

$$\tan^2 (45° - \tfrac{1}{2} A)$$

$$= \tan \tfrac{1}{2}(c - a) \cot \tfrac{1}{2}(c + a).$$

From [38],

$$\sin A = \frac{\sin a}{\sin c};$$

when, operating as in Example 4, we have

$$\frac{1-\sin A}{1+\sin A} = \frac{\sin c - \sin a}{\sin c + \sin a}.$$

If in [19], page 47, we substitute $90° + A$ for z, and remember that $\cos(90° + A) = -\sin A$, [19] reduces to the form

$$\frac{1-\sin A}{1+\sin A} = \cot^2(45° + \tfrac{1}{2}A)$$
$$= \tan^2(45° - \tfrac{1}{2}A),$$

(since $45° + \frac{1}{2}A$ and $45° - \frac{1}{2}A$ are complementary angles).

And by dividing [21] by [20], page 48, and writing c for A and a for B, we have

$$\frac{\sin c - \sin A}{\sin c + \sin A}$$
$$= \tan\tfrac{1}{2}(c-a)\cot\tfrac{1}{2}(c+a).$$
$$\therefore \tan^2(45° - \tfrac{1}{2}A)$$
$$= \tan\tfrac{1}{2}(c-a)\cot\tfrac{1}{2}(c+a).$$

6. Deduce from [37]–[42] the formula

$$\tan^2\tfrac{1}{2}B = \sin(c-a)\csc(c+a).$$

From [39], by operating as before,

$$\frac{1-\cos B}{1+\cos B} = \frac{\tan c - \tan a}{\tan c + \tan a}.$$

By [18], page 47,

$$\frac{1-\cos B}{1+\cos B} = \tan^2\tfrac{1}{2}B.$$

And we write on the second side $\frac{\sin c}{\cos c}$ in place of $\tan c$, and $\frac{\sin a}{\cos a}$ in place of $\tan a$, and reducing, we obtain

$$\frac{\tan c - \tan a}{\tan c + \tan a} = \frac{\sin(c-a)}{\sin(c+a)}.$$
$$\therefore \tan^2\tfrac{1}{2}B = \frac{\sin(c-a)}{\sin(c+a)}$$
$$= \sin(c-a)\csc(c+a).$$

7. Deduce from [37]–[42] the formula

$$\tan^2\tfrac{1}{2}c = -\cos(A+B)\sec(A-B).$$

By [42], $\cos c = \cot A \cot B$

$$= \frac{\cot A}{\tan B};$$

whence, as before,

$$\frac{1-\cos c}{1+\cos c} = \frac{\tan B - \cot A}{\tan B + \cot A}.$$

By [18], page 47,

$$\frac{1-\cos c}{1+\cos c} = \tan^2\tfrac{1}{2}c,$$

and

$$\frac{\tan B - \cot A}{\tan B + \cot A}$$
$$= \frac{\dfrac{\sin B}{\cos B} - \dfrac{\cos A}{\sin A}}{\dfrac{\sin B}{\cos B} + \dfrac{\cos A}{\sin A}}$$
$$= \frac{\sin A \sin B - \cos A \cos B}{\sin A \sin B + \cos A \cos B}$$
$$= \frac{-\cos(A+B)}{\cos(A-B)}.$$
$$\therefore \tan^2\tfrac{1}{2}c = -\cos(A+B)\sec(A-B).$$

8. Deduce from [37]–[42] the formula

$$\tan^2\tfrac{1}{2}A = \tan[\tfrac{1}{2}(A+B) - 45°]\tan[\tfrac{1}{2}(A-B) + 45°].$$

From [40]

$$\cos a = \cos A \csc B = \frac{\cos A}{\sin B},$$

whence, by proceeding as before,

$$\frac{1-\cos a}{1+\cos a}=\frac{\sin B-\cos A}{\sin B+\cos A}.$$

By [18], page 47,

$$\frac{1-\cos a}{1+\cos a}=\tan^2 \tfrac{1}{2}a.$$

If in [6], p. 44, we make $x = 45°$, we have

$$\tan(45°+y)=\frac{1+\dfrac{\sin y}{\cos y}}{1-\dfrac{\sin y}{\cos y}}$$

$$=\frac{\cos y+\sin y}{\cos y-\sin y}.$$

And, in like manner, making $y = 45°$ in [10], page 46, we have

$$\tan(x-45°)=\frac{\sin x-\cos x}{\sin x+\cos x}.$$

$$\therefore \tan(x-45°)\tan(45°+y)$$

$$=\frac{(\sin x-\cos x)(\sin y+\cos y)}{(\sin x+\cos x)(\cos y-\sin y)}$$

$$=\frac{\sin(x-y)-\cos(x+y)}{\sin(x-y)+\cos(x+y)}.$$

If in this equation, in which x and y may have any values, we make

$$x=\tfrac{1}{2}(A+B),\quad y=\tfrac{1}{2}(A-B),$$

whence

$$x-y=B,\qquad x+y=A,$$

the equation becomes

$$\tan[\tfrac{1}{2}(A+B)-45°]\tan[\tfrac{1}{2}(A-B)+45°]$$

$$=\frac{\sin B-\cos A}{\sin B+\cos A}.$$

$$\therefore \tan^2\tfrac{1}{2}a=\tan[\tfrac{1}{2}(A+B)-45°]\tan[\tfrac{1}{2}(A-B)+45°].$$

Exercise XXI. Page 109.

1. Show that Napier's Rules lead to the equations contained in Formulas [38], [39], [40], and [41].

$$\sin a = \cos \text{co. } c \cos \text{co. } A$$
$$= \sin c \sin A.$$
$$\sin b = \cos \text{co. } c \cos \text{co. } B$$
$$= \sin c \sin B.$$

$$\sin \text{co. } B = \tan a \tan \text{co. } c.$$
$$\cos B = \tan a \cot c.$$
$$\sin \text{co. } A = \tan b \tan \text{co. } c.$$
$$\cos A = \tan b \cot c.$$

$$\sin \text{co. } A = \cos a \cos \text{co. } B.$$
$$\cos A = \cos a \sin B.$$
$$\sin \text{co. } B = \cos b \cos \text{co. } A.$$
$$\cos B = \cos b \sin A.$$

$$\sin a = \tan \text{co. } B \tan b$$
$$= \cot B \tan b.$$
$$\sin b = \tan a \tan \text{co. } A$$
$$= \tan a \cot A.$$

2. What will Napier's Rules become, if we take as the five parts of the triangle, the hypotenuse, the two oblique angles, and the *complements* of the two legs.

(i.) Cosine of middle part equals product of cotangents of adjacent parts.

(ii.) Cosine of middle part equals product of sines of opposite parts.

Exercise XXII. Page 114.

1. Solve the right triangle, given $a = 36° \; 27'$, $b = 43° \; 32' \; 31''$.

Taking a as the middle part, we have, by Rule I.,

$$\sin a = \tan b \cot B,$$

whence $$\tan b = \sin a \tan B,$$

and $$\tan B = \frac{\tan b}{\sin a}.$$

$$\begin{aligned} \log \tan b &= 9.97789 \\ \text{colog} \sin a &= 0.22613 \\ \hline \log \tan B &= 10.20402 \\ B &= 57° \; 59' \; 19.3''. \end{aligned}$$

Taking b as the middle part, by Rule I.,

$$\sin b = \tan a \cot A.$$

$$\tan a = \sin b \tan A.$$

$$\tan A = \frac{\tan a}{\sin b}.$$

$$\begin{aligned} \log \tan a &= 9.86842 \\ \text{colog} \sin b &= 0.16185 \\ \hline \log \tan A &= 10.03027 \\ A &= 46° \; 59' \; 43.2''. \end{aligned}$$

Taking co. c as the middle part, by Rule II.,

$$\cos c = \cos a \cos b.$$

$$\begin{aligned} \log \cos a &= 9.90546 \\ \log \cos b &= 9.86026 \\ \hline \log \cos c &= 9.76572 \\ c &= 54° \; 20'. \end{aligned}$$

2. Solve the right triangle, given $a = 86° \; 40'$, $b = 32° \; 40'$.

$$\cos c = \cos a \cos b.$$

$$\begin{aligned} \log \cos a &= 8.76451 \\ \log \cos b &= 9.92522 \\ \hline \log \cos c &= 8.68973 \\ c &= 87° \; 11' \; 39.8''. \end{aligned}$$

$$\tan A = \tan a \csc b.$$

$$\begin{aligned} \log \tan a &= 11.23475 \\ \log \csc b &= 0.26781 \\ \hline \log \tan A &= 11.50256 \\ A &= 88° \; 11' \; 57.8''. \end{aligned}$$

$$\tan B = \tan b \csc a.$$

$$\begin{aligned} \log \tan b &= 9.80697 \\ \log \csc a &= 0.00074 \\ \hline \log \tan B &= 9.80771 \\ B &= 32° \; 42' \; 38.7''. \end{aligned}$$

3. Solve the right triangle, given $a = 50°$, $b = 36° \; 54' \; 49''$.

$$\cos c = \cos a \cos b.$$

$$\tan A = \tan a \csc b.$$

$$\tan B = \tan b \csc a.$$

$$\begin{aligned} \log \cos a &= 9.80807 \\ \log \cos b &= 9.90284 \\ \hline \log \cos c &= 9.71091 \\ c &= 59° \; 4' \; 25.7''. \end{aligned}$$

$$\begin{aligned} \log \tan a &= 10.07619 \\ \log \csc b &= 0.22141 \\ \hline \log \tan A &= 10.29760 \\ A &= 63° \; 15' \; 13.13''. \end{aligned}$$

$$\begin{aligned} \log \tan b &= 9.87575 \\ \log \csc a &= 0.11575 \\ \hline \log \tan B &= 9.99150 \\ B &= 44° \; 26' \; 21.6''. \end{aligned}$$

4. Solve the right triangle, given $a = 120° \; 10'$, $b = 150° \; 59' \; 44''$.

$$\cos c = \cos a \cos b.$$
$$\tan A = \tan a \csc b.$$
$$\tan B = \tan b \csc a.$$

$$\begin{aligned} \log \cos a &= 9.70115 \\ \log \cos b &= \underline{9.94180} \\ \log \cos c &= 9.64295 \\ c &= 63° \; 55' \; 43.3''. \end{aligned}$$

$$\begin{aligned} \log \tan a &= 10.23565 \\ \log \csc b &= \underline{0.31437} \\ \log \tan A &= 10.55002 \\ A &= 105° \, 44' 21.25''. \end{aligned}$$

$$\begin{aligned} \log \tan b &= 9.74383 \\ \log \csc a &= \underline{0.06320} \\ \log \tan B &= 9.80703 \\ B &= 147° \, 19' 47.14''. \end{aligned}$$

5. Solve the right triangle, given $c = 55° \; 9' \; 32''$, $a = 22° \; 15' \; 7''$.

$$\cos B = \tan a \cot c.$$

$$\begin{aligned} \log \tan a &= 9.61188 \\ \log \cot c &= \underline{9.84266} \\ \log \cos B &= 9.45454 \\ B &= 73° \; 27' \; 11.16''. \end{aligned}$$

$$\tan b = \sin a \tan B.$$

$$\begin{aligned} \log \sin a &= 9.57828 \\ \log \tan B &= \underline{10.52709} \\ \log \tan b &= 10.10537 \\ b &= 51° \; 53'. \end{aligned}$$

$$\cos A = \tan b \cot c.$$

$$\begin{aligned} \log \tan b &= 10.10537 \\ \log \cot c &= \underline{9.84266} \\ \log \cos A &= 9.94803 \\ A &= 27° \; 28' \; 25.71''. \end{aligned}$$

6. Solve the right triangle, given $c = 23° \; 49' \; 51''$, $a = 14° \; 16' \; 35''$.

$$\cos b = \cos c \sec a.$$

$$\begin{aligned} \log \cos c &= 9.96130 \\ \log \sec a &= \underline{0.01362} \\ \log \cos b &= 9.97492 \\ b &= 19° \; 17'. \end{aligned}$$

$$\sin A = \sin a \csc c.$$

$$\begin{aligned} \log \sin a &= 9.39199 \\ \log \csc c &= \underline{0.39358} \\ \log \sin A &= 9.78557 \\ A &= 37° \; 36' \; 49.4'' \end{aligned}$$

$$\cos B = \tan a \cot c.$$

$$\begin{aligned} \log \tan a &= 9.40562 \\ \log \cot c &= \underline{10.33488} \\ \log \cos B &= 9.76050 \\ B &= 54° \; 49' \; 23.3''. \end{aligned}$$

7. Solve the right triangle, given $c = 44° \; 33' \; 17''$, $a = 32° \; 9' \; 17''$.

$$\cos b = \cos c \sec a.$$

$$\begin{aligned} \log \cos c &= 9.85283 \\ \log \sec a &= \underline{0.07231} \\ \log \cos b &= 9.92514 \\ b &= 32° \; 41'. \end{aligned}$$

$$\sin A = \sin a \csc c.$$

$$\begin{aligned} \log \sin a &= 9.72608 \\ \log \csc c &= \underline{0.15391} \\ \log \sin A &= 9.87999 \\ A &= 49° \; 20' \; 16.3''. \end{aligned}$$

$$\cos B = \tan a \cot c.$$

$$\begin{aligned} \log \tan a &= 9.79840 \\ \log \cot c &= \underline{10.00675} \\ \log \cos B &= 9.80515 \\ B &= 50° \; 19' \; 16''. \end{aligned}$$

8. Solve the right triangle, given $c = 97° 13' 4''$, $a = 132° 14' 12''$.

$$\cos b = \cos c \sec a.$$

$$\begin{aligned} \log\cos c &= 9.09914 \\ \log\sec a &= 0.17250 \\ \hline \log\cos b &= 9.27164 \\ b &= 79° 13' 38.18''. \end{aligned}$$

$$\sin A = \sin a \csc c.$$

$$\begin{aligned} \log\sin a &= 9.86945 \\ \log\csc c &= 0.00345 \\ \hline \log\sin A &= 9.87290 \\ A &= 48° 16' 10''. \end{aligned}$$

But A and a must be of the same kind,

$$\therefore A = 131° 43' 50''.$$

$$\cos B = \tan a \cot c.$$

$$\begin{aligned} \log\tan a &= 10.04196 \\ \log\cot c &= 9.10259 \\ \hline \log\cos B &= 9.14455 \\ B &= 81° 58' 53.3''. \end{aligned}$$

9. Solve the right triangle, given $a = 77° 21' 50''$, $A = 83° 56' 40''$.

$$\sin c = \sin a \csc A.$$

$$\begin{aligned} \log\sin a &= 9.98935 \\ \log\csc A &= 0.00243 \\ \hline \log\sin c &= 9.99178 \\ c &= 78° 53' 20''. \end{aligned}$$

Since c is found from its sine, it may have two values which are supplements of each other.

Hence also $c = 101° 6' 40''$.

$$\sin b = \tan a \cot A.$$

$$\begin{aligned} \log\tan a &= 10.64939 \\ \log\cot A &= 9.02565 \\ \hline \log\sin b &= 9.67504 \\ b &= 28° 14' 31.3''. \\ \text{or} \quad &= 151° 45' 28.7''. \end{aligned}$$

$$\sin B = \sec a \cos A.$$

$$\begin{aligned} \log\sec a &= 0.66004 \\ \log\cos A &= 9.02323 \\ \hline \log\sin B &= 9.68327 \\ B &= 28° 49' 57.4'', \\ \text{or} \quad &= 151° 10' \ 2.6''. \end{aligned}$$

10. Solve the right triangle, given $a = 77° 21' 50''$, $A = 40° 40' 40''$.

$$\sin c = \sin a \csc A.$$

$$\begin{aligned} \log\sin a &= 9.98935 \\ \log\csc A &= 0.18588 \\ \hline \log\sin c &= 10.17523 \end{aligned}$$

$\therefore \sin c > 1$, which is impossible.

11. Solve the right triangle, given $a = 92° 47' 32''$, $B = 50° 2' 1''$.

$$\tan c = \tan a \sec B.$$

$$\begin{aligned} \log\tan a &= 11.31183 \\ \log\sec B &= 0.19223 \\ \hline \log\tan c &= 11.50406 \\ c &= 91° 47' 40''. \end{aligned}$$

$$\tan b = \sin a \tan B.$$

$$\begin{aligned} \log\sin a &= 9.99948 \\ \log\tan B &= 10.07671 \\ \hline \log\tan b &= 10.07619 \\ b &= 50°. \end{aligned}$$

$$\cos A = \cos a \sin B.$$

$$\begin{aligned} \log\cos a &= 8.68765 \\ \log\sin B &= 9.88447 \\ \hline \log\cos A &= 8.57212 \\ A &= 92° 8' 23''. \end{aligned}$$

12. Solve the right triangle, given $a = 2° 0' 55''$, $B = 12° 40'$.

$$\tan b = \sin a \tan B.$$

$$\begin{aligned} \log \sin a &= 8.54612 \\ \log \tan B &= \underline{9.35170} \\ \log \tan b &= 7.89782 \\ b &= 0° 27' 10.2''. \end{aligned}$$

$$\tan c = \tan a \sec B.$$

$$\begin{aligned} \log \tan a &= 8.54639 \\ \log \sec B &= \underline{0.01070} \\ \log \tan c &= 8.55709 \\ c &= 2° 3' 55.7''. \end{aligned}$$

$$\cos A = \cos a \sin B.$$

$$\begin{aligned} \log \cos a &= 9.99973 \\ \log \sin B &= \underline{9.34100} \\ \log \cos A &= 9.34073 \\ A &= 77° 20' 28.4''. \end{aligned}$$

13. Solve the right triangle, given $a = 20° 20' 20''$, $B = 38° 10' 10''$.

$$\tan b = \sin a \tan B.$$

$$\begin{aligned} \log \sin a &= 9.54104 \\ \log \tan B &= \underline{9.89545} \\ \log \tan b &= 9.43649 \\ b &= 15° 16' 50.4''. \end{aligned}$$

$$\tan c = \tan a \sec B.$$

$$\begin{aligned} \log \tan a &= 9.56900 \\ \log \sec B &= \underline{0.10448} \\ \log \tan c &= 9.67348 \\ c &= 25° 14' 38.2''. \end{aligned}$$

$$\cos A = \cos a \sin B.$$

$$\begin{aligned} \log \cos a &= 9.97204 \\ \log \sin B &= \underline{9.79098} \\ \log \cos A &= 9.76302 \\ A &= 54° 35' 16.7''. \end{aligned}$$

14. Solve the right triangle, given $a = 54° 30'$, $B = 35° 30'$.

$$\tan c = \tan a \sec B.$$

$$\begin{aligned} \log \tan a &= 10.14673 \\ \log \sec B &= \underline{0.08931} \\ \log \tan c &= 10.23604 \\ c &= 59° 51' 20.7''. \end{aligned}$$

$$\tan b = \sin a \tan B.$$

$$\begin{aligned} \log \sin a &= 9.91069 \\ \log \tan B &= \underline{9.85327} \\ \log \tan b &= 9.76396 \\ b &= 30° 8' 39.3''. \end{aligned}$$

$$\cos A = \cos a \sin B.$$

$$\begin{aligned} \log \cos a &= 9.76395 \\ \log \sin B &= \underline{9.76395} \\ \log \cos A &= 9.52790 \\ A &= 70° 17' 35''. \end{aligned}$$

15. Solve the right triangle, given $c = 69° 25' 11''$, $A = 54° 54' 42''$.

$$\sin a = \sin c \sin A.$$

$$\begin{aligned} \log \sin c &= 9.97136 \\ \log \sin A &= \underline{9.91289} \\ \log \sin a &= 9.88425 \\ a &= 50°. \end{aligned}$$

$$\tan b = \tan c \cos A.$$

$$\begin{aligned} \log \tan c &= 10.42541 \\ \log \cos A &= \underline{9.75954} \\ \log \tan b &= 10.18495 \\ b &= 56° 50' 49.3''. \end{aligned}$$

$$\cot B = \cos c \tan A.$$

$$\begin{aligned} \log \cos c &= 9.54595 \\ \log \tan A &= \underline{10.15335} \\ \log \cot B &= 9.69930 \\ B &= 63° 25' 4''. \end{aligned}$$

16. Solve the right triangle, given $c = 112° 48'$, $A = 56° 11' 56''$.

$$\cot B = \cos c \tan A.$$

$$\begin{aligned} \log \cos c &= 9.58829 \\ \log \tan A &= 10.17427 \\ \hline \log \cot B &= 9.76256 \\ B &= 120° 3' 50''. \end{aligned}$$

$$\sin a = \sin c \sin A.$$

$$\begin{aligned} \log \sin c &= 9.96467 \\ \log \sin A &= 9.91958 \\ \hline \log \sin a &= 9.88425 \\ a &= 50°. \end{aligned}$$

$$\tan b = \cos A \tan c.$$

$$\begin{aligned} \log \cos A &= 9.74532 \\ \log \tan c &= 10.37638 \\ \hline \log \tan b &= 10.12170 \\ b &= 127° 4' 30''. \end{aligned}$$

17. Solve the right triangle, given $c = 46° 40' 12''$, $A = 37° 46' 9''$.

$$\sin a = \sin A \sin c.$$

$$\begin{aligned} \log \sin A &= 9.78709 \\ \log \sin c &= 9.86178 \\ \hline \log \sin a &= 9.64887 \\ a &= 26° 27' 24''. \end{aligned}$$

$$\tan b = \tan c \cos A.$$

$$\begin{aligned} \log \tan c &= 10.02533 \\ \log \cos A &= 9.89789 \\ \hline \log \tan b &= 9.92322 \\ b &= 39° 57' 41.5''. \end{aligned}$$

$$\cot B = \tan A \cos c.$$

$$\begin{aligned} \log \cos c &= 9.83645 \\ \log \tan A &= 9.88920 \\ \hline \log \cot B &= 9.72565 \\ B &= 62° 0' 4''. \end{aligned}$$

18. Solve the right triangle, given $c = 118° 40' 1''$, $A = 128° 0' 4''$.

$$\sin a = \sin c \sin A.$$

$$\begin{aligned} \log \sin c &= 9.94321 \\ \log \sin A &= 9.89652 \\ \hline \log \sin a &= 9.83973 \\ a &= 136° 15' 32.3''. \end{aligned}$$

$$\tan b = \tan c \cos A.$$

$$\begin{aligned} \log \tan c &= 10.26222 \\ \log \cos A &= 9.78935 \\ \hline \log \tan b &= 10.05157 \\ b &= 48° 23' 38.4''. \end{aligned}$$

$$\cot B = \cos c \tan A.$$

$$\begin{aligned} \log \cos c &= 9.68098 \\ \log \tan A &= 10.10717 \\ \hline \log \cot B &= 9.78815 \\ B &= 58° 27' 4.3''. \end{aligned}$$

19. Solve the right triangle, given $A = 63° 15' 12''$, $B = 135° 33' 39''$.

$$\cos c = \cot A \cot B.$$

$$\begin{aligned} \log \cot A &= 9.70241 \\ \log \cot B &= 10.00850 \\ \hline \log \cos c &= 9.71091 \\ c &= 120° 55' 34.3''. \end{aligned}$$

$$\cos a = \cos A \csc B.$$

$$\begin{aligned} \log \cos A &= 9.65326 \\ \text{colog} \sin B &= 0.15480 \\ \hline \log \cos a &= 9.80806 \\ a &= 49° 59' 56''. \end{aligned}$$

$$\cos b = \cos B \csc A.$$

$$\begin{aligned} \log \cos B &= 9.85369 \\ \text{colog} \sin A &= 0.04915 \\ \hline \log \cos b &= 9.90284 \\ b &= 143° 5' 12''. \end{aligned}$$

20. Solve the right triangle, given $A = 116° 43' 12''$, $B = 116° 31' 25''$.

$$\cos a = \cos A \csc B.$$

$$\begin{aligned} \log \cos A &= 9.65286 \\ \log \csc B &= \underline{0.04830} \\ \log \cos a &= 9.70116 \\ a &= 120° \ 10' \ 3''. \end{aligned}$$

$$\cos b = \cos B \csc A.$$

$$\begin{aligned} \log \cos B &= 9.64988 \\ \log \csc A &= \underline{0.04904} \\ \log \cos b &= 9.69892 \\ b &= 119° \ 59' \ 46''. \end{aligned}$$

$$\log \cos c = \cot A \cot B.$$

$$\begin{aligned} \log \cot A &= 9.70190 \\ \log \cot B &= \underline{9.69818} \\ \log \cos c &= 9.40008 \\ c &= 75° \ 26' \ 58''. \end{aligned}$$

21. Solve the right triangle, given $A = 46° 59' 42''$, $B = 57° 59' 17''$.

$$\cos a = \cos A \csc B.$$

$$\begin{aligned} \log \cos A &= 9.83382 \\ \log \csc B &= \underline{0.07164} \\ \log \cos a &= 9.90546 \\ a &= 36° \ 27'. \end{aligned}$$

$$\cos b = \cos B \csc A.$$

$$\begin{aligned} \log \cos B &= 9.72435 \\ \log \csc A &= \underline{0.13591} \\ \log \cos b &= 9.86026 \\ b &= 43° \ 32' \ 37''. \end{aligned}$$

$$\cos c = \cot A \cot B.$$

$$\begin{aligned} \log \cot A &= 9.96973 \\ \log \cot B &= \underline{9.79599} \\ \log \cos c &= 9.76572 \\ c &= 54° \ 20'. \end{aligned}$$

22. Solve the right triangle, given $A = 90°$, $B = 88° 24' 35''$.

$$\cos c = \cot A \cot B.$$

But $\cot A = 0$.

