Functions

Jeremy Lane

December 8, 2021

Functions are fundamental tools for mathematical reasoning. You have likely already seen and worked with functions in several contexts, such as calculus and possibly also linear algebra. Those contexts are somewhat specific; they deal only with functions of real numbers or vectors. In fact, functions are something that can be defined much more generally. Adopting this general perspective can be incredibly useful as a problem solving tool. We will use the results from this section in a crucial way when we talk about infinite cardinalities later in the course.

The goal of these notes is to present functions from the general perspective where the *domain* and *codomain* can be any set. We introduce several of the most fundamental and basic concepts about functions in this general setting, namely: injective (one-to-one), surjective (onto), and bijective. These definitions involve quantifiers, so they need to be defined and understood carefully. We also define *function composition* and *function inverse*. To emphasize the generality of functions and these concepts, we give several running examples which are not functions of numbers or vectors. The key result at the end of these notes is the fact that these concepts are related: *a function is invertible if and only if it is a bijection.*

Note that although this material lines up nicely with Chapter 5 of Lakins, we omit a lot of the material covered by Lakins in Chapter 5. Here is a list of things in Lakins which we will NOT cover:

- Pre-image, image (of a set), and range.
- Graph of a function.
- Increasing or decreasing functions of real numbers.

These concepts are useful, to be sure, but we feel that they are not as essential as the core concepts we choose to focus on here. To help you focus on this material, we have listed relevant exercise from Lakins at the end of each section.

These notes would be improved by accompanying diagrams and figures. Some of these are provided in your textbook. The rest should have appeared in the lectures.

1 What is a function?

A function is three things: a correspondence, a domain, and a codomain.

Definition 1.1 (Lakins, Definition 5.1.1)**.** Let X and Y be sets. A *function from X to Y* is a correspondence that assigns to each element of X a unique element of Y. The set X is the *domain* and the set Y is the *codomain.*

Notation and terminology: We denote a function with domain X and codomain Y with the notation like "$f\colon X \to Y$" which is read out loud as "the function f from X to Y." Here f is the name of the correspondence (or "rule" or "formula"), which is often given with an explicit formula (see next example). If we have named the correspondence f, then we denote the element in Y to which $x \in X$ is assigned by $f(x)$. You should not confuse f (the correspondence/rule/formula) with $f\colon X \to Y$ (the function). The function consists of three things: the rule f, the domain X, and the codomain Y!

An exception to the rule: Although we have just emphasized that all three objects in the notation $f\colon X \to Y$ are needed to define a function, one often sees authors referring to a function simply as f. Although this is generally ambiguous, it is often convenient to abbreviate "$f\colon X \to Y$" to simply "f" for the sake of brevity. The general rule of thumb is that this should only be done when the rest of the definition of the function (the domain and codomain) are somehow clear from context. If they are not, then something is missing and you should be suspicious! Thus, you must read carefully.

Example 1.2. Consider the correspondence $f(x) = \sqrt{x}$ (where $\sqrt{x}$ means the non-negative square root of x). This alone does not define a function. We also need to specify what the domain and codomain are. There are many possible choices of domain and codomain that will produce a function from this correspondence. For example,

- $f\colon [0, \infty) \to \mathbb{R}$,
- $f\colon [0, \infty) \to [0, \infty)$, and
- $f\colon [0, 1] \to [0, 1]$.

When dealing with functions of real numbers, it is often the convention, especially in a calculus course, to take the largest possible domain and the largest possible codomain for the given rule[1]. What are those for $f(x) = \sqrt{x}$? As we will see below when we talk about inverses, there are good reasons why we may want to take a different domain. Thus, the domain is part of the definition of the function (and similarly for the codomain).

On the other hand, the following are not functions (for the correspondence $f(x) = \sqrt{x}$). Why not?

[1] A typical calculus exercise may ask you to find "the" domain of a given formula. To be more precise, those exercises mean to ask you to find "the largest possible" domain for the given formula. There is often no definitive domain for a given formula. Picking the domain so as to define a function from a given formula is a choice that we make and we do not have to choose the largest possible domain!

- $f\colon \mathbb{R} \to \mathbb{R}$
- $f\colon [0,2] \to [0,1]$

As we mentioned above, any sets can be the domains and/or codomains of functions, not just sets of numbers. Indeed, functions are much more ubiquitous than our calculus and linear algebra courses would lead us to believe.

