Graph genus and surfaces, part III

MATH 3V03

October 10, 2021

Abstract

The following is the third in a series of notes on graph genus. These were written while I was teaching MATH 3V03: Graph Theory at McMaster University in Fall, 2019. These notes are presented as-is and may contain errors. The textbook referred to in these notes is Pearls in Graph Theory by Ringel and Hartsfield (henceforth "Pearls"). These notes were written because I wanted to present the topological aspect of graph genus in more detail than in Pearls.

In part III we introduce orientations of surfaces constructed by gluing polygons (Section 1). Next, we define the Euler characteristic and genus of a closed surface in terms of a triangulation (Section 2). It becomes clear at this point that graph rotations are related to surface orientations, as we see in the proof of Corollary 2.5. Then we turn to graph embeddings in surfaces (Section 3). This is quite similar to how planar embeddings of graphs were defined earlier in the semester. We show, using stereographic projection, that a graph is planar iff it embeds into the 2-sphere (Lemma 3.2).

1 Orientations of closed surfaces

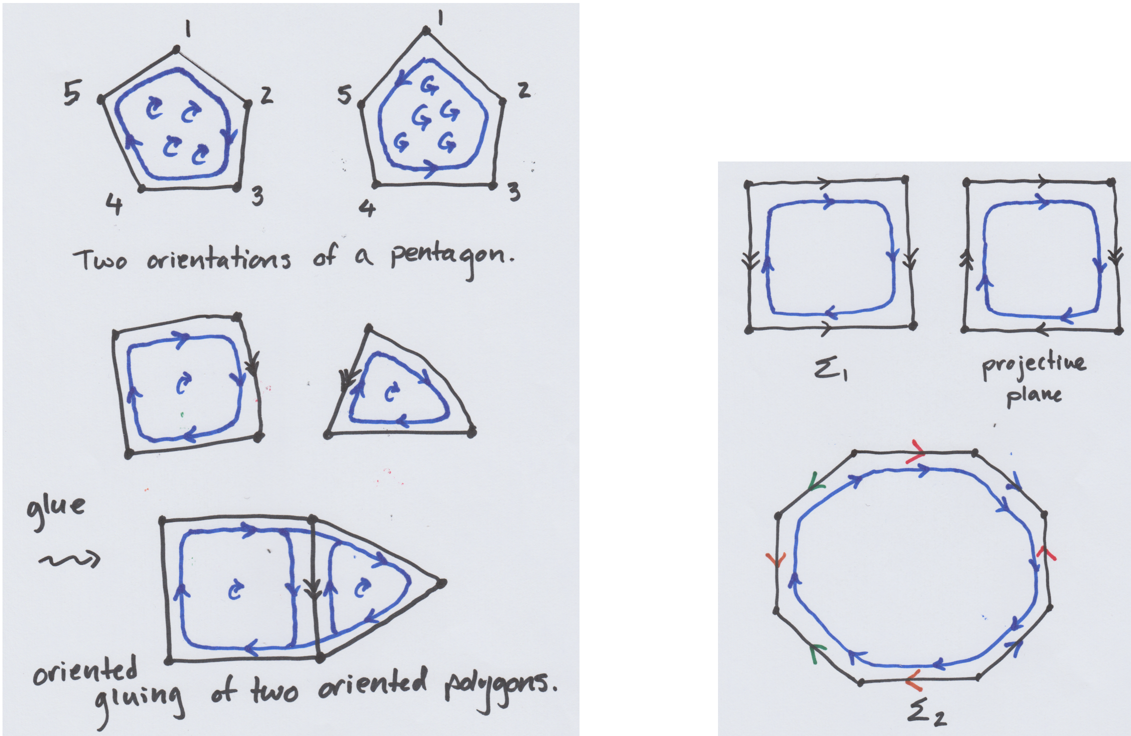
An *orientation* of a polygon is a consistent way of specifying what a "counterclockwise" rotation is at all points inside the polygon. Every polygon has two orientations.

The boundary of a polygon is a cycle. There are two ways to orient a cycle, i.e. to turn a cycle into a directed cycle. A *boundary orientation* of a polygon is a specific choice of direction of the boundary cycle.

An orientation of a polygon determines a compatible boundary orientation and vis versa, the same way as one orients the boundary of a surface in vector calculus in a compatible way before applying Stoke's theorem.

Figure 3(a) shows two orientations and boundary orientations of a pentagon that are compatible with each other.

When two polygons are glued together along a single edge, they create a new polygon. Suppose we glue together two oriented polygons. If we glue them along an edge so that the boundary orientations of the two polygons point in opposite directions, then we get a boundary orientation of the larger polygon that is consistent with the boundary orientations of the two smaller polygons. See Figure 3(a).



(a) Two different orientations of a polygon. An oriented gluing of two oriented polygons defines an orientation on the new surface.

(b) An orientation of Σ_1 and Σ_2. The projective plane cannot be oriented.

