1 $f(x) = \frac{1}{2}x^T Q x - b^T x$

$\nabla f(x) = Qx - b$

$\phi(\alpha) = f(x_k + \alpha p_k)$

$\phi'(\alpha) = (Q(x+\alpha p_k) - b)^T p_k = 0 \leftarrow \phi$ is minimized when $\phi' = 0$

$0 = (Qx + \alpha Q p_k - b)^T p_k$

$0 = ((Qx)^T + (\alpha Q p_k)^T - b^T) p_k$

$0 = x^T Q^T p_k + \alpha p_k^T Q^T p_k - b^T p_k$

$-\alpha p_k^T Q^T p_k = x^T Q^T p_k - b^T p_k$

$-\alpha p_k^T Q^T p_k = (x^T Q^T - b^T) p_k$

$-\alpha p_k^T Q^T p_k = ((Qx)^T - b^T) p_k$

$-\alpha p_k^T Q^T p_k = (Qx - b)^T p_k$

$-\alpha p_k^T Q^T p_k = \nabla f_k^T p_k$

$$\boxed{\alpha = -\frac{\nabla f_k^T p_k}{p_k^T Q p_k}}$$

2 $f(x) = 500 - x(x-20)^3$

$-f(x) = -500 + x(x-20)^3$

$\nabla(-f(x)) = x(3(x-20)^2) + (1)(x-20)^3$

$\nabla(-f(x)) = 3x(x-20)^2 + (x-20)^3$

$$
\begin{aligned}
\nabla^2(-f(x)) &= 3x(2(x-20)) + (3)(x-20)^2 + 3(x-20)^2 \\
&= 6x(x-20) + 6(x-20)^2 \\
&= 6x^2 - 120x + 6x^2 - 240x + 2400 \\
&= 12x^2 - 360x + 2400 \\
&= 12(x - 30x + 200)
\end{aligned}
$$

$\nabla^2(-f(x)) = 12(x-10)(x-20)$

$\nabla^2(-f(x))$ can be negative, therefore $-f(x)$ is not convex and $f(x)$ is not concave

5 $f(x) = \frac{1}{2}\|Jx - y\|^2$

$$= \frac{1}{2}(Jx - y)(Jx - y)$$
$$= \frac{1}{2}(x^T J^T Jx - 2y^T Jx + y^T y)$$

$$\nabla f(x) = \frac{1}{2}(J^T Jx - 2y^T J)$$

$$\nabla^2 f(x) = \frac{1}{2}(J^T J)$$

$$x^T(\tfrac{1}{2} J^T J)x$$
$$= \frac{1}{2}(x^T J^T Jx)$$
$$= \frac{1}{2}((Jx)(Jx))$$
$$= \frac{1}{2}\|Jx\|^2 \geq 0$$ ← Jx is squared so it will always be ≥ 0

$\therefore f(x)$ is positive-semi definite

$\therefore f(x)$ is convex

6a $P(Y=y) = \frac{1}{1+\exp(-ya^T x)}$

$$\int_{-1}^{1} \frac{1}{1+e^{(-ya^T x)}}\, dy$$

$$\int \frac{1}{1+e^{(-ya^T x)}}\, dy = \int \frac{1}{1+e^{(-ya^T x)}} \cdot \frac{e^{(ya^T x)}}{e^{(ya^T x)}}\, dy = \int \frac{e^{(ya^T x)}}{e^{(ya^T x)}+1}\, dy$$

$u = e^{(ya^T x)} + 1$

$\frac{du}{dy} = a^T x e^{(ya^T x)}$

$dy = \frac{e^{(-ya^T x)}}{a^T x}\, du$

$$= \int \frac{e^{(ya^T x)}}{u} \frac{e^{(-ya^T x)}}{a^T x}\, du = \int \frac{1}{u \cdot a^T x}\, du = \frac{1}{a^T x} \int u\, du$$

$$= \frac{1}{a^T x} \ln(u) = \frac{1}{a^T x} \ln(e^{(ya^T x)} + 1)$$

$$\frac{1}{a^T x} \ln(e^{(ya^T x)} + 1)\Big|_{-1}^{1}$$

$$= \frac{1}{a^T x}\left(\ln(e^{(a^T x)}+1) - \ln(e^{(-a^T x)}+1\right)$$

$$= \frac{1}{a^T x}\left(\ln\left(\frac{e^{(a^T x)}+1}{e^{(-a^T x)}+1}\right)\right)$$

$$= \frac{1}{a^T x}\left(\ln(e^{(a^T x)})\right) = \frac{1}{a^T x}(a^T x) = 1$$

$$\boxed{\int_{-1}^{1} \frac{1}{1+e^{(-ya^T x)}}\, dy = 1}$$

$\therefore$ the function is a valid probability distribution

6b letting $i=1,\dots,q$ be indices where $y=+1$
letting $i=q+1,\dots,m$ be indices where $y=-1$

log-likelihood function $= \boxed{\log\left[\prod_{i=1}^{q} \frac{1}{1+e^{(-a_i^T x)}} \prod_{i=q+1}^{m} \frac{1}{1+e^{(a_i^T x)}}\right]}$

6c
$$-\log\left[\prod_{i=1}^{q} \frac{1}{1+e^{(-a_i^T x)}} \prod_{i=q+1}^{m} \frac{1}{1+e^{(a_i^T x)}}\right]$$

$$= -\sum_{i=1}^{q} \log\left(\frac{1}{1+e^{(-a_i^T x)}}\right) - \sum_{i=q+1}^{m} \log\left(\frac{1}{1+e^{(a_i^T x)}}\right)$$

$$= \sum_{i=1}^{q} \log\left(1+e^{(-a_i^T x)}\right) + \sum_{i=q+1}^{m} \log\left(1+e^{(a_i^T x)}\right)$$

$$= \boxed{\sum_{i=1}^{m} \log\left(1+e^{(-y a_i^T x)}\right)}$$

$\therefore$ minimizing the negative of the log likelihood function leads to

$$\min_x \sum_{i=1}^{m} \log\left(1+e^{(-y a_i^T x)}\right)$$