M1C03 Lecture 22

Arbitrary unions/intersections and open/closed subsets of $\mathbb{R}$

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Announcement(s)

1. Assignment 3 due Friday.
2. Quiz 3 due Friday.
3. Test 1 solutions are posted.

Overview

Arbitrary unions and intersections.

Examples and properties.

Open and closed subsets of $\mathbb{R}$.

Reference: Lakins, 4.2, 4.3.

Finite unions and intersections

Let $n \geq 1$ and let $A_1, \ldots, A_n$ be sets. Then,

$$\bigcup_{i=1}^{n} A_i = \{x \mid \text{ exists } i \in \{1, \ldots, n\}, \text{ such that } x \in A_i\}$$

and

$$\bigcap_{i=1}^{n} A_i = \{x \mid \text{ for all } i \in \{1, \ldots, n\}, x \in A_i\}.$$

Arbitrary unions and intersections

Let J be a non-empty set and for every $j \in J$ let A_j be a set.

$$\bigcup_{j \in J} A_j = \{x \mid \text{ exists } j \in J, \text{ such that } x \in A_j\} \qquad \bigcap_{j \in J} A_j = \{x \mid \text{ for all } j \in J, x \in A_j\}$$

Example

Describe the union for different index sets $J \subset \mathbb{R}$.

$$\bigcup_{j \in J} [2j, 2j+1]$$

Example

$$\bigcup_{j=1}^{\infty} [-j, j] = \mathbb{R}$$

The Archimedean property of real numbers

For every real number $x > 0$, there exists a positive integer n such that $x < n$.

Example

$$\bigcap_{j=1}^{\infty}(-1/j, 1/j) = \{0\}$$

Example

$$\bigcup_{j=1}^{\infty}[1/j, 1-1/j] = (0,1)$$

Arbitrary unions and intersections

Theorem (Lakins, Theorem 4.3.7)

Let J be a non-empty set and let A_j, $j \in J$ be a family of sets indexed by J. Let B be a set. Then:

1. *For all $i \in J$,*
$$\bigcap_{j \in J} A_j \subseteq A_i \subseteq \bigcup_{j \in J} A_j.$$
2. *($\cup$ distributes over arbitrary $\cap$)* $B \cup \left(\bigcap_{j \in J} A_j\right) = \bigcap_{j \in J}(B \cup A_j)$.
3. *($\cap$ distributes over arbitrary $\cup$)* $B \cap \left(\bigcup_{j \in J} A_j\right) = \bigcup_{j \in J}(B \cap A_j)$.
4. *(arbitrary de Morgan 1)* $\overline{\left(\bigcup_{j \in J} A_j\right)} = \bigcap_{j \in J} \overline{A_j}$.
5. *(arbitrary de Morgan 2)* $\overline{\left(\bigcap_{j \in J} A_j\right)} = \bigcup_{j \in J} \overline{A_j}$.

Arbitrary unions and intersections

$B \cap \left(\bigcup_{j \in J} A_j \right) \subseteq \bigcup_{j \in J} (B \cap A_j)$.

Open and closed sets in $\mathbb{R}$ [Lakins, Exercise 4.2.24, 4.3.11, 4.3.12]

Definition: A set $S \subseteq \mathbb{R}$ is *open* if for all $x \in S$, there exists a real number $r > 0$ such that $(x - r, x + r) \subseteq S$.

Definition: A set $S \subseteq \mathbb{R}$ is *closed* if $\overline{S} = \mathbb{R} - S$ is open.

Open and closed sets in $\mathbb{R}$ [Lakins, Exercise 4.2.24, 4.3.11, 4.3.12]