GDV II - Uebung 2

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Aufgabe 1 Bezier-Kurven und Hermite-Interpolation

a)

$$
\begin{aligned}
P_0' &= \frac{3!}{(3-1)!}\left(-\mathbf{b}_{30} + \mathbf{b}_{21}\right) && (1)\\
P_0' &= 3\left(-\mathbf{b}_{30} + \mathbf{b}_{21}\right) && (2)\\
P_0' + 3\mathbf{b}_{30} &= 3\mathbf{b}_{21} && (3)\\
1/3P_0' + \mathbf{b}_{30} &= \mathbf{b}_{21} && (4)
\end{aligned}
$$

$$
\begin{aligned}
P_1' &= \frac{3!}{(3-1)!}\left(-\mathbf{b}_{03} + \mathbf{b}_{12}\right) && (5)\\
P_1' &= 3\left(-\mathbf{b}_{03} + \mathbf{b}_{12}\right) && (6)\\
P_1' + 3\mathbf{b}_{03} &= 3\mathbf{b}_{12} && (7)\\
1/3P_1' + \mathbf{b}_{03} &= \mathbf{b}_{12} && (8)
\end{aligned}
$$

Einsetzen von $\mathbf{b}_{30}$ in (4) und $\mathbf{b}_{03}$ in (8) ergibt dann:

$$
\begin{aligned}
\mathbf{b}_{30} &= P_0 && (9)\\
\mathbf{b}_{21} &= 1/3P_0' + P_0 && (10)\\
\mathbf{b}_{12} &= 1/3P_1' + P_1 && (11)\\
\mathbf{b}_{03} &= P_1 && (12)
\end{aligned}
$$

$P(t) = B_{30}(t)P_0 + B_{21}(t)(1/3P_0' + P_0) + B_{12}(t)(1/3P_1' + P_1) + B_{03}(t)P_1$

b)

$$
\begin{aligned}
H_0^3 &= B_{30}(t) + B_{21}(t) && (13)\\
H_1^3 &= 1/3 B_{21}(t) && (14)\\
H_2^3 &= 1/3 B_{12}(t) && (15)\\
H_3^3 &= B_{03}(t) + B_{12}(t) && (16)
\end{aligned}
$$

c)

$$
\begin{aligned}
P(a) = a_0 + a_1 a + a_2 a^2 + a_3 a^3 &= \mathbf{b}_{03} && (17)\\
P^{(1)}(a) = a_1 + 2a_2 a + 3a_3 a^2 &= \frac{3}{b-a}\left(\mathbf{b}_{21} - \mathbf{b}_{30}\right) && (18)\\
P^{(2)}(a) = 2a_2 + 6a_3 a &= \frac{6}{(b-a)^2}\left(\mathbf{b}_{12} - 2\mathbf{b}_{21} + \mathbf{b}_{30}\right) && (19)\\
P^{(3)}(a) = 6a_3 &= \frac{6}{(b-a)^2}\left(\mathbf{b}_{03} - 3\mathbf{b}_{12} + 3\mathbf{b}_{21} - \mathbf{b}_{30}\right) && (20)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{b}_{30} &= a_0 + a_1 a + a_2 a^2 + a_3 a^3 && (21)\\
\mathbf{b}_{21} &= \frac{b-a}{3}(a_1 + 2a_2 a + 3a_3 a^2) + \mathbf{b}_{30} && (22)\\
\mathbf{b}_{12} &= \frac{(b-a)^2}{6}(2a_2 + 6a_3 a) + 2\mathbf{b}_{21} - \mathbf{b}_{30}\\
\mathbf{b}_{03} &= (b-a)^3 a_3 + 3\mathbf{b}_{12} - 3\mathbf{b}_{21} + \mathbf{b}_{30} && (23)
\end{aligned}
$$