Figure 1

Definition 1.1. A gluing of oriented polygons is an *oriented gluing* if at each edge where they are glued together the boundary orientations point in opposite directions.

Example 1.2. In Figure 3(b) a boundary orientation of a square is drawn in blue. One checks that the gluings of the square used to define Σ_1 are oriented. Similarly, the gluings used to define Σ_2 are oriented. In fact, in both cases the gluings would also be oriented if we started with the opposite orientation of the square and octagon.

The gluings used to define the Mobius strip, projective plane, and Klein bottle are not oriented with respect to either choice of orientation of the square. In fact, it is impossible to construct these surfaces by oriented gluings.

Definition 1.3. A closed surface S is *oriented* if it is constructed by an oriented gluing of oriented polygons. A closed surface S is *orientable* if it can be oriented and *non-orientable* if it is impossible to orient the surface.

If S is oriented, then the orientations of the polygons combine to consistently define "counterclockwise" everywhere on the surface. For the Mobius strip, projective plane, and Klein bottle this is impossible as we saw in class. For this reason, these surfaces are non-orientable.

From the previous example we see that Σ_1 and Σ_2 are orientable. In fact they can be oriented precisely two different ways.

2 Triangulations and Euler characteristic of closed surfaces

A triangulation of a surface is a way of constructing the surface by gluing together triangles. This is the same idea as gluing polygons, except that we add several additional rules.

Definition 2.1. A *triangulation* of a surface S is a construction of S by gluing together triangles such that:

1. Edges of a triangle are not glued to each other.
2. Any two triangles are glued along at most one edge.

Every polygon can be triangulated by "barycentric subdivision."

By definition, every closed surface is constructed by gluing together polygons. Performing barycentric subdivision twice on every polygon in the construction of S results in a triangulation of S. Thus every closed surface can be triangulated.

Note that only performing subdivision once might not result in a triangulation.

Remark 2.2. If one takes the more standard definition of closed surface mentioned in part II, then the fact that every closed surface can be triangulated is a non-trivial theorem in topology due to Radó.

Suppose we have a closed surface S with a triangulation $\mathcal{T}$. Let v denote the number of vertices, e the number of edges, and t the number of triangles in $\mathcal{T}$. Define

$$X(\mathcal{T}) = v - e + t.$$

We will take for granted the following fact from topology.

Theorem 2.3 (Topological lemma)**.** *For any two triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of a closed surface S,*

$$X(\mathcal{T}_1) = X(\mathcal{T}_2).$$

This allows us to make the following definition.

Definition 2.4. The *Euler characteristic*, $X(S)$, of a closed surface S is the Euler characteristic of any triangulation of S.

The Main Theorem for Rotations implies the following fact (and thus we see the connection between rotations and orientations).

Corollary 2.5. *If S is a closed oriented surface, then $X(S)$ is even and $X(S) \leq 2$.*

Proof. Let S be an arbitrary closed, oriented surface. Take any triangulation $\mathcal{T}$ of S. Let G be the connected graph whose vertices and edges are the vertices and edges of $\mathcal{T}$.

The orientation of the surface induces a rotation ρ of the graph. The circuits induced by this rotation are the directed cycles indicated by the boundary orientations of the triangles. Thus,

$$v = |V|, \quad e = |E|, \quad t = r(\rho).$$

and we have that

$$X(S) = X(\mathcal{T}) = v - e + t = |V| - |E| + r(\rho)$$

By The Main Theorem for Rotations, this number is even and ≤ 2. $\square$

Note that the rotation in this proof was a maximal rotation since all the circuits are triangles. Thus, the Euler characteristic of a closed oriented surface S equals the Euler characteristic of the graph of any triangulation of S.

The possible values of $X(S)$ for a closed oriented surface S are

$$2, 0, -2, -4, -6, -8, \ldots$$

We can also define the *genus* of a closed oriented surface as the number

$$\gamma(S) = \frac{2 - X(S)}{2}.$$

Similar to the genus of a graph, the possibles values of $\gamma(S)$ are

$$0, 1, 2, 3, 4, 5, \ldots$$

Example 2.6.

- The tetrahedral graphs defines a triangulation of the sphere with

$$v - e + t = 2$$

- Two different triangulations of Σ_1. See Pearls, Figure 10.3.7. for one of them. The computation for that triangulation is

$$X(\Sigma_1) = v - e + t = 7 - 21 + 14 = 0.$$

- Using the "same" triangulation for N_1 as for the previous example produces different values, $X(N_1) = 1$, because the four vertices of the rectangle are not all identified. Using the triangulation from Pearls, Figure 10.3.7.,

$$X(N_1) = v - e + t = 8 - 21 + 14 = 1.$$

Example 2.7. One can compute $\gamma(\Sigma_g) = g$. We write $\Sigma_0 = S^2$ by convention. There are also non-orientable closed surfaces N_k defined for all integers $k \geq 1$.