$\therefore \cos c = 0.$

$\therefore \quad c = 90°.$

$$\cos a = \cos A \csc B.$$

But $\cos A = 0$.

$\therefore \cos a = 0.$

$\therefore \quad a = 90°.$

$$\cos b = \cos B \csc A.$$

$\csc A = 1.$

$\therefore \quad b = B.$

$\therefore \quad b = 88° \ 24' \ 35''.$

23. Define a quadrantal triangle, and show how its solution may be reduced to that of the right triangle.

A quadrantal triangle is a triangle having one or more of its sides equal to a quadrant.

Let $A'B'C'$ be a quadrantal triangle with side $A'B' = 90°$, or a quadrant.

Let ABC be its polar triangle.

Then since

$$A'B' + C = 180°, \ C = 90°.$$

$\therefore$ ABC is a right triangle.

$\therefore$ all parts of the polar triangle may be found by formulas for right triangle. The parts of $A'B'C'$ may then be found by subtracting proper parts of ABC from 180°.

24. Solve the quadrantal triangle whose sides are:

$$\begin{aligned} a &= 174° \ 12' \ 49.1'', \\ b &= 94° \ \ 8' \ 20'', \\ c &= 90°. \end{aligned}$$

Let A', B', C', a', b', c' represent the corresponding angles and sides of the polar triangle.

Then $A' = 5° \ 47' \ 10.9''$,

$B' = 85° \ 51' \ 40''$,

$C' = 90°$.

$$\tan^2 \tfrac{1}{2} c' = -\cos(B'+A') \sec(B'-A').$$

$B' + A' = 91° \ 38' \ 50.9''$.

$B' - A' = 80° \ 4' \ 29.1''$.

$$\begin{aligned}
\log\cos(B'+A') &= 8.45863 \\
\log\sec(B'-A') &= \underline{0.76356} \\
2)\,&\underline{9.22219} \\
\log\tan\tfrac{1}{2}c' &= 9.61110 \\
\tfrac{1}{2}c' &= 22° \ 12' \ 56\tfrac{2}{3}''. \\
c' &= 44° \ 25' \ 53''. \\
C &= 135° \ 34' \ 7''.
\end{aligned}$$

$$\tan^2 \tfrac{1}{2} b' = \tan\left[\tfrac{1}{2}(B'+A') - 45°\right] \tan\left[45° + \tfrac{1}{2}(B'-A')\right].$$

$\frac{1}{2}(A'+B') - 45° = 49' \ 25.5''$.

$45° + \frac{1}{2}(B'-A') = 85° \ 2' \ 14.6''$.

$$\begin{aligned}
\log\tan 0° \ 49' \ 25.5'' &= 8.15770 \\
\log\tan 85° \ 2' \ 14.6'' &= \underline{11.06133} \\
2)\,&\underline{9.21903} \\
\log\tan\tfrac{1}{2}b' &= 9.60952 \\
\tfrac{1}{2}b' &= 22° \ 8' \ 35''. \\
b' &= 44° \ 17' \ 10''. \\
B &= 135° \ 42' \ 50''.
\end{aligned}$$

$$\tan^2 \tfrac{1}{2} a' = \tan\left[\tfrac{1}{2}(B'+A') - 45°\right] \tan\left[45° - \tfrac{1}{2}(B'-A')\right].$$

$\frac{1}{2}(B'+A') - 45° = 0° \ 49' \ 25.5''$.

$45° - \frac{1}{2}(B'-A') = 4° \ 57' \ 45.4''$.

$$\begin{aligned}
\log\tan 0° \ 49' \ 25.5'' &= 8.15770 \\
\log\tan 4° \ 57' \ 45.4'' &= \underline{8.93867} \\
2)\,&\underline{7.09637} \\
\log\tan\tfrac{1}{2}a' &= 8.54819 \\
\tfrac{1}{2}a' &= 2° \ 1' \ 25''. \\
a' &= 4° \ 2' \ 50''. \\
A &= 175° \ 57' \ 10''.
\end{aligned}$$

25. Solve the quadrantal triangle in which

$c = 90°$,

$A = 110° \ 47' \ 50''$,

$B = 135° \ 35' \ 34.5''$.

Let A', B', C', a', b', c' represent the corresponding angles and sides of the polar triangle.

Then $a' = 69° \ 12' \ 10''$.

$b' = 44° \ 24' \ 25.5''$.

$C' = 90°$.

$$\tan A' = \tan a' \csc b'.$$

$$\begin{aligned}
\log\tan a' &= 10.42043 \\
\log\csc b' &= \underline{0.15505} \\
\log\tan A' &= 10.57548 \\
A' &= 75° \ 6' \ 58''. \\
a &= 104° \ 53' \ 2''.
\end{aligned}$$

$$\tan B' = \tan b' \csc a'.$$

$$\begin{aligned}
\log\tan b' &= 9.99101 \\
\log\csc a' &= \underline{0.02926} \\
\log\tan B' &= 10.02027 \\
B' &= 46° \ 20' \ 12''. \\
b &= 133° \ 39' \ 48''.
\end{aligned}$$

$$\cos c' = \cot A' \cot B'.$$

$$\begin{aligned}\log\cot A' &= 9.42452\\ \log\cot B' &= \underline{9.97973}\\ \log\cos c' &= 9.40425\\ c' &= 75°\ 18'\ 21''.\\ C &= 104°\ 41'\ 39''.\end{aligned}$$

26. Given in a spherical triangle A, C, and $c = 90°$; solve the triangle.

$$\begin{aligned}\sin a &= \sin c \sin A.\\ &= 1 \times 1.\\ \therefore a &= 90°.\\ \tan b &= \tan c \cos A\\ &= \infty \times 0.\\ \therefore b &= 45°.\\ \cot B &= \cos c \tan A.\\ &= 0 \times \infty.\\ \therefore B &= 45°.\end{aligned}$$

27. Given $A = 60°$, $C = 90°$, and $c = 90°$; solve the triangle.

$$\begin{aligned}\sin\ a &= \sin c \sin A.\\ \tan\ b &= \tan c \cos A.\\ \cot\ B &= \cos c \tan A.\\ \sin\ c &= 1.\\ \therefore \sin\ a &= \sin A.\\ \therefore\ a &= A = 60°.\\ \tan\ c &= \infty.\\ \therefore \tan\ b &= \infty.\\ b &= 90°.\\ \cos\ c &= 0.\\ \therefore \cot B &= 0.\\ B &= 90°.\end{aligned}$$

28. Given in a right spherical triangle, $A = 42°\ 24'\ 9''$, $B = 9°\ 4'\ 11''$; solve the triangle.

$$\begin{aligned}\cos c &= \cot A \cot B.\\ \log\cot A &= 10.03943\\ \log\cot B &= \underline{10.79688}\\ \log\cos c &= 10.83631\end{aligned}$$

which is impossible.

$\therefore$ triangle is impossible.

29. In a right triangle, given $a = 119°\ 11'$, $B = 126°\ 54'$; solve the triangle.

$$\begin{aligned}\tan c &= \tan a \sec B.\\ \log\tan a &= 10.25298\\ \log\cos B &= \underline{\ \ 0.22154}\\ \log\tan c &= 10.47452\\ c &= 71°\ 27'\ 43''.\\ \tan b &= \sin a \tan B.\\ \log\sin a &= \ \ 9.94105\\ \log\tan B &= \underline{10.12446}\\ \log\tan b &= 10.06551\\ b &= 130°\ 41'\ 42''.\\ \cos A &= \cos a \sin B.\\ \log\cos a &= 9.68807\\ \log\sin B &= \underline{9.90292}\\ \log\cos A &= 9.59099\\ A &= 112°\ 57'\ 2''.\end{aligned}$$

30. In a right triangle, given $c = 50°$, $b = 44°\ 18'\ 39''$; solve the triangle.

$$\begin{aligned}\cos a &= \cos c \sec b.\\ \log\cos c &= 9.80807\\ \text{colog}\cos b &= \underline{0.14535}\\ \log\cos a &= 9.95342\\ a &= 26°\ 3'\ 51''.\end{aligned}$$

$$\sin A = \sin a \csc c.$$

$$\begin{aligned} \log \sin a &= 9.64284 \\ \log \csc c &= 0.11575 \\ \log \sin A &= 9.75859 \\ A &= 35°. \end{aligned}$$

$$\tan B = \tan b \csc a.$$

$$\begin{aligned} \log \tan b &= 9.98955 \\ \log \csc a &= 0.35716 \\ \log \tan B &= 10.34671 \\ B &= 65° \, 46' \, 7''. \end{aligned}$$

31. In a right triangle, given $A = 156° \, 20' \, 30''$, $a = 65° \, 15' 45''$; solve the triangle.

It is impossible, because a and A are unlike in kind. And, in Case III., they must be alike in kind; otherwise, impossible.

32. If the legs a and b of a right spherical triangle are equal, prove that $\cos a = \cot A = \sqrt{\cos c}$.

$$\cos c = \cos a \cos b.$$

But $\cos a = \cos b$.

$$\therefore \cos c = \cos^2 a,$$

and $\therefore \cos a = \sqrt{\cos c}$.

$$\sin a = \cos a \sin b \tan A.$$

Since $\sin a = \sin b$,

$$1 = \cos a \tan A.$$

$$\cos a = \frac{1}{\tan A}.$$

$$\therefore \cos a = \cot A.$$

33. In a right triangle prove that $\cos^2 A \times \sin^2 c = \sin (c - a) \sin (c + a)$.

$$\cos A \sin c = \cos a \sin b.$$

$$\therefore \cos^2 A \sin^2 c = \cos^2 a \sin^2 b. \quad (1)$$

$$\begin{aligned} \sin^2 b &= \sin^2 c \sin^2 B \\ &= \sin^2 c \, (1 - \cos^2 B). \quad (2) \end{aligned}$$

$$\begin{aligned} \cos^2 B &= \tan^2 a \cot^2 c \\ &= \frac{\sin^2 a}{\cos^2 a} \times \frac{\cos^2 c}{\sin^2 c}. \end{aligned}$$

Substitute in (2),

$$\begin{aligned} \sin^2 b &= \sin^2 c - \frac{\sin^2 a \cos^2 c}{\cos^2 a} \\ &= \frac{\sin^2 c \cos^2 a - \sin^2 a \cos^2 c}{\cos^2 a}. \end{aligned}$$

Substitute in (1),

$$\begin{aligned} \cos^2 a \sin^2 b &= \sin^2 c \cos^2 a - \sin^2 a \cos^2 c \\ &= (\sin c \cos a + \sin a \cos c) \\ &\quad (\sin c \cos a - \sin a \cos c) \\ &= \sin (c + a) \sin (c - a). \end{aligned}$$

Substitute in (1),

$$\cos^2 A \sin^2 c = \sin (c + a) \sin (c - a).$$

34. In a right triangle prove that $\tan a \cos c = \sin b \cot B$.

$$\sin b = \tan a \cot A.$$

$$\cot A = \frac{\sin b}{\tan a}.$$

$$\cos c = \cot A \cot B.$$

$$\cot A = \frac{\cos c}{\cot B}.$$

$$\therefore \frac{\cos c}{\cot B} = \frac{\sin b}{\tan a}.$$

$$\tan a \cos c = \sin b \cot B.$$

35. In a right triangle prove that $\sin^2 A = \cos^2 B + \sin^2 a \sin^2 B$.

$$\sin a = \sin A \sin c.$$

$$\sin^2 A = \frac{\sin^2 a}{\sin^2 c}.$$

$$\cos B = \tan a \cot c.$$

$$\cos^2 B = \tan^2 a \cot^2 c.$$

$\sin^2 A - \cos^2 B$

$$= \frac{\sin^2 a}{\sin^2 c} - \tan^2 a \cot^2 c$$

$$= \frac{\sin^2 a}{\sin^2 c} - \frac{\sin^2 a}{\cos^2 a} \times \frac{\cos^2 c}{\sin^2 c}$$

$$= \frac{\sin^2 a \cos^2 a - \sin^2 a \cos^2 c}{\cos^2 a \sin^2 c}$$

$$= \frac{\sin^2 a (\cos^2 a - \cos^2 c)}{\cos^2 a \sin^2 c}. \qquad (1)$$

Now

$$\cos c = \cos a \cos b.$$

$$\cos^2 a = \frac{\cos^2 c}{\cos^2 b}.$$

$$\cos^2 c = \cos^2 a \cos^2 b.$$

$$\sin c = \frac{\sin b}{\sin B}.$$

$$\sin^2 c = \frac{\sin^2 b}{\sin^2 B}.$$

Substitute these values in (1),

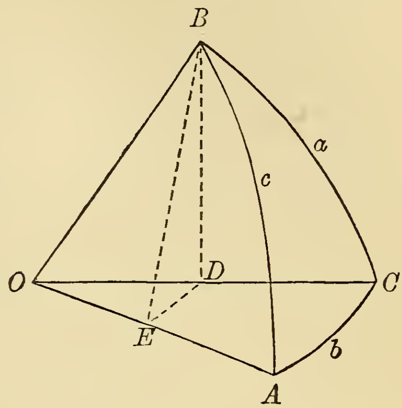
$\sin^2 a - \cos^2 B$

$$= \sin^2 a \left[\frac{\dfrac{\cos^2 c}{\cos^2 b} - \cos^2 a \cos^2 b}{\dfrac{\cos^2 c}{\cos^2 b} \times \dfrac{\sin^2 b}{\sin^2 B}} \right]$$

$$= \sin^2 a \left[\frac{\dfrac{\cos^2 c - \cos^2 a \cos^4 b}{\cos^2 b}}{\dfrac{\cos^2 c \sin^2 b}{\cos^2 b \sin^2 B}} \right]$$

$$= \sin^2 a \left[\sin^2 B \left(\frac{\cos^2 c - \cos^2 a \cos^4 b}{\cos^2 c \sin^2 b} \right) \right]$$

$$= \sin^2 a \sin^2 B \left(\frac{\cos^2 c - \cos^2 a \cos^4 b}{\cos^2 c \sin^2 b} \right).$$



But $\dfrac{\cos^2 c - \cos^2 a \cos^4 b}{\cos^2 c \sin^2 b}$

$$= \left[\frac{\overline{OE}^2 - \dfrac{\overline{OD}^2 \times \overline{OE}^4}{\overline{OD}^4}}{\dfrac{\overline{OE}^2 \times \overline{DE}^2}{\overline{OD}^2}} \right]$$

$$= \frac{\overline{OE}^2 \times \overline{OD}^2 - \overline{OE}^4}{\overline{OE}^2 \times \overline{ED}^2}$$

$$= \frac{\overline{OD}^2 - \overline{OE}^2}{\overline{ED}^2}$$

$$= 1.$$

$\therefore \sin^2 A - \cos^2 B = \sin^2 a \sin^2 B.$

$\therefore \sin^2 A = \cos^2 B + \sin^2 a \sin^2 B.$

36. In a right triangle prove that $\sin (b + c) = 2 \cos^2 \frac{1}{2} a \cos b \sin c$.

$\sin (b + c)$

$$= \sin b \cos c + \cos b \sin c$$

$$= \left(\frac{\sin b \cos c}{\cos b \sin c} + 1 \right) \cos b \sin c$$

$$= (\tan b \cot c + 1) \cos b \sin c. \qquad (1)$$

But $\tan b \cot c = \cos A$,

$\therefore \tan b \cot c + 1 = \cos A + 1. \qquad (2)$

$$\sqrt{\frac{\cos A+1}{2}}=\cos\tfrac{1}{2}A.$$

Substitute in (2),

$$(\tan b\cot c+1)=2\cos^2\tfrac{1}{2}A.$$

Substitute in (1),

$$\sin(b+c)=2\cos^2\tfrac{1}{2}A\cos b\sin c.$$

37. In a right triangle prove that $\sin(c-b)=2\sin^2\frac{1}{2}a\cos b\sin c$.

$$\begin{aligned}\sin(c-b)&\\ &=\sin c\cos b-\cos c\sin b\\ &=\sin c\cos b\left(1-\frac{\cos c\sin b}{\sin c\cos b}\right)\quad(1)\\ &=\sin c\cos b(1-\cot c\tan b).\quad(2)\end{aligned}$$

$$\cot c\tan b=\cos A.$$

$$1-\cot c\tan b=2\left(\frac{1-\cos A}{2}\right).\quad(3)$$

$$\sin\tfrac{1}{2}A=\sqrt{\frac{1-\cos A}{2}}.$$

Substitute in (3),

$$1-\cot c\tan b=2\sin^2\tfrac{1}{2}A.$$

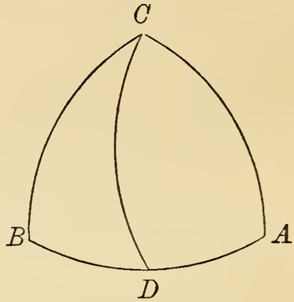
Substitute in (2),

$$\sin(c-b)=2\sin^2\tfrac{1}{2}a\sin c\cos b.$$

38. If, in a right triangle, p denote the arc of the great circle passing through the vertex of the right angle and perpendicular to the hypotenuse, m and n the segments of the hypotenuse made by this arc adjacent to the legs a and b respectively, prove that

(i.) $\tan^2 a=\tan c\tan m$,

(ii.) $\sin^2 p=\tan m\tan n$.



In triangle BCA

$$\cos B=\tan a\cot c.$$

$$\therefore\ \tan a=\frac{\cos B}{\cot c}.$$

In triangle CBD

$$\begin{aligned}\tan BC&=\tan BD\sec B.\\ \tan a&=\tan m\sec B\\ &=\frac{\tan m}{\cos B}.\end{aligned}$$

Multiplying the two equations,

$$\begin{aligned}\tan^2 a&=\frac{\tan m}{\cos B}\times\frac{\cos B}{\cot c}\\ &=\tan m\tan c.\end{aligned}$$

2d. In triangle CBD

$$\sin p=\tan m\cot M;$$

and in triangle CAD

$$\sin p=\tan n\cot N.$$

But $\cot M\times\cot N=1$.

$$\therefore\ M+N=90^\circ.$$

$$\therefore\ \sin^2 p=\tan m\tan n.$$

Exercise XXIII. Page 116.

1. In an isosceles spherical triangle, given the base b and the side a; find A the angle at the base, B the angle at the vertex, and h the altitude.

Let ABA' be an isosceles triangle, A and A' being the equal angles, a and a' the equal sides.

Let h the arc of a great circle be drawn from B perpendicular to AA'.

Let b and co. c be the given parts.

$b = \frac{1}{2} b$ in triangle ABA',

$c = \ a$ in triangle ABA'.

$B = \frac{1}{2} B$ in triangle ABA'.

$$\cos A = \cot a \tan \tfrac{1}{2} b.$$

$$\sin \tfrac{1}{2} b = \sin a \sin \tfrac{1}{2} B.$$

$$\sin \tfrac{1}{2} B = \csc a \sin \tfrac{1}{2} b.$$

$$\cos h = \cos a \sec \tfrac{1}{2} b.$$

2. In an equilateral spherical triangle, given the side a; find the angle A.

In the equilat. triangle $AA'A''$ draw arc $AC \perp$ to $A'A''$.

Then in right triangle $AA'C$, given a.

$$\sin \tfrac{1}{2} A = \sin \tfrac{1}{2} a \csc a$$

$$= \sqrt{\frac{1 - \cos a}{2}} \times \frac{1}{\sin a}$$

$$= \sqrt{\frac{1 - \cos a}{2\,(1 - \cos^2 a)}}$$

$$= \sqrt{\frac{1}{2(1 + \cos a)}}$$

$$= \frac{1}{2}\sqrt{\frac{2}{1 + \cos a}}$$

$$= \tfrac{1}{2} \sec \tfrac{1}{2} a.$$

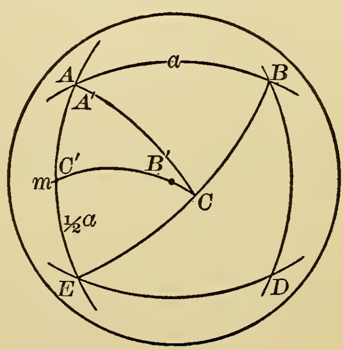
Also, $\cos A = \tan \frac{1}{2} a \cot a$

$$= \sqrt{\frac{1 - \cos a}{1 + \cos a} \times \frac{\cos^2 a}{1 - \cos^2 a}}$$

$$= \cos a \times \frac{1}{1 + \cos a}$$

$$= \cos a \tfrac{1}{2} \sec^2 \tfrac{1}{2} a.$$

3. Given the side a of a regular spherical polygon of n sides; find the angle A of the polygon, the distance R from the centre of the polygon to one of its vertices, and the distance r from the centre to the middle point of one of its sides.



In the regular polygon $ABDE$ draw arcs from the vertices A, B, etc., through the centre C, and from C to M, the middle of one side.

Then $ACB = \frac{360°}{n}$;

also $ACM = \frac{180°}{n}$,

and $CAM = \frac{1}{2} A$,

$AM = \frac{1}{2} a$,

$AC = R$,

and $MC = r$.

In equilateral right triangle CMA represent the parts by letters A', B', C', etc.

$$\sin A' = \cos B' \sec b'.$$

Substituting known values,

$$\sin \tfrac{1}{2} A = \sec \tfrac{1}{2} a \cos \frac{180°}{n}.$$

Or, $\sin c' = \sin b' \csc B'$,

$$\sin R = \sin \tfrac{1}{2} a \csc \frac{180°}{n}.$$

Or, $\sin a' = \tan b' \cot B'$,

$$\sin r = \tan \tfrac{1}{2} a \cot \frac{180°}{n}.$$

4. Compute the dihedral angles made by the faces of the five regular polyhedrons.

Let the figure $ABCD$ represent a tetrahedron.

It is required to find the dihedral angle $ADCB$ or the corresponding plane angle DFO.

$$DAF = 60°$$

(since it is an angle of an equilateral triangle).

$$DFA = \text{rt. triangle}$$

(DF being $\perp$ to AC, since it is a side of the plane angle DFO corresponding to the dihedral angle $ADCB$).

