M1C03 Lecture 25

Function composition

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Nov 15, 2021

Announcement(s)

1. Quiz due Friday
2. Assignment 4 due Friday

Overview

Properties of function composition

Definition of function inverse

Reference: Notes on functions (Avenue), Lakins Chapter 5.

Recall

A *function* consists of three things:

- a set X called the *domain*,
- a set Y called the *codomain*, and
- a *correspondence* (or rule, or formula) that assigns to every element of the domain a unique element of the codomain.

A function $f\colon X \to Y$ is:

- *injective* if for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
- *surjective* if for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$.
- *bijective* if it is surjective and injective.

Function composition [Lakins, Definition 5.2.1]

Let $f\colon X \to Y$ and $g\colon A \to B$ be functions with $Y \subseteq A$.

The *composition of $f\colon X \to Y$ and $g\colon A \to B$* is the function

$$g \circ f\colon X \to B$$

defined by

$$g \circ f(x) = g(f(x)) \qquad \forall x \in X.$$

Functions on binary sequences

- $n\colon B_4 \to B_4$ bit flip.
- $r\colon B_4 \to B_4$ right shift.
- $l\colon B_4 \to B_4$ left shift.
- $t\colon B_4 \to B_3$ removes the leftmost digit.
- $a\colon B_3 \to B_4$ appends 0 as the leftmost digit.

Compute:

- $a \circ t(1010)$
- $t \circ a(111)$
- $r \circ l(1010)$
- $l \circ r(1111)$
- $r \circ (r \circ r)(1010)$
- $n \circ n(1010)$

The identity function

Definition: The *identity function* on a set X is the function $I_X \colon X \to X$ defined by

$$I_X(x) = x \qquad \forall x \in X.$$

Theorem (Lakins, Proposition 5.2.5 (2))

Let $f \colon X \to Y$ *be a function. Then*

$$f \circ I_X = f = I_Y \circ f$$

as functions from X *to* Y*.*

Function composition and injective/surjective properties

Theorem (Lakins, Theorem 5.3.10)

Let $f: X \to Y$ *and* $g: Y \to Z$ *be functions.*

1. *If* $f: X \to Y$ *and* $g: Y \to Z$ *are both injective, then* $g \circ f: X \to Z$ *is injective.*
2. *If* $f: X \to Y$ *and* $g: Y \to Z$ *are both surjective, then* $g \circ f: X \to Z$ *is surjective.*
3. *If* $f: X \to Y$ *and* $g: Y \to Z$ *are both bijective, then* $g \circ f: X \to Z$ *is bijective.*
4. *If* $g \circ f: X \to Z$ *is injective, then* $f: X \to Y$ *is injective.*
5. *If* $g \circ f: X \to Z$ *is surjective, then* $g: Y \to Z$ *is surjective.*

If $f\colon X \to Y$ and $g\colon Y \to Z$ are both surjective, then $g \circ f\colon X \to Z$ is surjective.

If $g \circ f \colon X \to Z$ is injective, then $f \colon X \to Y$ is injective.

Consider $Low\colon \mathcal{A} \to \mathcal{L}$ and $Cap\colon \mathcal{L} \to \mathcal{A}$. Then

- $Low \circ Cap\colon \mathcal{L} \to \mathcal{L}$ is injective.
- $Cap\colon \mathcal{L} \to \mathcal{A}$ is injective.
- $Low\colon \mathcal{A} \to \mathcal{L}$ is not injective.

Consider $r\colon B_4 \to B_4$ (right shift) and $t\colon B_4 \to B_3$. Then

- $t \circ r\colon B_4 \to B_3$ is surjective.
- $t\colon B_4 \to B_3$ is surjective.
- $r\colon B_4 \to B_4$ is not surjective.

Function inverse

Definition: $g\colon Y \to X$ is *inverse* to $f\colon X \to Y$ if:

$$\forall x \in X, y \in Y, \quad y = f(x) \iff g(y) = x.$$

- $g\colon Y \to X$ is inverse to $f\colon X \to Y$ if and only if $f\colon X \to Y$ is inverse to $g\colon Y \to X$.
- $f\colon X \to Y$ is *invertible* if there exists $g\colon Y \to X$ inverse to $f\colon X \to Y$.
- If $f\colon X \to Y$ is invertible, then the inverse is unique.

Example

- $Low\colon \mathcal{U} \to \mathcal{L}$ is the inverse of $Cap\colon \mathcal{L} \to \mathcal{U}$
- $Low\colon \mathcal{A} \to \mathcal{A}$ is not the inverse of $Cap\colon \mathcal{A} \to \mathcal{A}$

Function inverse

Definition: $g\colon Y \to X$ is *inverse* to $f\colon X \to Y$ if:

$$\forall x \in X, y \in Y, \quad y = f(x) \iff g(y) = x.$$

Proposition (Lakins, Proposition 5.4.3)

Let $f\colon X \to Y$ and $g\colon Y \to X$ be functions.

Then $g\colon Y \to X$ is the inverse of $f\colon X \to Y$ if and only if

$$g \circ f = I_X \qquad \text{and} \qquad f \circ g = I_Y\,.$$