M1C03 Lectures 9 and 10

Quantifiers

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Announcement(s)

1. Quiz 3 due Friday
2. Thursday lecture is cancelled.

Overview of this week

- Quantifiers
- Natural numbers, integer, rational numbers, real numbers, prime numbers
- Direct proofs and counter-examples involving quantifiers

Reference: Lakins, 1.1.2, 1.1.3, 1.2, 2.1.

Predicates, universes, and quantifiers

A *predicate* is a statement whose truth value depends on part of the statement that is variable. The set of possible values of the variable is the *universe*.

Example: $x + 1 > 3$. *This is a predicate, which we may denote* $P(x)$*.*

Predicates can be turned into propositions in several different ways.

Example: $3 + 1 > 3$. *This is* $P(3)$*.*

Example: For all x, $x + 1 > 3$. *This is the universal quantifier* $(\forall x)P(x)$*.*

Example: There exists x such that $x + 1 > 3$. *This is the existential quantifier* $(\exists x)P(x)$*.*

Exercise

Definition: An integer n is *even* if there exists an integer j such that $n = 2j$.

Definition: An integer n is *odd* if there exists an integer j such that $n = 2j + 1$.

Show that if m and n are even, then $m + n$ is even.

Rough work:

- *Understand the definition. Some examples might help. 6 is even because $6 = 2 \cdot 3$. 5 is not even because it cannot be written as $2 \cdot j$ for an integer j.*
- *Given: $(\exists j)(m = 2j)$, $(\exists j)(n = 2j)$. Probably the j that satisfies the condition for m and n is different. Let's call the second one k instead.*
- *Goal: Show that $m + n$ can be written as $2 \cdot j$ for some integer j.*
- $m + n = 2j + 2k = 2(j + k)$

Exercise

Definition: An integer n is *even* if there exists an integer j such that $n = 2j$.

Definition: An integer n is *odd* if there exists an integer j such that $n = 2j + 1$.

Show that if m and n are even, then $m + n$ is even.

Proof.

Assume that m and n are even integers.

We will show from the definition that $m + n$ is even.

Since m is even, we can fix an integer j such that $m = 2j$.

Similarly, since n is even, we can fix an integer k such that $n = 2k$.

Then,

$$m + n = 2j + 2k = 2(j + k)$$

where we have used the distributive property of multiplication in the second step.

Thus, we have shown that $m + n = 2(j + k)$, so $m + n$ is even. $\square$

Exercise

Definition: A function $f\colon \mathbb{R} \to \mathbb{R}$ is *strictly increasing* if for all $a, b \in \mathbb{R}$, if $b > a$, then $f(b) > f(a)$.

Show that $f(x) = x^3$ is strictly increasing.

Rough work:

- *Understand the definition. Draw the picture of what a strictly increasing function looks like and think about how the quantifier and the implication in the definition is related to the shape of the graph.*
- *We will need to show that $b^3 > a^3$ when $b > a$. We can't use derivatives. Let's try basic algebra.*
- *We remember that $b^3 - a^3 = (b-a)(b^2 + ab + a^2)$ (difference of cubes). First term is > 0. Remains to explain why second term is > 0.*
- *Second term is $b^2 + ab + a^2 = \frac{(b+a)^2}{2} + \frac{b^2+a^2}{2}$ (check for yourself).*
- *We know $x^2 \geq 0$ with equality if and only if $x = 0$ (Prove it yourself: Lakins Exercise 2.1.6)*
- *Since a and b cannot both be 0, $b^2 + a^2 > 0$*

Exercise

Show that $f(x) = x^3$ is strictly increasing.

Proof.

We show that $f(x) = x^3$ is strictly increasing from the definition. Fix a, b arbitrary real numbers. Assume that $b > a$. We want to show that $b^3 > a^3$.

By the difference of cubes formula and some basic algebra, we have

$$b^3 - a^3 = (b-a)(b^2 + ab + a^2) = (b-a)\left(\frac{(b+a)^2}{2} + \frac{b^2 + a^2}{2}\right).$$

We show that the expression on the right is > 0. We have $b - a > 0$ by assumption, so it will suffice to show that the second term is also positive. For the second term,

$$\frac{(b+a)^2}{2} + \frac{b^2 + a^2}{2} \geq \frac{1}{2}(a^2 + b^2)$$

since $x^2 \geq 0$ for any number x. Finally, since $x^2 = 0$ if and only if $x = 0$ and a and b cannot both be 0, we have that

$$\frac{1}{2}(a^2 + b^2) > 0.$$

Putting this all together, we have shown that $b^3 > a^3$. Thus f is strictly increasing. $\square$

Proof by cases

In the previous proof we used some algebra trickery to show that the following statement is true.

If $b > a$, then $b^2 + ab + a^2 > 0$.

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Exercise

Definition: An integer n is *divisible* by a non-zero integer m if there exists an integer k such that $n = km$. We write $m \mid n$ and say that m is a *divisor* of n.

Show that if n is divisible by m, then n^2 is divisible by m^2.

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Negating statements with quantifiers

Negate the following statements.

- n is even.

- n is divisible by m.

- f is strictly increasing.

Negating statements with quantifiers

Theorem

For any predicate $P(x)$,

- $\neg(\forall x)P(x)$ *is logically equivalent to* $(\exists x)\neg P(x)$.
- $\neg(\exists x)P(x)$ *is logically equivalent to* $(\forall x)\neg P(x)$.

Exercise

Definition: A function $f\colon \mathbb{R} \to \mathbb{R}$ is *strictly increasing* if for all $a, b \in \mathbb{R}$, if $b > a$, then $f(b) > f(a)$.

Show that $f(x) = x^2 - 1$ is not strictly increasing.

Exercise

Definition: A function $f\colon \mathbb{R} \to \mathbb{R}$ is *strictly increasing* if for all $a, b \in \mathbb{R}$, if $b > a$, then $f(b) > f(a)$.

Show that $f(x) = x^2 - 1$ is not strictly increasing.

Exercise

Definition: An integer n is *divisible* by a non-zero integer m if there exists an integer k such that $n = km$. We write $m \mid n$ and say that m is a *divisor* of n.

Show that the following statement is false.

For all positive integers p, n, and m, if $p \mid nm$, then $p \mid n$ or $p \mid m$.

Exercise

Definition: An integer n is ***divisible*** by a non-zero integer m if there exists an integer k such that $n = km$. We write $m \mid n$ and say that m is a ***divisor*** of n.

Show that the following statement is false.

For all positive integers p, n, and m, if $p \mid nm$, then $p \mid n$ or $p \mid m$.

Summary

- In math, Defintions are always biconditional statements (even though we only write "if"). (Lakins, pp. 18)
- If we want to prove something satisfies a definition, and all we know is the definition, then we must give a proof using the definition. (Lakins, pp. 21)
- When using a statement of the form $(\exists x)P(x)$ in a proof, we *fix* x such that $P(x)$ (this is called *existential instantiation*). (Lakins, pp. 21)
- When proving a statement of the form $(\forall x)P(x)$, we begin by *fixing* an *arbitrary* x.
- Negate a statement involving quantifiers by switching $\exists/\forall$ and negating predicates (be careful, more on this next week).
- One way to show a statement is false is to prove its negation is true.
- Showing a universal statement $(\forall x)P(x)$ is false by finding an x such that $\neg P(x)$ (and thus proving the negation $(\exists x)\neg P(x)$ is true) is called *proof by counterexample*. (Lakins, 2.1.1)
- To give a *proof by cases*, you must find a way to enumerate all possible cases AND give a valid proof for each case. (Lakins, 2.1.2)