Let $AD = 1$,

Then $\frac{DF}{AD} = \sin 60°$.

$$DF = AD \sin 60°.$$

$$\begin{aligned} \log AD &= 0.00000 \\ \log \sin 60° &= \underline{9.93753 - 10} \\ \log DF &= 9.93753 - 10 \end{aligned}$$

$$\frac{AF}{AD} = \cos 60°.$$

$$AF = AD \cos 60°.$$

$$\begin{aligned} \log AD &= 0.00000 \\ \log \cos 60° &= \underline{9.69897 - 10} \\ \log AF &= 9.69897 - 10 \end{aligned}$$

Draw OA from centre of triangle ABC.

$$OAF = \tfrac{1}{2} CAB = 30°$$

(since OA produced to the middle of the opposite side CB will pass through the centre O and will also bisect the angle CAB, CA and AB being equal),

$$OFA = \text{rt. triangle}$$

(since OF is $\perp$ to AC).

$$\therefore \frac{OF}{AF} = \tan 30°.$$

$$OF = AF \tan 30°.$$

$$\begin{aligned} \log AF &= 9.69897 \\ \log \tan 30° &= \underline{9.76144 - 10} \\ \log OF &= 9.46041 - 10 \end{aligned}$$

$$\frac{OF}{DF} = \cos DFO.$$

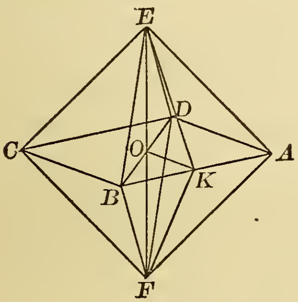
$$\begin{aligned} \log OF &= 9.46041 - 10 \\ \text{colog } DF &= \underline{0.06247} \\ \log \cos DFO &= 9.52288 - 10 \\ DFO &= 70° \ 31' \ 43''. \end{aligned}$$

Let AC' be a cube. Required the dihedral angle $AB'D'$.

The lines AA' and BB' determine the plane $ABB'A'$, and the lines $A'D'$ and $B'C'$ determine the plane $A'D'C'B'$; and as AA' and BB' are $\perp$ to $A'B'$ and $B'C'$, respectively, the planes must be perpendicular.

$\therefore$ the angle required is 90°.

Let E–$ABCD$–F be an octahedron. Required the dihedral angle E–BA–F.



Draw EK and $FK \perp$ to AB, also OK from intersection of axes. Then is EKF the plane angle required.

Let $\quad AB = 1,$

Then $\quad OK = 0.5$

(since OA, OB, and OE are perpendicular and equal),

and $\quad OE(= OA) = \sin 45° = 0.7071.$

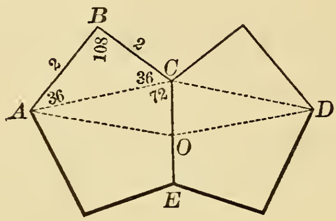
$$\cot EKO = \frac{OK}{OE}.$$

$$\begin{aligned} \log OK &= 9.69897 \\ \operatorname{colog} OE &= 0.15051 \\ \hline \log\cot EKO &= 9.84948 \end{aligned}$$

$$EKO = 54°\ 44'\ 9''.$$
$$EKF = 2\ EKO = 109°\ 28'\ 18''.$$

Let AE and DE be two faces of a dodecagon.

It is required to find the dihedral angle AOD.



Take each side of the pentagons AE and $DE = 2$.

$\cdot$ $B = 108°$, and $A = 36°$.

$$\log AC = \log 2 + \log\sin 108° + \operatorname{colog}\sin 36°$$

$$\begin{aligned} \log 2 &= 0.40103 \\ \log\sin 108° &= 9.97821 \\ \operatorname{colog}\sin 36° &= 0.23078 \\ \hline \log AC &= 0.51002 \\ AC &= 3.2861. \end{aligned}$$

$$ACO = 72°.$$

$$\begin{aligned} \log AC &= 0.51002 \\ \log\sin 72° &= 9.97821 \\ \hline \log AO &= 0.48823 \\ AO &= 3.0777. \end{aligned}$$

$CD = AC$ and $AO = DO$.

$ACD = 108°$, being an angle of a regular pentagon.

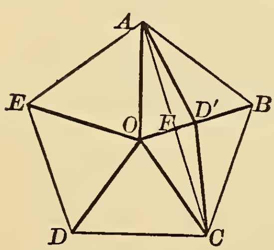
$$CDA = 36°.$$

$$\log AD = \log AC + \log\sin 108° + \operatorname{colog}\sin 36°.$$

$$\begin{aligned}
\log AC &= 0.51002\\
\log \sin 108° &= 9.97821\\
\text{colog} \sin 36° &= 0.23078\\
\log AD &= 0.71901\\
AD &= 5.2361.\\
\tfrac{1}{2} AD &= 2.61805.
\end{aligned}$$

$$\log \sin \tfrac{1}{2} AOD = \log \tfrac{1}{2} AD + \text{colog}\, AO.$$

$$\begin{aligned}
\log \tfrac{1}{2} AD &= 0.41798\\
\text{colog}\, AO &= 9.51177\\
\log \sin \tfrac{1}{2} AOD &= 9.92975\\
\tfrac{1}{2} AOD &= 58° \, 16' \, 52''.\\
AOD &= 116° \, 33' \, 44''.
\end{aligned}$$



Let AOB, BOC, COD, etc., be equilateral triangles forming five of the surfaces of a regular icosahedron, and let AB, BC, CD, etc. $= 1$.

Regarding $ABCDE$ as a plane pentagon, each angle $= 108°$.

$$\therefore ABC = 108°,$$
$$BAC = 36°.$$

In a triangle of which sides are AB, BC, and ADC (regarding ADC as a straight line joining centres of bases of triangles AOB and BOC),

$$\sin DCB : AB :: \sin ABC : ADC.$$

$$\therefore ADC = \frac{AB \sin ABC}{\sin DCB}.$$

$$\begin{aligned}
\log AB &= 0.00000\\
\log \sin ABC &= 9.93753\\
\text{colog} \sin DCB &= 0.30103\\
\log ADC &= 0.23856\\
ADC &= 1.73204.
\end{aligned}$$

But $\quad AD = \frac{1}{2} ADC.$
$\qquad\quad = 0.86602.$

AC is a diagonal of plane pentagon $ABCDE$.

$$\therefore \sin FCB : AB :: \sin ABC : AC.$$

$$\therefore AC = \frac{AB \sin ABC}{\sin FCB}.$$

$$\begin{aligned}
\log AB &= 0.00000\\
\log \sin ABC &= 9.97821\\
\text{colog} \sin FBC &= 0.23078\\
\log AC &= 0.20899\\
AC &= 1.61804.
\end{aligned}$$

But $\quad AF = \frac{1}{2} AC.$
$\qquad\quad = 0.80902.$

In right triangle AFD

$$\sin ADF = \frac{AF}{AD},$$

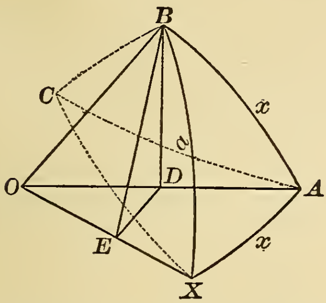
or $\log \sin ADF = \log AF + \text{colog}\, AD.$

$$\begin{aligned}
\log AF &= 9.90796\\
\text{colog}\, AD &= 0.06247\\
\log \sin ADF &= 9.97043\\
ADF &= 69° \, 5' \, 48''.
\end{aligned}$$

But $\quad ADF = \frac{1}{2} A\text{-}OB\text{-}C.$

$$\therefore A\text{-}OB\text{-}C = 138° \, 11' \, 36''.$$

5. A spherical square is a spherical quadrilateral which has equal sides and equal angles. Its two diagonals divide it into four equal right triangles. Find the angle A of the square, having given the side a.



Let $ABCX$ be a spherical square, and CA, BX the two diagonals; also, let the side $BA = x$, then XA will $= x$ (being equal sides).

By [37], $\cos a = \cos x \times \cos x$

$$= \cos^2 x.$$

$$\cos x = \sqrt{\cos a}.$$

Also $\sin x = \frac{DE}{OD}$,

$$\tan x = \frac{BD}{OD}.$$

Dividing first equation by the second,

$$\cos x = \frac{DE}{BD} = \cot X.$$

But $X = \frac{1}{2} A$

(diagonals of a square bisect the angles).

Substitute $\cot \frac{1}{2} A$ for $\cos x$;

$$\cot \tfrac{1}{2} A = \sqrt{\cos a}.$$

Exercise XXIV. Page 119.

1. What do Formulas [43] become if $A = 90°$? if $B = 90°$? if $C = 90°$? if $a = 90°$? if $A = B = 90°$? if $a = b = 90°$?

If $A = 90°$,

$$\sin a \sin B = \sin b,$$
$$\sin a \sin C = \sin c.$$

If $B = 90°$,

$$\sin a = \sin b \sin A,$$
$$\sin b \sin C = \sin c.$$

If $C = 90°$,

$$\sin a = \sin c \sin A,$$
$$\sin b = \sin c \sin B.$$

If $a = 90°$,

$$\sin B = \sin b \sin A,$$
$$\sin C = \sin c \sin A.$$

If $A = B = 90°$,

$$\sin a = \sin b.$$

If $a = b = 90°$,

$$\sin B = \sin A.$$

2. What does the first of [44] become if $A = 0°$? if $A = 90°$? if $A = 180°$?

If $A = 0°$,

$$\cos a = \cos b \cos c + \sin b \sin c.$$

If $A = 90°$,

$$\cos a = \cos b \cos c.$$

If $A = 180°$,

$$\cos a = \cos b \cos c - \sin b \sin c.$$

3. From Formulas [44] deduce Formulas [45], by means of the relations between polar triangles (§ 45).

Substituting in Formulas [44] for a, b, and c, their equals, $180° - A'$, $180° - B'$, $180° - C'$, we obtain

$$\cos(180° - A')$$
$$= \cos(180° - B')\cos(180° - C')$$
$$+ \sin(180° - B')\sin(180° - C')$$
$$\cos(180° - A').$$

Substituting for $\cos(180° - A')$, etc., their equals $-\cos A'$, etc., we obtain

$$-\cos A' = \cos B' \cos C' - \sin B' \sin C' \cos a'.$$

Multiply by -1,

$$\cos A' = -\cos B' \cos C' + \sin B' \sin C' \cos a';$$

and similarly,

$$\cos B' = -\cos A' \cos C' + \sin A' \sin C' \cos b'.$$

$$\cos C' = -\cos A' \cos B' + \sin A' \sin B' \cos c'.$$

Exercise XXV. Page 124.

1. Write formulas for finding, by Napier's Rules, the side a when b, c, and A are given, and for finding the side b when a, c, and B are given.

(i.) By Rule I.,

$$\cos A = \cot b \tan m,$$

whence $\tan m = \tan b \cos A$.

By Rule II.,

$$\cos a = \cos n \cos p,$$

whence $\cos p = \cos a \sec n$.

$$\cos b = \cos m \cos p,$$

whence $\cos p = \cos b \sec m$.

$$\therefore \cos a \sec n = \cos b \sec m.$$

Or, since $n = (c - m)$,

$$\cos a = \cos b \sec m \cos(c - m).$$

(ii.) By Rule I.,

$$\cos B = \tan n \cot a,$$

whence $\tan n = \tan a \cos B$.

By Rule II.,

$$\cos b = \cos m \cos p,$$

whence $\cos p = \cos b \sec m$.

$$\cos a = \cos n \cos p,$$

whence $\cos p = \cos a \sec n$.

$$\therefore \cos b \sec m = \cos a \sec n.$$

Or, since $m = (c - n)$,

$$\cos b = \cos a \sec n \cos(c - n).$$

2. Given | find

$a = 88° 12' 20''$, $A = 63° 15' 11''$,
$b = 124° 7' 17''$, $B = 132° 17' 59''$,
$C = 50° 2' 1''$; $c = 59° 4' 18''$.

$$\tfrac{1}{2}(b - a) = 17° 57' 28.5''.$$
$$\tfrac{1}{2}(a + b) = 106° 9' 48.5''.$$
$$\tfrac{1}{2} C = 25° 1' 0.5''.$$

$$\begin{aligned}
\log \cos \tfrac{1}{2}(b - a) &= 9.97831\\
\log \sec \tfrac{1}{2}(a + b) &= 0.55536\\
\log \cot \tfrac{1}{2} C &= 9.33100\\
\hline
\log \tan \tfrac{1}{2}(A + B) &= 0.86467
\end{aligned}$$

$$\begin{aligned}
\log \sec \tfrac{1}{2}(A + B) &= 0.86868\\
\log \cos \tfrac{1}{2}(a + b) &= 9.44464\\
\log \sin \tfrac{1}{2} C &= 9.62622\\
\hline
\log \cos \tfrac{1}{2} c &= 9.93954\\
\tfrac{1}{2} c &= 29° 32' 9''.
\end{aligned}$$

$$\begin{aligned}
\log \sin \tfrac{1}{2}(b - a) &= 9.48900\\
\log \csc \tfrac{1}{2}(a + b) &= 0.01751\\
\log \cot \tfrac{1}{2} C &= 0.33100\\
\hline
\log \tan \tfrac{1}{2}(B - A) &= 9.83751\\
\tfrac{1}{2}(B - A) &= 34° 31' 24''.\\
\tfrac{1}{2}(A + B) &= 97° 46' 35''.\\
A &= 63° 15' 11''.\\
B &= 132° 17' 59''.\\
c &= 59° 4' 18''.
\end{aligned}$$

3. Given find

$a = 120° 55' 35''$, $A = 129° 58' \ 3''$,
$b = \ 88° 12' 20''$, $B = \ 63° 15' \ 9''$,
$C = \ 47° 42' \ 1''$; $c = \ 55° 52' 40''$.

$\frac{1}{2}(a - b) = \ 16° \ 21' \ 37.5''$.
$\frac{1}{2}(a + b) = 104° \ 33' \ 57.5''$
$\frac{1}{2} C = \ 23° \ 51' \ 0.5''$

$$\begin{aligned} \log \cos \tfrac{1}{2}(a-b) &= 9.98205 \\ \log \sec \tfrac{1}{2}(a+b) &= 0.59947 \\ \log \cot \tfrac{1}{2} C &= \underline{0.35448} \\ \log \tan \tfrac{1}{2}(A+B) &= 10.93600 \\ \tfrac{1}{2}(A+B) &= 96° \ 36' \ 36''. \end{aligned}$$

$$\begin{aligned} \log \sin \tfrac{1}{2}(a-b) &= 9.44976 \\ \log \csc \tfrac{1}{2}(a+b) &= 0.01419 \\ \log \cot \tfrac{1}{2} C &= \underline{0.35448} \\ \log \tan \tfrac{1}{2}(A-B) &= 9.81843 \\ \tfrac{1}{2}(A-B) &= 33° \ 21' \ 27''. \\ \tfrac{1}{2}(A+B) &= 96° \ 36' \ 36''. \end{aligned}$$

$$\begin{aligned} A &= 129° \ 58' \ 3''. \\ B &= \ 63° \ 15' \ 9''. \end{aligned}$$

$$\begin{aligned} \log \sec \tfrac{1}{2}(A+B) &= 0.93890 \\ \log \cos \tfrac{1}{2}(a+b) &= 9.40053 \\ \log \sin \tfrac{1}{2} C &= \underline{9.60675} \\ \log \cos \tfrac{1}{2} c &= 9.94618 \\ \tfrac{1}{2} c &= 27° \ 56' \ 20''. \\ c &= 55° \ 52' \ 40''. \end{aligned}$$

4. Given find

$b = 63° 15' 12''$, $B = 88° 12' 24''$,
$c = 47° 42' \ 1''$, $C = 55° 52' 42''$,
$A = 59° \ 4' 25''$; $a = 50° \ 1' 40''$.

$\frac{1}{2}(b + c) = \ 55° \ 28' \ 36.5''$.
$\frac{1}{2}(b - c) = \ \ 7° \ 46' \ 35.5''$.
$\frac{1}{2} A = \ 29° \ 32' \ 12.5''$.

$$\begin{aligned} \log \cos \tfrac{1}{2}(b-c) &= 9.99599 \\ \text{colog} \cos \tfrac{1}{2}(b+c) &= 0.24662 \\ \log \cot \tfrac{1}{2} A &= \underline{10.24671} \\ \log \tan \tfrac{1}{2}(B+C) &= 10.48932 \\ \tfrac{1}{2}(B+C) &= 72° \ 2' \ 33''. \end{aligned}$$

$$\begin{aligned} \log \sin \tfrac{1}{2}(b-c) &= 9.13133 \\ \text{colog} \sin \tfrac{1}{2}(b+c) &= 0.08413 \\ \log \cot \tfrac{1}{2} A &= \underline{10.24671} \\ \log \tan \tfrac{1}{2}(B-C) &= 9.46217 \\ \tfrac{1}{2}(B-C) &= 16° \ \ 9' \ 51''. \\ \tfrac{1}{2}(B+C) &= 72° \ \ 2' \ 33''. \\ B &= 88° \ 12' \ 24''. \\ C &= 55° \ 52' \ 42''. \end{aligned}$$

$$\begin{aligned} \log \cos \tfrac{1}{2}(b+c) &= 9.75338 \\ \text{colog} \cos \tfrac{1}{2}(B+C) &= 0.51101 \\ \log \sin \tfrac{1}{2} A &= \underline{9.69284} \\ \log \cos \tfrac{1}{2} a &= 9.95723 \\ \tfrac{1}{2} a &= 25° \ 0' \ 50''. \\ a &= 50° \ 1' \ 40''. \end{aligned}$$

5. Given find

$b = \ 69° 25' 11''$, $B = \ 56° 11' 57''$,
$c = 109° 46' 19''$, $C = 123° 21' 12''$,
$A = \ 54° 54' 42''$; $a = \ 67° 13'$.

$\frac{1}{2}(c - b) = 20° \ 10' \ 34''$.
$\frac{1}{2}(c + b) = 89° \ 35' \ 45''$.
$\frac{1}{2} A \qquad = 27° \ 27' \ 21''$.

$$\begin{aligned} \log \cos \tfrac{1}{2}(c-b) &= 9.97250 \\ \text{colog} \cos \tfrac{1}{2}(c+b) &= 2.15157 \\ \log \cot \tfrac{1}{2} A &= \underline{10.28434} \\ \log \tan \tfrac{1}{2}(C+B) &= 12.40841 \\ \tfrac{1}{2}(C+B) &= 89° \ 46' \ 34.5''. \end{aligned}$$

$$\begin{aligned}
\log\sin\tfrac{1}{2}(c-b) &= 9.53770\\
\text{colog}\sin\tfrac{1}{2}(c+b) &= 0.00001\\
\log\cot\tfrac{1}{2}A &= 10.28434\\
\log\tan\tfrac{1}{2}(C-B) &= 9.82205\\
\tfrac{1}{2}(C-B) &= 33°\ 34'\ 37.8''.\\
C &= 123°\ 21'\ 12''.\\
B &= 56°\ 11'\ 57''.
\end{aligned}$$

$$\begin{aligned}
\log\cos\tfrac{1}{2}(c+b) &= 7.84843\\
\text{colog}\cos\tfrac{1}{2}(C+B) &= 2.40837\\
\log\sin\tfrac{1}{2}A &= 9.66376\\
\log\cos\tfrac{1}{2}a &= 9.92056\\
\tfrac{1}{2}a &= 33°\ 36'\ 30''.\\
a &= 67°\ 13'.
\end{aligned}$$

Exercise XXVI. Page 126.

1. What are the formulas for computing A when B, C, and a are given; and for computing B when A, C, and b are given?

By Rule I.,

$$\cos a = \cot y \cot B,$$
$$\therefore \cot y = \tan B \cos a.$$

By Rule II.,

$$\cos A = \cos p \sin x,$$
$$\therefore \cos p = \cos A \csc x.$$
$$\cos B = \cos p \sin y,$$
$$\therefore \cos p = \cos B \csc y.$$
$$\therefore \cos A \csc x = \cos B \csc y.$$

Or, since $x = C - y$,

$$\cos A = \cos B \csc y \sin(C-y).$$

(ii.) By Rule I.,

$$\cos b = \cot x \cot A,$$
$$\therefore \cot x = \tan A \cos b.$$

By Rule II.,

$$\cos A = \cos p \sin x,$$
$$\therefore \cos p = \cos A \csc x.$$
$$\cos B = \cos p \sin y,$$
$$\therefore \cos p = \cos B \csc y.$$
$$\therefore \cos B \csc y = \cos A \csc x.$$

Or, since $y = C - x$,

$$\cos B = \cos A \csc x \sin(C-x).$$

2. Given / find

$A = 26° 58' 46''$, $a = 37° 14' 10''$.
$B = 39° 45' 10''$, $b = 121° 28' 10''$,
$c = 154° 46' 48''$, $C = 161° 22' 11''$.

$$\begin{aligned}
\tfrac{1}{2}(B-A) &= 6°\ 23'\ 12''.\\
\tfrac{1}{2}(B+A) &= 33°\ 21'\ 58''.\\
\tfrac{1}{2}c &= 77°\ 23'\ 24''.
\end{aligned}$$

$$\begin{aligned}
\log\cos\tfrac{1}{2}(B-A) &= 9.99730\\
\log\sec\tfrac{1}{2}(B+A) &= 0.07823\\
\log\tan\tfrac{1}{2}c &= 10.65032\\
\log\tan\tfrac{1}{2}(b+a) &= 10.72585
\end{aligned}$$

$$\begin{aligned}
\log\sin\tfrac{1}{2}(B+A) &= 9.74035\\
\log\sec\tfrac{1}{2}(b-a) &= 0.12972\\
\log\cos\tfrac{1}{2}c &= 9.33908\\
\log\cos\tfrac{1}{2}C &= 9.20915\\
\tfrac{1}{2}C &= 80°\ 41'\ 5.4''.
\end{aligned}$$

$$\begin{aligned}
\log\sin\tfrac{1}{2}(B-A) &= 9.04625\\
\log\csc\tfrac{1}{2}(B+A) &= 0.25965\\
\log\tan\tfrac{1}{2}c &= 10.65032\\
\log\tan\tfrac{1}{2}(b-a) &= 9.95622\\
\tfrac{1}{2}(b-a) &= 42°\ 7'.\\
\tfrac{1}{2}(b+a) &= 79°\ 21'\ 10''.\\
b &= 121°\ 28'\ 10''.\\
a &= 37°\ 14'\ 10''.\\
C &= 161°\ 22'\ 11''.
\end{aligned}$$

3. Given find

$A = 128° 41' 49''$, $a = 125° 41' 44''$,
$B = 107° 33' 20''$, $b = 82° 47' 34''$,
$c = 124° 12' 31''$; $C = 127° 22'$.

$$\tfrac{1}{2}(A - B) = 10° \; 34' \; 14.5''.$$
$$\tfrac{1}{2}(A + B) = 118° \; 7' \; 34.5''.$$
$$\tfrac{1}{2}c = 62° \; 6' \; 15.5''.$$

$$\begin{aligned}\log\cos\tfrac{1}{2}(A - B) &= 9.99257\\ \text{colog}\cos\tfrac{1}{2}(A + B) &= 0.32660\\ \log\tan\tfrac{1}{2}c &= 10.27624\\ \hline \log\tan\tfrac{1}{2}(a + b) &= 10.59541\\ \tfrac{1}{2}(a + b) &= 104° 14' 38.5''.\end{aligned}$$

$$\begin{aligned}\log\sin\tfrac{1}{2}(A - B) &= 9.26351\\ \text{colog}\sin\tfrac{1}{2}(A + B) &= 0.05457\\ \log\tan\tfrac{1}{2}c &= 10.27624\\ \hline \log\tan\tfrac{1}{2}(a - b) &= 9.59432\\ \tfrac{1}{2}(a - b) &= 21° 27' \; 5''.\\ a &= 125° 41' 44''.\\ b &= 82° 47' 34''.\end{aligned}$$

$$\begin{aligned}\log\sin\tfrac{1}{2}(A + B) &= 9.94543\\ \text{colog}\cos\tfrac{1}{2}(a - b) &= 0.03118\\ \log\cos\tfrac{1}{2}c &= 9.67012\\ \hline \log\cos\tfrac{1}{2}C &= 9.64673\\ \tfrac{1}{2}C &= 63° \; 41'.\\ C &= 127° \; 22'.\end{aligned}$$

4. Given find

$B = 153° 17' \; 6''$, $b = 152° 43' 51''$,
$C = 78° 43' 36''$, $c = 88° 12' 21''$,
$a = 86° 15' 15''$; $A = 78° 15' 48''$.

$$\tfrac{1}{2}(B + C) = 116° \; 0' \; 21''.$$
$$\tfrac{1}{2}(B - C) = 37° \; 16' \; 45''.$$
$$\tfrac{1}{2}a = 43° \; 7' \; 37.5''.$$

$$\begin{aligned}\log\cos\tfrac{1}{2}(B - C) &= 9.90074\\ \log\sec\tfrac{1}{2}(B + C) &= 0.35807\\ \log\tan\tfrac{1}{2}a &= 9.97159\\ \hline \log\tan\tfrac{1}{2}(b + c) &= 0.23040\\ \tfrac{1}{2}(b + c) &= 120° \; 28' \; 6''.\end{aligned}$$

$$\begin{aligned}\log\sin\tfrac{1}{2}(B - C) &= 9.78226\\ \log\csc\tfrac{1}{2}(B + C) &= 0.04636\\ \log\tan\tfrac{1}{2}a &= 9.97159\\ \hline \log\tan\tfrac{1}{2}(b - c) &= 9.80021\\ \tfrac{1}{2}(b - c) &= 32° \; 15' \; 45''.\end{aligned}$$

$$\begin{aligned}\log\sin\tfrac{1}{2}(B + C) &= 9.95364\\ \log\sec\tfrac{1}{2}(b - c) &= 0.07283\\ \log\cos\tfrac{1}{2}a &= 9.86322\\ \hline \log\cos\tfrac{1}{2}A &= 9.88969\\ \tfrac{1}{2}A &= 39° \; 7' \; 54''.\\ b &= 152° \; 43' \; 51''.\\ c &= 88° \; 12' \; 21''.\\ A &= 78° \; 15' \; 48''.\end{aligned}$$

5. Given find

$A = 125° 41' 44''$, $a = 128° 31' 46''$,
$C = 82° 47' 35''$, $c = 107° 33' 20''$,
$b = 52° 37' 57''$; $B = 55° 47' 40''$.

$$\tfrac{1}{2}(A + C) = 104° \; 14' \; 39.5''.$$
$$\tfrac{1}{2}(A - C) = 21° \; 27' \; 4.5''.$$
$$\tfrac{1}{2}b = 26° \; 18' \; 58.5''.$$

$$\begin{aligned}\log\cos\tfrac{1}{2}(A - C) &= 9.96883\\ \log\sec\tfrac{1}{2}(A + C) &= 0.60896\\ \log\tan\tfrac{1}{2}b &= 9.69424\\ \hline \log\tan\tfrac{1}{2}(a + c) &= 0.27203\\ \tfrac{1}{2}(a + c) &= 118° \; 7' \; 33''.\end{aligned}$$

$$\begin{aligned}\log\sin\tfrac{1}{2}(A + C) &= 9.98644\\ \log\sec\tfrac{1}{2}(a - c) &= 0.00743\\ \log\cos\tfrac{1}{2}b &= 9.95248\\ \hline \log\cos\tfrac{1}{2}B &= 9.94635\\ \tfrac{1}{2}B &= 27° \; 53' \; 50''.\end{aligned}$$

$\log \sin \frac{1}{2}(A-C) = 9.56313$
$\log \csc \frac{1}{2}(A+C) = 0.01356$
$\log \tan \frac{1}{2} b \quad = 9.69424$
$\log \tan \frac{1}{2}(a-c) \quad = 9.27093$

$\frac{1}{2}(a-c) = 10° \, 24' \, 13''.$
$a = 128° \, 31' \, 46''.$
$c = 107° \, 33' \, 20''.$
$B = 55° \, 47' \, 40''.$

Exercise XXVII. Page 128.

