Graph genus and surfaces, part IV

MATH 3V03

October 10, 2021

Abstract

The following is the fourth and final note in a series of notes on graph genus. These were written while I was teaching MATH 3V03: Graph Theory at McMaster University in Fall, 2019. These notes are presented as-is and may contain errors. The textbook referred to in these notes is Pearls in Graph Theory by Ringel and Hartsfield (henceforth "Pearls"). These notes were written because I wanted to present the topological aspect of graph genus in more detail than in Pearls.

In part IV we start by introducing the classification of closed surfaces. We highlight the analogy between the classification problem for graphs (up to isomorphism) and the classification problem for surfaces and emphasize that the latter is solved whereas the former is not. Next we present the two definitions of graph genus (one in terms of graph embeddings in closed oriented surfaces and the other in terms of graph rotations). We prove that these definitions are equivalent. The key idea of this proof is to finally understand that a rotation of a graph defines an embedding into a closed oriented surface by gluing together polygons. The other direction of the proof is somewhat more technical and we are content to sketch the main idea (which uses handlebodies).

1 Classification of closed orientable surfaces

The best way to introduce the classification of closed orientable surfaces is by analogy and comparison with what we have learned about graphs. Let's recall some things we learned in the course already.

- We defined *graph isomorphisms*. Two graphs have the same structure if they are isomorphic. In other words, two isomorphic graphs are really the same graph, just represented in different ways.
- We introduced *graph invariants*. A graph invariant is any quantity associated to graphs with the property:

  $$\text{Two graphs are isomorphic} \Rightarrow \text{they have the same invariant.}$$

  We defined many different graph invariants.

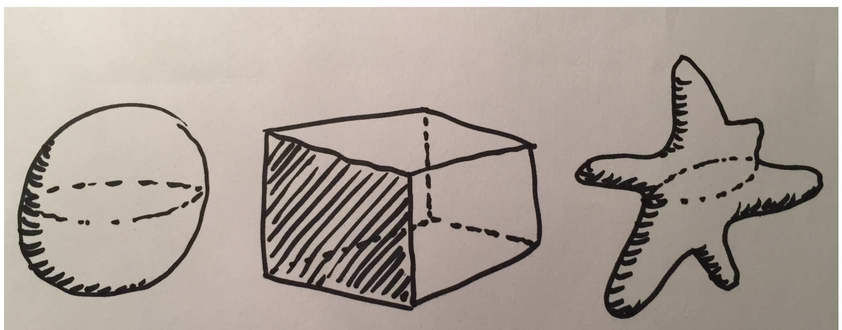
- We briefly discussed the *classification problem* for graphs: to somehow list all graphs up to isomorphism. One way to solve this problem would be to find a "master" graph invariant with the property that the converse of the statement above is true:

  Two graphs have the same master invariant $\Rightarrow$ they are isomorphic.

  No one has found a graph invariant with this property. In other words, no one has solved the classification problem for graphs.

There is a similar story for closed surfaces with a very different conclusion.

- Two closed surfaces are *homeomorphic* if there is a continuous bijection between them.



Three homeomorphic surfaces.

- Genus is a *surface invariant* for closed orientable surfaces. It has the property:

  Two closed orientable surfaces are homeomorphic $\Rightarrow$ they have the same genus.

  (this isn't hard to see: a homeomorphism will send a triangulation of one surface to a triangulation of the other surface without changing the number of vertices, edges, and triangles)

- The *classification problem* for closed orientable surfaces is to list all closed orientable surfaces up to homeomorphism. In fact, genus is a master invariant for closed orientable surfaces! The converse of the above statement is a theorem in topology:

  Two closed orientable surfaces have the same genus $\Rightarrow$ they are homeomorphic.

  In other words, $\Sigma_0 = S^2, \Sigma_1, \Sigma_2, \ldots$ is a complete list of closed orientable surfaces up to homeomorphism.

2 The definition of $\gamma(G)$ revisited

We now have two ways to define the genus of a graph.