Für $P(t) = 3t - \frac{9}{2}t^2 + \frac{7}{4}t^3$:

$$
\begin{aligned}
\mathbf{b}_{30} &= 0 && (24)\\
\mathbf{b}_{21} &= 1 && (25)\\
\mathbf{b}_{12} &= 3.5 && (26)\\
\mathbf{b}_{03} &= 9.25 && (27)
\end{aligned}
$$

Aufgabe 2 Approximation mit Bernstein-Polynomen

a)

Eine Bezierkurve von Grad 2 hat 3 Kontrollpunke und 3 Bernstein-Polynome:

$$
\begin{aligned}
B_{20} &= \frac{2!}{2!}\,(1-t)^2 = (1-t)^2 && (28)\\
B_{11} &= \frac{2!}{1!1!}\,(1-t)\,t = 2\,(1-t)\,t && (29)\\
B_{02} &= \frac{2!}{2!}t^2 = t^2 && (30)
\end{aligned}
$$

b)

$\mathbf{d}_k = \sum_{i+j=q} B_{ij}(t_k)\mathbf{b}_{ij}$

$$\begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\ \begin{pmatrix} 8 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 12 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} B_{20}(0) & B_{11}(0) & B_{02}(0) \\ B_{20}(1/3) & B_{11}(1/3) & B_{02}(1/3) \\ B_{20}(2/3) & B_{11}(2/3) & B_{02}(2/3) \\ B_{20}(1) & B_{11}(1) & B_{02}(1) \end{pmatrix} \begin{pmatrix} \mathbf{b}_{20} \\ \mathbf{b}_{11} \\ \mathbf{b}_{02} \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\ \begin{pmatrix} 8 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 12 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 4/9 & 4/9 & 1/9 \\ 1/9 & 4/9 & 4/9 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{b}_{20} \\ \mathbf{b}_{11} \\ \mathbf{b}_{02} \end{pmatrix}$$

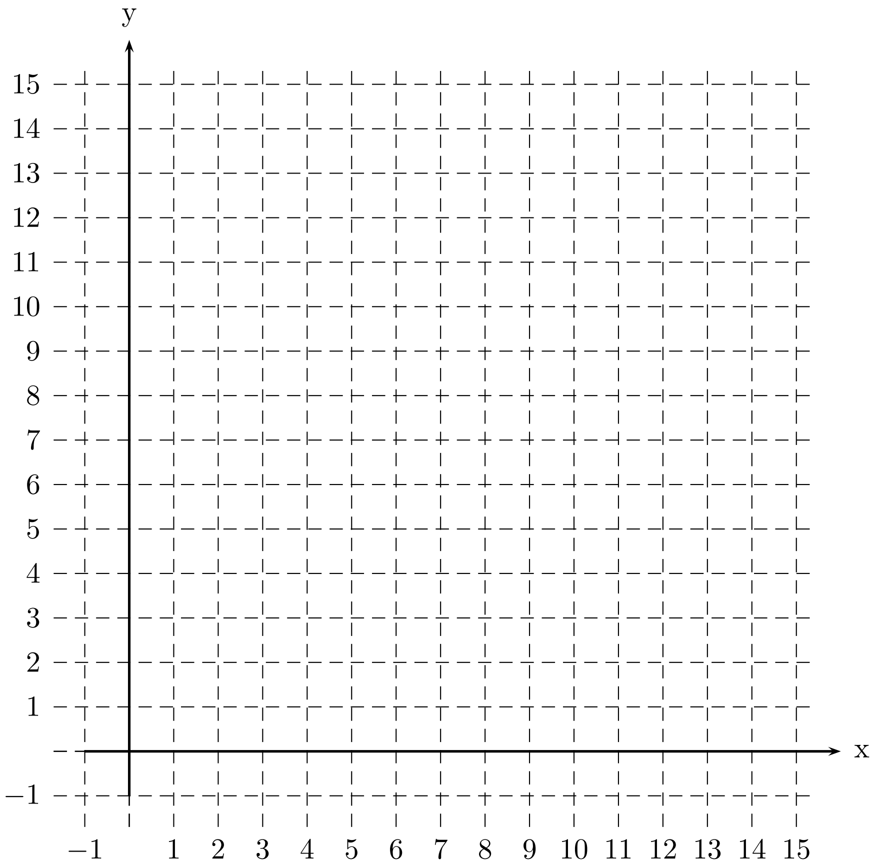
Dies ist ein überbestimmtes Gleichungssystem und es lässt sich nicht lösen, da die Zeilenvektoren alle linear unabhängig voneinander sind.