1. Given $a = 73° 49' 38''$, $b = 120° 53' 35''$, $A = 88° 52' 42''$; find $B = 116° 42' 30''$, $c = 120° 57' 27''$, $C = 116° 47' \, 4''$.

$\log \sin A \quad = 9.99992$
$\log \sin b \quad = 9.93355$
$\log \csc a \quad = 0.01753$
$\log \sin B \quad = 9.95100$
$B = [180° - (63° \, 17' \, 30'')]$
$= 116° \, 42' \, 30''.$

$\frac{1}{2}(B+A) = 102° \, 47' \, 36''.$
$\frac{1}{2}(B-A) = 13° \, 54' \, 54''.$
$\frac{1}{2}(b+a) = 97° \, 21' \, 36.5''.$
$\frac{1}{2}(b-a) = 23° \, 31' \, 58.5''.$

$\log \sin \frac{1}{2}(B+A) = 9.98908$
$\log \csc \frac{1}{2}(B-A) = 0.61892$
$\log \tan \frac{1}{2}(b-a) = 9.63898$
$\log \tan \frac{1}{2} c \quad = 10.24698$
$\frac{1}{2} c = 60° \, 28' 43.5''.$
$c = 120° \, 57' \, 27''.$

$\log \sin \frac{1}{2}(b+a) = 9.99641$
$\log \csc \frac{1}{2}(b-a) = 0.39873$
$\log \tan \frac{1}{2}(B-A) = 9.39401$
$\log \cot \frac{1}{2} C \quad = 9.78915$
$\frac{1}{2} C = 58° \, 23' \, 32''.$
$C = 116° \, 47' \, 4''.$

2. Given $a = 150° \, 57' \, 5''$, $b = 134° \, 15' \, 54''$, $A = 144° \, 22' \, 42''$; find $B_1 = 120° \, 47' \, 45''$, $B_2 = 59° \, 12' \, 15''$, $c_1 = 55° \, 42' \, 8''$, $c_2 = 23° \, 57' \, 17.4''$, $C_1 = 97° \, 42' \, 55''$, $C_2 = 29° \, 8' \, 39''$.

$A > 90°$, $(a+b) > 180°$, $a > b$; hence two solutions.

I. $\sin B_1 = \sin A \sin b \csc a.$

$\log \sin A \quad = 9.76524$
$\log \sin b \quad = 9.85498$
$\text{colog} \sin a \quad = 0.31377$
$\log \sin B_1 \quad = 9.93399$
$B_1 = 120° \, 47' \, 45''.$
$B_2 = 59° \, 12' \, 15''.$

$\frac{1}{2}(a-b) = 8° \, 20' \, 35.5''.$
$\frac{1}{2}(a+b) = 142° \, 36' \, 28.5''.$
$\frac{1}{2}(A-B_1) = 11° \, 47' \, 28.5''.$
$\frac{1}{2}(A-B_2) = 42° \, 35' \, 13.5''.$

$\log \sin \frac{1}{2}(a+b) = 9.78338$
$\log \csc \frac{1}{2}(a-b) = 0.83833$
$\log \tan \frac{1}{2}(A-B_1) = 9.31963$
$\log \cot \frac{1}{2} C_1 \quad = 9.94134$
$\frac{1}{2} C_1 = 48° \, 51' \, 27.7''.$
$C_1 = 97° \, 42' \, 55.4''.$

$\log\sin\frac{1}{2}(a+b) = 9.78338$
$\log\csc\frac{1}{2}(a-b) = 0.83833$
$\log\tan\frac{1}{2}(A-B_2) = 9.96338$
$\log\cot\frac{1}{2}C_2 = 10.58509$
$\frac{1}{2}C_2 = 14°\ 34'\ 19.6''.$
$C_2 = 29°\ 8'\ 39''.$

$\frac{1}{2}(A+B_1) = 132°\ 35'\ 13.5''.$
$\frac{1}{2}(A+B_2) = 101°\ 47'\ 28.5''.$

$\log\sin\frac{1}{2}(A+B_1) = 9.86703$
$\text{colog}\sin\frac{1}{2}(A-B_1) = 0.68963$
$\log\tan\frac{1}{2}(a-b) = 9.16629$
$\log\tan\frac{1}{2}c_1 = 9.72295$
$\frac{1}{2}c_1 = 27°\ 51'\ 4''.$
$c_1 = 55°\ 42'\ 8''.$

$\log\sin\frac{1}{2}(A+B_2) = 9.99074$
$\text{colog}\sin\frac{1}{2}(A-B_2) = 0.16960$
$\log\tan\frac{1}{2}(a-b) = 9.16629$
$\log\tan\frac{1}{2}c_2 = 9.32663$
$\frac{1}{2}c_2 = 11°\ 58'\ 38.7''.$
$c_2 = 23°\ 57'\ 17.4''.$

3. Given find
$a = 79°\ 0'\ 54.5''$, $B = 90°$,
$b = 82°\ 17'\ 4''$, $c = 45°\ 12'\ 19''$,
$A = 82°\ 9'\ 25.8''$; $C = 45°\ 44'$.

$\log\sin A = 9.99592$
$\log\sin b = 9.99605$
$\text{colog}\sin a = 0.00803$
$\log\sin B = 0.00000$
$B = 90°.$

$\tan c = \cos A\tan b.$
$\cot C = \tan A\cos b.$

$\log\cos A = 9.13499$
$\log\tan b = 10.86812$
$\log\tan c = 10.00311$
$c = 45°\ 12'\ 19''.$

$\log\tan A = 0.86092$
$\log\cos b = 9.12793$
$\log\cot C = 9.98885$
$C = 45°\ 44'.$

4. Given $a = 30°\ 52'\ 36.6''$, $b = 31°\ 9'\ 16''$, $A = 87°\ 34'\ 12''$; show that the triangle is impossible.

From [48]
$\sin B = \sin A\sin b\csc a.$

$\log\sin A = 9.99961$
$\log\sin b = 9.71378$
$\log\csc a = 0.28972$
$\log\sin B = 0.00311$
$\sin B = 1.009.$

$\therefore$ impossible, since $\sin B > 1$.

Exercise XXVIII. Page 129.

1. Given find
$A = 110°\ 10'$, $b = 155°\ 5'\ 18''$,
$B = 133°\ 18'$, $c = 33°\ 1'\ 36''$,
$a = 147°\ 5'\ 32''$; $C = 70°\ 20'\ 40''$.

$\sin b = \sin a\sin B\csc A.$

$\log\sin a = 9.73503$
$\log\sin B = 9.86200$
$\text{colog}\sin A = 0.02748$
$\log\sin b = 9.62451$
$b = 155°\ 5'\ 18''.$

$\frac{1}{2}(B+A) = 121° \; 44'.$
$\frac{1}{2}(B-A) = 11° \; 34'.$
$\frac{1}{2}(b-a) = 3° \; 59' \; 53''.$
$\frac{1}{2}(b+a) = 151° \; 5' \; 25''.$

$$\begin{aligned}
\log \sin \tfrac{1}{2}(B+A) &= 9.92968 \\
\text{colog} \sin \tfrac{1}{2}(B-A) &= 0.69787 \\
\log \tan \tfrac{1}{2}(b-a) &= \underline{8.84443} \\
\log \tan \tfrac{1}{2} c &= 9.47198 \\
\tfrac{1}{2} c &= 16° \; 30' \; 48''. \\
c &= 33° \; 1' \; 36''.
\end{aligned}$$

$$\begin{aligned}
\text{colog} \sin \tfrac{1}{2}(b-a) &= 0.15663 \\
\log \sin \tfrac{1}{2}(b+a) &= 9.68433 \\
\log \tan \tfrac{1}{2}(B-A) &= \underline{9.31104} \\
\log \cot \tfrac{1}{2} C &= 9.15200 \\
\tfrac{1}{2} C &= 35° \; 10' \; 20''. \\
C &= 70° \; 20' \; 40''.
\end{aligned}$$

2. Given find
$A = 113° \, 39' \, 21''$, $b = 124° \; 7' \, 20''$,
$B = 123° \, 40' \, 18''$, $c = 159° \, 53' \; 2''$,
$a = 65° \, 39' \, 46''$; $C = 159° \, 43' \, 35''$.

$$\begin{aligned}
\log \sin a &= 9.95959 \\
\log \sin B &= 9.92024 \\
\text{colog} \sin A &= \underline{0.03812} \\
\log \sin b &= 9.91795 \\
b &= 124° \; 7' \; 20''.
\end{aligned}$$

$\frac{1}{2}(B+A) = 118° \; 39' \; 49.5''.$
$\frac{1}{2}(B-A) = 5° \; 0' \; 28.5''.$
$\frac{1}{2}(b-a) = 29° \; 13' \; 52''.$
$\frac{1}{2}(b+a) = 94° \; 53' \; 33''.$

$$\begin{aligned}
\log \sin \tfrac{1}{2}(B+A) &= 9.94422 \\
\text{colog} \sin \tfrac{1}{2}(B-A) &= 1.05901 \\
\log \tan \tfrac{1}{2}(b-a) &= \underline{9.74789} \\
\log \tan \tfrac{1}{2} c &= 10.75112 \\
\tfrac{1}{2} c &= 79° \; 56' \; 51''. \\
c &= 159° \; 53' \; 2''.
\end{aligned}$$

$$\begin{aligned}
\log \sin \tfrac{1}{2}(b+a) &= 9.99842 \\
\text{colog} \sin \tfrac{1}{2}(b-a) &= 0.31128 \\
\log \tan \tfrac{1}{2}(B-A) &= \underline{8.94264} \\
\log \cot \tfrac{1}{2} C &= 9.25234 \\
\tfrac{1}{2} C &= 79° \, 51' \, 47.7''. \\
C &= 159° \, 43' \, 35''.
\end{aligned}$$

3. Given find
$A = 100° \; 2' \, 11.3''$, $b = 90°$,
$B = 98° \, 30' \, 28''$, $c = 147° \, 41' \, 43''$,
$a = 95° \, 20' \, 38.7''$; $C = 148° \; 5' \, 33''$.

$$\begin{aligned}
\log \sin a &= 9.99811 \\
\log \sin B &= 9.99519 \\
\log \csc A &= \underline{0.00670} \\
\log \sin b &= 0.00000 \\
b &= 90°.
\end{aligned}$$

$\frac{1}{2}(A+B) = 99° \; 16' \; 19.7''.$
$\frac{1}{2}(A-B) = 0° \; 45' \; 51.7''.$
$\frac{1}{2}(a-b) = 2° \; 40' \; 19.4''.$

$$\begin{aligned}
\log \sin \tfrac{1}{2}(A+B) &= 9.99428 \\
\text{colog} \sin \tfrac{1}{2}(A-B) &= 1.87484 \\
\log \tan \tfrac{1}{2}(a-b) &= \underline{8.66904} \\
\log \tan \tfrac{1}{2} c &= 10.53816 \\
\tfrac{1}{2} c &= 73° \, 50' \, 51.7''. \\
c &= 147° \, 41' \, 43''.
\end{aligned}$$

$$\begin{aligned}
\log \sin \tfrac{1}{2}(a+b) &= 9.99953 \\
\text{colog} \sin \tfrac{1}{2}(a-b) &= 1.33144 \\
\log \tan \tfrac{1}{2}(A-B) &= \underline{8.12520} \\
\log \cot \tfrac{1}{2} C &= 9.45617 \\
\tfrac{1}{2} C &= 74° \; 2' \; 46.3''. \\
C &= 148° \; 5' \; 33''.
\end{aligned}$$

4. Given $A = 24° \; 33' \; 9''$, $B = 38° \; 0' \; 12''$, $a = 65° \; 20' \; 13''$; show that the triangle is impossible.

$\cot x = \cos a \tan B.$

$$\begin{aligned}
\log \cos a &= 9.62042\\
\log \tan B &= 9.89286\\
\hline
\log \cot x &= 9.51328\\
x &= 71^\circ\ 56'\ 30''.
\end{aligned}$$

$\sin (c - x) = \cos A \sec B \sin x.$

$$\begin{aligned}
\log \cos A &= 9.95884\\
\log \sec B &= 0.10349\\
\log \sin x &= 9.97806\\
\hline
\log \sin (c - x) &= 0.04039\\
\sin (c - x) &= 1.0974.
\end{aligned}$$

Since $\sin (c - x) > 1$, the angle C is impossible.

$\therefore$ the triangle is impossible.

Exercise XXIX. Page 131.

1. Given $a = 120^\circ 55' 35''$, $b = 59^\circ 4' 25''$, $c = 106^\circ 10' 22''$; find $A = 116^\circ 44' 49''$, $B = 63^\circ 15' 14''$, $C = 91^\circ 7' 21''$.

$$\begin{aligned}
a &= 120^\circ\ 55'\ 35''\\
b &= 59^\circ\ 4'\ 25''\\
c &= 106^\circ\ 10'\ 22''\\
\hline
2s &= 286^\circ\ 10'\ 22''\\
s &= 143^\circ\ 5'\ 11''.\\
s - a &= 22^\circ\ 9'\ 36''.\\
s - b &= 84^\circ\ 0'\ 46''.\\
s - c &= 36^\circ\ 54'\ 49''.
\end{aligned}$$

$$\begin{aligned}
\log \sin (s - a) &= 9.57657\\
\log \sin (s - b) &= 9.99763\\
\log \sin (s - c) &= 9.77859\\
\log \csc s &= 0.22141\\
\hline
\log \tan^2 r &= 19.57420\\
\log \tan r &= 9.78710.
\end{aligned}$$

$\tan \frac{1}{2} A = \tan r \csc (s - a).$

$$\begin{aligned}
\log \tan \tfrac{1}{2} A &= 10.21053\\
\log \tan \tfrac{1}{2} B &= 9.78948\\
\log \tan \tfrac{1}{2} C &= 10.00851\\
\hline
\tfrac{1}{2} A &= 58^\circ\ 22'\ 24.8''.\\
\tfrac{1}{2} B &= 31^\circ\ 37.2'.\\
\tfrac{1}{2} C &= 45^\circ\ 33'\ 40.8''.
\end{aligned}$$

$$\begin{aligned}
A &= 116^\circ\ 44'\ 49''.\\
B &= 63^\circ\ 15'\ 14''.\\
C &= 91^\circ\ 7'\ 21''.
\end{aligned}$$

2. Given $a = 50^\circ 12' 4''$, $b = 116^\circ 44' 48''$, $c = 129^\circ 11' 42''$; find $A = 59^\circ 4' 28''$, $B = 94^\circ 23' 12''$, $C = 120^\circ 4' 52''$.

$$\begin{aligned}
a &= 50^\circ\ 12'\ 4''\\
b &= 116^\circ\ 44'\ 48''\\
c &= 129^\circ\ 11'\ 42''\\
\hline
2s &= 296^\circ\ 8'\ 34''\\
s &= 148^\circ\ 4'\ 17''.\\
s - a &= 97^\circ\ 52'\ 13''.\\
s - b &= 31^\circ\ 19'\ 29''.\\
s - c &= 18^\circ\ 52'\ 35''.
\end{aligned}$$

$$\begin{aligned}
\log \sin (s - a) &= 9.99589\\
\log \sin (s - b) &= 9.71591\\
\log \sin (s - c) &= 9.50992\\
\log \csc s &= 0.27666\\
\hline
\log \tan^2 r &= 9.49838\\
\log \tan r &= 9.74919.
\end{aligned}$$

$\log\tan\frac{1}{2}A = 9.75330$
$\log\tan\frac{1}{2}B = 0.03328$
$\log\tan\frac{1}{2}C = 0.23927$

$\frac{1}{2}A = 29°\ 32'\ 14''.$
$\frac{1}{2}B = 47°\ 11'\ 36''.$
$\frac{1}{2}C = 60°\ 2'\ 26''.$
$A = 59°\ 4'\ 28''.$
$B = 94°\ 23'\ 12''.$
$C = 120°\ 4'\ 52''.$

3. Given $a = 131°\,35'\,4''$, $b = 108°\,30'\,14''$, $c = 84°\,46'\,34''$; find $A = 132°\,14'\,21''$, $B = 110°\,10'\,40''$, $C = 99°\,42'\,24''$.

$a = 131°\ 35'\ 4''$
$b = 108°\ 30'\ 14''$
$c = 84°\ 46'\ 34''$

$2s = 324°\ 51'\ 52''$
$s = 162°\ 25'\ 56''.$
$s - a = 30°\ 50'\ 52''.$
$s - b = 53°\ 55'\ 42''.$
$s - c = 77°\ 39'\ 22''.$

$\log\sin(s-a) = 9.70991$
$\log\sin(s-b) = 9.90756$
$\log\sin(s-c) = 9.98984$
$\log\csc s = 0.52023$

$\log\tan^2 r = 10.12754$
$\log\tan r = 10.06377.$

$\log\tan\frac{1}{2}A = 0.35386$
$\log\tan\frac{1}{2}B = 0.15621$
$\log\tan\frac{1}{2}C = 0.07393$

$\frac{1}{2}A = 66°\ 7'\ 10.6''.$
$\frac{1}{2}B = 55°\ 5'\ 20''.$
$\frac{1}{2}C = 49°\ 51'\ 12''.$

$A = 132°\ 14'\ 21''.$
$B = 110°\ 10'\ 40''.$
$C = 99°\ 42'\ 24''.$

4. Given $a = 20°\,16'\,38''$, $b = 56°\,19'\,40''$, $c = 66°\,20'\,44''$; find $A = 20°\,9'\,54''$, $B = 55°\,52'\,31''$, $C = 114°\,20'\,17''$.

$a = 20°\ 16'\ 38''$
$b = 56°\ 19'\ 40''$
$c = 66°\ 20'\ 44''$

$2s = 142°\ 57'\ 2''$
$s = 71°\ 28'\ 31''.$
$s - a = 51°\ 11'\ 53''.$
$s - b = 15°\ 8'\ 51''.$
$s - c = 5°\ 7'\ 47''.$

$\log\sin(s-a) = 9.89172$
$\log\sin(s-b) = 9.41715$
$\log\sin(s-c) = 8.95139$
$\log\csc s = 0.02311$

$\log\tan^2 r = 8.28337$
$\log\tan r = 9.14168.$

$\log\tan\frac{1}{2}A = 9.24996$
$\log\tan\frac{1}{2}B = 9.72453$
$\log\tan\frac{1}{2}C = 10.19029$

$\frac{1}{2}A = 10°\ 4'\ 56.8''.$
$\frac{1}{2}B = 27°\ 56'\ 15.5''.$
$\frac{1}{2}C = 57°\ 10'\ 8.6''.$
$A = 20°\ 9'\ 54''.$
$B = 55°\ 52'\ 31''.$
$C = 114°\ 20'\ 17''.$

Exercise XXX. Page 132.

1. Given $A = 130°$, $B = 110°$, $C = 80°$; find $a = 139° 21' 22''$, $b = 126° 57' 52''$, $c = 56° 51' 48''$.

$$\begin{aligned} A &= 130° \\ B &= 110° \\ C &= 80° \\ \hline 2S &= 320° \\ S &= 160°. \\ S-A &= 30°. \\ S-B &= 50°. \\ S-C &= 80°. \end{aligned}$$

$$\begin{aligned} \log\cos S &= 9.97299 \\ \log\sec(S-A) &= 0.06247 \\ \log\sec(S-B) &= 0.19193 \\ \log\sec(S-C) &= 0.76033 \\ \hline \log\tan^2 R &= 10.98772 \\ \log\tan R &= 10.49386. \end{aligned}$$

$$\begin{aligned} \log\tan\tfrac{1}{2}a &= 10.43139 \\ \log\tan\tfrac{1}{2}b &= 10.30193 \\ \log\tan\tfrac{1}{2}c &= 9.73353 \\ \hline \tfrac{1}{2}a &= 69° 40' 41''. \\ \tfrac{1}{2}b &= 63° 28' 56''. \\ \tfrac{1}{2}c &= 28° 25' 54''. \\ a &= 139° 21' 22''. \\ b &= 126° 57' 52''. \\ c &= 56° 51' 48''. \end{aligned}$$

2. Given $A = 59° 55' 10''$, $B = 85° 36' 50''$, $C = 59° 55' 10''$; find $a = 128° 42' 29''$, $b = 64° 2' 47''$, $c = 128° 42' 29''$.

$$\begin{aligned} A &= 59° 55' 10'' \\ B &= 85° 36' 50'' \\ C &= 59° 55' 10'' \\ \hline 2S &= 205° 27' 10'' \\ S &= 102° 43' 35''. \\ S-A &= 42° 48' 25''. \\ S-B &= 17° 6' 45''. \\ S-C &= 42° 48' 25''. \end{aligned}$$

$$\begin{aligned} \log\cos S &= 9.34301 \\ \log\sec(S-A) &= 0.13451 \\ \log\sec(S-B) &= 0.01967 \\ \log\sec(S-C) &= 0.13451 \\ \hline \log\tan^2 R &= 9.63170 \\ \log\tan R &= 9.81585. \end{aligned}$$

$$\begin{aligned} \log\tan\tfrac{1}{2}a &= 9.68134 \\ \log\tan\tfrac{1}{2}b &= 9.79618 \\ \log\tan\tfrac{1}{2}c &= 9.68134 \\ \hline \tfrac{1}{2}a &= 25° 38' 45.5''. \\ \tfrac{1}{2}b &= 32° 1' 23.6''. \\ \tfrac{1}{2}c &= 25° 38' 45.5''. \\ a &= 51° 17' 31''. \\ b &= 64° 2' 47''. \\ c &= 51° 17' 31''. \end{aligned}$$

3. Given $A = 102° 14' 12''$, $B = 54° 32' 24''$, $C = 89° 5' 46''$; find $a = 104° 25' 9''$, $b = 53° 49' 25''$, $c = 97° 44' 24''$.

$$\begin{aligned} A &= 102° 14' 12'' \\ B &= 54° 32' 24'' \\ C &= 89° 5' 46'' \\ \hline 2S &= 245° 52' 22'' \\ S &= 122° 56' 11''. \end{aligned}$$

$S - A = 20° \; 41' \; 59''$.
$S - B = 68° \; 23' \; 47''$.
$S - C = 33° \; 50' \; 25''$.

