Graph genus and surfaces, part I

MATH 3V03

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Abstract

The following is the first in a series of notes on graph genus. These were written while I was teaching MATH 3V03: Graph Theory at McMaster University in Fall, 2019. These notes are presented as-is and may contain errors. The textbook referred to in these notes is Pearls in Graph Theory by Ringel and Hartsfield (henceforth "Pearls"). These notes were written because I wanted to present the topological aspect of graph genus in more detail than in Pearls.

1 Preface

The primary goal of these notes is to introduce graph genus from a topological perspective. In Pearls graph genus is described via a technical gadget called a "rotation" which is useful for computations and proofs involving graph genus but, in my opinion, does not impart the same level of intuition and conceptual understand as the topological definition. Moreover, this topic is an excellent excuse to give a first exposure to combinatorial topology which is an active and interesting area of research that relates naturally to the focus of this course and thus felt like an appropriate capstone.

We give an undergraduate-level introduction to the topology of oriented surfaces from a combinatorial perspective (i.e., taking for granted the fact that every closed surface can be triangulated) including a combinatorial definition of orientation of a triangulated surface, Euler characteristic of a closed surface, and the classification of closed surfaces. We then discuss embedding graphs in surfaces and give the topological definition of graph genus and show that it is equivalent to the definition via rotations (the construction of a rotation from a graph embedding involves handles on surfaces, so we cannot give the details but the idea is quite simple). As a capstone, we conclude that a graph has genus 0 if and only if it has genus 0 (giving a proof using stereographic projection). The course ended shortly after these notes by covering the Heawood theorem for colouring graphs with non-zero genus (this material is at the end of Pearls and is not repeated here).

2 Introduction

The four-colour theorem is one of the most well-known results in graph theory: the vertices of a planar graph can be coloured with at most 4 colours so that no two adjacent vertices have the same colour. But what can we say about colourings of graphs that aren't planar?

To prove colour theorems for planar graphs, we used Euler's formula,

$$|V| - |E| + |F| = 2.$$

Are there "Euler's formulas" for those graphs? In fact, there are "Euler's formulas" for non-planar graphs as well, but they are slightly different because we need to account for the graph's "genus." As we will see at the end of the course, these Euler formulas allow us to prove "colour theorems" for non-planar graphs!

In order to understand the Euler formula for non-planar graphs, we need to understand "rotations" which are defined in Pearls. Although rotations - as described in Pearls - sound like a very formal gadget, they actually have a visual interpretation. In order to make more sense of rotations and Eulers formula, we introduce surfaces and graph embeddings on surfaces. To this end, we give a completely elementary description of closed surfaces, sweeping all the details from topology under the rug.

We start by recalling rotations from Pearls in Sections 2 and 3. In the remaining sections we cover topology of surfaces, triangulations, and graph embeddings. In Section 10 we give our alternate definition of graph genus (using embeddings rather than rotations) and outline the proof that the definitions are equivalent.

3 The Euler characteristic and genus of a graph

This section summarizes Pearls, Section 10.1.

Definition 3.1. A *rotation* of a vertex v in a graph is a cyclic ordering of the edges incident to v. A rotation of every vertex in a graph G defines a *rotation* of G.

A vertex of degree d has $(d-1)!$ possible rotations. For example, K_8 has $(6!)^8$ different rotations.

In a drawing of a graph, two possible rotations of a vertex can be defined by specifying clockwise or counter-clockwise. (these are only two of many possible rotations of v) A rotation of G can be defined by labelling all the vertices in a drawing as black (clockwise) or white (counterclockwise).

The idea of rotations is that they are "directions" that tell us how to walk around the graph. Suppose you are walking and you approach vertex v along an edge e. The rotation of v tells you what the next edge after e in the cyclic order is. Let's say the next edge is e'. The directions tell us to turn onto edge e' at vertex v and continue along edge e'.

If we start on a particular edge going in a particular direction and we follow the directions indicated by a rotation, then our walk will eventually return us to that same edge, going in the same direction (because the graph is finite). In this way, the rotation of G determines a circuit through e, travelling in a given direction.

Example 3.2. See the examples at the beginning of Pearls, Section 10.1.

Note that since travelling along e in the opposite direction also determines a circuit, every edge in G occurs exactly twice in all the circuits defined by a rotation. Either e occurs once in two different circuits, or it occurs twice in one circuit.

Definition 3.3. The total number of circuits defined by a rotation ρ is the *rotation number*, which we denote $r(\rho)$.

The examples in Pearls, Section 10.1 show that different rotations of a graph may have different rotation numbers.

The following theorem will allow us to define "Euler equations" for arbitrary graphs.

Theorem 3.4 (Main theorem for rotations)**.** *For any rotation ρ of a connected graph G,*

$$|V| - |E| + r(\rho) \leq 2.$$

Moreover, $|V| - |E| + r(\rho)$ is even.

Proof. See Pearls, page 214. □

Definition 3.5. The *Euler characteristic* of a connected graph G is

$$X(G) = \max\{|V| - |E| + r(\rho)\}.$$

The Euler characteristic of G is a graph invariant. By Theorem 3.4, $X(G)$ is even and ≤ 2. The possible values of $X(G)$ are

$$2, 0, -2, -4, -6, -8, \ldots$$

As with any other graph invariant, we would like to compute it. However, the set of all possible rotations on G may be very large, so computing $X(G)$ by brute force is a lengthy chore.