c)

$$\mathbf{B}^* = \mathbf{B}^T\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 4/9 & 4/9 & 1/9 \\ 1/9 & 4/9 & 4/9 \\ 0 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 \\ 4/9 & 4/9 & 1/9 \\ 1/9 & 4/9 & 4/9 \\ 0 & 0 & 1 \end{pmatrix} = 1/81 \begin{pmatrix} 98 & 20 & 8 \\ 20 & 32 & 20 \\ 8 & 20 & 98 \end{pmatrix}$$

$$\mathbf{d}^* = B^T\mathbf{d} = \begin{pmatrix} \begin{pmatrix} 8/3 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 16/3 \\ 8/3 \end{pmatrix} \\ \begin{pmatrix} 16 \\ 4/3 \end{pmatrix} \end{pmatrix}$$

$$b = (\mathbf{B}*)^{-1}\,\mathbf{d}^* = \begin{pmatrix} \begin{pmatrix} 0 \\ 0.3 \end{pmatrix} \\ \begin{pmatrix} 6 \\ 6.75 \end{pmatrix} \\ \begin{pmatrix} 12 \\ -0.3 \end{pmatrix} \end{pmatrix}$$



Aufgabe 3 B-Splines

a)

Anhand des Dreiecksschemas sieht man schnell, dass nur $B^3_{-2}(t), B^2_{-2}(t), B^2_{-1}(t), B^1_{-1}(t)$, $B^1_0(t), B^0_0(t), B^0_1(t)$ benoetig werden.

$$B^0_0(t) = 1\ ;\ t \in [0,1] \tag{31}$$

$$B^0_1(t) = 1\ ;\ t \in [1,2] \tag{32}$$

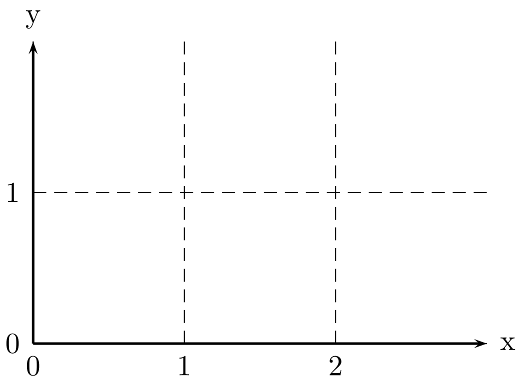
$$B^1_{-1}(t) = 1 - t\ ;\ t \in [0,1] \tag{33}$$

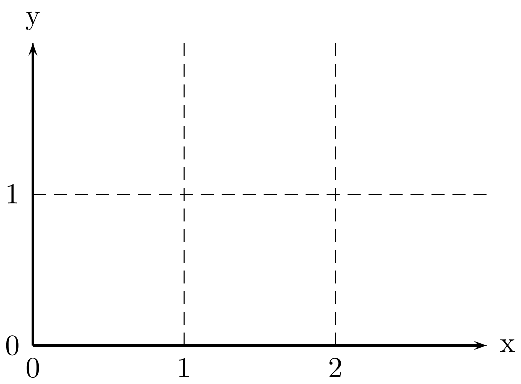
$$B^1_0(t) = \begin{cases} t\ ;\ t \in [0,1] \\ 2 - t\ ;\ t \in [1,2] \end{cases} \tag{34}$$