$\log \cos S = 9.73536$
$\log \sec (S - A) = 0.02898$
$\log \sec (S - B) = 0.43394$
$\log \sec (S - C) = 0.08061$
$\log \tan^2 R = 0.27889$
$\log \tan R = 0.13945$.

$\log \tan \frac{1}{2} a = 0.11047$
$\log \tan \frac{1}{2} b = 9.70551$
$\log \tan \frac{1}{2} c = 0.05885$

$\frac{1}{2} a = 52° \; 12' \; 34.6''$.
$\frac{1}{2} b = 26° \; 54' \; 42.5''$.
$\frac{1}{2} c = 48° \; 52' \; 12''$.
$a = 104° \; 25' \; 9''$.
$b = 53° \; 49' \; 25''$.
$c = 97° \; 44' \; 24''$.

4. Given find
$A = 4° 23' 35''$, $a = 31° \; 9' 11''$,
$B = 8° 28' 20''$, $b = 84° 18' 23''$,
$C = 172° 17' 56''$; $c = 115° \; 9' 56''$.

$A = 4° \; 23' \; 35''$
$B = 8° \; 28' \; 20''$
$C = 172° \; 17' \; 56''$
$2S = 185° \; 9' \; 51''$
$S = 92° \; 34' \; 55.5''$.
$S - A = 88° \; 11' \; 20.5''$.
$S - B = 84° \; 6' \; 35.5''$.
$S - C = -(79° \; 43' \; 0.5'')$.

$\log \cos S = 8.65368$
$\log \sec (S - A) = 1.50029$
$\log \sec (S - B) = 0.98876$
$\log \sec (S - C) = 0.74833$
$\log \tan^2 R = 11.89106$
$\log \tan R = 10.94553$.

$\log \tan \frac{1}{2} a = 9.44524$
$\log \tan \frac{1}{2} b = 9.95677$
$\log \tan \frac{1}{2} c = 10.19720$

$\frac{1}{2} a = 15° \; 34' \; 35.5''$.
$\frac{1}{2} b = 42° \; 9' \; 11.5''$.
$\frac{1}{2} c = 57° \; 34' \; 58''$.
$a = 31° \; 9' \; 11''$.
$b = 84° \; 18' \; 23''$.
$c = 115° \; 9' \; 56''$.

Exercise XXXI. Page 135.

1. Given find
$A = 84° \; 20' \; 19''$, $E = 26159''$,
$B = 27° \; 22' \; 40''$, $F = 0.12685 \, R^2$.
$C = 75° \; 33'$;

$E = A + B + C - 180°$.

$A = 84° \; 20' \; 19''$
$B = 27° \; 22' \; 40''$
$C = 75° \; 33'$

$187° \; 15' \; 59''$
$180°$

$E = 7° \; 15' \; 59''$
$= 26159''$.

$\log 26159 = 4.41762$
$\operatorname{colog} 648000 = 4.18842 - 10$
$\log 3.14159 = 0.49715$
$\log F = 9.10319 - 10$
$F = 0.12685 \, R^2$.

2. Given find
$a = 69° 15' \; 6''$, $E = 216° 40' 18''$.
$b = 120° 42' 47''$,
$c = 159° 18' 33''$;

$$
\begin{aligned}
a &= 69^\circ\ 15'\ \ 6'' \\
b &= 120^\circ\ 42'\ 47'' \\
c &= 159^\circ\ 18'\ 33'' \\
\hline
2s &= 349^\circ\ 16'\ 26'' \\
s &= 174^\circ\ 38'\ 13''. \\
s-a &= 105^\circ\ 23'\ \ 7''. \\
s-b &= 53^\circ\ 55'\ 26''. \\
s-c &= 15^\circ\ 19'\ 40''.
\end{aligned}
$$

$$
\begin{aligned}
\tfrac{1}{2}s &= 87^\circ\ 19'\ \ 6.5''. \\
\tfrac{1}{2}(s-a) &= 52^\circ\ 41'\ 33.5''. \\
\tfrac{1}{2}(s-b) &= 26^\circ\ 57'\ 43''. \\
\tfrac{1}{2}(s-c) &= 7^\circ\ 39'\ 50''.
\end{aligned}
$$

$$
\begin{aligned}
\log\tan\tfrac{1}{2}s &= 11.32942 \\
\log\tan\tfrac{1}{2}(s-a) &= 10.11804 \\
\log\tan\tfrac{1}{2}(s-b) &= 9.70645 \\
\log\tan\tfrac{1}{2}(s-c) &= 9.12893 \\
\hline
\log\tan^2\tfrac{1}{4}E &= 10.28284 \\
\log\tan\tfrac{1}{4}E &= 10.14142. \\
\tfrac{1}{4}E &= 54^\circ\ 10'\ \ 4.6''. \\
E &= 216^\circ\ 40'\ 18''.
\end{aligned}
$$

$$
\begin{aligned}
a &= 33^\circ\ \ 1'\ 45'' \\
b &= 155^\circ\ \ 5'\ 18'' \\
c &= 147^\circ\ \ 5'\ 30'' \\
\hline
2s &= 335^\circ\ 12'\ 33'' \\
s &= 167^\circ\ 36'\ 16.5''. \\
s-a &= 134^\circ\ 34'\ 31.5''. \\
s-b &= 12^\circ\ 30'\ 58.5''. \\
s-c &= 20^\circ\ 30'\ 46.5''. \\
\tfrac{1}{2}s &= 83^\circ\ 48'\ \ 8.25''. \\
\tfrac{1}{2}(s-a) &= 67^\circ\ 17'\ 15.75''. \\
\tfrac{1}{2}(s-b) &= 6^\circ\ 15'\ 29.25''. \\
\tfrac{1}{2}(s-c) &= 10^\circ\ 15'\ 23.25''.
\end{aligned}
$$

$$
\begin{aligned}
\log\tan\tfrac{1}{2}s &= 0.96419 \\
\log\tan\tfrac{1}{2}(s-a) &= 0.37824 \\
\log\tan\tfrac{1}{2}(s-b) &= 9.04005 \\
\log\tan\tfrac{1}{2}(s-c) &= 9.25755 \\
\hline
\log\tan^2\tfrac{1}{4}E &= 9.64003 \\
\log\tan\tfrac{1}{4}E &= 9.82002. \\
\tfrac{1}{4}E &= 33^\circ\ 27'\ 13\tfrac{1}{3}''. \\
E &= 133^\circ\ 48'\ 53''.
\end{aligned}
$$

3. Given find

$a = 33^\circ\ 1'45''$, $E = 133^\circ 48' 53''$.
$b = 155^\circ\ 5'18''$,
$C = 110^\circ 10'$;

$\tan m = \tan a \cos C.$ (§ 54)

$\cos c = \cos a \sec m \cos(b-m).$ (§ 54)

$$
\begin{aligned}
\log\tan a &= 9.81300 \\
\log\cos c &= 9.53751 \\
\hline
\log\tan m &= 9.35051 \\
m &= 167^\circ 22'. \\
b-m &= -(12^\circ 16' 42'').
\end{aligned}
$$

$$
\begin{aligned}
\log\cos a &= 9.92345 \\
\log\sec m &= 0.01064 \\
\log\cos(b-m) &= 9.98995 \\
\hline
\log\cos c &= 9.92404 \\
c &= 147^\circ\ 5'\ 30''.
\end{aligned}
$$

4. Find the spherical excess of a triangle on the earth's surface (regarded as spherical), if each side of the triangle is equal to 1°.

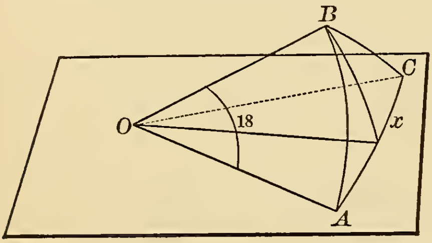
Given a, b, and c each $= 1^\circ$; then

$$
\begin{aligned}
2s &= 3^\circ. & \tfrac{1}{2}s &= 45'. \\
s &= 1^\circ\ 30'. & \tfrac{1}{2}(s-a) &= 15'. \\
s-a &= 30'. & \tfrac{1}{2}(s-b) &= 15'. \\
s-b &= 30'. & \tfrac{1}{2}(s-c) &= 15'. \\
s-c &= 30'.
\end{aligned}
$$

$$
\begin{aligned}
\log\tan\tfrac{1}{2}s &= 8.11696 \\
\log\tan\tfrac{1}{2}(s-a) &= 7.63982 \\
\log\tan\tfrac{1}{2}(s-b) &= 7.63982 \\
\log\tan\tfrac{1}{2}(s-c) &= 7.63982 \\
\hline
\log\tan^2\tfrac{1}{4}E &= 11.03642 \\
\log\tan\tfrac{1}{4}E &= 5.51821. \\
\tfrac{1}{4}E &= 6.814''. \\
E &= 27.25''.
\end{aligned}
$$

EXERCISE XXXII. PAGE 148.

1. Find the dihedral angle made by the lateral faces of a regular ten-sided pyramid; given the angle $A = 18°$, made at the vertex by two adjacent lateral edges.



About O the vertex of the pyramid, describe a sphere. It will intersect the lateral surface, forming a regular spherical decagon, of which the sides $= 18°$, being measured by the plane angles at the centre.

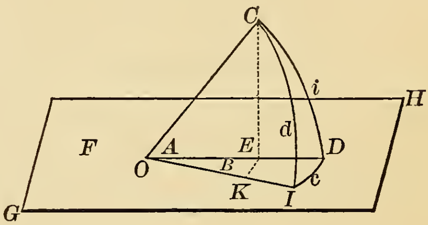
Pass a plane through O and A and C, forming an isosceles spherical triangle ABC.

Then (by Prob. 3, Ex. XXIII.) given side $a = 18°$, $n = 10$, to find angle A of the polygon.

$$\sin \tfrac{1}{2} A = \sec \tfrac{1}{2} A \cos \frac{180°}{n}.$$

$$\begin{aligned} \log \cos 18° &= 9.97821 \\ \text{colog} \cos 9° &= \underline{0.00538} \\ \log \sin \tfrac{1}{2} A &= 9.98359 \\ \tfrac{1}{2} A &= 74° \; 21'. \\ A &= 148° \; 42'. \end{aligned}$$

2. Through the foot of a rod which makes the angle A with a plane, a straight line is drawn in the plane. This line makes the angle B with the projection of the rod upon the plane. What angle does this line make with the rod?



Let CO be a straight line, making the angle A with the plane GH; IF a straight line passing through the foot of CO, making the angle B with the projection DO of CO upon the plane GH.

It is required to find the angle $COI = x$.

With a radius equal to unity, from O as a centre, construct the spherical triangle DCI.

Then
$$\begin{aligned} i &= A, \\ c &= B, \\ d &= COI = x \\ CEK &= \text{rt. angle.} \end{aligned}$$

Since OE is the projection of CO on the plane GH; CE drawn from C to E, is perpendicular to OE.

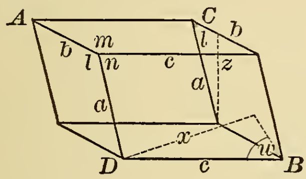
$$\therefore \; CDI = \text{rt. triangle.}$$

By Formula [37],

$$\cos d = \cos i \cos c.$$

$$\therefore \; \cos x = \cos A \cos B.$$

3. Find the volume V of an oblique parallelopipedon; given the three unequal edges a, b, c, and the three angles l, m, n, which the edges make with one another.



Let AB be a parallelopipedon, and l, m, and n, the angles which the unequal edges a, b, and c make.

We are to find V, the volume.

Let $w =$ the inclination of the edge to the plane of a and b.

$V =$ area base $\times$ altitude.

Area base $= bz$, when $z = a \sin l$.

Altitude $= x$, when $x = c \sin w$.

$\therefore V = abc \sin l \sin w$.

Suppose a sphere to be constructed having for its centre the vertex of the trihedral angle whose edges are a, b, and c. The spherical triangle whose vertices are the points where a, b, and c meet the surface has for its sides l, m, n; and $w =$ perpendicular arc from side l to the opposite vertex.

Let L, M, N denote the angles of the triangle.

Then by [38] and [47],

$$\sin w = \sin m \sin N$$
$$= 2 \sin m \sin \tfrac{1}{2} N \cos \tfrac{1}{2} N.$$

Or if $s = \frac{1}{2}(l + m + n)$,

$$\sin w = \frac{2}{\sin l}\sqrt{\sin s \sin(s-l)\sin(s-m)\sin(s-n)}$$

$$\therefore V = 2abc\sqrt{\sin s \sin(s-l)\sin(s-m)\sin(s-n)}$$

4. The continent of Asia has nearly the shape of an equilateral triangle, the vertices being the East Cape, Cape Romania, and the Promontory of Baba. Assuming each side of this triangle to be 4800 geographical miles, and the earth's radius to be 3440 geographical miles, find the area of the triangle:

(i.) regarded as a plane triangle;

(ii.) regarded as a spherical triangle.

Area $= \frac{1}{2}$ (base $\times$ altitude).

Altitude $= \sqrt{4800^2 - 2400^2}$.

$= \sqrt{17280000}$.

$$\begin{aligned} \log \sqrt{17280000} &= 3.61877 \\ \log 2400 &= 3.38021 \\ \log \text{area} &= 6.99898 \end{aligned}$$

Area $= 9976500$.

(ii.) $$F = \frac{E}{180^\circ}\pi R^2.$$

$$a, b, \text{ and } c = \frac{4800^\circ}{60} = 80^\circ.$$

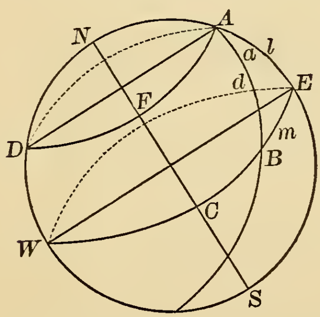
$$2s = 240^\circ.$$
$$\tfrac{1}{2}(s-a) = 20^\circ.$$
$$\tfrac{1}{2}(s-b) = 20^\circ.$$
$$\tfrac{1}{2}(s-c) = 20^\circ.$$

$$\begin{aligned} \log \tan \tfrac{1}{2} s &= 10.23856 \\ \log \tan \tfrac{1}{2}(s-a) &= 9.56107 \\ \log \tan \tfrac{1}{2}(s-b) &= 9.56107 \\ \log \tan \tfrac{1}{2}(s-c) &= 9.56107 \\ \log \tan^2 \tfrac{1}{4} E &= 8.92177 \end{aligned}$$

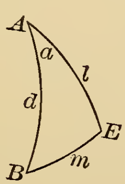
$$\tfrac{1}{4}E = 16^\circ\ 7'\ 8.1''.$$
$$E = 64^\circ\ 28'\ 32.5''.$$
$$= 232112.5''.$$

$$\log E = 5.36570$$
$$\log \frac{\pi}{648000} = 4.68557$$
$$\log R^2 = 7.07312$$
$$\log F = 7.12439$$
$$F = 13316560.$$

5. A ship sails from a harbor in latitude l, and keeps on the arc of a great circle. Her *course* (or angle between the direction in which she sails and the meridian) at starting is a. Find where she will cross the equator, her course at the equator, and the distance she has sailed.



Let $NESW$ be the earth, WCE the equator, N and S the north and south poles. Let A be the point from which the ship starts, AFD the parallel of latitude the ship starts from, and AB the great circle of its course.



Then

$BAE = a$ = course of ship.

$AE = l$ = latitude of its starting-place.

$BE = m$ = place of crossing the equator.

B = course at equator.

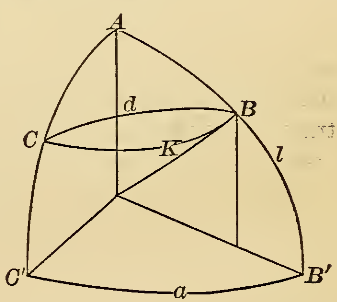
$AB = d$ = distance sailed.

By Napier's rule,

$$\sin l = \tan m \cot a;$$

whence
$$\tan m = \sin l \tan a.$$
$$\cos B = \cos l \sin a.$$
$$\cot d = \cot l \cos a.$$

6. Two places have the same latitude l, and their distance apart, measured on an arc of a great circle, is d. How much greater is the arc of the parallel of latitude between the places than the arc of the great circle? Compute the results for $l = 45°$, $d = 90°$.



In isosceles spherical triangle ABC

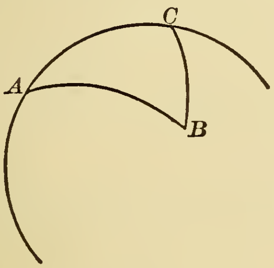
$$\sin \tfrac{1}{2} A = \sin \tfrac{1}{2} d \csc(90° - l)$$
$$= \sin \tfrac{1}{2} d \sec l.$$

Let $l = 45°$, $d = 90°$.

$$\log \sin \tfrac{1}{2} d = 9.84949$$
$$\log \sec l = 0.15051$$
$$\log \sin \tfrac{1}{2} A = 10.00000 - 10$$
$$\tfrac{1}{2} A = 90°.$$
$$A = 180°.$$
$$\text{Arc } a = 180°.$$

$$\text{Arc } K = a \times \cos l$$
$$= \tfrac{1}{2} a\sqrt{2} = 90°\sqrt{2}.$$
$$90°\sqrt{2} - 90° = 90°(\sqrt{2} - 1).$$

7. The shortest distance d between two places and their latitudes l and l' are known. Find the difference between their longitudes.



Let C represent the north pole, A the position of one city, B the position of the other. Then $AB = d$, and if m represent the longitude of one city, m' that of the other, l the latitude of one city, and l' that of the other, angle C will be equal to $(m - m')$, and BC will equal $90° - l$, and AC will be equal to $90° - l'$.

Therefore we have an oblique spherical triangle with three sides given to find the angle C.

Now from Formula [44],

$$\cos c = \cos a \cos b + \sin a \sin b \cos C;$$

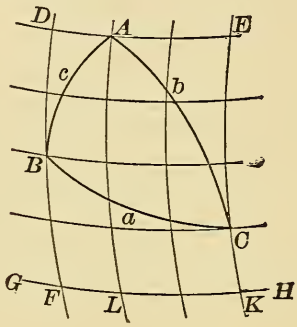
then by substituting,

$$\cos d = \cos(90° - l)\cos(90° - l') + \sin(90° - l) \times \sin(90° - l')\cos(m - m');$$

or $\cos d = \sin l \sin l' + \cos l \cos l' \cos(m - m')$.

$$\therefore \cos(m - m') = (\cos d - \sin l \sin l') \times \sec l \sec l'.$$

8. Given the latitudes and longitudes of three places on the earth's surface, and also the radius of the earth; show how to find the area of the spherical triangle formed by arcs of great circles passing through the places.



Let A, B, and C represent the positions of three places on the earth's surface.

§ 62 shows how to find the distance between two places when the latitudes and difference in longitude are given.

In this case we have the latitudes given, and also the longitudes; so that we can find the difference in longitude.

Let $GH =$ equator;

then, from § 54, in triangle ABC,

$$\tan m = \cot a \cos b,$$

and from § 62,

$$\cos BC = \sin a \sec m \sin(b + m),$$

and the same with the distance between the other places.

Therefore, we have given the three sides of a spherical triangle, to find the area.

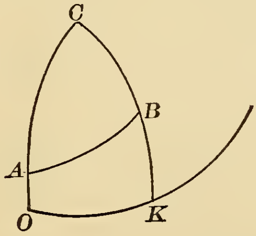
By [51],

$$\tan^2 \tfrac{1}{4} E = \tan \tfrac{1}{2} s \tan \tfrac{1}{2}(s-a)$$
$$\times \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c).$$

Then, since we have the radius of the sphere given $(=R)$ and the spherical excess $= E$, from Formula [50],

$$F = \frac{E}{180^\circ} \pi R^2.$$

9. The distance between Paris and Berlin (that is, the arc of a great circle between these places) is equal to 472 geographical miles. The latitude of Paris is 48° 50′ 13″; that of Berlin, 52° 30′ 16″. When it is noon at Paris what time is it at Berlin?



Let AO represent the latitude of Paris, and BK the latitude of Berlin. Then C represents the difference in longitude.

$$CA = b = 41^\circ\ 9'\ 47''$$
$$CB = a = 37^\circ\ 29'\ 44''$$
$$AB = c = 7^\circ\ 52' \qquad (472 \div 60)$$
$$2s = 86^\circ\ 31'\ 31''$$
$$s = 43^\circ\ 15'\ 45.5''.$$
$$s-a = 5^\circ\ 46'\ 1.5''.$$
$$s-b = 2^\circ\ 5'\ 58.5''.$$
$$s-c = 35^\circ\ 23'\ 45.5''.$$

$$\tan^2 \tfrac{1}{2} C = \csc s$$
$$\sin(s-a) \sin(s-b) \csc(s-c).$$

$$\log \csc s = 0.16409$$
$$\log \sin(s-a) = 9.00210$$
$$\log \sin(s-b) = 8.56391$$
$$\log \csc(s-c) = 0.23716$$

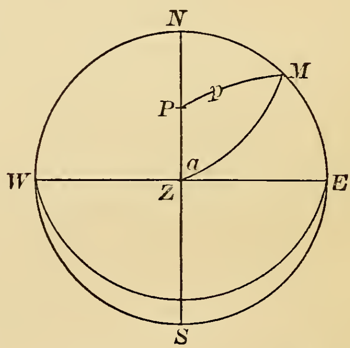
$$\log \tan^2 \tfrac{1}{2} C = 17.96726$$
$$\tan \tfrac{1}{2} C = 8.98363$$
$$\tfrac{1}{2} C = 5^\circ\ 30'\ 2''.$$
$$C = 11^\circ\ 0'\ 4''.$$
$$= 660.$$

Difference of time

$$= \tfrac{1}{15}(660) \text{ minutes}$$
$$= 44 \text{ min.}$$

Time at Berlin, 12 h. 44 min.

10. The altitude of the pole being 45°, I see a star on the horizon and observe its azimuth to be 45°; find its polar distance.



$NP = 45^\circ$. $\qquad PM = p$.

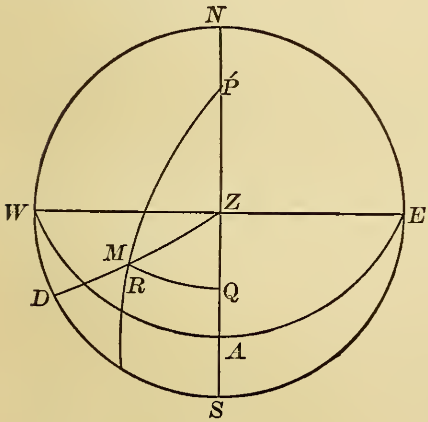
$\therefore PZ = 45^\circ = l$. $\qquad PZM = a$.

