M1C03 Lecture 32

Introduction to Real Numbers

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Dec 1, 2021

Announcement(s)

1. Quiz due friday
2. Assignment 5 posted

Overview

Last time we recalled that a decimal expansion defines a real number.

In fact, this is something that needs to be justified with an axiom.

The *completeness axiom*.

Reference: Lakins, Section 9.1 and 9.2.

Basic properties of real numbers

For all real numbers a, b, and c,

(Closure under $+$ and $\cdot$)	$a+b$ and ab are real numbers.
(Associativity)	$(a+b)+c = a+(b+c)$ and $(ab)c = a(bc)$.
(Commutativity)	$a+b = b+a$ and $ab = ba$.
(Distributivity)	$a(b+c) = ab+ac$.
(Identities)	$0 \neq 1$, $a+0 = a$, $a \cdot 1 = a$, and $a \cdot 0 = 0$.
(Additive inverses)	There is a unique real number $-a = (-1)\cdot a$ such that $a+(-a) = 0$.
(Subtraction)	$a-b$ is defined to be $a+(-b)$.
(Multiplicative inverses)	If $a \neq 0$, then there is a unique real number $a^{-1} = \frac{1}{a}$ such that $a \cdot a^{-1} = 1$.
(Division)	When $a \neq 0$, $\frac{b}{a}$ is defined to be $b \cdot a^{-1}$.
(Transitive property of $<$)	If $a < b$ and $b < c$, then $a < c$.
(Trichotomy)	Exactly one of $a < b$, $a = b$, or $a > b$ holds.
(Order property 1)	If $a < b$, then $a + c < b + c$.
(Order property 2)	If $c > 0$, then $a < b$ iff $ac < bc$.
(Order property 3)	If $c < 0$, then $a < b$ iff $ac > bc$.

Things the basic properties do not tell us:

1. The equation
$$x^2 = 2$$
has a solution that is a real number.

2. The infinite series
$$\sum_{k=0}^{\infty} \frac{1}{k!}$$
has a limit that is a real number.

3. The limit
$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$
exists and is a real number.

4. The infinite decimal expansion
$$2.718281828459045235360287471352662497757247093699 95\ldots$$
is a real number.

5. The equation
$$\int_1^x \frac{1}{t}\,dt = 1.$$
has a solution that is a real number.

Upper and lower bounds of sets

Let $A \subseteq \mathbb{R}$.

Definition: $u \in \mathbb{R}$ is an *upper bound* of A if for all $a \in A$, $a \leq e$.

Definition: $\ell \in \mathbb{R}$ is a *lower bound* of A if for all $a \in A$, $\ell \leq a$.

Definition: A is *bounded above* if there exists an upper bound of A.

Definition: A is *bounded below* if there exists a lower bound of A.

Definition: A is *bounded* if it is both bounded above and bounded below.

Maximum and minimum

Let $A \subseteq \mathbb{R}$.

Definition: M is a *maximum element* of A if $M \in A$ and M is an upper bound for A.

Definition: m is a *minimum element* of A if $m \in A$ and m is a lower bound of A.

Supremum and infemum

Let $A \subseteq \mathbb{R}$.

Definition: β is a *supremum (least upper bound)* of A if β is an upper bound of A and if u is an upper bound of A, then $\beta \leq u$.

Definition: β is a *infimum (greatest lower bound)* of A if α is a lower bound of A and if ℓ is a lower bound of A, then $\ell \leq \alpha$.

The completeness axiom

Let $A \subseteq \mathbb{R}$.

If A is non-empty and A is bounded above, then A has a supremum.

If A is non-empty and A is bounded below, then A has an infimum.

Example

Recall, the equation

$$x^2 = 2$$

Let

$$A = \{x \in \mathbb{R} \mid x \geq 0,\, x^2 \leq 2\}.$$

Example

Recall, the infinite decimal expansion

$$2.71828182845904523536028747135266249775724709369995\ldots$$

Let

$$A = \{2,\ 2.7,\ 2.71,\ 2.718,\ 2.7182, \ldots\}.$$

Example

Recall the infinite series

$$\sum_{k=0}^{\infty} \frac{1}{k!}$$

Let

$$A = \{1,\, 2,\, 2.5,\, 2.66667,\, 2.70833, \dots\}.$$

Example

Recall the limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}.$$

Example

Recall the equation

$$\int_1^x \frac{1}{t} dt = 1.$$

Let

$$A = \left\{ x \in \mathbb{R} \mid x > 0, \int_1^x \frac{1}{t} dt \leq 1 \right\}.$$