M1C03 Lecture 2

What is math?

Jeremy Lane

Sept 9, 2021

Overview

Today:

1. What is math? Propositions, predicates, and deductive reasoning. *Lakins, p. 1–2*.
2. Euclidean geometry. *Geometry notes*.

Some other references:

1. Mathematician comes to defence of TikTok teen blasted for saying math isn't real (link)
2. BBC In Our Time podcast: Euclid's Elements (link).

What is math?

Math uses deductive reasoning to justify claims about mathematical objects.

- Lakins, page 1, roughly paraphrasing.

Let's unpack this:

1. mathematical objects: numbers, shapes, etc.
2. claims: statements that we hope to show are true (or false)
3. deductive reasoning: using logic to make conclusions based on given assumptions

Example

1. $n + 1$.
2. $a^2 + b^2 = c^2$.
3. $3 < \pi$.

A *statement* is a declarative sentence with a subject and a verb.

A *proposition* is a statement that is either true or false (but not both).

A *predicate* is a statement whose truth value depends on parts of the statement that are variable. Something is variable if it may take different values/meanings, depending on the context.

Which of the examples above are propositions, predicates, or neither? It helps to read the sentence out loud or write it in words.

Example

Which of the following examples are propositions, predicates, or neither?

1. James Cordon is taller than Taylor Swift.
2. We are all taller than James Cordon.
3. Do you like James Cordon?

Example

We saw that the following statement is a predicate.

$$a^2 + b^2 = c^2.$$

How can we turn this predicate into a proposition?

If c is the hypotenuse of a right triangle and a and b are the other two sides, then

$$a^2 + b^2 = c^2.$$

There is more than one way!

There equation $a^2 + b^2 = c^2$ has infinitely many solutions where a, b, c are all integers.

Common Terminology

A proposition whose truth value is not known yet is a *claim* or *conjecture*. Often we use claim or conjecture to refer to propositions that we expect to be true (whereas we might not use them in reference to propositions we expect are false).

A proposition that is known to be true is a *theorem*. Depending on the relative importance of a theorem, or its role in the current setting, we sometimes use other words such as *lemma*, *corollary*, and *proposition*.

WARNING!!!

Different types of reasoning

Example

1. We checked that $a^2 + b^2 = c^2$ for 1000 different right triangles, so $a^2 + b^2 = c^2$ for any right triangle.
2. Using polling data, pollsters predict that the most likely outcome of the upcoming Canadian federal election is a Liberal minority government.
3. We are shown two doors and told that a prize is behind one of them. We check the door on the left and find that there is no prize. By process of elimination, we conclude the prize is behind the door on the right.

Deductive reasoning starts from a list of facts that are accepted as true (*axioms* or *assumptions*) and combines them with logic to reach conclusions. A conclusion reached by deductive reasoning is *always true*, as long as the assumptions are true.

By comparison, *inductive reasoning* provides an argument, typically with supporting evidence, for why a claim is true. Conclusions reached by inductive reasoning might not be true.

Which of the examples above are deductive reasoning?

The goal of Euclidean geometry is to use *deductive reasoning* to justify *claims* about *geometric objects* (lines, circles, triangles, angles, etc.).

Euclidean geometry starts with a list of assumptions. They are divided into two categories:

- *Common notions:* assumptions we agree on about how quantities work.
- *Postulates:* geometric constructions that we agree are possible.

Common notions

[A1] Things that are equal to the same thing are equal to each other.

[A2] If equals are added to equals, the sums are equal.

[A3] If equals are subtracted from from equals, the remainders are equal.

[A4] If equals are added to unequals, the sums are unequal. The greater sum is obtained from the greater unequal.

[A5] If equals are taken from unequals, the remainders are unequal. The greater remainder is obtained from the greater unequal.

[A6] If equals are doubled, the results are equal.

[A7] If equals are halved, the results are equal.

[A8] If the whole consists of more than one part, then the whole is greater than each of its parts and is equal to the sum of its parts.

[A9] Things that can be made to coincide with one another are equal to one another.

Example

Claim: If two straight lines intersect, the opposite angles are equal.

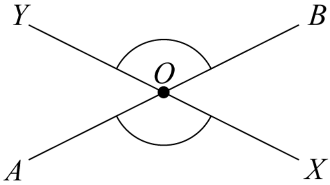

Figure 1: Angles ∠YOB and ∠AOX are opposite, as are ∠AOY and ∠BOX.

Let's show that $\angle AOY$ equals $\angle BOX$.

1. AOB and YOX are both straight lines, so $\angle AOB = \angle YOX$.
2. $\angle AOB = \angle AOY + \angle YOB$.
3. $\angle YOX = \angle YOB + \angle BOX$.
4. Combining 1-3, we have that

$$\angle AOY + \angle YOB = \angle YOB + \angle BOX.$$

5. Subtracting $\angle YOB$ from both sums, we are left with

$$\angle AOY = \angle BOX.$$

Example

Claim: If two straight lines intersect, the opposite angles are equal.

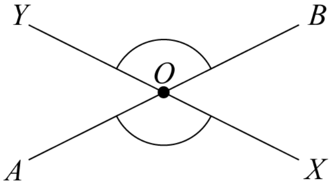

Figure 2: Angles ∠YOB and ∠AOX are opposite, as are ∠AOY and ∠BOX.

Let's show that $\angle YOB$ equals $\angle AOX$. *Try to justify each step with a common notion.*

1. AOB and YOX are both straight lines, so $\angle AOB = \angle YOX$.
2. *[A8]* $\angle AOB = \angle AOY + \angle YOB$.
3. *[A8]* $\angle YOX = \angle YOB + \angle BOX$.
4. *[A1]* Combining 1-3, we have that

$$\angle AOY + \angle YOB = \angle YOB + \angle BOX.$$

5. *[A3]* Subtracting $\angle YOB$ from both sums, we are left with

$$\angle AOY = \angle BOX.$$