We have given two parts of the triangle,

$$a = 45^\circ.$$
$$(90^\circ - l) = 45^\circ.$$
$$\cos p = \cos a \cos l.$$
$$\cos a = \sqrt{\tfrac{1}{2}}.$$
$$\cos l = \sqrt{\tfrac{1}{2}}.$$
$$\therefore \cos p = \tfrac{1}{2}.$$
$$\therefore p = 60^\circ.$$

11. Given the latitude l of the observer, and the declination d of the sun; find the local time (apparent solar time) of sunrise and sunset, and also the azimuth of the sun at these times (refraction being neglected). When and where does the sun rise on the longest day of the year (at which time $d = +23° 27'$) in Boston ($l = 42° 21'$), and what is the length of the day from sunrise to sunset? Also, find when and where the sun rises in Boston on the shortest day of the year (when $d = -23° 27'$), and the length of this day.

To find the hour angle t when the sun is on the horizon.



$$PM = 90° - d.$$
$$ZQ = 90°.$$
$$\therefore PQ = 90° - l + 90°$$
$$= 180° - l.$$

Then in triangle PMQ, by [39],

$$\cos QPM = \tan PQ \cot PM,$$

or $$\cos t = \tan(180° - l)\cot(90° - d),$$
$$\cos t = -\tan l \tan d.$$

Also to find azimuth a.

In triangle PMQ, MQ is measured by angle $QZM = 180° - a$.

Then, by [37],

$$\cos PM = \cos PQ \cos MQ.$$

$$\cos(90° - d) = \cos(180° - l)\cos(180° - a).$$
$$\sin d = -\cos l\,(-\cos a),$$

or $$\cos a = \sin d \sec l.$$

Now $\cos t = -\tan d \tan l.$

Time of sunrise $= 12 - \frac{t}{15}$ o'clk A.M.

Time of sunset $= \frac{t}{15}$ o'clk P.M.

$$\log \tan d = 9.63726$$
$$\log \tan l = 9.95977$$
$$\log \cos t = 9.59703$$
$$t = 66° 42' 26''.$$
$$12 - \frac{t}{15} = 7 \text{ h. } 33 \text{ min. } 10 \text{ sec.}$$
$$\frac{t}{15} = 4 \text{ h. } 26 \text{ min. } 50 \text{ sec.}$$

$\therefore$ shortest day

$= 2 \times 4$ h. 26 min. 50 sec.

$= 8$ h. 53 min. 40 sec.

Again,

$$\cos a = \sin d \sec l.$$
$$\log \sin d = 9.59983$$
$$\log \sec l = 0.13133$$
$$\log \cos a = 9.73116$$
$$a = 57° 25' 15''.$$
$$\therefore a' = 122° 34' 45''.$$

Longest day $= 12$ hrs.

$+ [(7$ h. 33 min. 10 sec.

$- 4$ h. 26 min. 50 sec.$)]$

$= 15$ h. 6 min. 20 sec.

12. When is the solution of the problem in Example 11 impossible, and for what places is the solution impossible?

d has for its maximum value $23° 27'$.

Suppose $l = 66° 33'$.

Then $\tan(180° - l) = -\tan l$

$$= -\cot d.$$

Formula $\cos t = -\tan l \tan d$

becomes $\cos t = -\cot d \tan d$

$$= -1.$$

$$\therefore\ t = 180°;$$

that is, the sun just appears in the south on shortest day.

For places within the Arctic circle, $l > 66° 33'$, and $-\tan l$ numerically greater than $-\cot d$.

Hence $-\tan l \tan d > -1$ (numerically), or

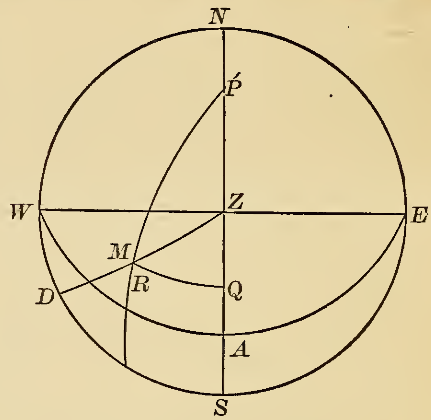
$$\cos t = \pm 1+,$$

which is not possible. Hence the sun may fail to rise during 24 hours.

13. Given the latitude of a place and the sun's declination; find his altitude and azimuth at 6 o'clock A.M. (neglecting refraction). Compute the results for the longest day of the year at Munich ($l = 48° 9'$).

$$PZM = a.$$
$$PZ = 90° - l.$$
$$PM = 90° - d = p.$$
$$ZPM = t.$$
$$ZM = 90° - h.$$
$$l = 48° 9'.$$

Sun's declination on longest day, $23° 27'$.



By Napier's Rules,

$$\sin h = \sin l \sin d.$$

$$\log \sin l = 9.87209$$
$$\log \sin d = 9.59983$$
$$\log \sin h = 9.47192$$

Altitude $= h = 17° 14' 35''$.

By Napier's Rules,

$$\cot a = \cos l \tan d.$$

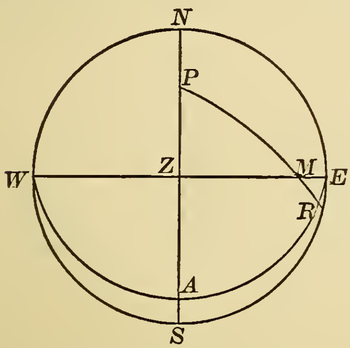
$$\log \cos l = 9.82424$$
$$\log \tan d = 9.63726$$
$$\log \cot a = 9.46150$$

Azimuth $= a = 73° 51' 34''$.

14. How does the altitude of the sun at 6 A.M. *on a given day* change as we go from the equator to the pole? At what time of the year is it a maximum *at a given place?* (Given $\sin h = \sin l \sin d$.)

The farther the place from the equator, the greater the sun's altitude at 6 A.M. in summer. At the equator it is 0°. At the north pole it is equal to the sun's declination. At a given place, the sun's altitude at 6 A.M. is a maximum on the longest day of the year, and then $\sin h = \sin l \sin e$ (where $e = 23° 27'$).

15. Given the latitude of a place north of the equator, and the declination of the sun; find the time of day when the sun bears due east and due west. Compute the results for the longest day at St. Petersburg ($l = 59° 56'$).



Let $NESW$ be the horizon, Z the zenith, NZS the meridian, WZE the prime vertical, WAE the equinoctial, P the elevated pole, M the position of the sun when due east, MR its declination, and ZPM its hour angle.

Then we have the right spherical triangle PZM, with PM and PZ known, to find ZPM.

Given l and d, to find t.

$$PM = 90° - d.$$

$$PZ = 90° - l.$$

From § 48, Case II.,

$$\cos B = \tan a \cot c.$$

Substitute in this equation,

$$\cos t = \tan PZ \cot PM,$$

or $\cos t = \tan (90° - l) \cot (90° - d)$.

$\therefore \cos t = \cot l \tan d.$

And from § 65, the times of bearing due east and west are

$$12 - \frac{t}{15} \text{ A.M. and } \frac{t}{15} \text{ P.M.},$$

respectively.

Since the day given is the longest day of the year, the declination of the sun $= 23° 27'$.

$\therefore$ we have given $d = 23° 27'$ and $l = 59° 56'$, to find t.

Now $\cos t = \cot l \tan d$.

$$\begin{aligned} \log \cot l &= 9.76261 \\ \log \tan d &= \underline{9.63726} \\ \log \cos t &= 9.39987 \\ t &= 75° 27' 24''. \end{aligned}$$

$$\therefore 12 - \frac{t}{15} = 6 \text{ hrs. } 58 \text{ min. A.M.},$$

and $$\frac{t}{15} = 5 \text{ hrs. } 2 \text{ min. P.M.}$$

16. Apply the general result in Example 15 ($\cos t = \cot l \tan d$) to the case when the days and nights are equal in length (that is, when $d = 0°$). Why can the sun in summer never be due east before 6 A.M., or due west after 6 P.M.? How does the time of bearing due east and due west change with the declination of the sun? Apply the general result to the cases where $l < d$ and $l = d$. What does it become at the north pole?

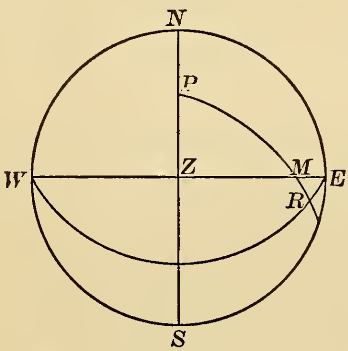
When the days and nights are equal, $d = 0°$, $\cos t = 0°$, and $t = 90°$; that is, the sun is due east at 6 A.M. and due west at 6 P.M. Since l and d must both be less than 90°, $\cos t$ cannot be negative; therefore t cannot be greater than 90°. As d

increases, t decreases; that is, the times in question both approach noon.

If $l < d$, then $\cos t > 1$; therefore this case is impossible. If $l = d$, then $\cos t = 1$, and $t = 0°$; that is, the times both coincide with noon.

The explanation of this result is, that the sun at noon is in the zenith; hence, on the prime vertical, at the pole, $l = 90°$, $\cos t = 0°$, $t = 90°$; therefore the sun in summer always bears due east at 6 A.M. and due west at 6 P.M.

17. Given the sun's declination and his altitude when he bears due east; find the latitude of the observer.



In the figure let Z be the zenith of observer, P the elevated pole, M the position of the sun.

Then $ZM = 90° - h$.

$PM = 90° - d$.

$PZ = 90° - l$.

Since the sun M bears due east, MZP is a right angle.

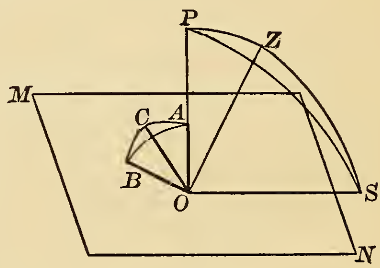
$\therefore$ by Napier's Rules,

$$\cos PM = \cos PZ \cos MZ.$$

$$\therefore \sin d = \sin l \sin h.$$

$$\sin l = \sin d \csc h.$$

18. At a point O in a horizontal plane MN a staff OA is fixed, so that its angle of inclination AOB with the plane is equal to the latitude of the place, 51° 30′ N., and the direction OB is due north. What angle will OB make with the shadow of OA on the plane, at 1 P.M.



Given direction of OB due north, $AOB = 51° 30' = l$, and plane MN horizontal; to find BOC.

Produce OA; it will pass through the pole. The sun being on the equinoctial, $POS = 90°$, and the shadow OC will lie in the plane of this angle. Draw $OZ \perp$ to plane MN; it will lie in the plane of OB and OA.

$SPZ =$ hour angle of sun at 1 P.M.
$= 15°$.

$SPZ = CAB$, being vertical angles.

$\therefore CAB = 15°$.

$ABC = 90°$, since OB is the projection of OA on plane MN.

Arc $AB = 51° 30'$, being the measure of plane angle AOB.

Then in right spherical triangle ABC, by [41],

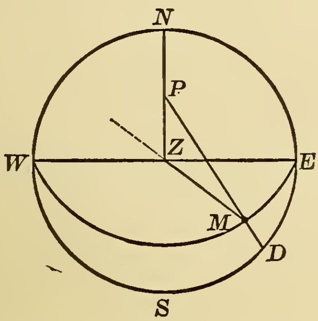
$$\tan BC = \tan BAC \sin AB.$$

$\log \tan 15° \quad = 9.42805$
$\log \sin 51° 30' = 9.89354$
$\log \tan BC \quad = 9.32159$
Arc $BC \quad = 11° 50' 35''$.

Arc BC measures plane angle BOC.

$\therefore\ BOC = 11° 50' 35''$.

19. What is the direction of a wall in latitude 52° 30′ N. which casts no shadow at 6 A.M. on the longest day of the year.



The wall must lie in the line passing through the sun, in order that it may cast no shadow.

In the figure,

$$PZ = 90° - l.$$
$$PM = 90° - d.$$
$$ZPD = 6 \times 15° = 90°.$$

To find $MZP = x$.

By [41],

$$\sin(90° - l) = \tan(90° - d) \cot MZP.$$
$$\cos l = \cot d \cot x.$$

Or, $\cot x = \cos l \tan e.$

$\log \cos l \quad = 9.78445$
$\log \tan e \quad = 9.63726$
$\log \cot x \quad = 9.42171$
$x \quad = 75° 12' 38''$.

20. At a certain place the sun is observed to rise exactly in the north-east point on the longest day of the year; find the latitude of the place.

When the sun rises in the north-east on the longest day of the year, $a = 45°$, $d = 23° 27'$.

To find l. In the formula

$$\cos a = \sin d \sec l.$$
$$\log \sec l = \log \cos a + \log \csc d.$$

$\log \cos 45° \quad = 9.84949$
$\log \csc 23° 27' = 0.40017$
$\log \sec l \quad = 0.24966$
$l = 55° 45' 6''$.

21. Find the latitude of the place at which the sun sets at 10 o'clock on the longest day.

Since the sun sets at 10 o'clock, the hour angle of the sun is equal to $15° \times 10 = 150°$.

The declination of the sun is equal to 23° 27′.

$\therefore$ we have in the triangle two parts given; viz., the angle ZPM, and $MP = 90° - d$.

In the formula,

$$\cot l = \cos t \cot d.$$
$$t = 150°.$$
$$d = 23° 27'.$$

$\log \cos t = 9.93753$
$\log \cot d = 0.36274$
$\log \cot l = 0.30027$
$l = 63° 23' 41''$.

22. To what does the general formula for the hour angle, in § 67, become when (i.) $h = 0°$, (ii.) $l = 0°$ and $d = 0°$, (iii.) l or $d = 90°$.

In the general formula, § 67, let $h=0$. Then

$$\sin\tfrac{1}{2}t = \pm\left[\cos\tfrac{1}{2}(l+p) \times \sin\tfrac{1}{2}(l+p)\sec l\csc p\right]^{\frac{1}{2}}.$$

$$\sin\tfrac{1}{2}t = \pm\sqrt{\frac{1-\cos t}{2}}.$$

$$\cos\tfrac{1}{2}(l+p) = \pm\sqrt{\frac{1+\cos(l+p)}{2}}.$$

$$\sin\tfrac{1}{2}(l+p) = \pm\sqrt{\frac{1-\cos(l+p)}{2}}.$$

$$\sec l\csc p = \frac{1}{\cos l\sin p}.$$

Substitute these values in the first equation,

$$\frac{1-\cos t}{2} = \sqrt{\frac{1+\cos(l+p)}{2}} \times \sqrt{\frac{1-\cos(l+p)}{2}} \times \frac{1}{\cos l\sin p}.$$

$$1-\cos t = \sqrt{1-\cos^2(l+p)} \times \frac{1}{\cos l\sin p}.$$

$$1-\cos t = \sin(l+p) \times \frac{1}{\cos l\sin p}$$

$$= (\sin l\cos p + \cos l\sin p) \times \frac{1}{\cos l\sin p}$$

$$= \tan l\cot p + 1.$$

$$\cos t = -\tan l\cot p.$$

But $\quad p = 90° - d.$

$$\therefore \cot p = \tan d,$$

and $\quad \cos t = -\tan l\tan d.$

(ii.)

$$\sin\tfrac{1}{2}A = \sqrt{\sin(s-b)\sin(s-c)\csc b\csc c}.$$

$$l = 0.$$

$$d = 0.$$

$$s = \tfrac{1}{2}a + \tfrac{1}{2}b + \tfrac{1}{2}c.$$

$$PZ = 90° - l = 90° = c.$$

$$PM = 90° - d = 90° = b.$$

$$ZM = 90° - h = a.$$

$$A = t.$$

$$s - b = \tfrac{1}{2}(90° - h).$$

$$s - c = \tfrac{1}{2}(90° - h).$$

$$\csc b = \csc 90° = 1.$$

$$\csc c = \csc 90° = 1.$$

Substitute

$$\sin\tfrac{1}{2}t = \sqrt{\sin\tfrac{1}{2}(90°-h)\sin\tfrac{1}{2}(90°-h)}$$

$$= \sin\tfrac{1}{2}(90°-h).$$

$$\sin t = \sin(90° - h).$$

$$t = 90° - h$$

$$= z.$$

(iii.)

l or $d = 90°$.

$$PZ = 90° - l = 0° = a.$$

$\therefore$ no triangle will be made.

$\therefore$ answer indeterminate.

$$PM = 90° - d = 0° = b.$$

$\therefore$ no triangle formed.

$\therefore$ result indeterminate.

23. What does the general formula for the azimuth of a celestial body, in § 68, become when $t = 90°$ $= 6$ hours?

When $t = 90°$, $m = 0$, and we have a right spherical triangle with the two legs given to find the angle opposite one of the legs.

Then by substituting the values of the given parts in the formula,

$$\tan B = \tan b \csc a,$$

we have

$$\tan a = \tan(90° - d)\csc(90° - l)$$

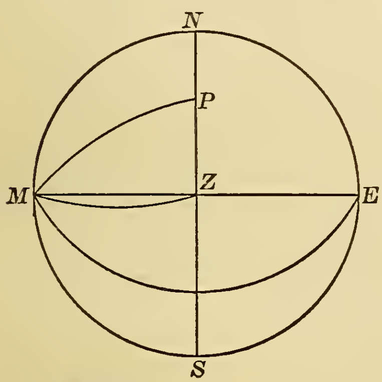
or $\tan a = \cot d \sec l.$

$$\therefore \frac{1}{\tan a} = \frac{1}{\cot d} \times \frac{1}{\sec l},$$

and $\cot a = \tan d \cos l.$

24. Show that the formulas of § 69, if $t = 90°$, lead to the equation $\sin l = \sin h \csc d$; and that if $d = 0°$, they lead to the equation

$$\cos l = \sin h \sec t.$$



I. If $t = 90°$.

From § 69,

$$\sin h = \cos n \cos MQ. \quad (1)$$

$$\sin d = \cos m \cos MQ. \quad (2)$$

Divide (1) by (2),

$$\frac{\sin h}{\sin d} = \frac{\cos n}{\cos m};$$

but now $n = ZP = 90° - l,$

and $m = 0°.$

$$\therefore \frac{\sin h}{\sin d} = \sin l.$$

$$\therefore \sin l = \sin h \csc d.$$

II. If $d = 0°$.

From § 69,

$$\cos l = \cos m \sin h \csc d. \quad (3)$$

$$\tan d = \frac{\cos t}{\tan m}. \quad (4)$$

Multiply (3) by (4),

$$\frac{\cos l \cos t}{\tan m} = \frac{\cos m \sin h}{\cos d}.$$

$$\cos l \cos t = \frac{\sin m \sin h}{\cos d}.$$

But if $d = 0°,$

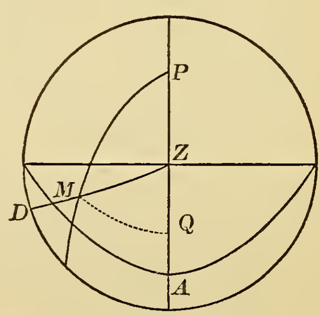
$$PM = 90°,$$

and $m = 0.$

$$\therefore \cos l \cos t = \sin h.$$

$$\therefore \cos l = \sin h \sec t.$$

25. Given latitude of place 52° 30′ 16″, declination of star 38°, its hour angle 28° 17′ 15″; find its altitude.



Given $PZ = 90° - l.$

$PM = 90° - d.$

$ZPM = t;$

required $ZM = 90° - h.$

Let $PQ = m.$

By Napier's Rules,

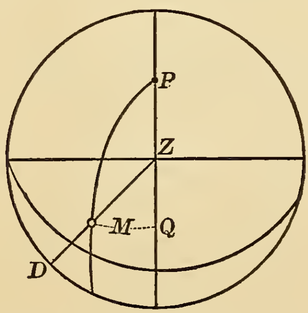
$$\tan m = \cot d \cos d.$$

$$\log\cot d = 0.10719$$
$$\log\cos t = 9.94477$$
$$\log\tan m = 10.05196$$
$$m = 48°\ 25'\ 10''.$$

$$\sin h = \sin(l+m)\sin d \sec m.$$

$$\log\sin(l+m) = 9.99206$$
$$\log\sin d = 9.78934$$
$$\log\sec m = 0.17804$$
$$\log\sin h = 9.95944 - 10$$
$$h = 65°\ 37'\ 20'.$$

26. Given latitude of place 51° 19′ 20″, polar distance of star 67° 59′ 5″, its hour angle 15° 8′ 12″; find its altitude and its azimuth.



Given PM = polar distance of star.

ZPM = hour angle of star.

PZ = co-latitude of observer.

Find PZM = azimuth of star,

and DM its altitude.

Let $d = 90° - PM.$

Let $PQ = m.$

$$\tan m = \cot d \cos t.$$
$$\sin h = \sin(l+m)\sin d \sec m.$$
$$\tan a = \sec(l+m)\tan t \sin m.$$
$$l = 51°\ 19'\ 20''.$$

$$d = 90° - (67°\ 59'\ 5'').$$
$$= 22°\ 0'\ 55''.$$
$$t = 15°\ 8'\ 12''.$$

$$\log\cot d = 10.39326$$
$$\log\cos t = 9.98466$$
$$\log\tan m = 10.37792$$
$$m = 67°\ 16'\ 22''.$$

$$\log\sin(l+m) = 9.94351$$
$$\log\sin d = 9.57387$$
$$\log\sec m = 0.41302$$
$$\log\sin h = 9.93040$$
$$h = 58°\ 25'\ 15''.$$

$$\log\sec(l+m) = 0.32001$$
$$\log\tan t = 9.43218$$
$$\log\sin m = 9.96490$$
$$\log\tan a = 9.71709$$
$$a = 152°\ 28'.$$

27. Given the declination of a star 7° 54′, its altitude 22° 45′ 12″, its azimuth 129° 45′ 37″; find its hour angle and the latitude of the observer.

$$\sin t = \sin a \cos h \sec d.$$

$$\log\sin a = 9.88577$$
$$\log\cos h = 9.96482$$
$$\text{colog}\cos d = 0.00414$$
$$\log\sin t = 9.85473$$
$$t = 45°\ 42'.$$

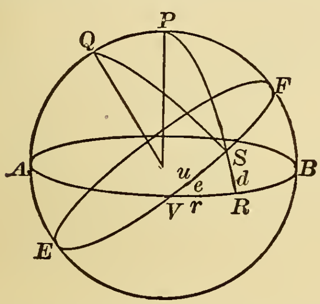
$$\tan m = \cot d \cos t.$$
$$\cos n = \cos m \sin h \csc d.$$
$$l = 90° - (m \pm n).$$

$$\log\cot d = 10.85773$$
$$\log\cos t = 9.84411$$
$$\log\tan m = 10.70184$$
$$m = 78°\ 45'\ 45''.$$

$$\log \cos m = 9.28976$$
$$\log \sin h = 9.58745$$
$$\log \csc d = 0.86187$$
$$\log \cos n = 9.73908$$
$$n = 56° \ 44' \ 39''.$$

$$m - n = 12° \ 1' \ 6''.$$
$$90° - (m - n) = 67° \ 58' \ 54''.$$
$$\therefore \ l = 67° \ 58' \ 54''.$$

28. Given the longitude u of the sun, and the obliquity of the ecliptic $e = 23° \ 27'$; find the declination d, and the right ascension r.



In the figure let P represent the pole of the equinoctial AVB, S the position of the sun, and Q the pole of the ecliptic EVF.

Then $VS = u$.

$$VR = r.$$
$$SR = d.$$
$$RVS = e.$$

Then in the right triangle RVS, by [38],

$$\sin SR = \sin VS \times \sin RVS,$$

or $\sin d = \sin u \sin e$.

Also by [39],

$$\cos RVS = \tan RV \cot VS,$$

or $\cos e = \tan r \cot u$.

$$\tan r = \tan u \cos e.$$

29. Given the obliquity of the ecliptic $e = 23° \ 27'$, the latitude of a star $51°$, its longitude $315°$; find its declination and its right ascension.

In Fig. 47, given

$$VT = 315° \text{ or } -45°,$$
$$TM = 51°,$$
$$RVT = 23° \ 27',$$

to find $VR = r$

and $RM = d$.

