HW1

1. For all integers $n \geq 1$, let $b(n)$ denote the number of sequences $(b_1, \ldots, b_n)$ with $b_i \in \{0, 1\}$ such that

$$b_1 \leq b_2 \geq b_3 \leq b_4 \geq \ldots. \tag{1}$$

Give a direct combinatorial proof of the following identity.

$$\text{For all integers } n \geq 1, \quad 1 + \sum_{k=1}^{n} b(3k-1) = \frac{1}{2} b(3n+1). \tag{2}$$

Hint: Experiment and look for patterns!

2. Prove the following proposition without using the notion of cardinality introduced in class, nor Theorem 1.4.4 from How To Count.

Proposition 0.1. *For all integers $n, m \geq 1$, if there exists an injective function $f \colon [n] \to [m]$, then $n \leq m$.*

Hint: Induction!

HW2

1. Recall that $P(n, k)$ denotes n place k. Give *a combinatorial proof* of the following identity.

Theorem 0.2. *For all integers $n, k \geq 3$,*

$$\begin{aligned} P(n,k) = P(n-3,k) + 3kP(n-3,k-1) \\ + 3k(k-1)P(n-3,k-2) + k(k-1)(k-2)P(n-3,k-3). \end{aligned}$$

2. Evaluate the following proofs according to the rubric provided on Kritik and provide comments.

Theorem 0.3. *For all integers $n \geq 1$, the number of ways to partition the set $[2n] = \{1, 2, \ldots, 2n\}$ by subsets of cardinality 2 equals*

$$\prod_{k=1}^{n} (2k-1) = 1 \cdot 3 \cdot 5 \cdots (2n-1).$$

The first proof is adapted from a solution posted to math.stackexchange[1].

Proof 1. Let $p(n)$ denote the number of ways to partition $[2n]$ by subsets of cardinality 2. Then $p(n+1)$ equals $p(n) + 2n(2n-1)p(n-1)$ because either every partition contains $\{2n+1, 2n+2\}$ in which case there are $p(n)$ ways to finish, or it does not, in which case there are $2n(2n-1)p(n-1)$ ways to finish. By induction, it is easy to see that $p(n) = \prod_{k=1}^{n}(2k-1)$. $\square$

Proof 2. We give a proof by induction on n. Let $p(n)$ denote the number of ways to partition $[2n]$ by subsets of cardinality 2.

Base case ($n = 1$): The only way to partition $[2]$ by subsets of cardinality 2 is to take the rather uninteresting partition by 1 subset, $[2]$ itself, so $p(1) = 1$. Comparing this with the formula, we have

$$\prod_{k=1}^{1}(2k-1) = (2(1)-1) = 1.$$

Thus, we have verified the base case.

Induction step: Assume we know that

$$p(n) = \prod_{k=1}^{n}(2k-1) = 1 \cdot 3 \cdot 5 \cdots (2n-1).$$

We want to show that

$$p(n+1) = \prod_{k=1}^{n+1}(2k-1) = 1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot (2n+1).$$

Every partition of $[2n+2]$ by subsets of cardinality 2 falls into one of the following two cases:

[1]https://math.stackexchange.com/questions/1828364/number-of-ways-to-partition-a-set-with-2n-elements-into-unordered-pairs

- **Case 1 (The partition contains $\{2n+1, 2n+2\}$):** Every partition of $[2n+2]$ by subsets of size 2 which contains $\{2n+1, 2n+2\}$ defines a partition of $[2n]$ by subsets of size 2 (and vis versa). Thus, the number of partitions which contain $\{2n+1, 2n+2\}$ equals $p(n)$.
- **Case 2 (The partition does not contain $\{2n+1, 2n+2\}$):** If the partition does not contain $\{2n+1, 2n+2\}$, then it must contain $\{a, 2n+1\}$ and $\{b, 2n+2\}$ for some $a, b \in [2n]$, $a \neq b$. There are $P(2n, 2)$ ways to choose a and b from $[2n]$. Once the subsets $\{a, 2n+1\}$ and $\{b, 2n+2\}$ are fixed, we must partition the remaining $2n-2$ elements of $[2n+2]$. This is equivalent to partitioning $[2n-2]$ by subsets of cardinality 2, so the number of ways to do this equals $p(n-1)$. By the multiplication principle, there are
$$2n(2n-1)p(n-1)$$
partitions which do not contain the set $\{2n+1, 2n+2\}$.

