M1C03 Lecture 26

Function inverse

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Announcement(s)

1. Quiz due Friday
2. Assignment 4 due Friday

Overview

Function inverses.

Reference: Notes on functions (Avenue), Lakins Chapter 5.

Recall

A *function* consists of three things:

- a set X called the *domain*,
- a set Y called the *codomain*, and
- a *correspondence* (or rule, or formula) that assigns to every element of the domain a unique element of the codomain.

A function $f\colon X \to Y$ is:

- *injective* if for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
- *surjective* if for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$.
- *bijective* if it is surjective and injective.

The *composition of $f\colon X \to Y$ and $g\colon A \to B$* is the function

$$g \circ f\colon X \to B, \qquad g \circ f(x) = g(f(x)) \qquad \forall x \in X.$$

The *identity function* on a set X is

$$I_X\colon X \to X, \qquad I_X(x) = x \qquad \forall x \in X.$$

Function inverse

Definition: $g\colon Y \to X$ is *inverse* to $f\colon X \to Y$ if:

$$\forall x \in X, y \in Y, \quad y = f(x) \iff g(y) = x.$$

- $g\colon Y \to X$ is inverse to $f\colon X \to Y$ if and only if $f\colon X \to Y$ is inverse to $g\colon Y \to X$.
- $f\colon X \to Y$ is *invertible* if there exists $g\colon Y \to X$ inverse to $f\colon X \to Y$.
- If $f\colon X \to Y$ is invertible, then the inverse is unique.

Example

- $g\colon [0,\infty) \to [0,\infty)$, $g(y) = y^2$, is the inverse of $f\colon [0,\infty) \to [0,\infty)$, $f(x) = \sqrt{x}$

- $g\colon \mathbb{R} \to [0,\infty)$ is NOT the inverse of $f\colon [0,\infty) \to [0,\infty)$

- $n\colon B_4 \to B_4$ is the inverse of itself.

Function inverse

Definition: $g\colon Y \to X$ is *inverse* to $f\colon X \to Y$ if:

$$\forall x \in X, y \in Y, \quad y = f(x) \iff g(y) = x.$$

Proposition (Lakins, Proposition 5.4.3)

Let $f\colon X \to Y$ and $g\colon Y \to X$ be functions.

Then $g\colon Y \to X$ is the inverse of $f\colon X \to Y$ if and only if

$$g \circ f = I_X \qquad \text{and} \qquad f \circ g = I_Y.$$

Example

Consider $t\colon B_4 \to B_3$ and $a\colon B_3 \to B_4$.

Theorem (Lakins, Theorem 5.4.7 (1))

Let X and Y be sets and let $f\colon X \to Y$ be a function. Then, $f\colon X \to Y$ is invertible if and only if $f\colon X \to Y$ is a bijection.

This theorem has some straightforward but very useful consequences. For instance:

- If $f\colon X \to Y$ is not injective, then $f\colon X \to Y$ is not invertible.
- If $f\colon X \to Y$ is not surjective, then $f\colon X \to Y$ is not invertible.
- If $f\colon X \to Y$ is invertible, then $f\colon X \to Y$ is injective and surjective.
- On the other hand, if we know $f\colon X \to Y$ is not invertible, then $f\colon X \to Y$ is not injective OR $f\colon X \to Y$ is not surjective.