2.1 $f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$

$\frac{\partial f}{\partial x_1} = 100(2)(x_2 - x_1^2)(-2x_1) + (2)(1 - x_1)(-1)$

$= -400x_1x_2 + 400x_1^3 - 2 + 2x_1$

$\frac{\partial f}{\partial x_2} = 100(2)(x_2 - x_1^2)$

$= 200x_2 - 200x_1^2$

$$\boxed{\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -400x_1x_2 + 400x_1^3 - 2 + 2x_1 \\ 200x_2 - 200x_1^2 \end{bmatrix}}$$

$\frac{\partial^2 f}{\partial x_1^2} = -400x_2 + 1200x_1^2 + 2$

$\frac{\partial^2 f}{\partial x_1 \partial x_2} = -400x_1$

$\frac{\partial^2 f}{\partial x_2 \partial x_1} = -400x_1$

$\frac{\partial^2 f}{\partial x_2^2} = 200$

$$\boxed{\nabla^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -400x_2 + 1200x_1^2 + 2 & -400x_1 \\ -400x_1 & 200 \end{bmatrix}}$$

For x^* to be a possible local minimizer $\nabla f = 0$ at the point

Setting $\frac{\partial f}{\partial x_2} = 0$: $200x_2 - 200x_1^2 = 0 \rightarrow 200x_2 = 200x_1^2 \rightarrow x_2 = x_1^2$

Substituting $x_2 = x_1^2$ into $\frac{\partial f}{\partial x_1}$ & setting $\frac{\partial f}{\partial x_1} = 0$:

$-400x_1x_2 + 400x_1^3 - 2 + 2x_1 = 0$

$-400x_1(x_1^2) + 400x_1^3 - 2 + 2x_1 = 0$

$-400x_1^3 + 400x_1^3 - 2 + 2x_1 = 0$

$-2 + 2x_1 = 0 \rightarrow -2 = -2x_1 \rightarrow 1 = x_1 \rightarrow x_2 = 1^2 \rightarrow x_2 = 1$

$\therefore$ $x^* = (1,1)$ is the only point at which $\nabla f(x^*) = 0$

$\therefore$ x^* is the only stationary point and possible candidate for a local minimizer

Proof that $\nabla^2 f(x^*)$ is positive definite:

$x^{*T} \nabla f(x^*) x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -400x_2 + 1200x_1^2 + 2 & -400x_1 \\ -400x_1 & 200 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -400 + 1200 + 2 & -400 \\ -400 & 200 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 802 & -400 \\ -400 & 200 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 802 - 400 & -400 + 200 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 402 & -200 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 202 > 0$

$\therefore$ $\nabla^2 f$ is positive definite at x^*

$\therefore$ Because $\nabla f(x^*) = 0$ and $\nabla^2 f(x)$ is positive definite, by SOSC <u>x^* is a strict local minimizer.</u>

(and because x^* is the only stationary point it is the only local minimizer)

2.2 $f(x) = 8x_1 + 12x_2 + x_1^2 - 2x_2^2$

$\nabla f = \begin{bmatrix} 8 + 2x_1 \\ 12 - 4x_2 \end{bmatrix}$

For x^* to be a stationary point, $\nabla f(x^*) = 0$

Setting $\frac{\partial f}{\partial x_1} = 0$: $8 + 2x_1 = 0 \rightarrow \boxed{x_1^* = -4}$

Setting $\frac{\partial f}{\partial x_2} = 0$: $12 - 4x_2 = 0 \rightarrow \boxed{x_2^* = 3}$

$\therefore x^* = (-4, 3)^T$ is the only stationary point

Checking if x^* is a local minimizer:

$\nabla^2 f = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}$

It can be seen the $\nabla^2 f(x^*)$ has eiganvalues of $\lambda = 2$ and $\lambda = -4$

$\therefore$ Because there is one positive eiganvalue and one negative eiganvalue, moving in any direction from x^* will result in an increase or decrease. Therefore x^* is a saddlepoint.

2.3 $f_1(x) = a^T x$

$$= [a_1 \dots a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= a_1 x_1 + \dots + a_n x_n$$

$$\boxed{\nabla f_1 = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = a \qquad \nabla^2 f = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} = 0}$$

$f_2(x) = x^T A x$

$$= [x_1 \cdots x_n] \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= [a_{11}x_1 + \dots + a_{nn}x_n \quad a_{12}x_1 + \dots + a_{n2}x_n \quad \cdots \quad a_{1n}x_1 + \dots + a_{nn}x_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= a_{11}x_1^2 + \dots + a_{n1}x_1x_n + a_{12}x_2x_1 + \dots + a_{n2}x_2x_n + \dots + a_{1n}x_nx_1 + \dots + a_{nn}x_n^2$$

$\frac{\partial f_2}{\partial x_1} = 2a_{11}x_1 + a_{21}x_2 + \dots + a_{n1}x_n + a_{12}x_2 + \dots + a_{1n}x_n$

Because A is symmetric: $a_{12} = a_{21} \dots a_{1n} = a_{n1}$

$\therefore \frac{\partial f_2}{\partial x_1} = 2a_{11}x_1 + 2a_{12}x_2 + \dots + 2a_{1n}x_n$

$\frac{\partial f_2}{\partial x_2} = a_{21}x_1 + a_{12}x_1 + 2a_{22}x_2 + \dots + a_{n2}x_n + a_{23}x_3 + \dots + a_{2n}x_n$

$\frac{\partial f_2}{\partial x_2} = 2a_{21}x_1 + 2a_{22}x_2 + \dots + 2a_{2n}x_n$ (by symmetry of A)

$\vdots$

$\frac{\partial f_2}{\partial x_n} = 2a_{n1}x_1 + 2a_{n2}x_2 + \dots + 2a_{nn}x_n$

$$\boxed{\nabla f_2 = \begin{bmatrix} 2a_{11}x_1 + 2a_{12}x_2 + \dots + 2a_{1n}x_n \\ 2a_{21}x_1 + 2a_{22}x_2 + \dots + 2a_{2n}x_n \\ \vdots \\ 2a_{n1}x_1 + 2a_{n2}x_2 + \dots + 2a_{nn}x_n \end{bmatrix} = 2Ax \qquad \nabla^2 f_2 = \begin{bmatrix} 2a_{11} & 2a_{12} & \cdots & 2a_{1n} \\ 2a_{21} & 2a_{22} & \cdots & 2a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 2a_{n1} & 2a_{n2} & \cdots & 2a_{nn} \end{bmatrix} = 2A}$$

4 $f(x_1,x_2) = 3x_1^2 - x_1x_2 + x_2^2 - 11x_1$

$$\nabla f(x) = \begin{bmatrix} 6x_1 - x_2 - 11 \\ -x_1 + 2x_2 \end{bmatrix}_x = \begin{bmatrix} 6(2)-(1)-11 \\ -2+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\nabla^2 f = \begin{bmatrix} 6 & -1 \\ -1 & 2 \end{bmatrix}$$

$$x^T \nabla^2 f\, x = [x_1\ x_2] \begin{bmatrix} 6 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [2\ 1] \begin{bmatrix} 6 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= [12-1\quad -2+2] \begin{bmatrix} 2 \\ 1 \end{bmatrix} = [11\ 0] \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 22+1$$

$$= 23 > 0$$

$\therefore$ $\nabla^2 f(x)$ is positive definite

$\therefore$ Because $\nabla f(x) = 0$ and $\nabla^2 f(x)$ is positive definite, by SOSC x is a strict local minimizer

$\boxed{B}$