IE 3013

Taylor approximation with a search direction. Consider the function $f(x_1, x_2) = x_1^3 - 5x_1x_2 + 6x_2^2$ at the point $x = (0, 2)^T$ and search direction $p = (1, -1)^T$.

1. Derive the first-order Taylor approximation to $f(x + \alpha p)$, where α is the step size.
2. Derive the second-order Taylor approximation to $f(x + \alpha p)$.
3. Plot the original function and both Taylor series approximations as functions of α. Which approximation appears to be better?

1. $f(x+\alpha p) \simeq f(x) + \alpha \nabla f(x)^T p$

$$\simeq x_1^3 - 5x_1x_2 + 6x_2^2 + \alpha \begin{bmatrix} 3x_1^2 - 5x_2 & -5x_1 + 12x_2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\simeq x_1^3 - 5x_1x_2 + 6x_2^2 + \alpha(3x_1^2 - 5x_2 + 5x_1 - 12x_2)$$

at $(0,2)^T$ $\quad \simeq (0)^3 - 5(0)(2) + 6(2)^2 + \alpha(3(0)^2 - 5(2) + 5(0) - 12(2))$

$$\simeq 24 - 34\alpha$$

2. $f(x+\alpha p) \simeq f(x) + \alpha \nabla f(x)^T p + \alpha^2 \left(\frac{1}{2}\right) p^T \nabla^2 f(x) p$

$$\simeq f(x) + \alpha \nabla f(x)^T \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \alpha^2 \left(\tfrac{1}{2}\right) \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 6x_1 & -5 \\ -5 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\simeq x_1^3 - 5x_1x_2 + 6x_2^2 + \alpha \begin{bmatrix} 3x_1^2 - 5x_2 & -5x_1 + 12x_2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\simeq x_1^3 - 5x_1x_2 + 6x_2^2 + \alpha(3x_1^2 - 5x_2 + 5x_1 - 12x_2) + \alpha^2\left(\tfrac{1}{2}\right) \begin{bmatrix} 6x_1 + 5 & -5 - 12 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\simeq x_1^3 - 5x_1x_2 + 6x_2^2 + \alpha(3x_1^2 - 5x_2 + 5x_1 - 12x_2) + \alpha^2\left(\tfrac{1}{2}\right)(6x_1 + 5 + 5 + 12)$$

$$\simeq x_1^3 - 5x_1x_2 + 6x_2^2 + \alpha(3x_1^2 - 5x_2 + 5x_1 - 12x_2) + \alpha^2\left(\tfrac{1}{2}\right)(6(0) + 5 + 5 + 12)$$

at $(0,2)^T$ $\quad \simeq (0)^3 - 5(0)(2) + 6(2)^2 + \alpha(3(0)^2 - 5(2) + 5(0) - 12(2)) + \alpha^2\left(\tfrac{1}{2}\right)(6(0) + 5 + 5 + 12)$

$$\simeq 24 - 34\alpha + 11\alpha^2$$

$$f(x+\alpha p) = (x_1 + \alpha p_1)^3 - 5(x_1 + \alpha p_1)(x_2 + \alpha p_2) + 6(x_2 + \alpha p_2)^2$$

$$= (0 + \alpha(1))^3 - 5(0 + \alpha(1))(2 + \alpha(-1)) + 6(2 + \alpha(-1))^2$$

$$= \alpha^3 - 5(\alpha)(2 - \alpha) + 6(2 - \alpha)^2$$

3. See Jupyter Notebook

The second order approximation appears to be better.