M1C03 Lecture 21

More set operations

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Announcement(s)

1. Assignment 3 due Friday.
2. Quiz 3 due Friday.
3. Test 1 solutions are posted.

Overview

Basic identities.

Associative and distributive properties (and non-property)

Finite intersections and unions.

Reference: Lakins, 4.2, 4.3.

Definitions Recap

Let A and B be sets.

$A \subseteq B$ if $(\forall x)(x \in A \implies x \in B)$.

$A = B$ if $(\forall x)(x \in A \Longleftrightarrow x \in B)$. Equivalently, $A \subseteq B$ and $B \subseteq A$.

The *union of A and B* is $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

The *intersection of A and B* is $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

The *complement of A in B* is $B - A = \{x \mid x \in B \text{ and } x \notin A\}$.

The *complement of A* (in the universe $\mathcal{U}$) is $\overline{A} = \mathcal{U} - A = \{x \mid x \notin A\}$.

The *Cartesian product of A and B* is $A \times B = \{(x, y) \mid x \in A \text{ and } y \in A\}$.

Example

Let $A = \{1\}$, $B = \{2\}$, $C = \{3\}$, $D = \{4\}$. Draw the following sets.

- $(A \times B) \cup (C \times D)$.
- $(A \cup C) \times (B \cup D)$.

Properties of Cartesian product

Theorem (Lakins, Theorem 4.2.9)

Let A, B, C, and D be sets. Then:

1. $A \times \emptyset = \emptyset = \emptyset \times A$.
2. *($\times$ distributes over $\cup$)* $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
3. *($\times$ distributes over $\cap$)* $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
4. *($\times$ commutes with $\cap$)* $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
5. $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

Basic identities for union, intersection, and complement

Theorem (cf. Lakins, Theorem 4.2.6)

Let A and B be subsets of some universal set $\mathcal{U}$. Then:

1. $A \cup A = A \cap A = A$.
2. $A \cup \emptyset = A$.
3. $A \cap \emptyset = \emptyset$.
4. $A \cup \overline{A} = \mathcal{U}$.
5. $A \cap \overline{A} = \emptyset$.
6. $A \cap B \subseteq A \subseteq A \cup B$.
7. $\overline{\overline{A}} = A$.

“Order of operations” properties

Theorem (cf. Lakins, Theorem 4.2.6)

Let A, B, and C be subsets of some universal set $\mathcal{U}$. Then:

1. *($\cup$ is associative)* $A \cup (B \cup C) = (A \cup B) \cup C$.
2. *($\cap$ is associative)* $A \cap (B \cap C) = (A \cap B) \cap C$.
3. *($\cap$ distributes over $\cup$)* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
4. *($\cup$ distributes over $\cap$)* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
5. *(de Morgan 1)* $\overline{A \cap B} = \overline{A} \cup \overline{B}$.
6. *(de Morgan 2)* $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Finite unions and intersections

Let $n \geq 1$ and let $A_1, \ldots, A_n$ be sets.

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n \qquad \bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

Example

Let $A_i = [2i, 2i + 1]$.

Theorem (cf. Lakins, Theorem 4.3.2)

For all positive integers $n \geq 1$, if A and $B_1, \ldots, B_n$ are sets, then:

1. *($\cap$ distributes over finite $\cup$)*
$$A \cap \left(\bigcup_{i=1}^{n} B_i \right) = \bigcup_{i=1}^{n} (A \cap B_i).$$
2. *($\cup$ distributes over finite $\cap$)*
$$A \cup \left(\bigcap_{i=1}^{n} B_i \right) = \bigcap_{i=1}^{n} (A \cup B_i).$$

Proof of 1

Theorem (cf. Lakins, Theorem 4.3.2)

For all positive integers $n \geq 2$, if $A_1, \ldots, A_n$ are subsets of a universe $\mathcal{U}$, then:

1. *(finite de Morgan 1)*
$$\overline{\left(\bigcup_{i=1}^{n} A_i\right)} = \bigcap_{i=1}^{n} \overline{A_i}.$$
2. *(finite de Morgan 2)*
$$\overline{\left(\bigcap_{i=1}^{n} B_i\right)} = \bigcup_{i=1}^{n} \overline{A_i}.$$

Finite unions and intersections

Let $n \geq 1$ and let $A_1, \ldots, A_n$ be sets. Then,

$$\bigcup_{i=1}^{n} A_i = \{x \mid \text{ exists } i \in \{1, \ldots, n\}, \text{ such that } x \in A_i\}$$

and

$$\bigcap_{i=1}^{n} A_i = \{x \mid \text{ for all } i \in \{1, \ldots, n\}, x \in A_i\}.$$