Example 1.3. Let $\mathcal{L} = \{``a", ``b", \ldots, ``z"\}$ be the set of lower case letters[2]. Let $\mathcal{U} = \{``A", ``B", \ldots, ``Z"\}$ be the set of upper case letters. Let

$$\mathcal{A} = \mathcal{L} \cup \mathcal{U} = \{``a", ``b", \ldots, ``z", ``A", ``B", \ldots, ``Z"\}.$$

Let $Cap\colon \mathcal{A} \to \mathcal{A}$ be the function that sends a letter to its capitalization. For example, $Cap(``a") = ``A"$ and $Cap(``A") = ``A"$. The shift key on the keyboard of your laptop applies the capitalization function. Can you think of some natural ways to extend the domain and codomain of Cap beyond the set $\mathcal{A}$? Can you think of other functions defined on a set of characters?

Example 1.4. Let B_4 be the set of all binary sequences of length 4. In other words, B_4 is the set of all sequences of 1s and 0s of length 4:

$$B_4 = \left\{ \begin{array}{cccccccc} 0000, & 1000, & 0100, & 0010, & 0001, & 1100, & 1010, & 1001, \\ 0101, & 0011, & 0110, & 1110, & 1101, & 1011, & 0111, & 1111 \end{array} \right\}.$$

Here are some examples of functions with domain B_4 and codomain B_4.

- (bit flip) Let $n\colon B_4 \to B_4$ be the function that replaces every 0 with a 1 and every 1 with a 0. For example $n(0101) = 1010$ and $n(1111) = 0000$.
- (right shift) Let $r\colon B_4 \to B_4$ be the function that shifts everything to the right by one place so that the rightmost digit is deleted and the leftmost digit becomes a 0. For example, $r(1111) = 0111$ and $r(0101) = 0010$.
- (left shift) Let $l\colon B_4 \to B_4$ be the function that shifts to the left, deleting the leftmost digit and replacing the rightmost digit by 0. For example, $l(1111) = 1110$ and $l(0111) = 1110$.

There are many other interesting examples. Can you think of some?

If we defined B_3 to be the set of all binary sequences of length 3, then we could also define various functions from B_4 to B_3 or vis versa. For example:

- (truncate) Let $t\colon B_4 \to B_3$ be the function that sends a sequence of length 4 to a sequence of length 3 by removing the leftmost digit. For example, $t(1010) = 010$ and $t(0010) = 010$.
- Let $a\colon B_3 \to B_4$ be the function that sends a sequence of length 3 to a sequence of length 4 by adding a 0 as the leftmost digit. For example, $a(010) = 0010$ and $a(111) = 0111$.

[2]Here we really mean characters in the English alphabet, not to be confused with variables which are often denoted using letters!

Of course, the numbers 4 and 3 in these examples are not particularly important. Describe these functions more generally for binary sequences of arbitrary fixed length.

The example of binary sequences is a special case of sequences of characters in some given alphabet. For instance, we could consider sequences of characters from $\mathcal{A}$, or any other set of characters. Can you think of ways to extend the function *Cap* from individual characters to sequences? Can you think of other interesting functions on sequences of characters?

1.1 Exercises

Lakins, 5.1.6, 5.1.7 (a) and (b).

2 Injective, surjective, and bijective

The following three definitions are of fundamental importance. We will illustrate them with examples shortly.

Definition 2.1 (Lakins, Definition 5.3.1)**.** Let $f\colon X \to Y$ be a function.

1. $f\colon X \to Y$ is *injective* (or *one-to-one*) if for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

2. $f\colon X \to Y$ is *surjective* (or *onto*) if for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

3. $f\colon X \to Y$ is *bijective* if it is injective and surjective.

Note that it is sometimes easier to understand and work with the contrapositive of the definition of injective: for all $x_1, x_2 \in X$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Some people use the terms "one-to-one" to mean "injective" and "onto" to mean "surjective". One-to-one means the same thing as injective. Onto means the same thing as surjective.

Example 2.2. Consider all the examples from the previous section. Which are injective? Which are surjective? Which are bijective? In each case, if the function is not injective or not surjective, can you modify the domain and/or codomain so that it is?

- For example, with which choices of domain and codomain does the correspondence $f(x) = \sqrt{x}$ define a function that is injective, surjective, or bijective? What about for $g(x) = x^2$?

- For example, for the capitalization function $Cap\colon \mathcal{A} \to \mathcal{A}$, describe all possible subsets $X \subset \mathcal{A}$ so that $Cap\colon X \to \mathcal{A}$ is injective (be careful!). Describe all possible subsets $Y \subset \mathcal{A}$ so that $Cap\colon \mathcal{A} \to Y$ is surjective.