S	orientable	$X(S)$	$\gamma(S)$
$\Sigma_0 = S^2$	Yes	2	0
Σ_1	Yes	0	1
Σ_2	Yes	-2	2
$\vdots$			
Σ_g	Yes	$2-2g$	g
$\vdots$			
N_1	No	1	
N_2	No	0	
$\vdots$			
N_k	No	$2-k$	
$\vdots$			

3 Embedding graphs in surfaces

Definition 3.1. An *embedding* of a graph G in a surface S is a simple drawing of G on S with no crossings.

Lemma 3.2. *G embeds in the 2-sphere if and only if G is planar.*

Proof. ($\Rightarrow$) Take an embedding of G in the 2-sphere, i.e. a simple drawing without crossings. Let N be a point contained in a face of the embedding. Rotate the sphere so that N is the north pole, $(0, 0, 1)$.

Stereographic projection defines a map from $S^2 \setminus N$ to the xy-plane by the formula:

$$(x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

One can check that this map is a continuous bijection onto $\mathbb{R}^2$.

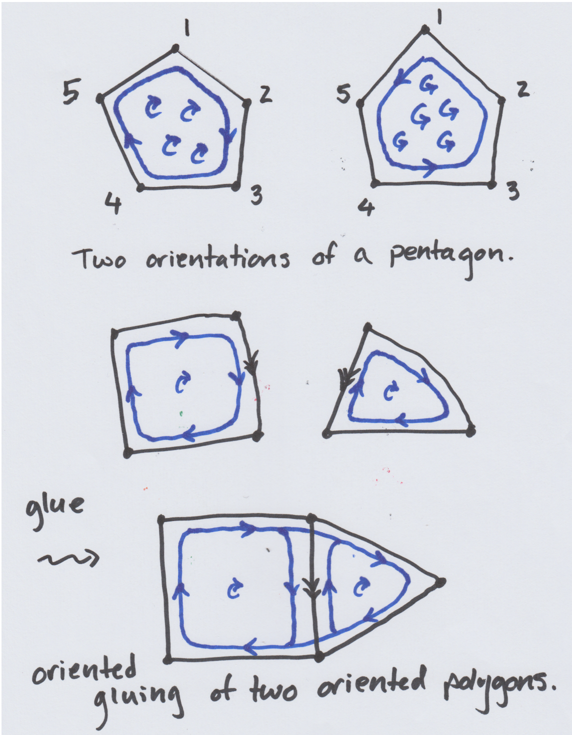
The image of our simple drawing of G on the sphere under stereographic projection is a drawing of G on the plane. Since this map is a continuous bijection, the new drawing on the plane is also a simple drawing without crossings. A simple drawing without crossings is a plane drawing, so we have produced a plane drawing of G. Since we have produced a plane drawing of G, G is planar.

($\Leftarrow$) The formula

$$(x, y) \mapsto \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2}\right)$$

defines a map $\mathbb{R}^2 \to S^2 \setminus N$. It is an exercise to check that this map is the inverse of stereographic projection, so it is a bijection onto $S^2 \setminus N$. Moreover, it is continuous since each coordinate is a continuous function of (x, y).

If G is planar, then it has a plane drawing. By the same reasoning as above, the image of the plane drawing under this map is an embedding of G in the 2-sphere. $\square$



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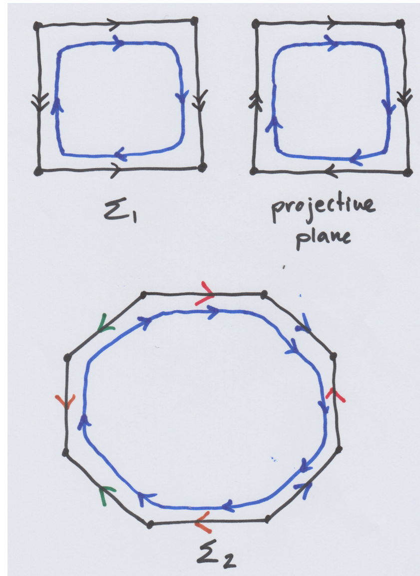

(b) An orientation of Σ_1 and Σ_2. The projective plane cannot be oriented.

Figure 2

Example 3.3.

- Embeddings of K_4 into Σ_0 and Σ_1. (See Pearls Figure 10.3.4. Remove the vertex 0 from the figure on the left.)
- Embedding of K_5 into Σ_1 constructed by adding a vertex and some edges to the embedding of K_4 from the previous example. See Pearls Figure 10.3.4. By the Lemma above, it is impossible to embed K_5 into the sphere because we know K_5 is not planar.
- Pearls Figure 10.3.4. again (now the drawing on the right): A different embedding of K_5 into Σ_1 with the property that the rotation determined by the orientation of Σ_1 is the precise rotation we saw earlier (the one that had five induced circuits). In fact, we can see the circuits induced by the rotation by looking at the embedding in Σ_1.
- An embedding of $K_{3,3}$ into Σ_1. See Pearls, Figure 10.3.5.
- An embedding of K_7 into Σ_1. See Pearls, Figure 10.3.7.