M1C03 Final Exam Proofs Practice

December, 2021

General approach to proving any statement (roughly in order):

- Recall the precise definitions used in the statement.
- Determine the logical form of the statement.
- Identify the most appropriate proof strategy.
- Write an outline/scaffold of your proof.
- Identify places where more details are needed. Work backwards in scratchwork when stuck.
- Identify steps that need to be broken into cases (e.g., proving an OR statement).
- Check that you have justified every step and cited facts that were used (e.g., from the handout of axioms etc.).

8.3.14 Show that if A is uncountable and $B \subseteq A$ is countable, then $A - B$ is uncountable.

2.3.3 Prove that $log_2 3$ is irrational. Note that $y = log_2 x$ is defined as the real number such that $2^y = x$.

6.3.9 Let $a, b \in \mathbb{Z}$. Prove that if $\gcd(ab, a+b) = 1$, then $\gcd(a, b) = 1$.

6.1.3 Prove that for all positive integers n, $6|n(n+1)(2n+1)$.

6.2.7 Let $a, b \in \mathbb{Z}$ not both 0. Prove that $\gcd(a, b) = \gcd(|a|, |b|)$.

5.5.8 Let $f\colon X \to Y$ and $g\colon Y \to Z$ be invertible functions. Prove that $g \circ f\colon X \to Z$ is invertible and the inverse function is $f^{-1} \circ g^{-1}$.

5.5.8 Let $f\colon X \to Y$ and $g\colon Y \to Z$ be invertible functions. Prove that $g \circ f\colon X \to Z$ is invertible and the inverse function is $f^{-1} \circ g^{-1}$.

5.3.3 Let

$$f\colon \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ 2x & \text{if } x < 0 \end{cases}$$

Is f injective? Is f surjective? Justify your answers.

4.4.12 Let $A_i \subseteq \mathbb{R}$ be a family of open sets indexed by a set $I \neq \emptyset$. Prove that

$$\bigcup_{i \in I} A_i$$

is open.

Let A be a subset of a universe $\mathcal{U}$. Prove that $A \cup \overline{A} = \mathcal{U}$.

Consider the sequence

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_n + F_{n+1}.$$

Show that for all integers $n \geq 1$,

$$(-1)^n = F_{n+1}F_{n-1} - F_n^2.$$

(Challenge) Consider the sequence

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_n + F_{n+1}.$$

Show that for all integers $n \geq 1$,

$$F_n = \sum_{k=0}^{n-1} \binom{n-1-k}{k}.$$

2.4.4 Prove there exists a real number M such that for all $x \in \mathbb{R}$ with $1 < x < 3$,

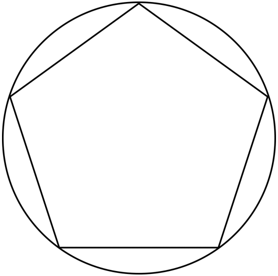
$$\left| \frac{5x^2 - 2x - 4}{5(x^2 + 1)} \right| \leq M.$$

2.2.9 Determine whether the following statements are true or false. If it is true, give a proof. If it is false, give a counterexample.

- If a and b are irrational, then ab is irrational.
- If a is irrational and b is rational, then ab is irrational.

Let P, Q, R be statements. Prove that $P \wedge (Q \vee R)$ is logically equivalent to $(P \wedge Q) \vee (P \wedge R)$ by showing that their truth tables are the same.

For every $n \geq 3$, we can draw a *regular n-gon* by placing n points equidistantly on a circle and drawing lines between them as illustrated in the following diagram for the case $n = 5$.



Assume that the radius of the circle is 1.

1. Prove that the sum of the interior angles is $(n-2)\pi$.
2. Let A denote the total area of the n-gon and let L denote the circumference of the n-gon. Without using trigonometry, show that
$$2A = L\sqrt{(1 - L^2/4)}.$$
3. Show that for $n \geq 3$ the number of *diagonals* (lines between vertices that are not on the circumference) is
$$\frac{n(n-3)}{2}.$$
Bonus: Can you interpret this result with a binomial coefficient or by counting certain subsets of a set of n elements?