M1C03 Lecture 12

Proofs with Integers and Real Numbers

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Announcement(s)

1. Test 1 is October 29. Details are forthcoming.
2. To prepare for Test 1 you should be doing exercises.
3. Quiz 4 due Friday.

Overview

- Square root
- Proof of uniqueness
- Proof by contradiction
- Absolute value

Reference: Lakins, chapter 2.

For all integers a, b, and c,

(Closure under $+$ and $\cdot$)	$a+b$ and ab are integers.
(Associativity)	$(a+b)+c=a+(b+c)$ and $(ab)c=a(bc)$.
(Commutativity)	$a+b=b+a$ and $ab=ba$.
(Distributivity)	$a(b+c)=ab+ac$.
(Identities)	$0\neq 1$, $a+0=a$, $a\cdot 1=a$, and $a\cdot 0=0$.
(Additive inverses)	There is a unique integer $-a=(-1)\cdot a$ such that $a+(-a)=0$.
(Subtraction)	$a-b$ is defined to be $a+(-b)$.
(No divisors of 0)	If $ab=0$, then $a=0$ or $b=0$.
(Cancellation)	If $ab=ac$ and $a\neq 0$, then $b=c$.
(Transitive property of $<$)	If $a<b$ and $b<c$, then $a<c$.
(Trichotomy)	Exactly one of $a<b$, $a=b$, or $a>b$ holds.
(Order property 1)	If $a<b$, then $a+c<b+c$.
(Order property 2)	If $c>0$, then $a<b$ iff $ac<bc$.
(Order property 3)	If $c<0$, then $a<b$ iff $ac>bc$.

Basic Properties of Real Numbers

For all real numbers a, b, and c,

(Closure under $+$ and $\cdot$)	$a+b$ and ab are real numbers.
(Associativity)	$(a+b)+c = a+(b+c)$ and $(ab)c = a(bc)$.
(Commutativity)	$a+b = b+a$ and $ab = ba$.
(Distributivity)	$a(b+c) = ab+ac$.
(Identities)	$0 \neq 1$, $a+0 = a$, $a \cdot 1 = a$, and $a \cdot 0 = 0$.
(Additive inverses)	There is a unique real number $-a = (-1) \cdot a$ such that $a + (-a) = 0$.
(Subtraction)	$a-b$ is defined to be $a+(-b)$.
(Multiplicative inverses)	If $a \neq 0$, then there is a unique real number $a^{-1} = \frac{1}{a}$ such that $a \cdot a^{-1} = 1$.
(Division)	When $a \neq 0$, $\frac{b}{a}$ is defined to be $b \cdot a^{-1}$.
(No divisors of 0)	If $ab = 0$, then $a = 0$ or $b = 0$.
(Cancellation)	If $ab = ac$ and $a \neq 0$, then $b = c$.
(Transitive property of $<$)	If $a < b$ and $b < c$, then $a < c$.
(Trichotomy)	Exactly one of $a < b$, $a = b$, or $a > b$ holds.
(Order property 1)	If $a < b$, then $a + c < b + c$.
(Order property 2)	If $c > 0$, then $a < b$ iff $ac < bc$.
(Order property 3)	If $c < 0$, then $a < b$ iff $ac > bc$.

Uniqueness

Theorem (1)

$(-1)^2 = 1.$

Roots

Theorem (2)

For every real number a with $a \geq 0$, there exists a unique non-negative real number x such that $x^2 = a$.

WARNING!!!!

Existence

Theorem (2.1)

For every real number a with $a \geq 0$, there exists a non-negative real number x such that $x^2 = a$.

Uniqueness (and a proof by contradiction)

Theorem (2.2)

For every real number a with $a \geq 0$, if x and z are non-negative real numbers with $x^2 = a$ and $z^2 = a$, then $x = z$.

Uniqueness (and a proof by contradiction)

Theorem (2.2')

For every real number a with $a > 0$, if $x > 0$ and $z > 0$ are real numbers with $x^2 = a$ and $z^2 = a$, then $x = z$.

Uniqueness (and a proof by contradiction)

Uniqueness (and a proof by contradiction)

Absolute value

Definition: The *absolute value* of a real number x, denoted $|x|$ is defined to be

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Absolute value

Theorem (3)

1. *For all real numbers* x, $|x|^2 = x^2$.
2. *For all real numbers* x, $|x| = \sqrt{x^2}$.

Absolute value

Absolute value

Theorem

For all real numbers a, b,

1. *If $a \geq 0$, $|b| \leq a$ if and only if $-a \leq b \leq a$.*
2. *If $a \geq 0$, $|b| = a$ if and only if $b = a$ or $b = -a$.*
3. $|a|^2 = a^2$.
4. $|a| = \sqrt{a^2}$.
5. $|a \cdot b| = |a||b|$.
6. *If $|a| = |b|$, then $a = b$ or $a = -b$.*
7. *(Triangle inequality)* $|a + b| \leq |a| + |b|$.
8. $|a + b| = |a| + |b|$ *if and only if a and b have the same sign.*
9. *(Reverse triangle inequality)* $||a| - |b|| \leq |a - b|$

Absolute value