$$B^2_{-2}(t) = (1-t)^2\ ;\ t \in [1,0] \tag{35}$$

$$B^2_{-1}(t) = \begin{cases} 2t - 1.5t^2\ ;\ t \in [0,1] \\ 2 - 2t + 0.5t^2\ ;\ t \in [1,2] \end{cases} \tag{36}$$

$$B^3_{-2}(t) = \begin{cases} 3t - 3.5t^2 - 0.25t^3\ ;\ t \in [0,1] \\ 2 - 3t + 1.5t^2 - 0.25t^3\ ;\ t \in [1,2] \end{cases} \tag{37}$$


y
1
0
x
0
1
2


y
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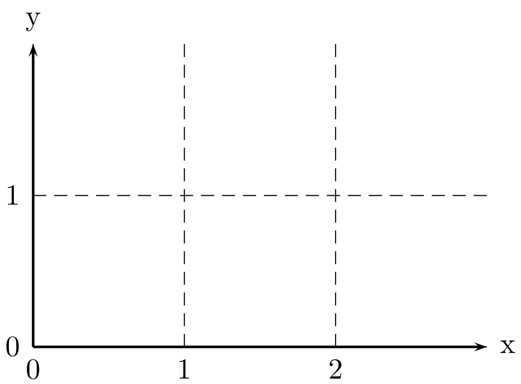
0
x
0
1
2


y
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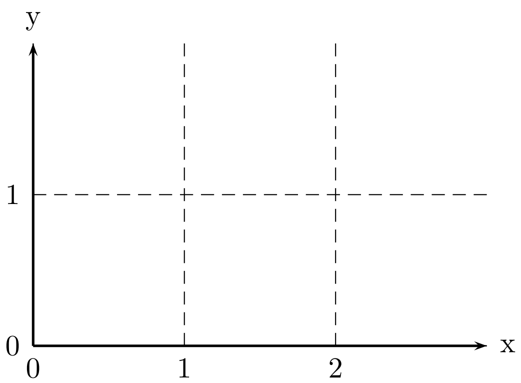
0
x
0
1
2


y
1
0
x
0
1
2

b)

$$\lambda_{0,1}^{[1]}(t) = \frac{3-t}{2} \qquad \lambda_{0,0}^{[1]}(t) = \frac{3-t}{3} \tag{38}$$

$$\lambda_{0,-1}^{[1]}(t) = \frac{2-t}{2} \qquad \lambda_{1,1}^{[1]}(t) = \frac{t-1}{2} \tag{39}$$

$$\lambda_{1,0}^{[1]}(t) = \frac{t}{3} \qquad \lambda_{1,-1}^{[1]}(t) = \frac{t}{2} \tag{40}$$

$$\lambda_{0,1}^{[2]}(t) = \frac{3-t}{2} \qquad \lambda_{0,0}^{[2]}(t) = \frac{2-t}{2} \tag{41}$$

$$\lambda_{1,1}^{[2]}(t) = \frac{t-1}{2} \qquad \lambda_{1,0}^{[2]}(t) = \frac{t}{2} \tag{42}$$

$$\lambda_{0,1}^{[3]}(t) = 2-t \qquad \lambda_{1,1}^{[3]}(t) = t-1 \tag{43}$$

$$b_{-2}^{[0]} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad b_{-1}^{[0]} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{44}$$

$$b_{0}^{[0]} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad b_{1}^{[0]} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{45}$$

$$b_{1}^{[1]} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \qquad b_{0}^{[1]} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{46}$$

$$b_{-1}^{[1]} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad b_{1}^{[2]} = \begin{pmatrix} 0.25 \\ 2.5 \end{pmatrix} \tag{47}$$

$$b_{0}^{[2]} = \begin{pmatrix} 0.25 \\ 1.5 \end{pmatrix} \qquad b_{1}^{[3]} = \begin{pmatrix} 0.25 \\ 2 \end{pmatrix} \tag{48}$$