M1C03 Lecture 13

Proof by Counterexample, Mixed Quantifiers, Limits

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Announcement(s)

1. Test 1 is October 29. See the Avenue announcement about time/location.
2. To prepare for Test 1 you should be doing exercises.
3. Quiz 4 is due Friday.
4. Assignment 2 will be posted tomorrow (due October 22).

Overview

- Proof by Counterexample
- Mixed quantifiers
- The limit definition

Reference: Lakins, Chapter 2.

Basic Properties of Integers

For all integers a, b, and c,

(Closure under $+$ and $\cdot$)	$a+b$ and ab are integers.
(Associativity)	$(a+b)+c = a+(b+c)$ and $(ab)c = a(bc)$.
(Commutativity)	$a+b = b+a$ and $ab = ba$.
(Distributivity)	$a(b+c) = ab+ac$.
(Identities)	$0 \neq 1$, $a+0 = a$, $a \cdot 1 = a$, and $a \cdot 0 = 0$.
(Additive inverses)	There is a unique integer $-a = (-1) \cdot a$ such that $a + (-a) = 0$.
(Subtraction)	$a-b$ is defined to be $a+(-b)$.
(No divisors of 0)	If $ab = 0$, then $a = 0$ or $b = 0$.
(Cancellation)	If $ab = ac$ and $a \neq 0$, then $b = c$.
(Transitive property of $<$)	If $a < b$ and $b < c$, then $a < c$.
(Trichotomy)	Exactly one of $a < b$, $a = b$, or $a > b$ holds.
(Order property 1)	If $a < b$, then $a + c < b + c$.
(Order property 2)	If $c > 0$, then $a < b$ iff $ac < bc$.
(Order property 3)	If $c < 0$, then $a < b$ iff $ac > bc$.

Basic Properties of Real Numbers

For all real numbers a, b, and c,

(Closure under $+$ and $\cdot$)	$a+b$ and ab are real numbers.
(Associativity)	$(a+b)+c = a+(b+c)$ and $(ab)c = a(bc)$.
(Commutativity)	$a+b = b+a$ and $ab = ba$.
(Distributivity)	$a(b+c) = ab+ac$.
(Identities)	$0 \neq 1$, $a+0 = a$, $a \cdot 1 = a$, and $a \cdot 0 = 0$.
(Additive inverses)	There is a unique real number $-a = (-1) \cdot a$ such that $a + (-a) = 0$.
(Subtraction)	$a-b$ is defined to be $a+(-b)$.
(Multiplicative inverses)	If $a \neq 0$, then there is a unique real number $a^{-1} = \frac{1}{a}$ such that $a \cdot a^{-1} = 1$.
(Division)	When $a \neq 0$, $\frac{b}{a}$ is defined to be $b \cdot a^{-1}$.
(No divisors of 0)	If $ab = 0$, then $a = 0$ or $b = 0$.
(Cancellation)	If $ab = ac$ and $a \neq 0$, then $b = c$.
(Transitive property of $<$)	If $a < b$ and $b < c$, then $a < c$.
(Trichotomy)	Exactly one of $a < b$, $a = b$, or $a > b$ holds.
(Order property 1)	If $a < b$, then $a + c < b + c$.
(Order property 2)	If $c > 0$, then $a < b$ iff $ac < bc$.
(Order property 3)	If $c < 0$, then $a < b$ iff $ac > bc$.

Basic properties of absolute value

For all real numbers a, b,

1. If $a \geq 0$, $|b| \leq a$ if and only if $-a \leq b \leq a$.
2. If $a \geq 0$, $|b| = a$ if and only if $b = a$ or $b = -a$.
3. $|a|^2 = a^2$.
4. $|a| = \sqrt{a^2}$.
5. $|a \cdot b| = |a||b|$.
6. If $|a| = |b|$, then $a = b$ or $a = -b$.
7. (Triangle inequality) $|a + b| \leq |a| + |b|$.
8. $|a + b| = |a| + |b|$ if and only if a and b have the same sign.
9. (Reverse triangle inequality) $||a| - |b|| \leq |a - b|$

Definition: An integer n is ***divisible*** by a non-zero integer m if there exists an integer k such that $n = km$. We write $m \mid n$ and say that m is a ***divisor*** of n.

Show that the following statement is false.

For all positive integers p, n, and m, if $p \mid nm$, then $p \mid n$ or $p \mid m$.

Proof by Counterexample

Mixed Quantifiers

For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x < y$.

There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $x < y$.

Definition: Let $f(x)$ be a function defined on an open interval containing a, except possibly at a itself.

The *limit of f at a equals L* if for all $\epsilon > 0$, there exists $\delta > 0$ such that: for all x, if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

In notation we write $\lim_{x \to a} f(x) = L$.

Example

The limit of $f(x) = 2x + 1$ at $a = 3$ is 7.

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Negating mixed quantifiers

$$\lim_{x \to 3} 2x + 1 \neq 8.$$

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Negating mixed quantifiers

In every intro to proofs course C, there exists a student S such that S can negate mixed quantifiers faster than every other student in C.

The limit of $f(x)$ at a exists.

The limit of $f(x)$ at a does not exist.