- For another example, consider the functions $B_4 \to B_4$. Which of them are injective, surjective, bijective?

We also note the following fact, which is very useful. We do not give a proof.

Theorem 2.3. *Let X be a set with n elements and let Y be a set with m elements (for n and m positive integers). Let $f\colon X \to Y$ be a function.*

1. *If $f\colon X \to Y$ is injective, then $n \leq m$.*
2. *If $f\colon X \to Y$ is surjective, then $n \geq m$.*
3. *If $f\colon X \to Y$ is bijective, then $n = m$.*

The contrapositive of the first item in the previous theorem commonly referred to as the Pigeonhole principle. It is illustrated by imagining n pigeons trying to sleep in m "pigeonholes" (aka birdhouses). If $n > m$, then there will always be a pigeonhole with at least two pigeons in it.

Theorem 2.4 (The Pigeonhole principle)**.** *Let X be a set with n elements and let Y be a set with m elements (for n and m positive integers). Let $f\colon X \to Y$ be a function.*

If $n > m$, then there exists $y \in Y$ such that y is the image of at least two distinct elements in X.

2.1 Exercises

Lakins, 5.3.1, 5.3.2, 5.3.3, 5.3.4, 5.3.6, 5.3.7.

3 Function composition

Definition 3.1 (Lakins, Definition 5.2.1)**.** Let $f\colon X \to Y$ and $g\colon A \to B$ be functions such that $Y \subseteq A$, i.e., the codomain of f is a subset of the domain of g. The *composition of $f\colon X \to Y$ and $g\colon A \to B$* is the function with domain X and codomain B that sends $x \in X$ to $g(f(x)) \in B$.

We denote this function with a circle, $g \circ f\colon X \to B$. Note that the order of f and g in the notation $g \circ f$ matters. By definition, $g \circ f(x) = g(f(x))$.

Example 3.2. Consider the function $f\colon [0, \infty) \to [0, \infty)$ with $f(x) = \sqrt{x}$. We can take the composition $f \circ f\colon [0, \infty) \to [0, \infty)$. What is $f(f(8))$? What is $f(f(4))$?

Example 3.3. Consider the function $Low\colon \mathcal{A} \to \mathcal{A}$ that converts a letter to lower case. What is $Low \circ Cap(\text{“}a\text{”})$? What is $Cap \circ Low(\text{“}a\text{”})$? What is interesting about these two calculations in comparison to each other?

Example 3.4. Consider the function $g\colon \mathbb{R} \to \mathbb{R}$ with $g(x) = x^2$ and $f\colon [0, \infty) \to [0, \infty)$. Can we compose $g \circ f$? What about $f \circ g$? If we cannot take one of these compositions, can we modify $g\colon \mathbb{R} \to \mathbb{R}$ so that we can?

Example 3.5. Consider the "bit flip" function $n\colon B_4 \to B_4$. What is $n \circ n$?

The function resulting from the composition in the previous example is known as the "identity function" on B_4. In general, the *identity function* on a set X is the function $I_X : X \to X$ defined by the correspondence $I_X(x) = x$. It maps every x *identically* to itself. As the name suggests, the identity function satisfies some identities.

Theorem 3.6 (Lakins, Proposition 5.2.5 (2))**.** *For any function $f\colon X \to Y$, $f \circ I_X = f = I_Y \circ f$.*

Note that two functions are equal when their domains, codomains, and correspondences are equal. In the statement above, we write $f \circ I_X = f$ but what we mean is that the functions $f \circ I_X : X \to Y$ and $f\colon X \to Y$ are equal.

Function composition is related to the properties of injective, surjective, and bijective.

Theorem 3.7 (Lakins, Theorem 5.3.10)**.** *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions.*

1. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are both injective, then $g \circ f\colon X \to Z$ is injective.*
2. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are both surjective, then $g \circ f\colon X \to Z$ is surjective.*
3. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are both bijective, then $g \circ f\colon X \to Z$ is bijective.*
4. *If $g \circ f\colon X \to Z$ is injective, then $f\colon X \to Y$ is injective.*
5. *If $g \circ f\colon X \to Z$ is surjective, then $g\colon Y \to Z$ is surjective.*

Example 3.8. Consider $Low\colon \mathcal{A} \to \mathcal{L}$ and $Cap\colon \mathcal{L} \to \mathcal{A}$. Then $Low \circ Cap\colon \mathcal{L} \to \mathcal{L}$ is injective. In agreement with item 4 from the theorem, $Cap\colon \mathcal{L} \to \mathcal{A}$ is injective (why?). However, $Low\colon \mathcal{A} \to \mathcal{L}$ is not injective (why?)!

