M1C03 Lecture 4

More Triangles!

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Announcement(s)

1. Office hours: Wednesdays after lecture, same room
2. Quiz due on Crowdmark by the end of Friday

Overview

1. Geogebra for geometry (link)
2. Isosceles triangles
3. A counter-example
4. Exterior vs. interior angles

Common notions

The axioms for working with and comparing geometric figures.

[A1] Things that are equal to the same thing are equal to each other.

[A2] If equals are added to equals, the sums are equal.

[A3] If equals are subtracted from from equals, the remainders are equal.

[A4] If equals are added to unequals, the sums are unequal. The greater sum is obtained from the greater unequal.

[A5] If equals are taken from unequals, the remainders are unequal. The greater remainder is obtained from the greater unequal.

[A6] If equals are doubled, the results are equal.

[A7] If equals are halved, the results are equal.

[A8] If the whole consists of more than one part, then the whole is greater than each of its parts and is equal to the sum of its parts.

[A9] Things that can be made to coincide with one another are equal to one another.

Postulates

The axioms for constructing geometric figures with a *compass* and *straightedge*.

[P1] A straight line may be drawn from any one point to another point.

[P2] A line segment can be extended any distance beyond each endpoint.

[P3] A circle can be constructed from any given point and radius.

[P4] All right angles are equal to one another.

[P5] Playfair's Axiom (see geometry notes).

A tour of Geogebra

Definition: An *isosceles triangles* is a triangle with two equal sides.

Proposition

In an isosceles triangle, the angles opposite the two equal sides are equal.

Proof.

Label our triangle ABC so that AB equals AC. We want to show $\angle ABC$ equals $\angle ACB$.

1. Construct the line that bisects the angle $\angle BAC$. Label with D the point where this line intersects the line BC.
2. (SAS) $\triangle ABD$ and $\triangle ACD$ are congruent.
3. $\angle ABD$ equals $\angle ACD$.

$\square$

Isosceles triangles

Definition: Two lines are *perpendicular* if they meet at right angles.

Proposition

The bisector AD from the previous proof is perpendicular to the line BC.

Definition: A *median* of a triangle is a line extending from a vertex that bisects the opposite side.

Proposition

The bisector AD from the previous proof is a median of $\triangle ABC$.

Application: equilateral triangles

Proposition

The angles of an equilateral triangle are equal.

Proof.

Label our equilateral triangle $\triangle ABC$. We want to show all three angles are equal to each other, i.e. we want to show that

$$\angle ABC = \angle BCA, \quad \angle BCA = \angle CAB, \quad \angle CAB = \angle ABC.$$

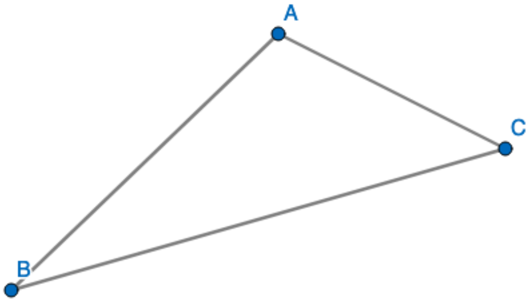
Let's start by showing that $\angle ABC = \angle BCA$.

1. Since $\triangle ABC$ is equilateral, AB equals AC.
2. Since $\triangle ABC$ is isosceles with AB equals AC, $\angle ABC = \angle BCA$.

The argument for the other two pairs of angles is the same. □

Application: non-congruent triangles

We are given a triangle $\triangle ABC$ as drawn below. Construct a triangle that has an angle equal to $\angle ABC$, a side equal to BC, and a side equal to CA, but is not congruent to $\triangle ABC$.



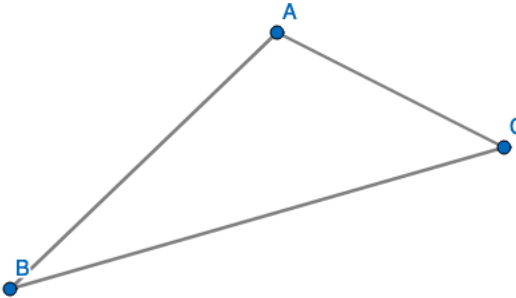
Application: non-congruent triangles

Claim

If two sides and an angle of a triangle are equal to two sides and an angle of another triangle, then the two triangles are congruent.

A *counter-example* to a claim is an example where the assumptions of the claim are all true, but the conclusion is false. If there is a counter-example to a claim, then the claim must be false. Thus, one way to show a claim is false is to find a counter-example.

We are given a triangle $\triangle ABC$ as drawn below. Construct a triangle that has an angle equal to $\angle ABC$, a side equal to BC, and a side equal to CA, but is not congruent to $\triangle ABC$.

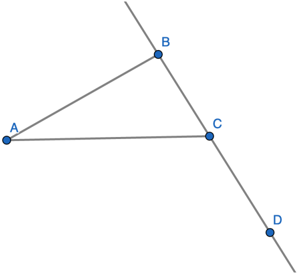


There are two situations where our construction does not work. What are they?

Proposition

An exterior angle of a triangle is greater than the two opposite interior angles.

In the following figure the exterior angle $\angle ACD$ is greater than the two interior angles $\angle BAC$ and $\angle ABC$.



Exterior vs interior angles

Proposition

An exterior angle of a triangle is greater than the two opposite interior angles.

Proof.

We show that $\angle ACD$ is greater than $\angle BAC$. First, we construct a second triangle.

1. Label the bisection of AC by E.
2. Draw BE and extend to BF so that $BE = EF$.
3. Draw CF.

Next, observe that:

1. $\angle AEB$ equals $\angle FEC$ since the angles are opposite.
2. By construction, FE equals EB and AE equals EC.
3. By SAS, $\triangle AEB$ is congruent to $\triangle FEC$.

By (A8), $\angle ACD > \angle ECF$. Since the triangles are congruent, $\angle ECF = \angle BAC$. Thus,

$$\angle ACD > \angle BAC.$$

To show that $\angle ACD$ is greater than $\angle ABC$ we do a similar construction, starting by bisecting BC instead of AC. $\square$