Since these are the only possible cases, it follows by the addition principle that
$$p(n+1) = p(n) + 2n(2n-1)p(n-1).$$
Doing some algebra and invoking our induction hypothesis, we see that
$$\begin{aligned}
p(n+1) &= p(n) + 2n(2n-1)p(n-1) \\
&= \prod_{k=1}^{n}(2k-1) + 2n(2n-1)\prod_{k=1}^{n-1}(2k-1) \\
&= \prod_{k=1}^{n}(2k-1) + 2n\prod_{k=1}^{n}(2k-1) \\
&= (2n+1)\prod_{k=1}^{n}(2k-1) \\
&= \prod_{k=1}^{n+1}(2k-1)
\end{aligned}$$

which completes the proof of the induction step. Thus, the theorem follows by induction. $\square$

HW3

1. A restaurant is preparing a large round table with $2n+1$ seats (the seats are unlabelled). The waiter has $n+1$ red napkins and n blue napkins. How many visually distinct ways are there for the waiter to place the napkins around the table? Prove that your solution is correct.

2. Recall that $\left[{n \atop k}\right]$ denotes the Stirling number of the first kind. Give a proof of the following using induction.

   Theorem 0.4. *For all integers $n \geq 1$, we have the following identity of polynomials in a real variable x*

   $$\prod_{j=0}^{n-1}(x+j) = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k.$$

Term Test

1. Give a **combinatorial proof** of **one** of the following identities. Clearly indicate which identity you are proving.

   (a) Recall that $t(n)$ denotes the number of ways to tile a $1 \times n$ board with square and domino tiles. For all $n \geq 1$,

   $$t(2n+2) = t(n)^2 + t(n+1)^2.$$

   (b) For all $n, k \geq 2$,

   $$k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2}.$$

2. There are $4n$ people. Half of them have blue eyes and half of them have brown eyes. How many ways are there to seat them at n round tables so that each table has 4 people, and eye colours alternate at each table? Both the seats and the tables are unlabelled. Justify your solution.

HW4

1. Consider the lattice cube in $\mathbb{R}^3$ with opposite vertices $(0,0,0)$ and (m,m,m) for some integer $m \geq 1$. Derive a formula for the number of lattice paths from $(0,0,0)$ to (m,m,m) which do not pass through any of the other vertices of the cube. Remember that our lattice paths are not allowed to go backwards: they can only step in the directions $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

   State your formula as a theorem and give your derivation in the format of a proof.

   Hint: PIE.

2. Multinomial coefficients satisfy a generalized version of the Vandermonde identity.

   Theorem 0.5. *For all non-negative integers n, m and $k_1, \ldots, k_p$ such that $n + m = k_1 + \cdots + k_p$,*

   $$\binom{n+m}{k_1, \ldots, k_p} = \sum_{\substack{(i_1,\ldots,i_p)\in\mathbb{N}^p,\\ i_1+\cdots+i_p=n}} \binom{n}{i_1, \ldots, i_p}\binom{m}{k_1 - i_1, \ldots, k_p - i_p}.$$

   Give a combinatorial proof of this identity.

HW5

1. The Lucas numbers L_n are defined by the recurrence relation

   $$L_0 = 2,\ L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n.$$

   Use the method of generating functions to derive a formula for L_n.

2. Use generating functions to prove the identity

   $$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

HW6

1. [2] The *Bernoulli numbers* b_n (Not to be confused with the Bell numbers!) are defined by the recurrence

$$b_0 = 1, \quad \sum_{k=0}^{n} \binom{n+1}{k} b_k = 0.$$

   (a) Prove that the exponential generating function of the Bernoulli numbers is
$$f(x) = \frac{x}{e^x - 1}.$$

   (b) Show that $f(x) + \frac{1}{2}x$ is an even function and deduce that $b_n = 0$ for all odd $n \geq 3$.

2. [3] Suppose $2n+1$ people sit at a round table. Suppose n of them have blue eyes and $n+1$ of them have brown eyes. Show that if $n \geq 2$, then there exists a person seated between two people with brown eyes.

[2]Taken from "Combinatorics: topics, techniques, algorithms" by Cameron.
[3]Taken from section 1.5 of "How To Count" by Beeler.