M1C03 Lecture 27

The Division Algorithm and the Well-Ordering Principle

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Announcement(s)

1. Quiz due Friday
2. Assignment 4 due Friday

Overview

Well-ordering principle.

The division algorithm.

Reference: Lakins, section 6.1.

Some simple questions

What is the largest integer q such that $2q \leq 11$? What is the remainder $11 - 2q$?

What is the largest integer q such that $3q \leq -7$? What is the remainder $-7 - 3q$?

Some simple questions

What is the smallest $0 \leq r = 11 - 2q$ where q can be any integer?

What is the smallest $0 \leq r = -7 - 3q$ where q can be any integer?

Division algorithm

Given integers a and b with $b > 0$.

The *quotient* of a by b is the largest integer q such that $bq \leq a$.

The *remainder* is $r = a - bq$. Note that $0 \leq r < b$.

By definition of q and r, $a = bq + r$.

Example: $a = 11$, $b = 2$.

Example: $a = -7$, $b = 3$.

The division algorithm

Theorem

Let $a, b \in \mathbb{Z}$ with $b > 0$. There exists a unique pair of integers q and r with $0 \leq r < b$ such that

$$a = qb + r.$$

Example

The square of any integer is equal to $3k$ or $3k + 1$, $k \in \mathbb{Z}$.

The Well-Ordering Principle: Let S be a non-empty subset of the set of non-negative integers, $\mathbb{Z}_{\geq 0}$.

Then S has a smallest element m.

The well-ordering principle

The Well-Ordering Principle: Let S be a non-empty subset of the set of non-negative integers, $\mathbb{Z}_{\geq 0}$.

Then S has a smallest element m.

Proof of the division algorithm (existence)

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Proof of the division algorithm (uniqueness)