M1C03 Lecture 11

Proofs with Integers and Real Numbers

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Announcement(s)

1. Test 1 is October 29. Details are forthcoming.
2. To prepare for Test 1 you should be doing exercises.
3. Quiz 4 due Friday.

Overview

- Proofs with properties of integers and real numbers.
- Proof by contrapositive.
- Proof of a biconditional.

Reference: Lakins, chapter 2.

Basic Properties of Integers

For all integers a, b, and c,

(Closure under $+$ and $\cdot$)	$a+b$ and ab are integers.
(Associativity)	$(a+b)+c = a+(b+c)$ and $(ab)c = a(bc)$.
(Commutativity)	$a+b = b+a$ and $ab = ba$.
(Distributivity)	$a(b+c) = ab+ac$.
(Identities)	$0 \neq 1$, $a+0 = a$, $a \cdot 1 = a$, and $a \cdot 0 = 0$.
(Additive inverses)	There is a unique integer $-a = (-1) \cdot a$ such that $a + (-a) = 0$.
(Subtraction)	$a - b$ is defined to be $a + (-b)$.
(No divisors of 0)	If $ab = 0$, then $a = 0$ or $b = 0$.
(Cancellation)	If $ab = ac$ and $a \neq 0$, then $b = c$.
(Transitive property of $<$)	If $a < b$ and $b < c$, then $a < c$.
(Trichotomy)	Exactly one of $a < b$, $a = b$, or $a > b$ holds.
(Order property 1)	If $a < b$, then $a + c < b + c$.
(Order property 2)	If $c > 0$, then $a < b$ iff $ac < bc$.
(Order property 3)	If $c < 0$, then $a < b$ iff $ac > bc$.

Basic Properties of Real Numbers

For all real numbers a, b, and c,

(Closure under $+$ and $\cdot$)	$a+b$ and ab are real numbers.
(Associativity)	$(a+b)+c = a+(b+c)$ and $(ab)c = a(bc)$.
(Commutativity)	$a+b = b+a$ and $ab = ba$.
(Distributivity)	$a(b+c) = ab+ac$.
(Identities)	$0 \neq 1$, $a+0 = a$, $a \cdot 1 = a$, and $a \cdot 0 = 0$.
(Additive inverses)	There is a unique real number $-a = (-1) \cdot a$ such that $a + (-a) = 0$.
(Subtraction)	$a-b$ is defined to be $a + (-b)$.
(Multiplicative inverses)	If $a \neq 0$, then there is a unique real number $a^{-1} = \frac{1}{a}$ such that $a \cdot a^{-1} = 1$.
(Division)	When $a \neq 0$, $\frac{b}{a}$ is defined to be $b \cdot a^{-1}$.
(No divisors of 0)	If $ab = 0$, then $a = 0$ or $b = 0$.
(Cancellation)	If $ab = ac$ and $a \neq 0$, then $b = c$.
(Transitive property of $<$)	If $a < b$ and $b < c$, then $a < c$.
(Trichotomy)	Exactly one of $a < b$, $a = b$, or $a > b$ holds.
(Order property 1)	If $a < b$, then $a + c < b + c$.
(Order property 2)	If $c > 0$, then $a < b$ iff $ac < bc$.
(Order property 3)	If $c < 0$, then $a < b$ iff $ac > bc$.

Theorem (1)

For all real numbers x, if $x \neq 0$, then $x^2 > 0$.

Theorem (2)

For all real numbers x, if $x^2 = 0$, then $x = 0$.

Proof of a biconditional

Theorem (3)

For all real numbers x, $x = 0$ if and only if $x^2 = 0$.

Theorem (4)

$0 < 1$.

An application

Theorem (5)

$-1 < 0.$