M1C03 Lecture 3

Triangles and congruence

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Before we start

1. Quiz due Friday on Crowdmark.
2. Tutorials start.

Overview

1. Euclid's postulates (compass and straightedge constructions)
2. Angles
3. Congruence
4. Triangles

Euclidean geometry

The goal of Euclidean geometry is to use *deductive reasoning* to justify *claims* about *geometric objects/figures* (lines, circles, triangles, angles, etc.).

Euclidean geometry starts with a list of assumptions. They are divided into two categories:

- *Common notions:* assumptions we agree on about how quantities work.
- *Postulates:* geometric constructions that we agree are possible.

We also need some definitions. We will introduce those as we need them.

Common notions

The axioms for working with and comparing geometric figures/magnitudes.

[A1] Things that are equal to the same thing are equal to each other.

[A2] If equals are added to equals, the sums are equal.

[A3] If equals are subtracted from from equals, the remainders are equal.

[A4] If equals are added to unequals, the sums are unequal. The greater sum is obtained from the greater unequal.

[A5] If equals are taken from unequals, the remainders are unequal. The greater remainder is obtained from the greater unequal.

[A6] If equals are doubled, the results are equal.

[A7] If equals are halved, the results are equal.

[A8] If the whole consists of more than one part, then the whole is greater than each of its parts and is equal to the sum of its parts.

[A9] Things that can be made to coincide with one another are equal to one another.

Postulates

The axioms for constructing geometric figures with a *compass* and *straightedge*.

[P1] A straight line may be drawn from any one point to another point.

[P2] A line segment can be extended any distance beyond each endpoint.

[P3] A circle can be constructed from any given point and radius.

[P4] All right angles are equal to one another.

[P5] Playfair's Axiom (see geometry notes).

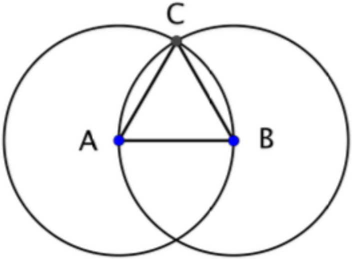
Definition: A *circle* is a plane figure with a center and a circumference such that all straight lines from the center to a point on the circumference are equal.

Constructing an equilateral triangle

Proposition

Given a straight line segment, we can construct an equilateral triangle with the line segment as one of its sides.

Let's call the two endpoints A and B. First we construct a triangle.



Constructing an equilateral triangle

Proposition

Given a straight line segment, we can construct an equilateral triangle with the line segment as one of its sides.

Proof.

Let's call the two endpoints A and B. First we construct a triangle.

1. (P3) Construct the circle with center A and radius AB.
2. (P3) Construct the circle with center B and radius AB.
3. Let C be one of the two points where the circles from 1 and 2 intersect.
4. (P1) Form the lines AC and BC. We now have a triangle ABC.

Second, we need to check ABC is equilateral.

5. Since AB and AC are both radiuses of circle 1, their lengths are equal.
6. Since AB and BC are both radiuses of circle 2, their lengths are equal.
7. (P1) Since AC and BC both equal AB, AC equals BC.

We have shown all three sides are equal to each other, so the triangle is equilateral. $\square$

How did we justify steps 3, 5, and 6?

Definition: An *angle* is a figure formed by two lines meeting at a point.

Definition: Two figures are *congruent* if one can be made to fit exactly over the other.

Definition: If two adjacent angles formed by intersecting lines are congruent, then they are *right angles*.

Congruent Triangles

Two triangles are congruent if their three sides and their three corresponding angles are all equal.

Checking 6 things to show two triangles are congruent sounds like a lot of work. There are shortcuts!

Proposition (Side-Side-Side)

If all three sides of a triangle are equal to all three sides of another triangle, then the triangles are congruent.

Proposition (Side-Angle-Side)

If two sides of a triangle, and the angle formed by those two sides, are equal to two sides of another triangle, and the angle formed by those two sides, then the triangles are congruent.

What about:

Claim

If two sides and an angle of a triangle are equal to two sides and an angle of another triangle, then the two triangles are congruent.

Claim

If all three angles of two triangles are the same, then the two triangles are congruent.

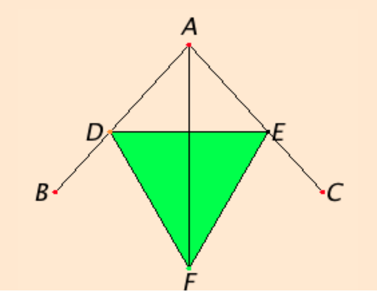
We will come back to these claims later, once we have some more tools to help us answer them.

Proposition

We can bisect any angle.

Bisecting an angle means to divide it into two smaller angles that are equal to each other.

Idea of the proof:



Bisecting angles

Proposition

We can bisect any angle.

Proof.

Let's call the angle $\angle BAC$. First we construct another line emanating from A.

1. (P3) Take a point D on AB and construct the circle with centre A and radius AD.
2. (P1) Let E be the point where AC intersects the circle and form the line segment DE.
3. Construct an equilateral triangle $\triangle DEF$.
4. (P1) Form the line AF (this is the line we wanted to construct).

Next, we show $\triangle ADF$ and $\triangle AEF$ are congruent.

5. Since $\triangle DEF$ is equilateral, DF equals EF.
6. Since AD and AE are both radii of the circle, AD equals AE.
7. The side AF is shared by both triangles.
8. Thus, $\triangle ADF$ and $\triangle AEF$ are congruent.

Finally, since $\triangle ADF$ and $\triangle AEF$ are congruent, $\angle BAF$ and $\angle CAF$ are equal. $\square$

Can you justify the remaining steps?

Bisecting lines

Proposition

We can bisect any line.

Can you do it?

Hint: use one of the constructions from earlier in this lecture.

Trisecting angles?

Is the following claim true?

Claim

We can trisect any angle.

The Greeks struggled with this question. They could not find a way to do it with only a compass and straightedge.

It was not until 1837 that someone was able to show trisection is not possible for every angle. This claim is false. The proof uses advanced math beyond the scope of this course to find a counter-example.

Nevertheless, there are some angles that can be trisected. For instance, it is possible to trisect a right angle. Can you do it?

Fun fact: one can trisect angles using origami (link). So one could say that origami is better than compass and straightedge! Sadly, Euclid did not know origami...