A rotation ρ of G is *maximal* if $r(\rho) \geq r(\rho')$ for any other rotation ρ'. Equivalently, ρ is maximal if and only if

$$X(G) = |V| - |E| + r(\rho).$$

This equation for maximal rotations is the generalized Euler's formula for G. If G is planar, then this formula reduces to the ordinary Euler formula for planar graphs, as the following theorem and its proof demonstrate.

Theorem 3.6. *If G is a connected planar graph, then $X(G) = 2$.*

Proof. Take any plane drawing of G and give every vertex the counterclockwise rotation. The circuits defined by this rotation are precisely walks along the boundaries of the faces of the plane drawing. Thus, for this rotation

$$r(\rho) = |F|$$

By Euler's formula for connected planar graphs,

$$|V| - |E| + r(\rho) = |V| - |E| + |F| = 2.$$

By Theorem 3.4, ρ is maximal, so

$$X(G) = |V| - |E| + r(\rho) = 2.$$ □

The converse of this theorem is true as well. We will see why later on.

Here are a couple useful theorems for identifying maximal rotations.

Theorem 3.7. *If every circuit induced by ρ has length 3, then ρ is a maximal rotation.*

Proof. If every circuit induced by ρ has length 3, then

$$3r(\rho) = \sum_{\text{circuit } c} \ell(c) = 2|E| \Longrightarrow r(\rho) = \frac{2|E|}{3}.$$

Assume for the sake of contradiction that ρ' is a rotation with $r(\rho') > r(\rho)$. The average length of circuits induced by ρ' is

$$\frac{\sum_{\text{circuit } c} \ell(c)}{r(\rho')} = \frac{2|E|}{r(\rho')} < \frac{2|E|}{2|E|/3} = 3.$$

The only way that the average circuit length can be less than 3 is if there exists circuits of length < 3. But every circuit in a finite simple graph has length at least 3 since there are no loops or multiple edges, so this is impossible. □

The following theorem has a similar proof.

Theorem 3.8. *If every circuit induced by ρ on a bipartite graph has length 4, then ρ is a maximal rotation.*

From Euler characteristic, we define another graph invariant called genus.

Definition 3.9. The *genus* of G is the number

$$\gamma(G) = \frac{2 - X(G)}{2}.$$

Since $X(G)$ is an even number, $\gamma(G)$ is an integer. Since $X(G) \leq 2$,

$$\gamma(G) = \frac{2 - X(G)}{2} \geq 0.$$

The possibles values of $\gamma(G)$ are

$$0, 1, 2, 3, 4, 5, \ldots$$

Theorem 3.6 can be rephrased to say that if G is planar then $\gamma(G) = 0$.

4 Some examples of rotations on K_5

Label the vertices of K_5 with the numbers 0 through 4. A rotation of each vertex is given by listing the other 4 vertices in a cyclic order. Here is table describing such a rotation:

$$\begin{array}{c|cccc}
0 & 1 & 3 & 2 & 4 \\
1 & 3 & 0 & 2 & 4 \\
2 & 1 & 4 & 0 & 3 \\
3 & 2 & 4 & 1 & 0 \\
4 & 1 & 0 & 3 & 2
\end{array}$$

The rule for reading a circuit from the table is as follows:

1. start by pick a directed edge ij that isn't already in one of the circuits you've found.
2. Look at row j and find the number k that comes after i.
3. The next edge in the circuit is jk.
4. Go back and repeat step 2 with jk. Stop when you get the same directed edge that you started with.

Doing this for the rotation of K_5, starting on 01 yields the circuit

$$01241302342043103214(01).$$

To explain how we got this:

1. Start on the edge 01.
2. In row 1 the number after 0 is 2. Turn onto edge 12.
3. In row 2 the number after 1 is 4. Turn onto edge 24.
4. etc.

This circuit has length $20 = 2|E|$, so in fact we have found all the circuits induced by the rotation. For this rotation,

$$|V| - |E| + r(\rho) = 5 - 10 + 1 = -4.$$

Here is another rotation of K_5:

$$\begin{array}{c|cccc} 0 & 1 & 3 & 4 & 2 \\ 1 & 2 & 4 & 0 & 3 \\ 2 & 3 & 0 & 1 & 4 \\ 3 & 4 & 1 & 2 & 0 \\ 4 & 0 & 2 & 3 & 1 \end{array}$$

It determines five circuits of length 4:

$$\begin{array}{cccc} 1 & 0 & 3 & 4 \\ 2 & 1 & 4 & 0 \\ 3 & 2 & 0 & 1 \\ 4 & 3 & 1 & 2 \\ 0 & 4 & 2 & 3 \end{array}$$

For this rotation,

$$|V| - |E| + r(\rho) = 5 - 10 + 5 = 0.$$

We will see later that this second rotation is in fact maximal. Currently, the only way we could determine this would be to compute $r(\rho)$ for all $(3!)^5 = 7776$ rotations of K_5 and observe that the maximal value is 0.

In general, there are no known efficient algorithms for computing genus of a graph; computing graph genus is roughly just as difficult as the travelling salesman problem, or determining whether a graph is Hamiltonian.