In right triangle VTM,

$$\cos VM = \cos VT \cos TM,$$

and $\tan MVT = \tan MT \csc VT$.

$$\log \cos 315° = 9.84949$$
$$\log \cos 51° = 9.79887$$
$$\log \cos VM = 9.64836$$
$$VM = 63° \ 34' \ 36''.$$

$$\log \tan 51° = 10.09163$$
$$\log \csc 315° = 0.15051 \ (n)$$
$$\log \tan MVT = 10.24214 \ (n)$$
$$MVT = -(60° \ 12' \ 14.5'').$$

In right triangle RVM,

$$RVM = RVT + TVM$$
$$= 23° \ 27' - (60° \ 12' \ 14.5'')$$
$$= -(36° \ 45' \ 14.5'').$$

By [38],

$$\sin RM = \sin VM \sin RVM.$$

$$\log \sin VM = 9.95208$$
$$\log \sin RVM = 9.77698$$
$$\log \sin RM = 9.72906$$
$$RM = d = 32° \ 24' \ 12''.$$

Also, by [41],

$$\sin VR = \tan RM \cot RVM.$$

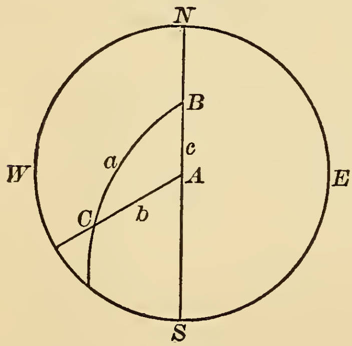
$$\begin{aligned} \log\tan RM &= 9.80257 \\ \log\cot RVM &= 0.12677\,(n) \\ \log\sin VR &= 9.92934\,(n) \end{aligned}$$

$$VR = -(58°\ 11'\ 43'').$$

$$\therefore\ VR = 360° - 58°\ 11'\ 43''$$

$$= 301°\ 48'\ 17''.$$

30. Given the latitude of a place 44° 50′ 14″, the azimuth of a star 138° 58′ 43″, and its hour angle 20°; find its declination.



Given $c = 90° - 44°\ 50'\ 14''$

$$= 45°\ 9'\ 46''.$$

$$A = 138°\ 58'\ 43''.$$

$$B = 20°.$$

$$\tfrac{1}{2}(A - B) = 59°\ 29'\ 22''.$$

$$\tfrac{1}{2}(A + B) = 79°\ 29'\ 22''.$$

$$\tfrac{1}{2}c = 22°\ 34'\ 53''.$$

$$\begin{aligned} \log\cos\tfrac{1}{2}(A-B) &= 9.70560 \\ \text{colog}\cos\tfrac{1}{2}(A+B) &= 0.73893 \\ \log\tan\tfrac{1}{2}c &= 9.61897 \\ \log\tan\tfrac{1}{2}(a+b) &= 0.06350 \\ \tfrac{1}{2}(a+b) &= 49°\ 10'\ 26''. \end{aligned}$$

$$\begin{aligned} \log\sin\tfrac{1}{2}(A-B) &= 9.93528 \\ \text{colog}\sin\tfrac{1}{2}(A+B) &= 0.00735 \\ \log\tan\tfrac{1}{2}c &= 9.61897 \\ \log\tan\tfrac{1}{2}(a-b) &= 9.56160 \\ \tfrac{1}{2}(a-b) &= 20°\ 1'\ 21.5''. \end{aligned}$$

$$\therefore\ a = 69°\ 11'\ 48''.$$

$$90° - 69°\ 11'\ 48'' = 20°\ 48'\ 12''.$$

31. Given latitude of place 51° 31′ 48″, altitude of sun west of the meridian 35° 14′ 27″, its declination +21° 27′; find the local apparent time.

By § 67,

$$PZ = 90° - l,$$

$$PM = 90° - d = p,$$

$$ZM = 90° - h;$$

required $t = ZPM.$

$$p = 68°\ 33'.$$

$$\tfrac{1}{2}(l + h + p) = 77°\ 39'\ 37.5''.$$

$$\tfrac{1}{2}(l - h + p) = 42°\ 25'\ 10.5''.$$

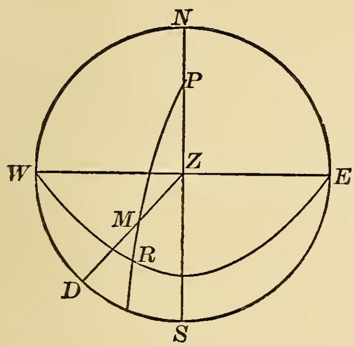
$$\begin{aligned} \log\cos\tfrac{1}{2}(l+p+h) &= 9.32982 \\ \log\sin\tfrac{1}{2}(l+p-h) &= 9.82901 \\ \text{colog}\cos l &= 0.20614 \\ \text{colog}\sin p &= 0.03117 \\ 2)\,&\underline{19.39614} \\ \log\sin\tfrac{1}{2}t &= 9.69807 \end{aligned}$$

$$\tfrac{1}{2}t = 29°\ 55'\ 55.5''.$$

$$t = 59°\ 51'\ 51''.$$

$$\frac{t}{15} = 3\text{ h. }59\text{ min. }27\tfrac{2}{5}\text{ sec. P.M.}$$

32. Given latitude of place l, the polar distance p of a star, and its altitude h; find its azimuth a.



Altitude	$= ZM = 90° - h.$
Co-latitude	$= PZ = 90° - l.$
Polar distance	$= PM$
	$= 90° - d = p.$
Azimuth	$= PZM$ or $a.$

$$\cos \tfrac{1}{2} A = \sqrt{\sin s \sin (s - a) \csc b \csc c}.$$

Let $A = PZM$ or a,

$a = p$,

$b = 90° - h$,

$c = 90° - l$.

Then

$$\sin s = \sin [90° - \tfrac{1}{2}(l + h - p)] = \cos \tfrac{1}{2}(h + l - p).$$

$$\sin (s - a) = \sin [90° - \tfrac{1}{2}(h + l + p)] = \cos \tfrac{1}{2}(h + l + p).$$

$$\csc b = \csc (90° - h) = \sec h.$$

$$\csc c = \csc (90° - l) = \sec l.$$

$$\therefore \cos \tfrac{1}{2} a = \sqrt{\cos \tfrac{1}{2}(p + h + l) \cos \tfrac{1}{2}(h + l - p) \sec l \sec h}$$

SURVEYING.

EXERCISE I. PAGE 143.

1. Required the area of a triangular field whose sides are respectively 13, 14, and 15 chains.

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}.$$

$$s = \tfrac{1}{2}(13 + 14 + 15) = 21, \qquad s - b = 21 - 14 = 7,$$
$$s - a = 21 - 13 = 8, \qquad s - c = 21 - 15 = 6.$$

$$\text{Area} = \sqrt{21 \times 8 \times 7 \times 6} = \sqrt{3^2 \times 7^2 \times 2^4} = 3 \times 7 \times 2^2$$
$$= 84 \text{ sq. ch.} = 8.4 \text{ A.} = 8 \text{ A. } 64 \text{ P.}$$

2. Required the area of a triangular field whose sides are respectively 20, 30, and 40 chains.

$$\text{Area} = \sqrt{45 \times 25 \times 15 \times 5} = \sqrt{3^3 \times 5^5} = 3 \times 5^2\sqrt{3 \times 5}$$
$$= 75\sqrt{15} = 290.4737+.$$

$290.4737 \text{ sq. ch.} = 29.04737 \text{ A.} = 29 \text{ A. } 7.579 \text{ P.} = 29 \text{ A. } 7\tfrac{3}{5} \text{ P.}$, nearly.

3. Required the area of a triangular field whose base is 12.60 chains, and altitude 6.40 chains.

$$\text{Area} = \tfrac{1}{2} \text{ base} \times \text{altitude}.$$
$$\text{Area} = \tfrac{1}{2} \times 12.6 \times 6.4 = 40.32 \text{ sq. ch.} = 4.032 \text{ A.} = 4 \text{ A. } 5\tfrac{3}{25} \text{ P.}$$

4. Required the area of a triangular field which has two sides 4.50 and 3.70 chains, respectively, and the included angle 60°.

$$\text{Area} = \tfrac{1}{2}\, bc \sin A.$$
$$\text{Area} = \tfrac{1}{2} \times 4.5 \times 3.7 \times 0.866 = 7.20945 \text{ sq. ch.} = 0.7209 \text{ A.}$$
$$= 115\tfrac{7}{20} \text{ P., nearly.}$$

5. Required the area of a field in the form of a trapezium, one of whose diagonals is 9 chains, and the two perpendiculars upon this diagonal from the opposite vertices 4.50 and 3.25 chains.

$$\text{Area} = \tfrac{1}{2} \times 9\,(4.5 + 3.25) = 34.875 \text{ sq. ch.} = 3.4875 \text{ A.}$$
$$= 3 \text{ A. } 78 \text{ P.}$$

6. Required the area of the field $ABCDEF$ (Fig. 19), if $AE = 9.25$ chains, $FF' = 6.40$ chains, $BE = 13.75$ chains, $DD' = 7$ chains, $DB = 10$ chains, $CC' = 4$ chains, and $AA' = 4.75$ chains.

$$\begin{aligned}
2 \text{ area } AFE &= 6.4 \times 9.25 &&= 59.2 \\
2 \text{ area } BDEA &= 13.75\,(4.75 + 7) &&= 161.5625 \\
2 \text{ area } BDC &= 10 \times 4 &&= 40 \\
\hline
2 \text{ area } ABCDEF & &&= 260.7625 \\
\text{area } ABCDEF & &&= 130.38125
\end{aligned}$$

130.38125 sq. ch. = 13.038125 A. = 13 A. $6\tfrac{1}{10}$ P.

7. Required the area of the field $ABCDEF$ (Fig. 20), if $AF' = 4$ chains, $FF' = 6$ chains, $EE' = 6.50$ chains, $AE' = 9$ chains, $AD = 14$ chains, $AC' = 10$ chains, $AB' = 6.50$ chains, $BB' = 7$ chains, $CC' = 6.75$ chains.

$$\begin{aligned}
2 \text{ area } AFF' &= 4 \times 6 &&= 24 \\
2 \text{ area } F'E'EF &= 5\,(6 + 6.5) &&= 62.5 \\
2 \text{ area } EE'D &= 6.5 \times 5 &&= 32.5 \\
2 \text{ area } ABB' &= 6.5 \times 7 &&= 45.5 \\
2 \text{ area } BCC'B' &= 3.5\,(7 + 6.75) &&= 48.125 \\
2 \text{ area } CDC' &= 6.75 \times 4 &&= 27 \\
\hline
2 \text{ area } ABCDEF & &&= 239.625 \\
\text{area } ABCDEF & &&= 119.8125
\end{aligned}$$

119.8125 sq. ch. = 11.98125 A. = 11 A. 157 P.

8. Required the area of the field $AGBCD$ (Fig. 15), if the diagonal $AC = 5$, BB' (the perpendicular from B to AC) = 1, DD' (the perpendicular from D to AC) = 1.60, $EE' = 0.25$, $FF' = 0.25$, $GG' = 0.60$, $HH' = 0.52$, $KK' = 0.54$, $AE' = 0.2$, $E'F' = 0.50$, $F'G' = 0.45$, $G'H' = 0.45$, $H'K' = 0.60$, and $K'B = 0.40$.

$$\begin{aligned}
2 \text{ area } ADCB &= 5\,(1 + 1.6) &&= 13. \\
2 \text{ area } AEE' &= 0.25 \times 0.2 &&= 0.05 \\
2 \text{ area } EE'F'F &= 0.5\,(0.25 + 0.25) &&= 0.25 \\
2 \text{ area } FF'G'G &= 0.45\,(0.25 + 0.6) &&= 0.3825 \\
2 \text{ area } GG'H'H &= 0.45\,(0.6 + 0.52) &&= 0.504 \\
2 \text{ area } HH'K'K &= 0.6\,(0.52 + 0.54) &&= 0.636 \\
2 \text{ area } KK'B &= 0.4 \times 0.54 &&= 0.216 \\
\hline
2 \text{ area } ADCBKHGFE & &&= 15.0385 \\
\text{area } ADCBKHGFE & &&= 7.51925.
\end{aligned}$$

9. Required the area of the field $AGBCD$ (Fig. 16), if $AD = 3$, $AC = 5$, $AB = 6$, angle $DAC = 45°$, angle $BAC = 30°$, $AE' = 0.75$, $AF' = 2.25$, $AH = 2.53$, $AG' = 3.15$, $EE' = 0.60$, $FF' = 0.40$, and $GG' = 0.75$.

$$\begin{array}{ll} 2 \text{ area } ADCB = 3 \times 5 \times 0.7071 + 5 \times 6 \times 0.5 & = 25.6065 \\ 2 \text{ area } HGB = 0.75 \times 3.47 & = 2.6025 \\ \hline 2 \text{ area } ADCBGH & = 28.2090 \\ 2 \text{ area } AEFH = 0.75 \times 0.6 + 1.5(0.6 + 0.4) + 0.4 \times 0.28 & = 2.062 \\ \hline 2 \text{ area } ADCBGHFE & = 26.147 \\ \text{area } ADCBGHFE & = 13.0735. \end{array}$$

10. Determine the area of the field $ABCD$ from two interior stations P and P', if $PP' = 1.50$ chains,

$$\begin{array}{llll} \text{angle } PP'C = & 89° \, 35', & \text{angle } P'PB = & 3° \, 35', \\ PP'B = & 185° \, 30', & P'PA = & 113° \, 45', \\ PP'A = & 309° \, 15', & P'PD = & 165° \, 40', \\ PP'D = & 349° \, 45', & P'PC = & 303° \, 15'. \end{array}$$

Area $= \triangle PAD + \triangle PCD + \triangle PBC + \triangle PAB$.

$$\begin{array}{lll} \angle PP'D = 10° \, 15', & \angle PP'A = 50° \, 45', & \angle PP'C = 89° \, 35', \\ \angle PDP' = 4° \, 5', & \angle PAP' = 15° \, 30', & \angle PCP' = 33° \, 40'. \\ \angle PP'B = 174° \, 30', & \angle PBP' = 1° \, 55', & \end{array}$$

$$PD = \frac{PP' \sin PP'D}{\sin PDP'}.$$

$$\begin{array}{ll} \log PP' & = 0.17609 \\ \log \sin PP'D & = 9.25028 \\ \text{colog} \sin PDP' & = 1.14748 \\ \hline \log PD & = 0.57385 \end{array}$$

$$PA = \frac{PP' \sin PP'A}{\sin PAP'}.$$

$$\begin{array}{ll} \log PP' & = 0.17609 \\ \log \sin PP'A & = 9.88896 \\ \text{colog} \sin PAP' & = 0.57310 \\ \hline \log PA & = 0.63815 \end{array}$$

$$PC = \frac{PP' \sin PP'C}{\sin PCP'}.$$

$$\begin{array}{ll} \log PP' & = 0.17609 \\ \log \sin PP'C & = 9.99999 \\ \text{colog} \sin PCP' & = 0.25621 \\ \hline \log PC & = 0.43229 \end{array}$$

$$PB = \frac{PP' \sin PP'B}{\sin PBP'}.$$

$$\begin{array}{ll} \log PP' & = 0.17609 \\ \log \sin PP'B & = 8.98157 \\ \text{colog} \sin PBP' & = 1.47566 \\ \hline \log PB & = 0.63332 \end{array}$$

$\angle APD = 51° \, 55'$, $\angle DPC = 137° \, 35'$, $\angle BPC = 60° \, 20'$, $\angle APB = 110° \, 10'$.

2 area $PAD = PD \times PA \sin APD$.

$$\begin{aligned} \log PD &= 0.57385 \\ \log PA &= 0.63815 \\ \log \sin APD &= 9.89604 \\ \hline \log 2\text{ area} &= 1.10804 \\ 2\text{ area } PAD &= 12.825. \end{aligned}$$

2 area $PCD = PD \times PC \sin DPC$.

$$\begin{aligned} \log PD &= 0.57385 \\ \log PC &= 0.43229 \\ \log \sin DPC &= 9.82899 \\ \hline \log 2\text{ area} &= 0.83513 \\ 2\text{ area } PCD &= 6.8412. \end{aligned}$$

2 area $PAB = PA \times PB \sin APB$.

$$\begin{aligned} \log PA &= 0.63815 \\ \log PB &= 0.63332 \\ \log \sin APB &= 9.97252 \\ \hline \log 2\text{ area} &= 1.24399 \\ 2\text{ area } PAB &= 17.538. \end{aligned}$$

2 area $PBC = PC \times PB \sin BPC$.

$$\begin{aligned} \log PC &= 0.43229 \\ \log PB &= 0.63332 \\ \log \sin PBC &= 9.93898 \\ \hline \log 2\text{ area} &= 1.00459 \\ 2\text{ area } PBC &= 10.106. \end{aligned}$$

$$\begin{aligned} 2\,\triangle\, PAD &= 12.825 \\ 2\,\triangle\, PCD &= 6.841 \\ 2\,\triangle\, PBC &= 10.106 \\ 2\,\triangle\, PAB &= 17.538 \\ \hline 2\, ABCD &= 47.310 \\ ABCD &= 23.655\text{ sq. ch.} \end{aligned}$$

23.655 sq. ch. = 2.3655 A. = 2 A. $58\frac{1}{2}$ P., nearly.

11. Determine the area of the field $ABCD$ from two exterior stations P and P', if $PP' = 1.50$ chains.

$$\begin{aligned} \text{angle } P'PB &= 41° 10', & \text{angle } PP'D &= 66° 45', \\ P'PA &= 55° 45', & PP'C &= 95° 40', \\ P'PC &= 77° 20', & PP'B &= 132° 15', \\ P'PD &= 104° 45', & PP'A &= 103° 0'. \end{aligned}$$

$$\text{Area} = (\triangle P'CB + \triangle P'CD) - (\triangle P'AB + \triangle P'AD).$$

$$\begin{aligned} \angle P'PB &= 41° 10', & \angle P'PD &= 104° 45', & \angle P'PC &= 77° 20', \\ \angle PBP' &= 6° 35', & \angle PDP' &= 8° 30', & \angle PCP' &= 7° 0'. \\ \angle P'PA &= 55° 45', & \angle PAP' &= 21° 15'. \end{aligned}$$

$$P'B = \frac{PP' \sin P'PB}{\sin PBP'}.$$

$$\begin{aligned} \log PP' &= 0.17609 \\ \log \sin P'PB &= 9.81839 \\ \text{colog} \sin PBP' &= 0.94063 \\ \hline \log P'B &= 0.93511 \end{aligned}$$

$$P'D = \frac{PP' \sin P'PD}{\sin PDP'}.$$

$$\begin{aligned} \log PP' &= 0.17609 \\ \log \sin P'PD &= 9.98545 \\ \text{colog} \sin PDP' &= 0.83030 \\ \hline \log P'D &= 0.99184 \end{aligned}$$

$$P'C = \frac{PP' \sin P'PC}{\sin PCP'}.$$

$\log PP'$	$= 0.17609$
$\log \sin P'PC$	$= 9.98930$
$\operatorname{colog} \sin PCP'$	$= 0.91411$
$\log P'C$	$= 1.07950$

$$P'A = \frac{PP' \sin P'PA}{\sin PAP'}.$$

$\log PP'$	$= 0.17609$
$\log P'PA$	$= 9.91729$
$\operatorname{colog} PAP'$	$= 0.44077$
$\log P'A$	$= 0.53415$

$\angle BP'C = 36° 35'$,
$\angle CP'D = 28° 55'$,

$\angle AP'B = 29° 15'$,
$\angle AP'D = 36° 15'$.

2 area $P'CB = P'C \times P'B \sin BP'C$.

$\log P'C$	$= 1.07950$
$\log P'B$	$= 0.93511$
$\log \sin BP'C$	$= 9.77524$
$\log 2$ area	$= 1.78985$
2 area $P'CB$	$= 61.639.$

2 area $P'CD = P'C \times P'D \sin CP'D$.

$\log P'C$	$= 1.07950$
$\log P'D$	$= 0.99184$
$\log \sin CP'D$	$= 9.68443$
$\log 2$ area	$= 1.75577$
2 area $P'CD$	$= 56.986.$

2 area $P'AB = P'B \times P'A \sin AP'B$.

$\log P'B$	$= 0.93511$
$\log P'A$	$= 0.53415$
$\log \sin AP'B$	$= 9.68897$
$\log 2$ area	$= 1.15823$
2 area $P'AB$	$= 14.396.$

2 area $P'AD = P'A \times P'D \sin AP'D$.

$\log P'A$	$= 0.53415$
$\log P'D$	$= 0.99184$
$\log \sin AP'D$	$= 9.77181$
$\log 2$ area	$= 1.29780$
2 area $P'AD$	$= 19.852.$

$2 \triangle P'CB$	$= 61.639$
$2 \triangle P'CD$	$= 56.986$
	118.625
	34.248
$2\,ABCD$	$= 84.377$
$ABCD$	$= 42.1885$

$2 \triangle P'AB$	$= 14.396$
$2 \triangle P'AD$	$= 19.852$
	34.248

42.1885 sq. ch. = 4.21885 A.
= 4 A. 35 P., nearly.

Exercise II. Page 152.

1.

			N.	*S.*	*E.*	*W.*	*M. D.*	*D.M.D.*	*N. A.*	*S. A.*
1	S. 75° E.	6.00	. . .	1.55	5.79 ~~5.80~~	. . .	5.79	5.79	. . . . .	8.9745
2	S. 15° E.	4.00	. . .	3.86	1.04	. . .	6.83	12.62	. . . . .	48.7132
3	S. 75° W.	6.93	. . .	1.80 ~~1.79~~	. . .	6.70 ~~6.69~~	0.13	6.96	. . . . .	12.5280
4	N. 45° E.	5.00	3.54	. . .	3.54	. . .	3.67	3.80	13.4520	. . . . .
5	N. 45° W.	5.19½	3.67	. . .	. . .	3.67	0	3.67	13.4689	. . . . .
									26.9209	70.2157
										26.9209
										43.2948
										21.6474

21.647 sq. ch. = 2.1647 A. = 2 A. 26 P., nearly.

2.

			N.	*S.*	*E.*	*W.*	*M. D.*	*D.M.D.*	*N. A.*	*S. A.*
1	N. 45° E.	10.00	7.07	. . .	7.07	. . .	7.07	7.07	49.9849	. . . . .
2	S. 75° E.	11.55	. . .	2.99	11.16	. . .	18.23	25.30	. . . . .	75.6470
3	S. 15° W.	18.21	. . .	17.59	. . .	4.71	13.52	31.75	. . . . .	558.4825
4	N. 45° W.	19.11	13.51	. . .	. . .	13.52 ~~13.51~~	0	13.52	182.6552	. . . . .
									232.6401	634.1295
										232.6401
										401.4894
										200.7447

200.74 sq. ch. = 20.074 A. = 20 A. 12 P., nearly

3.

			N.	*S.*	*E.*	*W.*	*M. D.*	*D.M.D.*	*N. A.*	*S. A.*
1	N. 15° E.	3.00	2.90	. . .	0.78	. . .	0.78	0.78	2.2620	. . . . .
2	N. 75° E.	6.00	1.55	. . .	5.79 ~~5.80~~	. . .	6.57	7.35	11.3925	. . . . .
3	S. 15° W.	6.00	. . .	5.80	. . .	1.55	5.02	11.59	. . . . .	67.2220
4	N. 75° W.	5.20	1.35	. . .	. . .	5.02	0	5.02	6.7770	. . . . .
									20.4315	67.2220
										20.4315
	23.395 sq. ch. = 2.3395 A. = 2 A. 54 P., nearly.									
										46.7905
										23.3953

4.

			N.	*S.*	*E.*	*W.*	*M. D.*	*D.M.D.*	*N. A.*	*S. A.*
1	N. 89° 45′ E.	4.94	0.00 ~~0.02~~	. . .	4.93 ~~4.94~~	. . .	4.93	4.93	. . . . .	. . . . .
2	S. 7° 00′ W.	2.30	. . .	2.29 ~~2.28~~	. . .	0.29 ~~0.28~~	4.64	9.57	. . . . .	21.9153
3	S. 28° 00′ E.	1.52	. . .	1.34	0.71	. . .	5.35	9.99	. . . . .	13.3866
4	S. 0° 45′ E.	2.57	. . .	2.58 ~~2.57~~	0.02 ~~0.03~~	. . .	5.37	10.72	. . . . .	27.6576
5	N. 84° 45′ W.	5.11	0.45 ~~0.47~~	. . .	. . .	5.10 ~~5.09~~	0.27	5.64	2.5380	. . . . .
6	N. 2° 30′ W.	5.79	5.76 ~~5.78~~	. . .	. . .	0.27 ~~0.25~~	0	0.27	1.5552	. . . . .
									4.0932	62.9595
										4.0932
	2.943 A. = 2 A. 151 P., nearly.									
										58.8663
										29.4332

5.