Definition 1: $\gamma(G)$ equals the smallest number g such that there exists a rotation ρ on G with

$$|V| - |E| + r(\rho) = 2 - 2g.$$

Definition 2: $\gamma(G)$ equals the smallest number g such that there exists an embedding of G into Σ_g.

The two definitions are equivalent because of the following facts:

Theorem 2.1. 1. *If ρ is a rotation on G with*

$$|V| - |E| + r(\rho) = 2 - 2g,$$

then G embeds into Σ_g.

2. *If G embeds into Σ_g, then there is a rotation ρ on G such that*

$$|V| - |E| + r(\rho) = 2 - 2g',$$

for some $g' \leq g$.

Proof of 1. Suppose ρ is a rotation on G with

$$|V| - |E| + r(\rho) = 2 - 2g.$$

The rotation ρ is a construction manual that tells us how to glue oriented polygons to create a closed oriented surface: Each circuit c of length $\ell(c) = n$ corresponds to a polygon with n sides. The direction of the circuit determines the boundary orientation of the polygon. Polygons are glued along edges where the corresponding circuits pass each other going opposite directions. Since the circuits defined by ρ go in opposite directions along each edge, this is an oriented gluing. So it defines a closed oriented surface S. The graph G is embedded in S by construction.

Now we must compute the genus of the surface S that we just constructed. Let $v = |V|$ be the number of vertices, $e = |E|$ the number of edges, and $f = r(\rho)$ the number of polygons, or faces. Applying barycentric subdivision to a single polygon with n edges will: increase v by 1, increase e by n, and increase f by $n - 1$. Thus the quantity

$$v - e + f$$

is unchanged by any barycentric subdivision. Applying barycentric subdivision twice to all the polygons results in a triangulation so

$$2 - 2g = |V| - |E| + r(\rho) = v - e + f = 2 - 2\gamma(S).$$

Thus $\gamma(S) = g$.

By the classification of closed orientable surfaces, $\gamma(S) = g$ implies that S is homeomorphic to Σ_g. It follows that G embeds into Σ_g. □

Sketch of the proof of 2. Suppose that G embeds into Σ_g. Pick an orientation of Σ_g. Define a rotation ρ on G by giving every vertex the counterclockwise rotation with respect to the orientation of Σ_g and the embedding of G. The circuits on G induced by ρ are precisely the boundaries of the faces in the embedding of G in Σ_g.

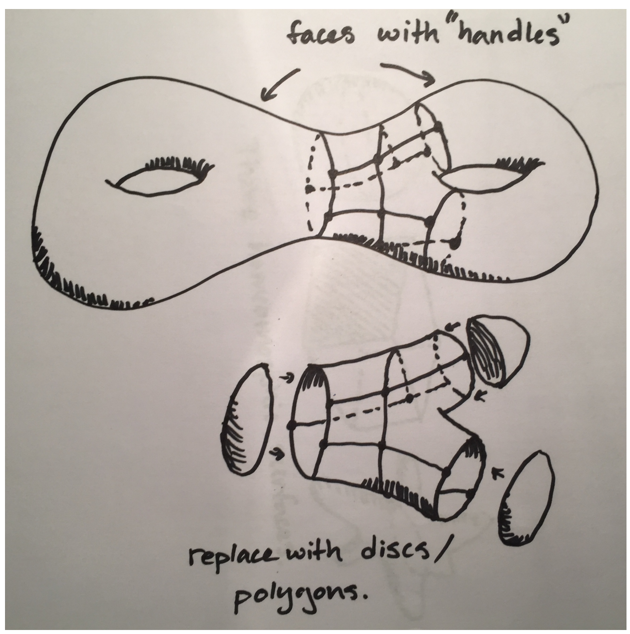
Now there are two possibilities. Either i) every face of the embedding is (up to homeomorphism) a polygon, or ii) there are some faces that are not polygons.

