M1C03 Lecture 31

Introduction to Cardinality

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Announcement(s)

1. Quiz due friday
2. Assignment 5 posted

Overview

For the rest of the course we will focus on real numbers.

Something remarkable about real numbers is how many there are.

To make sense of what this means, we need the notion of *cardinality*.

Reference: selected topics from Lakins, Sections 8.1 - 8.3.

Finite cardinality (Lakins, 8.1.2)

The *cardinality* of a set X is denoted $|X|$.

$|\emptyset| = 0$.

$|X| = n$ if there exists a bijection $f\colon X \to \{1, 2, \ldots, n\}$.

X is *finite* if there exists a non-negative integer n with $|X| = n$.

Finite cardinality

Example: Let B_4 be the set of binary sequences of length 4. What is $|B_4|$?

Cardinality

$|X| = |Y|$ if there exists a bijection $f\colon X \to Y$.

Example: $|\mathbb{N}| = |\mathbb{Z}|$.

Example: $|(0,1)| = |\mathbb{R}|$.

Example: $|\mathbb{N}| = |\mathbb{Q}|$.

The pigeonhole principle

Theorem

Let X and Y be finite sets with $|X| = m$ and $|Y| = n$ and let $f\colon X \to Y$ be a function.

- *If f is injective, then $m \leq n$.*
- *If f is surjective, then $m \geq n$.*

Theorem (The pigeonhole principle)

Let X and Y be finite sets with $|X| = m$ and $|Y| = n$ and let $f\colon X \to Y$ be a function.

If $m > n$, then f is not injective.

Example (Lakins, Corollary 8.2.4): $\mathbb{N}$ is infinite.

Example: $\mathbb{R}$ is infinite.

Infinite sets

Theorem

Let X and Y be sets and let $f\colon X \to Y$ be a function.

- *If Y is finite and f is injective, then X is finite.*
- *If X is finite and f is surjective, then Y is finite.*

Cantor's second diagonalization method (Lakins, Theorem 8.3.11)

Example: $|\mathbb{N}| \neq |\mathbb{R}|$.

Countable vs. uncountable

Definition: X is *countable* if X is finite or $|X| = |\mathbb{N}|$.

Definition: X is *countably infinite* (or *denumerable*) if $|X| = |\mathbb{N}|$.

Definition: X is *uncountable* if X is not countable.

Countable vs. uncountable

Theorem

Let X and Y be sets and let $f\colon X \to Y$ be a function.

- *If Y is countable and f is injective, then X is countable.*
- *If X is countable and f is surjective, then Y is countable.*

Further curiosities

Example: $|(0,1)| = |[0,1]|$.

Example: $|\mathcal{P}(\mathbb{R})| > |\mathbb{R}|$.