		N.	S.	E.	W.
N. 51° 45′ W.	2.39	1.48	. . .	. . .	1.88
S. 85° W.	6.47	. . .	0.56	. . .	6.45
S. 55° 10′ W.	1.62	. . .	0.93	. . .	1.33
			1.49	. . .	9.66
			1.48	. . .	. . .
			0.01		

			N.	S.	E.	W.	M. D.	D.M.D.	N. A.	S. A.
2	S. W.	. . .	. . .	0.03 ~~0.01~~	. . .	9.65 ~~9.66~~	9.65	9.65	. . . . .	0.2895
3	N. 3° 45′ E.	6.39	6.36 ~~6.38~~	. . .	0.43 ~~0.42~~	. . .	9.22	18.87	120.0132	. . . . .
4	S. 66° 45′ E.	1.70	. . .	0.67	1.56	. . .	7.66	16.88	. . . . .	11.3096
5	N. 15° E.	4.98	4.80 ~~4.81~~	. . .	1.29	. . .	6.37	14.03	67.3440	. . . . .
6	S. 82° 45′ E.	6.03	. . .	0.77 ~~0.76~~	5.98	. . .	0.39	6.76	. . . . .	5.2052
1	S. 2° 15′ E.	9.68	. . .	9.69 ~~9.67~~	0.39 ~~0.38~~	. . .	0	0.39	. . . . .	3.7791
									187.3572	20.5834
									20.5834	
	8.339 A. = 8 A. 54 P., nearly.								166.7738	
									83.3869	

6.

		N.	*S.*	*E.*	*W.*
S. 81° 20′ W.	4.28	. . .	0.65	. . .	4.23
N. 76° 30′ W.	2.67	0.62	. . .	. . .	2.60
			0.65	. . .	6.83
			0.62	. . .	. . .
			0.03		

		N.	*S.*	*E.*	*W.*
S. 7° E.	1.79	. . .	1.78	0.22	. . .
S. 27° E.	1.94	. . .	1.73	0.88	. . .
S. 10° 30′ E.	5.35	. . .	5.26	0.98	. . .
N. 76° 45′ W.	1.70	0.39	. . .	. . .	1.65
			8.77	2.08	. . .
			0.39	1.65	. . .
			8.38	0.43	

			N.	*S.*	*E.*	*W.*	*M. D.*	*D.M.D.*	*N. A.*	*S. A.*
1	S. W.	. . .	. . .	0.03	. . .	6.80 ~~6.83~~	6.80	6.80	. . . . .	0.2040
2	N. 5° E.	8.68	8.65	. . .	0.79 ~~0.76~~	. . .	6.01	12.81	110.8065	. . . . .
3	S. 87° 30′ E.	5.54	. . .	0.24	5.55 ~~5.53~~	. . .	0.46	6.47	. . . . .	1.5528
4	S. E.	. . .	. . .	8.38	0.46 ~~0.43~~	. . .	0	0.46	. . . . .	3.8548
									110.8065	5.6116
									5.6116	
									105.1949	
									52.597	

5.2597 A. = 5 A. 42 P., nearly.

7.

			N.	*S.*	*E.*	*W.*	*M. D.*	*D.M.D.*	*N. A.*	*S. A.*
3	S. 5° 00′ E.	5.86	. . .	5.83 ~~5.84~~	0.53 ~~0.51~~	. . .	0.53	0.53	. . . . .	3.0899
4	N. 88° 30′ E.	4.12	0.12 ~~0.11~~	. . .	4.14 ~~4.12~~	. . .	4.67	5.20	0.6240	. . . . .
1	N. 6° 15′ W.	6.31	6.28 ~~6.27~~	. . .	. . .	0.67 ~~0.69~~	4.00	8.67	54.4476	. . . . .
2	S. 81° 50′ W.	4.06	. . .	0.57 ~~0.58~~	. . .	4.00 ~~4.02~~	0	4.00	. . . . .	2.2800
									55.0716	5.3699
									5.3699	
	2.485 A. = 2 A. 78 P., nearly.								49.7017	
									24.8508	

8.

			N.	*S.*	*E.*	*W.*	*M. D.*	*D.M.D.*	*N. A.*	*S. A.*
3	S. 3° 00′ E.	5.33	. . .	5.29 ~~5.32~~	0.28	. . .	0.28	0.28	. . . . .	1.4812
4	E.	6.72	0.03 ~~0.00~~	. . .	6.73 ~~6.72~~	. . .	7.01	7.29	0.2187	. . . . .
1	N. 5° 30′ W.	6.08	6.08 ~~6.05~~	. . .	. . .	0.57 ~~0.58~~	6.44	13.45	81.7760	. . . . .
2	S. 82° 30′ W.	6.51	. . .	0.82 ~~0.85~~	. . .	6.44 ~~6.45~~	0	6.44	. . . . .	5.2808
									81.9947	6.7620
									6.7620	
	3.761 A. = 3 A. 122 P., nearly.								75.2327	
									37.6163	

9.

			N.	*S.*	*E.*	*W.*	*M. D.*	*D.M.D.*	*N. A.*	*S. A.*
1	N. 20° 00′ E.	4.62½	4.35	. . .	1.58	. . .	1.58	1.58	6.8730	. . . . .
2	N. 73° 00′ E.	4.16½	1.22	. . .	3.98	. . .	5.56	7.14	8.7108	. . . . .
3	S. 45° 15′ E.	6.18½	. . .	4.35	4.39	. . .	9.95	15.51	. . . . .	67.4685
4	S. 38° 30′ W.	8.00	. . .	6.26	. . .	4.98	4.97	14.92	. . . . .	93.3992
5	Wanting.	. . .	5.04	. . .	. . .	4.97	0	4.97	25.0488	. . . . .
									40.6326	160.8677
										40.6326
	6.012 A. = 6 A. 2 P., nearly.									120.2351
										60.1175

10.

			N.	*S.*	*E.*	*W.*	*M. D.*	*D.M.D.*	*N. A.*	*S. A.*
6	N. 32° 00′ E.	8.68	7.33 ~~7.36~~	. . .	4.61 ~~4.60~~	. . .	4.61	4.61	33.7913	. . . . .
7	S. 75° 50′ E.	6.38	. . .	1.58 ~~1.56~~	6.20 ~~6.19~~	. . .	10.81	15.42	. . . . .	24.3636
8	S. 14° 45′ W.	0.98	. . .	0.95	. . .	0.25	10.56	21.37	. . . . .	20.3015
9	S. 79° 15′ E.	4.52	. . .	0.86 ~~0.84~~	4.44	. . .	15.00	25.56	. . . . .	21.9816
1	S. 3° 00′ E.	4.23	. . .	4.23 ~~4.22~~	0.22	. . .	15.22	30.22	. . . . .	127.8306
2	S. 86° 45′ W.	4.78	. . .	0.29 ~~0.27~~	. . .	4.77	10.45	25.67	. . . . .	7.4443
3	S. 37° 00′ W.	2.00	. . .	1.60	. . .	1.20	9.25	19.70	. . . . .	31.5200
4	N. 81° 00′ W.	7.45	1.14 ~~1.17~~	. . .	. . .	7.35 ~~7.36~~	1.90	11.15	12.7110	. . . . .
5	N. 61° 00′ W.	2.17	1.04 ~~1.05~~	. . .	. . .	1.90	0	1.90	1.9760	. . . . .
									48.4783	233.4416
										48.4783
	9.248 A. = 9 A. 40 P., nearly.									184.9633
										92.48

EXERCISE III. PAGE 163.

1.

			N.	*S.*	*E.*	*W.*	*M. D.*	*D.M.D.*	*N. A.*	*S. A.*
AB	N.	4.000	4.000	. . .	. . . .	. . .	0	0	. . . .	. . . .
BC	S. 60° E.	4.000	. . .	2.000	3.464	. . .	3.464	3.464	. . . .	6.928
CD	S. 30° E.	6.928	. . .	6.000	3.464	. . .	6.928	10.392	. . . .	62.352
DA	N. 60° W.	8.000	4.000	. . .	. . . .	6.928	0	6.928	27.712	. . . .
									27.712	69.280
										27.712
20.784 sq. ch. = 2.0784 A. = 2 A. $12\frac{1}{2}$ P., nearly.										41.568
										20.784

2.

			N.	*S.*	*E.*	*W.*	*M. D.*	*D.M.D.*	*N. A.*	*S. A.*
AB	N.	79.86	79.860	. . .	. . . .	. . .	0	0	. . . .	. . . .
BC	N. 80° 20′ E.	121.13	20.338	. . .	119.410	. . .	119.410	119.410	2428.56	. . . .
CD	S. 40° 00′ E.	90.00	. . .	68.943	57.851	. . .	177.261	296.671	. . . .	20453.39
DE	S. 55° 52′ W.	100.65	. . .	59.350	. . . .	81.289	95.972	273.233	2696.33	16216.38
EA	N. 73° 41′ W.	100.00	28.095	. . .	. . . .	95.972	0	95.972	. . . .	. . . .
									5124.89	36669.77
										5124.89
15772.44 P. = 98 A. 92 P., nearly.										31544.88
										15772.44

Exercise IV. Page 161.

1. From the square $ABCD$, containing 6 A. 1 R. 24 P., part off 3 A. by a line EF parallel to AB.

$$6 \text{ A. } 1 \text{ R. } 24 \text{ P.} = 64 \text{ sq. ch.}; \quad \sqrt{64} = 8 \text{ ch.} = AB.$$
$$3 \text{ A.} = 30 \text{ sq. ch.}$$
$$AE = \frac{ABFE}{AB} = \frac{30}{8} = 3.75 \text{ ch.}$$

2. From the rectangle $ABCD$, containing 8 A. 1 R. 24 P., part off 2 A. 1 R. 32 P. by a line EF parallel to $AD = 7$ ch. Then, from the remainder of the rectangle part off 2 A. 3 R. 25 P. by a line GH parallel to EB.

$$8 \text{ A. } 1 \text{ R. } 24 \text{ P.} = 84 \text{ sq. ch.} = ABCD.$$
$$2 \text{ A. } 1 \text{ R. } 32 \text{ P.} = 24.5 \text{ sq. ch.} = AEFD.$$
$$2 \text{ A. } 3 \text{ R. } 25 \text{ P.} = 29.0625 \text{ sq. ch.} = EBHG.$$
$$AE = \frac{AEFD}{AD} = \frac{24.5}{7} = 3.5 \text{ ch.}$$
$$AB = \frac{ABCD}{AD} = \frac{84}{7} = 12 \text{ ch.}$$
$$EB = AB - AE = 12 - 3.5 = 8.5 \text{ ch.}$$
$$EG = \frac{EBHG}{EB} = \frac{29.0625}{8.5} = 3.42 \text{ ch., nearly.}$$

3. Part off 6 A. 3 R. 12 P. from a rectangle $ABCD$, containing 15 A. by a line EF parallel to AB; AD being 10 ch.

$$6 \text{ A. } 3 \text{ R. } 12 \text{ P.} = 68.25 \text{ sq. ch.} = ABFE.$$
$$15 \text{ A.} = 150 \text{ sq. ch.} = ABCD.$$
$$AB = \frac{ABCD}{AD} = \frac{150}{10} = 15 \text{ ch.}$$
$$AE = \frac{ABFE}{AB} = \frac{68.25}{15} = 4.55 \text{ ch.}$$

4. From a square $ABCD$, whose side is 9 ch., part off a triangle which shall contain 2 A. 1 R. 36 P., by a line BE drawn from B to the side AD.

$$2 \text{ A. } 1 \text{ R. } 36 \text{ P.} = 24.75 \text{ sq. ch.}$$
$$AE = \frac{2\,ABE}{AB} = \frac{2 \times 24.75}{9} = 5.50 \text{ ch.}$$

5. From $ABCD$, representing a rectangle, whose length is 12.65 ch., and breadth 7.58 ch., part off a trapezoid which shall contain 7 A. 3 R. 24 P., by a line BE drawn from B to the side DC.

$$7 \text{ A. } 3 \text{ R. } 24 \text{ P.} = 79 \text{ sq. ch.}$$

$$ABCD = 12.65 \times 7.58 = 95.887 \text{ sq. ch.}$$

$$\triangle BCE = 95.887 - 79 = 16.887 \text{ sq. ch.}$$

$$CE = \frac{2BCE}{BC} = \frac{2 \times 16.887}{7.58} = 4.456 \text{ ch., nearly.}$$

6. In the triangle ABC, $AB = 12$ ch., $AC = 10$ ch., and $BC = 8$ ch.; part off 1 A. 2 R. 16 P., by the line DE parallel to AB.

$$1 \text{ A. } 2 \text{ R. } 16 \text{ P.} = 16 \text{ sq. ch.}$$

$$CAB = \sqrt{15 \times 3 \times 5 \times 7} = 39.6863 \text{ sq. ch.}$$

$$CDE = CAB - ABED = 39.6863 - 16 = 23.6863 \text{ sq. ch.}$$

$$CAB : CDE :: \overline{CA}^2 : \overline{CD}^2$$
$$:: \overline{CB}^2 : \overline{CE}^2.$$

$$39.6863 : 23.6863 :: 10^2 : \overline{CD}^2. \quad \therefore CD = 7.725 \text{ ch.}$$
$$:: 8^2 : \overline{CE}^2. \quad \therefore CE = 6.18 \text{ ch.}$$

$$AD = CA - CD = 10 - 7.725 = 2.275 \text{ ch.}$$

$$BE = CB - CE = 8 - 6.18 = 1.82 \text{ ch.}$$

7. In the triangle ABC, $AB = 26$ ch., $AC = 20$ ch., and $BC = 16$ ch.; part off 6 A. 1 R. 24 P., by the line DE parallel to AB.

$$6 \text{ A. } 1 \text{ R. } 24 \text{ P.} = 64 \text{ sq. ch.}$$

$$CAB = \sqrt{31 \times 5 \times 11 \times 15} = 159.9218 \text{ sq. ch.}$$

$$CDE = CAB - ABED = 159.9218 - 64 = 95.9218 \text{ sq. ch.}$$

$$CAB : CDE :: \overline{CA}^2 : \overline{CD}^2$$
$$:: \overline{CB}^2 : \overline{CE}^2.$$

$$159.9218 : 95.9218 :: 20^2 : \overline{CD}^2. \quad \therefore CD = 15.49 \text{ ch.}$$
$$:: 16^2 : \overline{CE}^2. \quad \therefore CE = 12.39 \text{ ch.}$$

$$AD = CA - CD = 20 - 15.49 = 4.51 \text{ ch., nearly.}$$

$$BE = CB - CE = 16 - 12.39 = 3.61 \text{ ch., nearly.}$$

8. It is required to divide the triangular field ABC among three persons whose claims are as the numbers 2, 3, and 5, so that they may all have the use of a watering-place at C; $AB = 10$ ch., $AC = 6.85$ ch., and $CB = 6.10$ ch.

Since the triangles have the same altitude, they are to each other as their bases. Hence it is only necessary to divide the base 10 into the three parts, 2 ch., 3 ch., 5 ch.

9. Divide the five-sided field $ABCHE$ among three persons, X, Y, and Z, in proportion to their claims, X paying \$500, Y paying \$750, and Z paying \$1000, so that each may have the use of an interior pond, at P, the quality of the land being equal throughout. Given $AB = 8.64$ ch., $BC = 8.27$ ch., $CH = 8.06$ ch., $HE = 6.82$ ch., and $EA = 9.90$ ch. The perpendicular PD upon $AB = 5.60$ ch., PD' upon $BC = 6.08$ ch., PD'' upon $CH = 4.80$ ch., PD''' upon $HE = 5.44$ ch., and PD'''' upon $EA = 5.40$ ch. Assume PH as the divisional fence between X's and Z's shares; it is required to determine the position of the fences PM and PN between X's and Y's shares and Y's and Z's shares, respectively.

If P be joined to the vertices, the field is divided into triangles, whose bases are the sides, and the altitudes the given perpendiculars upon the sides from P.

$$\begin{array}{lll}
APB & = 8.64 \times 2.80 = & 24.1920 \text{ sq. ch.} \\
BPC & = 8.27 \times 3.04 = & 25.1408 \\
CPH & = 8.06 \times 2.40 = & 19.3440 \\
HPE & = 6.82 \times 2.72 = & 18.5504 \\
EPA & = 9.90 \times 2.70 = & 26.7300 \\
\hline
ABCHE & & = 113.9572
\end{array}$$

The whole area 113.9572 sq. ch. must be divided as the numbers 500, 750, 1000, or as 2, 3, 4. $2 + 3 + 4 = 9$.

$$\begin{aligned}
9 : 113.9572 : : 2 &: 25.3238 \text{ sq. ch.} = \text{X's share.} \\
: : 3 &: 37.9857 \text{ sq. ch.} = \text{Y's share.} \\
: : 4 &: 50.6476 \text{ sq. ch.} = \text{Z's share.}
\end{aligned}$$

PH is assumed as the line between X's and Z's shares. Since the triangle PHE is less than X's share by $25.3238 - 18.5504 = 6.7734$ sq. ch., this difference must be taken from the triangle PEA. The area of PEM is then 6.7734 sq. ch., and the altitude $PD'''' = 5.40$.

$$\therefore\ EM = \frac{2\,PEM}{PD''''} = \frac{2 \times 6.7734}{5.40} = 2.5087 \text{ ch.}$$

$$PMA = PEA - PEM = 26.7300 - 6.7734 = 19.9566. \text{ sq. ch.}$$

Since Y's share is greater than PMA (19.9566) and less than $PMA + PAB$ (44.1486), the point N is on AB.

Y's share diminished by PMA equals PAN; that is,

$$PAN = 37.9857 - 19.9566 = 18.0291 \text{ sq. ch.}$$

$$AN = \frac{2\,PAN}{PD} = \frac{2 \times 18.0291}{5.60} = 6.439 \text{ ch.}$$

10. Divide the triangular field ABC, whose sides AB, AC, and BC are 15, 12, and 10 ch., respectively, into three equal parts, by fences EG and DF parallel to BC.

$$ABC = \sqrt{18.5 \times 3.5 \times 6.5 \times 8.5} = 59.81169 \text{ sq. ch.}$$
$$ADF = \tfrac{1}{3} \text{ of } 59.81169 = 19.9372 \text{ sq. ch.}$$
$$AEG = \tfrac{2}{3} \text{ of } 59.81169 = 39.8744 \text{ sq. ch.}$$
$$ABC : AEG :: \overline{AB}^2 : \overline{AE}^2$$
$$:: \overline{AC}^2 : \overline{AG}^2.$$
$$59.81169 : 39.8744 :: 15^2 : \overline{AE}^2. \quad \therefore AE = 12.247 \text{ ch.}$$
$$:: 12^2 : \overline{AG}^2. \quad \therefore AG = 9.798 \text{ ch.}$$
$$ABC : ADF :: \overline{AB}^2 : \overline{AD}^2$$
$$:: \overline{AC}^2 : \overline{AF}^2.$$
$$59.81169 : 19.9372 :: 15^2 : \overline{AD}^2. \quad \therefore AD = 8.659 \text{ ch.}$$
$$:: 12^2 : \overline{AF}^2. \quad \therefore AF = 6.928 \text{ ch.}$$

11. Divide the triangular field ABC, whose sides AB, BC, and AC are 22, 17, and 15 ch., respectively, among three persons, A, B, and C, by fences parallel to the base AB, so that A may have 3 A., B 4 A., and C the remainder.

$$CAB = \sqrt{27 \times 5 \times 10 \times 12} = 127.2792 \text{ sq. ch.}$$
$$CDG = CAB - ABGD = 127.2792 - 30 = 97.2792 \text{ sq. ch.}$$
$$CEF = CAB - ABFE = 127.2792 - 70 = 57.2792 \text{ sq. ch.}$$
$$CAB : CDG :: \overline{CB}^2 : \overline{CG}^2$$
$$:: \overline{CA}^2 : \overline{CD}^2.$$
$$127.2792 : 97.2792 :: 17^2 : \overline{CG}^2. \quad \therefore CG = 14.862 \text{ ch.}$$
$$:: 15^2 : \overline{CD}^2. \quad \therefore CD = 13.113 \text{ ch.}$$
$$CAB : CEF :: \overline{CB}^2 : \overline{CF}^2$$
$$:: \overline{CA}^2 : \overline{CE}^2.$$
$$127.2792 : 57.2792 :: 17^2 : \overline{CF}^2. \quad \therefore CF = 11.404 \text{ ch.}$$
$$:: 15^2 : \overline{CE}^2. \quad \therefore CE = 10.062 \text{ ch.}$$

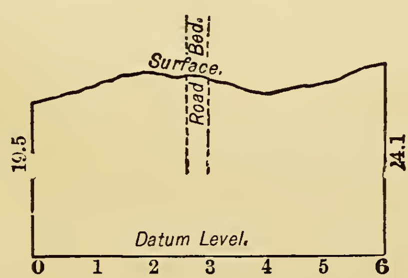
Exercise V.

1. Find the difference of level of two places from the following field notes: back-sights, 5.2, 6.8, and 4.0; fore-sights, 8.1, 9.5, and 7.9.

$$\begin{aligned} 8.1 + 9.5 + 7.9 &= 25.5 \\ 5.2 + 6.8 + 4 \quad &= 16 \\ \hline &\ 9.5 \end{aligned}$$

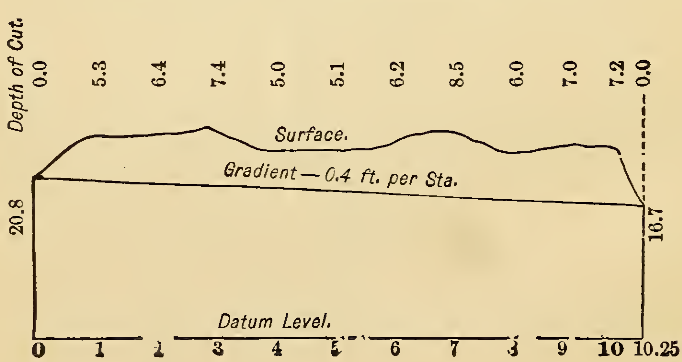
2. Write the proper numbers in the third and fifth columns of the following table of field notes, and make a profile of the section.

Station.	$+S.$	$H.I.$	$-S.$	$H.S.$	*Remarks.*
B	6.944	....	....	20.	Bench on post 22 feet north of 0.
0	....	26.944	7.4	19.5	
1	....	....	5.6	21.3	
2	....	....	3.9	23.0	
3	....	....	4.6	22.3	
t. p.	3.855	....	5.513	21.431	
4	....	25.286	4.9	20.4	
5	....	....	3.5	21.8	
6	....	....	1.2	24.1	



3. Stake 0 of the following notes stands at the lowest point of a pond to be drained into a creek; stake 10 stands at the edge of the bank, and 10.25 at the bottom of the creek. Make a profile, draw the grade line through 0 and 10.25, and fill out the columns *H.G.* and *C*, the former to show the height of grade line above the datum, and the latter, the depths of cut at the several stakes necessary to construct the drain.

Station.	*+S.*	*H.I.*	*−S.*	*H.S.*	*H.G.*	*C.*	*Remarks.*
B	6.000	....	....	25	....	....	Bench on rock 30 feet west of stake 1.
0	....	....	10.2	....	20.8	0.0	
1	....	....	5.3	....	20.4	5.3	
2	....	....	4.6	....	20.0	6.4	
3	....	....	4.0	....	19.6	7.4	
4	....	....	6.8	....	19.2	5.0	
5	4.572	....	7.090	....	18.8	5.1	
6	....	....	3.9	....	18.4	6.2	
7	....	....	2.0	....	18.0	8.5	
8	....	....	4.9	....	17.6	6.0	
9	....	....	4.3	....	17.2	7.0	
10	....	....	4.5	....	16.8	7.2	
10.25	....	....	11.8	....	16.7	0.0	



DATE DUE