Case i) In this case, the same idea as in the proof of 1. (performing barycentric subdivisions to compute the genus) can be used to show that

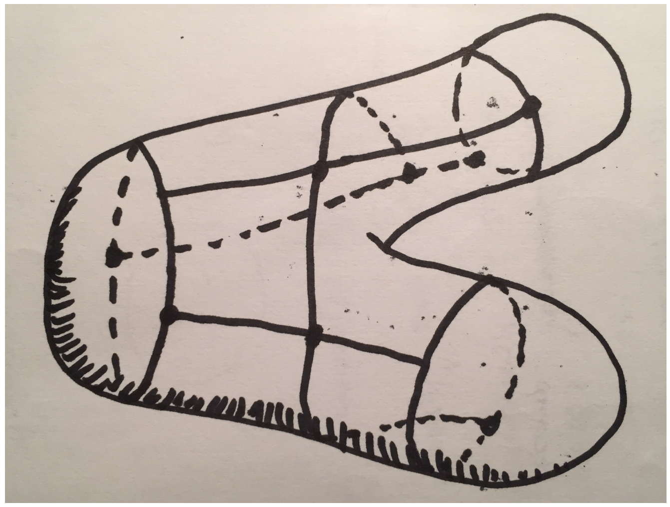
$$|V| - |E| + r(\rho) = 2 - 2g.$$

Case ii) Suppose there are faces of the embedding that are not polygons. In this case it is impossible to get a triangulation by performing barycentric subdivision of the faces.

With some topology, it is possible to show that faces that are not polygons must be "handles." See the image below which illustrates some things that can happen. This follows from a classification of "closed orientable surfaces with boundary" similar to the classification of closed oriented surfaces.



Modify Σ_g by replacing all the faces with "handles" with ordinary polygons. This creates a new surface S that comes with an embedding of G. The genus of S is $g' = g - k$ where k is the "number of handles" that were removed from Σ_g.



Now, using the same idea as in the proof of 1. we can define ρ via an orientation of S and show that

$$|V| - |E| + r(\rho) = 2 - 2g'.$$

$\square$

In particular, we get the following fact.

Corollary 2.2. *G is planar if and only if $\gamma(G) = 0$.*

Proof. Recall that we proved G is planar if and only if G embeds in the two-sphere, $\Sigma_0 = S^2$. By Definition 2, G embeds in $\Sigma_0 = S^2$ if and only if $\gamma(G) = 0$. $\square$

Example 2.3. Recall the rotation

$$\begin{array}{c|cccc} 0 & 1 & 3 & 4 & 2 \\ 1 & 2 & 4 & 0 & 3 \\ 2 & 3 & 0 & 1 & 4 \\ 3 & 4 & 1 & 2 & 0 \\ 4 & 0 & 2 & 3 & 1 \end{array}$$

on K_5. We computed that $r(\rho) = 5$, so

$$2 - 2\gamma(K_5) \geq |V| - |E| + r(\rho) = 5 - 10 + 5 = 0.$$

$$\Rightarrow 1 \geq \gamma(K_5).$$

Another way to conclude that

$$\Rightarrow 1 \geq \gamma(K_5).$$

is to use any of the examples of embeddings of K_5 into Σ_1.

On the other hand, we know that K_5 is not planar. By Corollary 9.1, it follows that

$$1 \leq \gamma(K_5).$$

Thus, we conclude that

$$1 = \gamma(K_5).$$

Example 2.4. We saw an embedding of $K_{3,3}$ into Σ_1, so

$$1 \geq \gamma(K_{3,3}).$$

On the other hand, we know that $K_{3,3}$ is not planar, so

$$1 \leq \gamma(K_{3,3}).$$

Thus $1 = \gamma(K_{3,3})$.

Example 2.5. We saw an embedding of K_7 into Σ_1, so

$$1 \geq \gamma(K_7).$$

On the other hand, we know that K_7 is not planar, so

$$1 \leq \gamma(K_7).$$

Thus $1 = \gamma(K_7)$.