Graph genus and surfaces, part II

MATH 3V03

October 10, 2021

Abstract

The following is the second in a series of notes on graph genus. These were written while I was teaching MATH 3V03: Graph Theory at McMaster University in Fall, 2019. These notes are presented as-is and may contain errors. The textbook referred to in these notes is Pearls in Graph Theory by Ringel and Hartsfield (henceforth "Pearls"). These notes were written because I wanted to present the topological aspect of graph genus in more detail than in Pearls.

In part II we introduce connected surfaces, the construction of surfaces by gluing polygons along their edges, and the definition of closed surfaces in terms of the gluing construction.

1 A crash course on surfaces

We introduce surfaces with some examples. Figure 1(a) contains pictures of various surfaces

Example 1.1. The plane $\mathbb{R}^2$ is a surface. So is the open disc

$$\mathbb{D} = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$$

and the closed disc

$$\overline{\mathbb{D}} = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}.$$

Another example is any polygon, considered with or without its boundary.

Example 1.2. The 2-sphere is the set of points of distance 1 from the origin in $\mathbb{R}^3$.

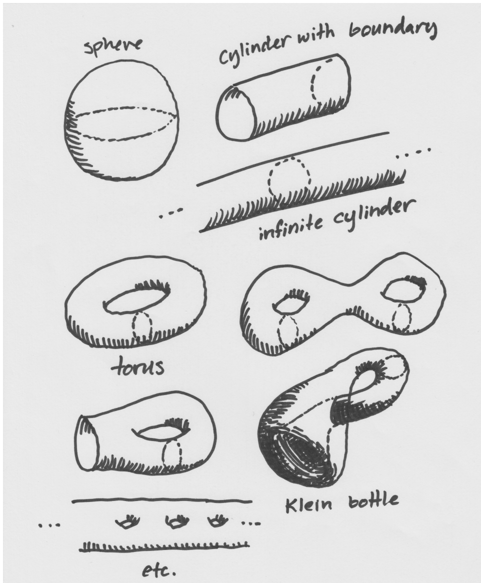
$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

Spherical or cylindrical coordinates can be used to continuously parameterize S^2 with two coordinates, so it is a surface.

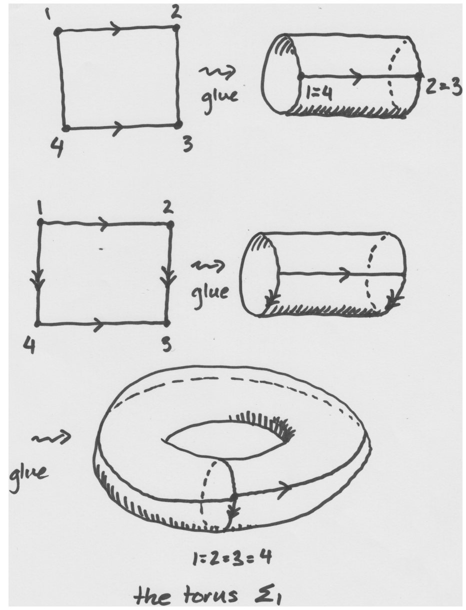
The boundary of a cube in $\mathbb{R}^3$ is also a surface. For example, the following set describes the boundary of a cube centred at 0 with edge lengths 2.

$$\{(x, y, z) \in \mathbb{R}^3 \mid \max\{|x|, |y|, |z|\} = 1\}.$$

This last example emphasizes that surfaces can have sharp corners like the graph of $|x|$. They don't need to be parameterized by smooth functions like the sphere, just continuous ones.



(a) The zoo of surfaces.



(b) The construction of the cylinder with boundary and the torus Σ_1 by gluing edges of a rectangle.

Figure 1

A surface is *connected* if it is possible to find a continuous path between any two points on the surface. For example, the 2-sphere is connected, but the surface formed by the union of the sphere of radius 1 centred at the origin, and the sphere of radius 1 centred at the point $(0, 0, 3)$ is not connected. We are only interested in connected surfaces. From this point onwards, when we say surface, we mean connected surface.

2 Closed surfaces

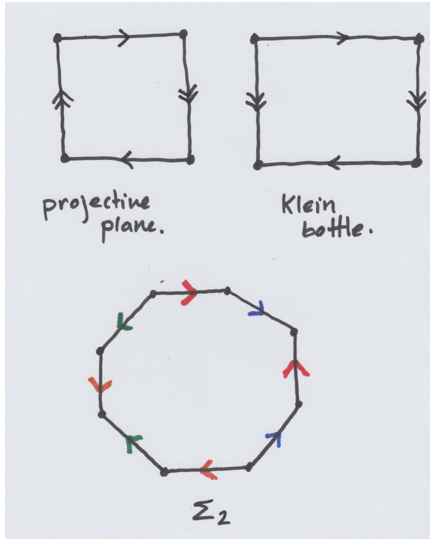
Surfaces can be constructed from polygons by gluing them together at their edges. The rules for gluing polygons:

- Each edge can only be glued to one other edge.
- Two edges, e_1 and e_2, are glued by sending the vertices of e_1 to the vertices of e_2 and identifying the points in between in a continuous and bijective manner.

In particular, it is not possible to glue together three edges, or to glue one edge onto another with "folds."

Example 2.1. Sphere constructed from two triangles viewed as upper and lower hemispheres.

Example 2.2. Tube, mobius strip, torus (Σ_1), projective plane (N_1), Klein bottle (N_2).



(a) Directions for constructing the projective plane, the Klein bottle, and the surface Σ_2 by gluing polygons.

Figure 2

Definition 2.3. A *closed surface* is a surface that can be constructed by gluing together finitely many polygons so that every edge is glued to another edge.

Example 2.4. The 2-sphere, torus, projective plane, Klein bottle are all closed surfaces. The plane, $\mathbb{R}^2$, and the open disc, $\mathbb{D}$, are not closed surfaces because one needs infinitely many polygons. The tube and mobius strip are not closed surfaces because they have boundaries: they can be constructed by gluing finitely many polygons, but some edges will inevitably remain unglued.

Example 2.5. The torus with one boundary component. Can glue together two copies to get a closed surface called Σ_2. Can also construct it by gluing edges of an octagon.

Remark 2.6. A more standard definition of closed surfaces is something like:

A *closed surface* is a compact, Hausdorff, second countable topological space that is locally homeomorphic to $\mathbb{R}^2$.

Our definition of closed surfaces is equivalent to this more standard definition by a theorem of Radó.