Example 3.9. Consider $r\colon B_4 \to B_4$ (right shift) and $t\colon B_4 \to B_3$. Then $t \circ r\colon B_4 \to B_3$ is surjective (why?). In agreement with item 5 from the theorem, $t\colon B_4 \to B_3$ is surjective. But $r\colon B_4 \to B_4$ is not surjective (why?).

3.1 Exercises

Lakins, 5.2.1, 5.2.2, 5.2.3.

4 Inverse functions

Definition 4.1 (Lakins, Defintion 5.4.1)**.** Let $f\colon X \to Y$ be a function. A function $g\colon Y \to X$ is the *inverse (function)* of $f\colon X \to Y$ if for all $x \in X$ and $y \in Y$,

$$y = f(x) \qquad \text{if and only if} \qquad x = g(y).$$

Proposition 4.2 (Lakins, Proposition 5.4.3). *Let $f: X \to Y$ and $g: Y \to X$ be functions. Then f is invertible and g is the inverse of f if and only if*

$$g \circ f = I_X \qquad \textit{and} \qquad f \circ g = I_Y.$$

Note that:

- Being inverse depends on the domain and codomain, not just the correspondence.
- Being inverse is symmetric: $g: Y \to X$ is the inverse of $f: X \to Y$ if and only if $f: X \to Y$ is the inverse of $g: Y \to X$. (See exercise 5.4.3)
- Not every function $f: X \to Y$ has an inverse. See the capitalization example when the domain is $\mathcal{A}$.
- If $f: X \to Y$ has an inverse, then the inverse is unique (Lakins, Theorem 5.4.7 (2)). The inverse function of f is sometimes denoted f^{-1}, but one should be careful not to confuse this with similar notation for other inverses (such as multiplicative inverse). For instance, f^{-1} does not mean $\frac{1}{f}$.
- If $f: X \to Y$ has an inverse, then we say that it is *invertible*.

Example 4.3. $t \circ a: B_3 \to B_3$ is the identity function $I_{B_3}: B_3 \to B_3$. However, $a \circ t: B_4 \to B_4$ is not the identity function $I_{B_4}: B_4 \to B_4$ (why?). Thus, t and a are not inverses of each other.

Example 4.4. The function $Low: \mathcal{U} \to \mathcal{L}$ is the inverse of $Cap: \mathcal{L} \to \mathcal{U}$ (why?). However, $Low: \mathcal{A} \to \mathcal{A}$ is not the inverse of $Cap: \mathcal{A} \to \mathcal{A}$ (why?).

Example 4.5. The function $g: [0,\infty) \to [0,\infty)$, $g(x) = x^2$, is the inverse of $f: [0,\infty) \to [0,\infty)$, $f(x) = \sqrt{x}$ (why?). However, $g: \mathbb{R} \to \mathbb{R}$ is NOT the inverse of $f: [0,\infty) \to [0,\infty)$ (why?).

Example 4.6. The function $n: B_4 \to B_4$ is the inverse of itself (why?). For example, $n(n(1010)) = n(0101) = 1010 = I_{B_4}(1010)$.

Finally, we see that being invertible is the same as being a bijection.

Theorem 4.7 (Lakins, Theorem 5.4.7 (1)). *Let X and Y be sets and let $f: X \to Y$ be a function. Then, $f: X \to Y$ is invertible if and only if $f: X \to Y$ is a bijection.*

Proof. See Lakins, page 117. The proof uses 4 and 5 from the previous theorem. □

This theorem has some straightforward but very useful consequences. For instance:

- If $f: X \to Y$ is not injective, then $f: X \to Y$ is not invertible.

- If $f\colon X \to Y$ is not surjective, then $f\colon X \to Y$ is not invertible.
- If $f\colon X \to Y$ is invertible, then $f\colon X \to Y$ is injective and surjective.
- On the other hand, if we know $f\colon X \to Y$ is not invertible, then $f\colon X \to Y$ is not injective or $f\colon X \to Y$ is not surjective.

4.1 Exercises

Lakins, 5.4.1, 5.4.2, 5.4.3, 5.4.8.