

M1C03 Lecture 21

More set operations

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Announcement(s)

- ① Assignment 3 due Friday.
- ② Quiz 3 due Friday.
- ③ Test 1 solutions are posted.

Basic identities.

Associative and distributive properties (and non-property)

Finite intersections and unions.

Reference: Lakins, 4.2, 4.3.

Definitions Recap

Let A and B be sets.

$A \subseteq B$ if $(\forall x)(x \in A \implies x \in B)$.

$A = B$ if $(\forall x)(x \in A \iff x \in B)$. Equivalently, $A \subseteq B$ and $B \subseteq A$.

The *union of A and B* is $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

The *intersection of A and B* is $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

The *complement of A in B* is $B - A = \{x \mid x \in B \text{ and } x \notin A\}$.

The *complement of A* (in the universe \mathcal{U}) is $\overline{A} = \mathcal{U} - A = \{x \mid x \notin A\}$.

The *Cartesian product of A and B* is $A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$.

Example

Let $A = \{1\}$, $B = \{2\}$, $C = \{3\}$, $D = \{4\}$. Draw the following sets.

- $(A \times B) \cup (C \times D)$.
- $(A \cup C) \times (B \cup D)$.

Theorem (Lakins, Theorem 4.2.9)

Let A , B , C , and D be sets. Then:

- ① $A \times \emptyset = \emptyset = \emptyset \times A$.
- ② (\times distributes over \cup) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
- ③ (\times distributes over \cap) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
- ④ (\times commutes with \cap) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
- ⑤ $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

Theorem (cf. Lakins, Theorem 4.2.6)

Let A and B be subsets of some universal set \mathcal{U} . Then:

① $A \cup A = A \cap A = A.$

② $A \cup \emptyset = A.$

③ $A \cap \emptyset = \emptyset.$

④ $A \cup \overline{A} = \mathcal{U}.$

⑤ $A \cap \overline{A} = \emptyset.$

⑥ $A \cap B \subseteq A \subseteq A \cup B.$

⑦ $\overline{\overline{A}} = A.$

“Order of operations” properties

Theorem (cf. Lakins, Theorem 4.2.6)

Let A , B , and C be subsets of some universal set \mathcal{U} . Then:

- ① (\cup is associative) $A \cup (B \cup C) = (A \cup B) \cup C$.
- ② (\cap is associative) $A \cap (B \cap C) = (A \cap B) \cap C$.
- ③ (\cap distributes over \cup) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- ④ (\cup distributes over \cap) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
- ⑤ (de Morgan 1) $\overline{A \cap B} = \overline{A} \cup \overline{B}$.
- ⑥ (de Morgan 2) $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Finite unions and intersections

Let $n \geq 1$ and let A_1, \dots, A_n be sets.

$$\bigcup_{i=1}^n A_i = A_1 \cup A_2 \cup \dots \cup A_n$$

$$\bigcap_{i=1}^n A_i = A_1 \cap A_2 \cap \dots \cap A_n$$

Example

Let $A_i = [2i, 2i + 1]$.

Theorem (cf. Lakins, Theorem 4.3.2)

For all positive integers $n \geq 1$, if A and B_1, \dots, B_n are sets, then:

① $(\cap \text{ distributes over finite } \cup)$

$$A \cap \left(\bigcup_{i=1}^n B_i \right) = \bigcup_{i=1}^n (A \cap B_i).$$

② $(\cup \text{ distributes over finite } \cap)$

$$A \cup \left(\bigcap_{i=1}^n B_i \right) = \bigcap_{i=1}^n (A \cup B_i).$$

Theorem (cf. Lakins, Theorem 4.3.2)

For all positive integers $n \geq 2$, if A_1, \dots, A_n are subsets of a universe \mathcal{U} , then:

❶ (finite de Morgan 1)

$$\overline{\left(\bigcup_{i=1}^n A_i\right)} = \bigcap_{i=1}^n \overline{A_i}.$$

❷ (finite de Morgan 2)

$$\overline{\left(\bigcap_{i=1}^n A_i\right)} = \bigcup_{i=1}^n \overline{A_i}.$$

Finite unions and intersections

Let $n \geq 1$ and let A_1, \dots, A_n be sets. Then,

$$\bigcup_{i=1}^n A_i = \{x \mid \text{exists } i \in \{1, \dots, n\}, \text{ such that } x \in A_i\}$$

and

$$\bigcap_{i=1}^n A_i = \{x \mid \text{for all } i \in \{1, \dots, n\}, x \in A_i\}.$$

